15. TRIGONOMETRIJA

15.1 Definicija trigonometrijskih funkcija

Naj jednostavnija definicija trigonometrijskih funkcija dobije se promatranjem pravokutnog trokuta. Svaki takav trokut, za promatrani kut α, ima: prilezecu stranicu (x), suprotnu stranicu (y) i hipotenuzu (r).

$$\sin\alpha = \frac{\text{suprotna stranica}}{\text{hipotenuza}} = \frac{y}{r} \qquad \cos\alpha = \frac{\text{prilezeca stranica}}{\text{hipotenuza}} = \frac{x}{r}$$

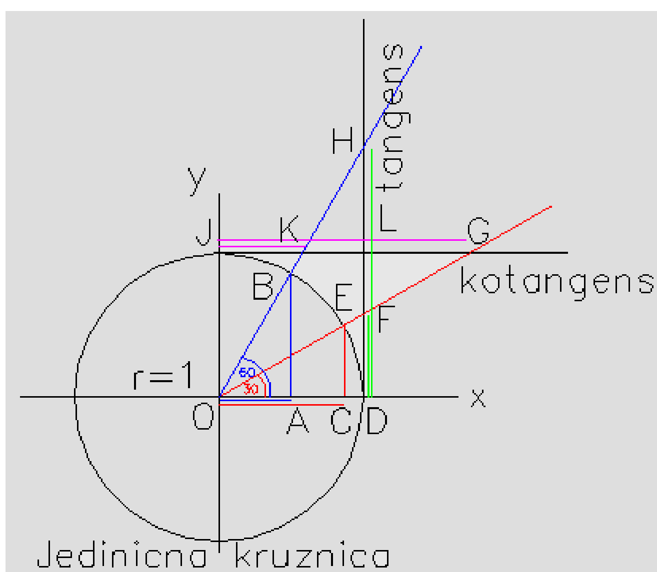
$$\tan\alpha = \frac{\text{suprotna stranica}}{\text{prilezeca stranica}} = \frac{y}{x} \qquad \cot\alpha = \frac{\text{prilezeca stranica}}{\text{suprotna stranica}} = \frac{x}{y}$$

$$\sec\alpha = \frac{\text{hipotenuza}}{\text{prilezeca stranica}} = \frac{r}{x} \qquad \operatorname{cosec}\alpha = \frac{\text{hipotenuza}}{\text{suprotna stranica}} = \frac{r}{y}$$

Na donjoj slici, prikazan je jedinicna kruznica, sa radijusom $r = 1$.
To je pomagalo pomocu kojeg se jednostavnim putem mogu prikazati vrijednosi za trigonometrijske funkcije. Promatrajmo slijedece trokute u kruznici:
Trokut $\triangle OCE$ za kut od $\alpha = 30°$ i trokut $\triangle OAB$ za kut $\alpha = 60°$

$$\sin 30° = \overline{CE} = \frac{1}{2} \qquad \cos 30° = \overline{OC} = \frac{\sqrt{3}}{2} \qquad \tan 30° = \overline{DF} = \frac{1}{\sqrt{3}} \qquad \cot 30° = \overline{JG} = \sqrt{3}$$

$$\sin 60° = \overline{AB} = \frac{\sqrt{3}}{2} \qquad \cos 60° = \overline{OA} = \frac{1}{2} \qquad \tan 60° = \overline{DH} = \sqrt{3} \qquad \cot 60° = \overline{JK} = \frac{1}{\sqrt{3}}$$



Vrijednosti za kut α=45° nije nacrtana radi preglednosti. Lako je zakljuciti, da je presjeciste tangens i kotangens pravaca u tocki L. Promatrajmo kvadrat □ODLJ. Dijagonala (nije nartana), je duzina $\overline{OL}$. Nagib dijagonale kvadrata je $\alpha = 45°$. Vrijednosti za funkcije su slijedece:

$\sin 45° = \frac{\sqrt{2}}{2}$ nije nacrtano $\cos 45° = \frac{\sqrt{2}}{2}$ $\tan 45° = \overline{DL} = 1$ $\cot 45° = \overline{JL} = 1$

U praksi se najcesce koriste funkcije : sin, cos i tan. Ostale funkcije se lako izvedu iz osnovnih.

15.2 Trigonometrijske funkcije specificnih kuteva

Promatrajmo donju jedinicnu kruznicu i odredimo pojednie trigonometrijske funkcije :

Analizirajmo kut $\varphi = \left(90^\circ + \alpha\right)$:

Iz sukladnosti trokuta △OCE i △OPT (vrijedi i △OVP) moze se definirati:

$$\overline{TP} = \overline{OC} \Rightarrow \quad \sin\varphi = \overline{TP} = \sin\left(90^\circ + \alpha\right) = \overline{OC} = \cos\alpha$$

$$\overline{CE} = -\overline{OT} \Rightarrow \; -\cos\varphi = -\overline{OT} = \cos\left(90^\circ + \alpha\right) = \overline{CE} = \sin\alpha$$

$$\overline{-DW} = \overline{JG} \Rightarrow -\tan\varphi = \overline{-DW} = \tan\left(90^\circ + \alpha\right) = \overline{JG} = \cot\alpha$$

$$\overline{DF} = -\overline{JM} \Rightarrow -\cot\varphi = -\overline{JM} = \cot\left(90^\circ + \alpha\right) = \overline{DF} = \tan\alpha$$

ili nakon sto sredimo, funkcije imaju slijedeci oblik:

$\sin\left(90^\circ + \alpha\right) = \cos\alpha$ odnosno: $\sin\left(90^\circ - \alpha\right) = \cos\alpha$

$\cos\left(90^\circ + \alpha\right) = -\sin\alpha$ $\quad \cos\left(90^\circ - \alpha\right) = \sin\alpha$

$\tan\left(90^\circ + \alpha\right) = -\cot\alpha$ $\quad \tan\left(90^\circ - \alpha\right) = \cot\alpha$

$\cot\left(90^\circ + \alpha\right) = -\tan\alpha$ $\quad \cot\left(90^\circ - \alpha\right) = \tan\alpha$

Slicnim putem dolazimo do slijedecih izraza:

$\sin\left(180^\circ \pm \alpha\right) = \mp\sin\alpha$ $\quad \sin\left(270^\circ \pm \alpha\right) = -\cos\alpha$ $\quad \sin\left(360^\circ - \alpha\right) = -\sin\alpha$

$\cos\left(180^\circ \pm \alpha\right) = -\cos\alpha$ $\quad \cos\left(270^\circ \pm \alpha\right) = \pm\sin\alpha$ $\quad \cos\left(360^\circ - \alpha\right) = \cos\alpha$

$\tan\left(180^\circ \pm \alpha\right) = \pm\tan\alpha$ $\quad \tan\left(270^\circ \pm \alpha\right) = \mp\cot\alpha$ $\quad \tan\left(360^\circ - \alpha\right) = -\tan\alpha$

$\cot\left(180^\circ \pm \alpha\right) = \pm\cot\alpha$ $\quad \cot\left(270^\circ \pm \alpha\right) = \mp\tan\alpha$ $\quad \cot\left(360^\circ - \alpha\right) = -\cot\alpha$

15.3 Funkcije slozenih kuteva, duzina luka, povrsina kruznog isjecka

1. $\sin 210^\circ = \sin\left(180^\circ + 30^\circ\right) = -\sin 30^\circ = -\frac{1}{2}$

2. $\cos 315^\circ = \cos\left(270^\circ + 45^\circ\right) = \sin 45^\circ = \frac{\sqrt{2}}{2}$

3. $\tan 110^\circ = \tan\left(180^\circ - 70^\circ\right) = -\tan 70^\circ = -2.74$

4. $30^\circ = 30^\circ \cdot \frac{\pi}{180} = \frac{\pi}{6}$ radijana se u pravilu ne pise

5. $45^\circ = 45^\circ \cdot \frac{\pi}{180} = \frac{\pi}{4}$

6. $\frac{\pi}{3} = \frac{\pi}{3} \cdot \frac{180}{\pi} = 60^\circ$

7. $\frac{3\pi}{4} = \frac{3\pi}{4} \cdot \frac{180}{\pi} = 135^\circ$

8. $\frac{5\pi}{6} = \frac{5\pi}{6} \cdot \frac{180}{\pi} = 150^\circ$

9. $\frac{7\pi}{4} = \frac{7\pi}{4} \cdot \frac{180}{\pi} = 315^\circ$

10. Nadji duzinu kruznog luka koji pripada kruznici radijusa $r = 3$ m i kuta $\alpha = \frac{\pi}{6}$

$$l = r \cdot \varphi (\text{u radijanima}) = 3\frac{\pi}{6} = \frac{\pi}{2} = 1.57\text{m}$$

11. Nadji radijus kruznice koja ima duzinu luka $l = 7.2$ cm, koji pripada kutu od $\alpha = \frac{\pi}{6}$

$$r = \frac{l}{\varphi} = \frac{7.2}{150^\circ \cdot \left(\frac{\pi}{180^\circ}\right)} = \frac{7.2}{\frac{\pi}{6}} = 2.618$$

12. Nadji pvrsinu kruznog isjecka, odredjenog kutem $\alpha = 218^\circ$ i radijusa $r = 5.25$ cm

$$P_{\sphericalangle} = \frac{1}{2} r^2 \varphi = \frac{1}{2} \cdot 5.25^2 \cdot 218^\circ \left(\frac{\pi}{180^\circ}\right) = 52.4\ cm^2$$

13. Odredi kut koji pripada kruznom isjecku povrsine $75.5\ cm^2$ i radijusa $r = 12.2\ cm$

$$P_{\sphericalangle} = \frac{1}{2} r^2 \varphi \Rightarrow \varphi = \frac{2P_{\sphericalangle}}{r^2} = \frac{2 \cdot 75.5}{12.2^2} = 1.01 \Rightarrow 1.01\frac{180^\circ}{\pi} = 57.869^\circ$$

14. Nadji duzinu centralne crte na autoputu, koji ima radijus $r = 320$ m i zatvara kut od $\alpha = 62^\circ$ $\Rightarrow$ $l = r \cdot \varphi = 320 \cdot 62^\circ \left(\frac{\pi}{180^\circ}\right) = 346\ m$

15. Nadji povrsinu koju poda koju zahvate vrata kada se otvore za $\alpha = 110^\circ$ a imaju sirinu r=75.2 cm.

$$P_{\measuredangle} = \frac{1}{2} r^2 \varphi = \frac{1}{2} \cdot 76.2^2 \cdot 110^\circ \left(\frac{\pi}{180^\circ} \right) = 5573.78\ cm^2$$

16. Plinovod duzine $l = 3.25\ km$ ima oblik kruznog luka radijusa $r = 8.5\ km$. Odredi koji kut zahvata taj luk.

$$\varphi = \frac{l}{r} = \frac{3.25}{8.5} = 0.382 \quad \Rightarrow 0.382 \left(\frac{180^\circ}{\pi} \right) = 21.91^\circ$$

17. Rotirajuci rasprsivac za zalijevanje trave zahvaca kut od $\alpha = 115^\circ$ i baca vodu na udaljenost od r=25 m. Odredi povrsinu trave koju zalijeva.

$$P_{\measuredangle} = \frac{1}{2} r^2 \varphi = \frac{1}{2} \cdot 25 \cdot 115^\circ \left(\frac{\pi}{180^\circ} \right) = 627.23\ m^2$$

18. Zeljeznicka pruga ima oblik luka, pod kutem od $\alpha = 28^\circ$.Ako je radijus unutarnjeg ruba $r = 28.55\ m$ a sirina pruge 1.44 m, nadji razmak u duzini izmedju vanjskog i unutarnjeg dijela pruge:

$$l_u = r_u \varphi = 28.55 \cdot 28^\circ \left(\frac{\pi}{180^\circ} \right) = 13.952 m$$

$$l_v = r_v \varphi = (28.55 + 1.44) \cdot 28^\circ \left(\frac{\pi}{180^\circ} \right) = 14.656 m$$

$$\triangle l = 14.656 - 13.952 = 0.704 m$$

19. Komunikacijski satelit je uvijek iznad iste tocke ekvatora, na visini od $h = 35920$ km. Ako je radijus zemlje r_z=6370 km odredi brzinu satelita.

Odnos obodne brzine(v) i kutne brzine(ω), je definirana sa

$v = \omega \cdot r \quad v[m/s], \quad \omega[rad/s]$

Odredimo kutnu brzinu zemlje: $\omega = \frac{1\ okret}{1\ dan} = \frac{2\pi}{24 sata} = 0.2618 rad/h$

Radijus na kome se krece satelit: $r = r_z + h = 6370 + 35920 = 42290 km$

Brzina satelita je: $v = \omega \cdot r = 0.2618 \cdot 42290 = 11070 km/h$

20. Automobil napravi "U" zaokret u vremenu $t = 6\ s$. Izracunaj kutnu brzinu ω automobila.

Put je jednak polovici opsega kruznice, $r\pi$: $r\pi = v \cdot t \Rightarrow v = \frac{r\pi}{t}$

Brzina iznosi: $v = \omega \cdot r \Rightarrow$ Izjednacimo: $\omega \cdot r = \frac{r\pi}{t} \Rightarrow \omega = \frac{\pi}{t} = \frac{\pi}{6} = 0.523$ rad/s

21. Dionica ceste ima oblik kruznog luka, radijusa $r = 285\ m$, i zatvara kut od $\alpha = 15.6^\circ$. Izracunaj kolicinu potrosenog asfalta, ako je sirina ceste 15.2 m a debljina asfalta $\delta = 0.305\ m$.

Povrsina isjecka iznosi: $P_{\measuredangle} = \frac{1}{2} r^2 \varphi$; Cesta ima dvije mjere: unutarnji radijus $r_u = 285\ m$ i vanjski radijus $r_v = 285 + 15.2 = 300.2\ m$

Povrsina isjecka: $P_{\measuredangle} = \frac{1}{2}\left(r_v^2 - r_u^2\right)\varphi = \frac{1}{2}\left(300.2 - 285\right)15.6\left(\frac{\pi}{180^\circ}\right) = 1210.932\ m^2$

Volumen asfalta iznosi: $V = P_{\measuredangle} \cdot \delta = 1210.932 \cdot 0.305 = 369.334\ m^3$

15.4 Trigonometrijski identiteti

Iz ranije izlozene jedinicne kruznice, mogu se izvesti slijedece identicnosti:

$$\sin^2\alpha + \cos^2\alpha = 1 \qquad \sin\alpha = \frac{1}{\csc\alpha} \qquad \cos\alpha = \frac{1}{\sec\alpha} \qquad \tan\alpha = \frac{\sin\alpha}{\cos\alpha}$$

$$\cot\alpha = \frac{\cos\alpha}{\sin\alpha} \qquad \tan\alpha \cdot \cot\alpha = 1 \qquad 1 + \tan^2\alpha = \sec^2\alpha \qquad 1 + \cot^2\alpha = \csc^2\alpha$$

Identicnosti za zbroj i razliku trigonometrijskih funkcija $\sin x$ i $\cos x$:

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \qquad \sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \qquad \cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

Identicnosti za produkt trigonometrijskih funkcija $\sin x$ i $\cos x$:

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin\left(\alpha+\beta\right) + \sin\left(\alpha-\beta\right)\right] \qquad \cos\alpha\sin\beta = \frac{1}{2}\left[\sin\left(\alpha+\beta\right) - \sin\left(\alpha-\beta\right)\right]$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos\left(\alpha+\beta\right) + \cos\left(\alpha-\beta\right)\right] \qquad \sin\alpha\sin\beta = -\frac{1}{2}\left[\cos\left(\alpha+\beta\right) - \cos\left(\alpha-\beta\right)\right]$$

Ne ulazeci u dokazivanje istinitosti, u nastavku su identiteti za gornje spomenute funkcije:

$$\sin\left(\alpha \pm \beta\right) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \qquad \cos\left(\alpha \pm \beta\right) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan\left(\alpha \pm \beta\right) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha \qquad \cos 2\alpha = \cos^2\alpha - \sin^2\alpha \qquad \tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}$$

$$\cos 2\alpha = 2\cos^2\alpha - 1 \qquad \cos 2\alpha = 1 - 2\sin^2\alpha$$

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}} \qquad \cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

$\sin\alpha \cdot \cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$ $\sin\alpha + \sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$\cos\alpha \cdot \sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta)-\sin(\alpha-\beta)\right]$ $\sin\alpha - \sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

$\cos\alpha \cdot \cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]$ $\cos\alpha + \cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$

$\sin\alpha \cdot \sin\beta = -\frac{1}{2}\left[\cos(\alpha+\beta)-\sin(\alpha-\beta)\right]$ $\cos\alpha - \cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

22. Dokazi da je: $\frac{\cos x \cdot \csc x}{\cot^2 x} = \underline{\tan x}$

$$\frac{\cos x \cdot \csc x}{\cot^2 x} = \frac{\cos x \frac{1}{\sin x}}{\frac{\cos^2 x}{\sin^2 x}} = \frac{\cos x \sin^2 x}{\sin x \cos^2 x} = \frac{\sin x}{\cos x} = \underline{\tan x}$$

23. $\frac{\sec^2\varphi}{\cot\varphi} - \tan^3\varphi = \underline{\tan\varphi}$ $\sec^2\varphi - \tan^2\varphi = 1$ po gornjoj definiciji

$$\frac{\sec^2\varphi}{\cot\varphi} - \tan^3\varphi = \frac{\sec^2\varphi}{\frac{1}{\tan\varphi}} - \tan^3\varphi = \sec^2\varphi\tan\varphi - \tan^3\varphi =$$

$$\tan\varphi\left(\sec^2\varphi - \tan^2\varphi\right) = \underline{\tan\varphi}$$

24. $\frac{1-\sin\varphi}{\sin\varphi\cot\varphi} = \underline{\frac{\cos\varphi}{1+\sin\varphi}}$

$$\frac{1-\sin\varphi}{\sin\varphi\cot\varphi} = \frac{1-\sin\varphi}{\sin\varphi\frac{\cos\varphi}{\sin\varphi}} = \frac{(1-\sin\varphi)\sin\varphi}{\sin\varphi\cos\varphi} = \frac{(1-\sin\varphi)}{\cos\varphi}\cdot\frac{(1+\sin\varphi)}{(1+\sin\varphi)} = \frac{1-\sin^2\varphi}{\cos\varphi(1+\sin\varphi)} =$$

$$= \frac{\cos^2\varphi}{\cos\varphi(1+\sin\varphi)} = \underline{\frac{\cos\varphi}{(1+\sin\varphi)}}$$

25. $\frac{\csc x}{\tan x + \cot x} = \underline{\cos x}$

$$\frac{\csc x}{\tan x + \cot x} = \frac{\frac{1}{\sin x}}{\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}} = \frac{\frac{1}{\sin x}}{\frac{\sin^2 x + \cos^2 x}{\sin x \cos x}} = \frac{\sin x \cos x}{1 \cdot \sin x} = \underline{\cos x}$$

26. $\sec^2 x \cdot \csc^2 x = \underline{\sec^2 x + \csc^2 x}$

$$\left(1+\tan^2 x\right)\csc^2 x = \csc^2 x + \csc^2 x \tan^2 x = \csc^2 x + \frac{\sin^2 x}{\cos^2 x}\cdot\frac{1}{\sin^2 x} = \csc^2 x + \frac{1}{\cos^2 x} =$$
$$= \underline{\sec^2 x + \csc^2 x}$$

27. $\sin x(\csc x - \sin x) = \underline{\cos^2 x}$

$$\sin x(\csc x - \sin x) = \sin x \csc x - \sin^2 x = \sin x \frac{1}{\sin x} - \sin^2 x = 1 - \sin^2 x = \underline{\cos^2 x}$$

28. $\sin x \tan x + \cos x = \underline{\sec x}$

$$\sin x \tan x + \cos x = \sin x \frac{\sin x}{\cos x} + \cos x = \frac{\sin^2 x + \cos^2 x}{\cos x} = \frac{1}{\cos x} = \underline{\sec x}$$

29. $\sec x \tan x \csc x = \underline{\tan^2 x + 1}$

$$\sec x \tan x \csc x = \sec x \csc x \frac{\sin x}{\cos x} = \sec x \csc x \frac{1}{\csc x} \sec x = \sec^2 x = \underline{\tan^2 x + 1}$$

30. $\tan^2 x \cos^2 x + \cot^2 x \sin^2 x = \underline{1}$

$$\tan^2 x \cos^2 x + \cot^2 x \sin^2 x = \frac{\sin^2 x}{\cos^2 x}\cos^2 x + \frac{\cos^2 x}{\sin^2 x}\sin^2 x = \sin^2 x + \cos^2 x = \underline{1}$$

31. $\dfrac{\sin(\alpha - \beta)}{\sin\alpha \sin\beta} = \underline{\cot\alpha - \cot\beta}$

$$\frac{\sin(\alpha-\beta)}{\sin\alpha\sin\beta} = \frac{\sin\alpha\cos\beta - \cos\beta\sin\alpha}{\sin\alpha\sin\beta} = \frac{\sin\alpha\cos\beta}{\sin\alpha\sin\beta} - \frac{\cos\beta\sin\alpha}{\sin\alpha\sin\beta} = \underline{\cot\beta - \cot\beta}$$

32. $\sin\left(\frac{\pi}{4}+x\right)\cos\left(\frac{\pi}{4}+x\right) = \underline{\frac{1}{2}\left(\cos^2 x - \sin^2 x\right)}$

$$\sin\left(\frac{\pi}{4}+x\right)\cos\left(\frac{\pi}{4}+x\right) = \left(\sin\frac{\pi}{4}\cos x + \cos\frac{\pi}{4}\sin x\right)\left(\cos\frac{\pi}{4}\cos x - \sin\frac{\pi}{4}\sin x\right) =$$
$$= \sin\frac{\pi}{4}\cos\frac{\pi}{4}\cos^2 x - \sin^2\frac{\pi}{4}\sin x\cos x + \cos^2\frac{\pi}{4}\sin x\cos x - \sin^2 x \sin\frac{\pi}{4}\cos\frac{\pi}{4} =$$
$$= \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}\cos^2 x - \left(\frac{\sqrt{2}}{2}\right)^2 \sin x\cos x + \left(\frac{\sqrt{2}}{2}\right)^2 \sin x\cos x - \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}\sin^2 x =$$
$$= \frac{1}{2}\cos^2 x - \frac{1}{2}\sin^2 x = \underline{\frac{1}{2}\left(\cos^2 x - \sin^2 x\right)}$$

33. $\frac{2}{1+\cos 2x}=\underline{\sec^2 x}$

$$\frac{2}{1+\cos 2x}=\frac{2}{1+\left(2\cos^2 x-1\right)}=\frac{2}{2+2\cos^2 x}=\frac{2}{2\cos^2 x}=\frac{1}{\cos^2 x}=\underline{\sec^2 x}$$

34. $\frac{\sin 3x}{\sin x}+\frac{\cos 3x}{\cos x}=\underline{4\cos 2x}$

$$\frac{\sin 3x}{\sin x}+\frac{\cos 3x}{\cos x}=\frac{\sin 3x\cos x+\cos 3x\sin x}{\sin x\cos x}=\frac{\sin(3x+x)}{\frac{1}{2}\sin 2x}=\frac{2\sin 4x}{\sin 2x}=$$

$$=\frac{2\left(2\sin 2x\cos 2x\right)}{\sin 2x}=\underline{4\cos 2x}$$

35. $\frac{2\left(\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}\right)}{\sin\alpha}=\underline{\sec\frac{\alpha}{2}+\csc\frac{\alpha}{2}}$

$$\frac{2\left(\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}\right)}{\sin\alpha}=\frac{2\left(\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}\right)}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}=\frac{\sin\frac{\alpha}{2}}{\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}+\frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}=\frac{1}{\cos\frac{\alpha}{2}}+\frac{1}{\sin\frac{\alpha}{2}}=$$

$$=\underline{\sec\frac{\alpha}{2}+\csc\frac{\alpha}{2}}$$

36. $\frac{\sec x}{\cos x}-\frac{\tan x}{\cot x}=\underline{1}$

$$\frac{\sec x}{\cos x}-\frac{\tan x}{\cot x}=\frac{\sec x}{\frac{1}{\sec x}}-\frac{\tan x}{\frac{1}{\tan x}}=\sec^2 x-\tan^2 x=\underline{1}\Rightarrow \text{po definiciji } \sec^2 x=1+\tan^2 x$$

37. $\frac{1-2\cos^2 x}{\sin x\cos x}=\underline{\tan x-\cot x}$

$$\frac{1-2\cos^2 x}{\sin x\cos x}=\frac{\left(\sin^2 x+\cos^2 x\right)-2\cos^2 x}{\sin x\cos x}=\frac{\sin^2 x-\cos^2 x}{\sin x\cos x}=\frac{\sin^2 x}{\sin x\cos x}-\frac{\cos^2 x}{\sin x\cos x}=$$

$$=\frac{\sin x}{\cos x}-\frac{\cos x}{\sin x}=\underline{\tan x-\cot x}$$

38. $\cos^3 x\csc^3 x\tan^3 x=\csc^2 x-\cot^2 x$

$$\cos^3 x\csc^3 x\tan^3 x = \cos^3 x\frac{1}{\sin^3 x}\frac{\sin^3 x}{\cos^3 x} = 1 \quad \Rightarrow \text{po definiciji } \csc^2 x = 1 + \cot^2 x$$

39. $\sec x + \tan x + \cot x = \underline{\dfrac{1+\sin x}{\sin x\cos x}}$

$$\sec x + \tan x + \cot x = \frac{1}{\cos x} + \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{\sin x + \sin^2 x + \cos^2 x}{\sin x\cos x} = \underline{\frac{1+\sin x}{\sin x\cos x}}$$

40. $\dfrac{\cos x + \sin x}{1+\tan x} = \underline{\cos x}$

$$\frac{\cos x + \sin x}{1+\tan x} = \frac{\cos x + \sin x}{1 + \dfrac{\sin x}{\cos x}} = \frac{\cos x + \sin x}{\dfrac{\cos x + \sin x}{\cos x}} = \underline{\cos x}$$

41. $(\tan x + \cot x)\sin x\cos x = \underline{1}$

$$(\tan x + \cot x)\sin x\cos x = \left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right)\sin x\cos x = \frac{\sin^2 x + \cos^2 x}{\sin x\cos x}\sin x\cos x =$$
$$= \sin^2 x + \cos^2 x = \underline{1}$$

42. $\dfrac{\sin^4 x - \cos^4 x}{1-\cot^4 x} = \underline{\sin^4 x}$

$$\frac{\sin^4 x - \cos^4 x}{1-\cot^4 x} = \frac{\sin^4 x - \cos^4 x}{1 - \dfrac{\cos^4 x}{\sin^4 x}} = \frac{\sin^4 x - \cos^4 x}{\dfrac{\sin^4 x - \cos^4 x}{\sin^4 x}} = \frac{\sin^4 x - \cos^4 x}{\sin^4 x - \cos^4 x}\sin^4 x = \underline{\sin^4 x}$$

43. $\sec x(\sec x - \cos x) + \dfrac{\cos x - \sin x}{\cos x} + \tan x = \underline{\sec^2 x}$

$$\sec^2 x - \sec x\cos x + \frac{\cos x - \sin x}{\cos x} + \frac{\sin x}{\cos x} = \frac{1}{\cos^2 x} - \frac{1}{\cos x}\cos x + \frac{\cos x - \sin x}{\cos x} +$$
$$+ \frac{\sin x}{\cos x} = \frac{1 - \cos^2 x + \cos^2 x + \cos x\sin x + \cos x\sin x}{\cos^2 x} = \frac{1}{\cos^2 x} = \underline{\sec^2 x}$$

44. $\cos(x+y)\cos y + \sin(x+y)\sin y$

$$\cos(x+y)\cos y + \sin(x+y)\sin y = (\cos x\cos y - \sin x\sin y)\cos y +$$
$$+(\sin x\cos y + \cos x\sin y)\sin y = \cos x\cos^2 y - \sin x\sin y\cos y + \sin x\sin y\cos y +$$
$$+\cos x\sin^2 y = \cos x\left(\cos^2 y + \sin^2 y\right) = \cos x$$

45. $\sin 3x \cos(3x-\pi) - \cos 3x \sin(3x-\pi)$

$\sin 3x \cos(3x-\pi) - \cos 3x \sin(3x-\pi) = \sin 3x(\cos 3x \cos\pi + \sin 3x \sin\pi) -$
$-\cos 3x(\sin 3x \cos\pi - \cos 3x \sin\pi) = -\sin 3x \cos 3x + \cos 3x \sin 3x = 0$

$\cos\pi = -1 \Rightarrow \quad \sin\pi = 0$

46. $\sin 122^\circ \cos 32^\circ - \cos 122^\circ \sin 32^\circ = \sin(122^\circ - 32^\circ) = \sin 90^\circ = 1$

47. $\cos 312^\circ \cos 48^\circ - \sin 312^\circ \sin 48^\circ = \cos(312^\circ + 48^\circ) = \cos 360^\circ = 1$

48. $\sin\left(\frac{\pi}{4}+x\right) = \underline{\frac{\sin x + \cos x}{\sqrt{2}}}$

$\sin\left(\frac{\pi}{4}+x\right) = \sin\frac{\pi}{4}\cos x + \cos\frac{\pi}{4}\sin x = \frac{\sqrt{2}}{2}\cos x + \frac{\sqrt{2}}{2}\sin x = \frac{\sqrt{2}}{2}(\cos x + \sin x)\frac{\sqrt{2}}{\sqrt{2}} =$
$= \underline{\frac{\cos x + \sin x}{\sqrt{2}}}$

49. $\sin(x+y)\sin(x-y) = \underline{\sin^2 x - \sin^2 y}$

$\sin(x+y)\sin(x-y) = (\sin x \cos y + \cos x \sin y)(\sin x \cos y - \cos x \sin y) =$
$= \sin^2 x \cos^2 y - \sin x \cos y \cos x \sin y + \sin x \cos y \cos x \sin y - \cos^2 x \sin^2 y =$
$= \sin^2 x \cos^2 y - \cos^2 x \sin^2 y = \sin^2 x(1-\sin^2 x) - \sin^2 y(1-\sin^2 y) =$
$= \sin^2 x - \sin^2 x \sin^2 y - \sin^2 y + \sin^2 y \sin^2 x = \underline{\sin^2 x - \sin^2 y}$

50. $\frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$

$\frac{\sin x}{1+\cos x} + \frac{1+\cos x}{\sin x} = 2\csc x$

$\frac{\sin^2 x + 1 + \cos x + \cos x + \cos^2 x}{(1+\cos x)\sin x} = \frac{2+2\cos x}{(1+\cos x)\sin x} = \frac{2(1+\cos x)}{(1+\cos x)\sin x} = \frac{2}{\sin x} = 2\csc x$

51. Izrazi $\sin 3x$ sa faktorima jednostrukog kuta x:

$\sin 3x = \sin(2x+x) = \sin 2x \cos x + \cos 2x \sin x =$

$= 2\sin x \cos x \cos x + \overbrace{(1-2\sin^2 x)}^{\cos 2x}\sin x = 2\sin x \cos^2 + (1-2\sin^2 x)\sin x =$
$= 2\sin x(1-\sin^2 x) + (1-2\sin^2 x)\sin x = 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x =$
$= 3\sin x - 4\sin^3 x$

$51-1.$ Dokazi identicnost: $1+\cos 2\alpha+\cos 4\alpha+\cos 6\alpha=\underline{4\cos\alpha\cos 2\alpha\cos 3\alpha}$

$$1+\cos 2\alpha+\cos 4\alpha+\cos 6\alpha=\underbrace{1+\cos 6\alpha}_{1+\cos 2x=2\cos^2 x}+2\cos\frac{4\alpha+2\alpha}{2}\cos\frac{4\alpha-2\alpha}{2}=$$

$$2\cos 3\alpha\cos\alpha+2\cos^2 3\alpha=2\cos 3\alpha\left(\cos 3\alpha+\cos\alpha\right)=$$

$$=2\cos 3\alpha\, 2\cos\frac{3\alpha+\alpha}{2}\cos\frac{3\alpha-\alpha}{2}=\underline{4\cos 3\alpha\cos 2\alpha\cos\alpha}$$

$51-2.$ Dokazi identicnost: $\dfrac{\sin 4x+\sin 2x}{\cos 4x+\cos 2x}=\tan 3x$

$$\frac{\sin 4x+\sin 2x}{\cos 4x+\cos 2x}=\frac{2\sin\frac{4x+2x}{2}\cos\frac{4x-2x}{2}}{2\cos\frac{4x+2x}{2}\cos\frac{4x-2x}{2}}=\frac{2\sin 3x\cos 2x}{2\cos 3x\cos 2x}=\frac{\sin 3x}{\cos 3x}=\underline{\tan 3x}$$

$51-3.$ Dokazi identicnost: $\cos^3 x\sin^2 x=\dfrac{1}{16}\left(2\cos x-\cos 3x-\cos 5x\right)$

$$\cos^3 x\sin^2 x=\underbrace{\left(\sin x\cos x\right)^2}_{\sin 2x=2\sin x\cos x}\cos x=\left(\frac{1}{2}\sin 2x\right)^2\cos x=\frac{1}{4}\sin^2 2x\cos x=$$

$$\frac{1}{4}\sin 2x\left(\sin 2x\cos x\right)=\qquad\frac{1}{4}\sin 2x\frac{1}{2}\left(\sin 3x+\sin x\right)=\frac{1}{8}\left(\sin 2x\sin 3x+\sin 2x\sin x\right)=$$

$$=\frac{1}{8}\left\{-\frac{1}{2}\left(\cos 5x-\cos x\right)+\left[-\frac{1}{2}\left(\cos 3x-\cos x\right)\right]\right\}=$$

$$=\frac{1}{16}\left(-\cos 5x+\cos x-\cos 3x+\cos x\right)=\underline{\frac{1}{16}\left(2\cos x-\cos 3x-\cos 5x\right)}$$

$51-4.$ Dokazi identicnost: $\dfrac{\sin x-\sin y}{\sin x+\sin y}=\dfrac{\tan\frac{x-y}{2}}{\tan\frac{x+y}{2}}$

$$\frac{\sin x-\sin y}{\sin x+\sin y}=\frac{\not{2}\cos\frac{x+y}{2}\sin\frac{x-y}{2}}{\not{2}\sin\frac{x+y}{2}\cos\frac{x-y}{2}}=\cot\frac{x+y}{2}\tan\frac{x-y}{2}=\frac{\tan\frac{x-y}{2}}{\tan\frac{x+y}{2}}=\underline{\frac{\tan\frac{x-y}{2}}{\tan\frac{x+y}{2}}}$$

$51-5.$ Dokazi jednakost: $\cos 465^\circ-\cos 165^\circ=-\dfrac{\sqrt{6}}{2}$

$$\cos 465^\circ-\cos 165^\circ=2\cos\frac{465^\circ+165^\circ}{2}\cos\frac{465^\circ-165^\circ}{2}=2\cos 315^\circ\cos 150^\circ$$

$$2\cos\left(270^\circ-45^\circ\right)\cos\left(180^\circ-30^\circ\right)=2\cos 45^\circ\cos 30^\circ=2\frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2}=\underline{\frac{\sqrt{6}}{2}}$$

51 – 6. Dokazi jednakost: $\frac{\sin 75^\circ - \sin 15^\circ}{\cos 75^\circ + \cos 15^\circ} = \frac{\sqrt{3}}{3}$

$$\frac{\sin 75^\circ - \sin 15}{\cos 75^\circ + \cos 15^\circ} = \frac{2\cos\frac{75^\circ + 15^\circ}{2}\sin\frac{75^\circ - 15^\circ}{2}}{2\cos\frac{75^\circ + 15^\circ}{2}\cos\frac{75^\circ - 15^\circ}{2}} = \frac{\sin 30^\circ}{\cos 30^\circ} = \tan 30^\circ = \frac{\sqrt{3}}{3}$$

15.5 Trigonometrijske jednadzbe

Rijesiti trigonometrijsku jednadzbu podrazumijeva pronaci odgovarajuce vrijednosti funkcije za zadani interval nezavisne promijenjive, obicno zadane u radijanima.
Dobijena rjesenja imaju, opcenito gledano, beskonacno mnogo rjesenja, jer su trigonometrijske funkcije periodicne, sa slijedecom periodom: Funkcije $\sin x$ i $\cos x$ imaju periodu 2π odnosno $2k\pi;(k = 0,1,2,...n)$, funkcije $\tan x$ i $\cot x$ imaju periodu π odnosno $k\pi;(k = 0,1,2,...n)$
U nastavku, sva rjesenja trigonometrijskih jednadzbi, biti ce za interval $0 \le x < 2\pi$.

52. Rijesi jednadzbu: $2\cos\varphi - 1 = 0$

$$2\cos\varphi = 1 \Rightarrow \cos\varphi = \frac{1}{2} \Rightarrow \varphi = \frac{\pi}{3}(60^\circ), \frac{5\pi}{3}(300^\circ)$$

Rjesenje jednadzbe je: $\varphi = \frac{\pi}{3}(60^\circ), \frac{5\pi}{3}(300^\circ)$ jer zadovoljava postavljene uvjete.

53. Rijesi jednadzbu: $2\cos^2\varphi - \sin x - 1 = 0$

$$2(1-\sin^2 x) - \sin x - 1 = 0 \Rightarrow 2 - 2\sin^2 x - \sin x - 1 = 0 \quad /\times(-1)$$

$$2\sin^2 x + \sin x - 1 = 0 \Rightarrow \underline{(2\sin x - 1)(\sin x + 1) = 0}$$

$$(2\sin x - 1) = 0 \Rightarrow \sin x_1 = \frac{1}{2} \Rightarrow x_1 = \frac{\pi}{6}(30^\circ), \frac{5\pi}{6}(150^\circ)$$

$$(\sin x + 1) = 0 \Rightarrow \sin x_2 = -1 \Rightarrow x_2 = \frac{2\pi}{3}(270^\circ)$$

Rjesenje jednadzbe je: $\underline{x = \frac{\pi}{6}(30^\circ), \frac{5\pi}{6}(150^\circ), \frac{2\pi}{3}(270^\circ)}$

54. Rijesi jednadzbu: $\sec^2 x + 2\tan x - 6 = 0$

$$1 + \tan^2 x + 2\tan x - 6 = 0 \Rightarrow \tan^2 x + 2\tan x - 5 = 0$$

$$\tan x_{1,2} = \frac{-2 \pm \sqrt{2^2 - 4\cdot 1\cdot(-4)}}{2} = \frac{-2 \pm \sqrt{24}}{2} = -1 \pm 2.4495$$

$\tan x_1 = 1.4495 \Rightarrow x_1 = 0.9669\left(55.39^\circ\right), 4.108\left(235.39^\circ\right)$

$\tan x_2 = -3.4495 \Rightarrow x_2 = 1.853\left(-73.83^\circ\right), 4.995\left(106.161^\circ\right)$

Rjesenje jednadzbe je:

$\underline{x = 0.9669\left(55.39^\circ\right), 4.108\left(235.39^\circ\right), 1.853\left(-73.83^\circ\right), 4.995\left(106.161^\circ\right)}$

55. Rijesi jednadzbu: $7\sin x - 2 = 3\left(2 - \sin x\right)$

$7\sin x - 2 - 6 + 3\sin x = 0 \Rightarrow 10\sin x - 8 = 0 \Rightarrow \sin x = 0.8$

$x = 45.837^\circ \Rightarrow$ Rjesenje jednadzbe je: $\underline{x = 45.837^\circ}$

56. Rijesi jednadzbu: $4\sin^2 x - 3 = 0$

$\sin^2 x = \frac{3}{4} \Rightarrow \sin x_{1,2} = \pm\frac{\sqrt{3}}{2}$

$\sin x_1 = \frac{\sqrt{3}}{2} = \frac{\pi}{3}\left(60^\circ\right), \frac{2\pi}{3}\left(120^\circ\right) \quad \sin x_2 = -\frac{\sqrt{3}}{2} \Rightarrow x_2 = \frac{4\pi}{3}\left(240^\circ\right), \frac{5\pi}{3}\left(300^\circ\right)$

Rjesenje jednadzbe je: $\underline{x = \frac{\pi}{3}\left(60^\circ\right), \frac{2\pi}{3}\left(120^\circ\right), \frac{4\pi}{3}\left(240^\circ\right), \frac{5\pi}{3}\left(300^\circ\right)}$

57. Rijesi jednadzbu: $\sin 4x - \cos 2x = 0$

$2\sin 2x\cos 2x - \cos 2x = 0 \Rightarrow \cos 2x\left(2\sin 2x - 1\right) = 0$

$\cos 2x_1 = 0 \Rightarrow 2x_1 = \frac{\pi}{2}, \frac{3\pi}{2} \Rightarrow x_1 = \frac{\pi}{4}\left(45^\circ\right), \frac{3\pi}{4}\left(135^\circ\right)$

$2\sin 2x_2 - 1 = 0 \Rightarrow \sin 2x_2 = \frac{1}{2} \qquad \Rightarrow 2x_2 = \frac{\pi}{6}, \frac{5\pi}{6} \Rightarrow x_2 = \frac{\pi}{12}\left(15^\circ\right), \frac{5\pi}{12}\left(75^\circ\right)$

Rjesenje jednadzbe je: $\underline{x = \frac{\pi}{3}\left(60^\circ\right), \frac{2\pi}{3}\left(120^\circ\right), \frac{4\pi}{3}\left(240^\circ\right), \frac{5\pi}{3}\left(300^\circ\right)}$

58. Rijesi jednadzbu: $\sin 2x\sin x + \cos x = 0$

$2\sin x\cos x\sin x + \cos x = 0 \Rightarrow \qquad \cos x\left(2\sin^2 x + 1\right) = 0$

$\cos x_1 = 0 \Rightarrow x_1 = \frac{\pi}{2}\left(90^\circ\right), \frac{3\pi}{2}\left(270^\circ\right)$

$2\sin^2 x + 1 = 0 \Rightarrow \sin x_{2,3} = \pm\sqrt{-\frac{1}{2}} \Rightarrow$ Nema smisla. Rjesenje je imaginarni broj.

Rjesenje jednadzbe je: $\underline{x = \frac{\pi}{2}\left(90^\circ\right), \frac{3\pi}{2}\left(270^\circ\right)}$

59. Rijesi jednadzbu: $2\cos^2 x - 2\cos 2x - 1 = 0$

$$2\cos^2 x - 2\left(\cos^2 - \sin^2 x\right) - 1 = 0 \Rightarrow 2\cos^2 x - 2\cos^2 + 2\sin^2 x - 1 = 0 \Rightarrow 2\sin^2 x - 1 = 0$$

$$\sin x_{1,2} = \pm\sqrt{\frac{1}{2}} = \pm\frac{\sqrt{2}}{2} \Rightarrow x_1 = \frac{\pi}{4}\left(45^\circ\right), \frac{3\pi}{4}\left(135^\circ\right) \qquad x_1 = \frac{5\pi}{4}\left(225^\circ\right), \frac{7\pi}{4}\left(315^\circ\right)$$

Rjesenje jednadzbe je: $\underline{x = \frac{\pi}{2}\left(90^\circ\right), \frac{3\pi}{2}\left(270^\circ\right)}$

60. Rijesi jednadzbu: $\cos 2x - 3\sin x + 1 = 0$

$$\left(1 - 2\sin^2 x\right) - 3\sin x + 1 = 0 \Rightarrow \cos 2x = 1 - 2\sin^2 x \Rightarrow \qquad 2\sin^2 x - 3\sin x - 2 = 0$$

$$\sin x = k \Leftrightarrow 2k^2 - 3k - 2 = 0 \Rightarrow k_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$k_1 = -2 \Rightarrow \sin x = -2$ nema smisla

$$\mathrm{k}_2 = \frac{1}{2} \Rightarrow \sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}\left(30^\circ\right), \frac{5\pi}{6}\left(150^\circ\right)$$

Rjesenje jednadzbe je: $\underline{x = \frac{\pi}{6}\left(30^\circ\right), \frac{5\pi}{6}\left(150^\circ\right)}$

61. Rijesi jednadzbu: $2\sin x - \csc x = 1$

$$2\sin x - \frac{1}{\sin x} - 1 = 0 \Big/ \cdot \sin x \Rightarrow \qquad 2\sin^2 x - \sin x - 1 = 0$$

$$\sin x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \begin{cases} \sin x_1 = -\frac{1}{2} \sin x_1 = -\frac{1}{2} \Rightarrow x_1 = \frac{7\pi}{6}\left(210^\circ\right), \frac{11\pi}{6}\left(330^\circ\right) \\ \sin x_2 = 1 \sin x_2 = 1 \quad \Rightarrow x_2 = \frac{\pi}{2}\left(90^\circ\right) \end{cases}$$

Rjesenje jednadzbe je: $\underline{x = \frac{7\pi}{6}\left(210^\circ\right), \frac{11\pi}{6}\left(330^\circ\right)\frac{\pi}{2}\left(90^\circ\right)}$

62. Rijesi jednadzbu: $\tan 2x + 2\sin x = 0$

$$\frac{\sin 2x}{\cos 2x} + 2\sin x = 0 \Rightarrow \frac{2\sin x\cos x}{\cos 2x} + \sin x = 0$$

$$2\sin x\left(\frac{\cos x}{\cos 2x} + 1\right) = 2\sin x\left(\frac{\cos x + \cos 2x}{\cos 2x}\right) = 0 \Rightarrow \begin{cases} \sin x_1 = 0 \Rightarrow x_1 = 0, \pi\left(180^\circ\right) \\ \sin x_2 \Rightarrow \cos x + \cos 2x = 0 \end{cases}$$

$$\cos x + 2\cos^2 x - 1 = 0 \Rightarrow \cos x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \begin{cases} \cos x_1 = -\frac{1}{2} \\ \cos x_2 = 1 \end{cases} \Rightarrow$$

$$x_{2,3} = \frac{\pi}{3}\left(60^\circ\right), \frac{5\pi}{3}\left(300^\circ\right)$$

Rjesenje jednadzbe je: $\underline{x = 0, \pi\left(180^\circ\right), \frac{\pi}{3}\left(60^\circ\right), \frac{5\pi}{3}\left(300^\circ\right)}$

15.6 Graficki prikaz trigonometrijskih funkcija

Funkcije $\mathbf{y = a\sin(dx + c)}$ $\mathbf{y = a\cos(bx + c)}$

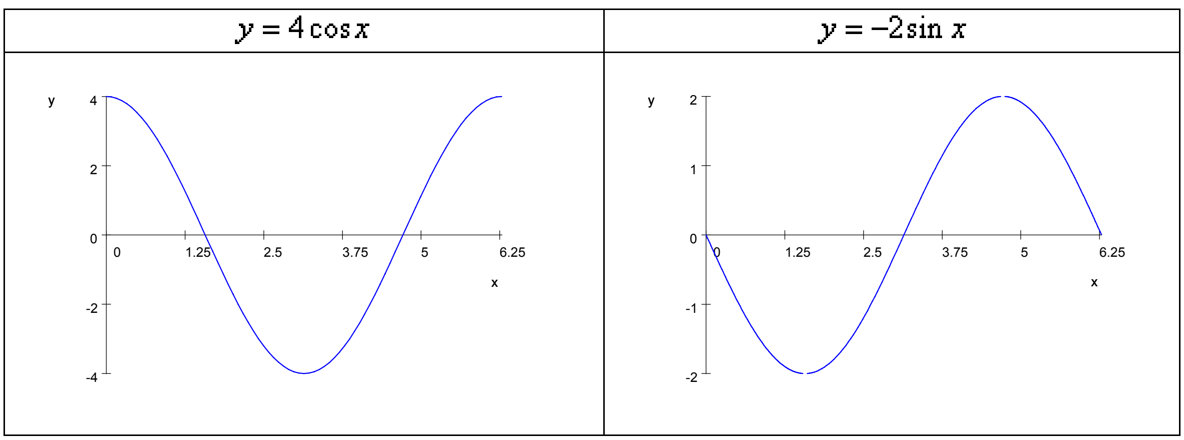
Graficki prikaz trigonometrijskih funkcija se u pravilu daje u pravokutnom koordinatnom sistemu, u radijanima, kao jedinici mjere. Na taj nacin funkcijske vrijednosti, zavisne promijenjive, poprimaju realne vrijednosti.

Svaka trigonometrijska funkcija ima osnovne karakteristike, za podrucje koje se obicno zadaje u domeni nezavisne promijenjive od $0 \le x \le 2\pi$:

AMPLITUDA $|a|$ Maksimalna vrijednost funkcije koja moze biti pozitivna ili negativna. Ta je vrijednost jednaka apsolutnoj vrijednosti $|a|$.

Promatrajmo funkcije $y = a\cos x = 4\cos x$ Amplituda funkcije iznosi $|a| = |4|$

$y = a\sin x = -2\sin x$ Amplituda funkcije iznosi $|a| = |-2|$

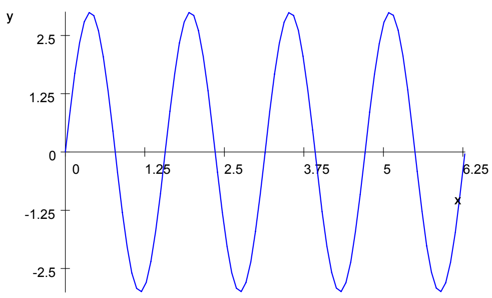


PERIODA P Perioda funkcije je definirana kao udaljenost dviju tocaka nezavisno promijenjive, kada funkcija ponavlja svoju vrijednost. Za funkcije $\sin x$ *i* $\cos x$, perioda iznosi 2π. To znaci da se vrijednosti funkcije ponavljaju svakih 2π odnosno nakon punog okretaja.

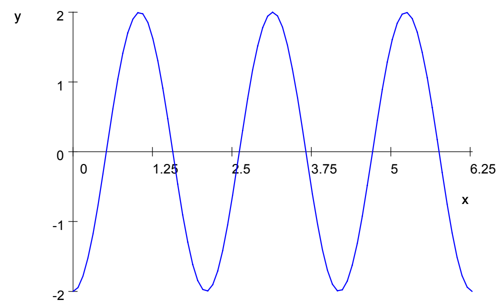
Promatranmo funkcije: $y = a\sin bx = 3\sin 4x \Rightarrow \quad |a| = |3|, \; P = \frac{2\pi}{b} = \frac{2\pi}{4} = \frac{\pi}{2}$

$$y = a\cos bx = -2\cos 3x \Rightarrow |a| = |2|, \; P = \frac{2\pi}{b} = \frac{2\pi}{3}$$

$y = 3\sin 4x$



$y = -2\cos 3x$



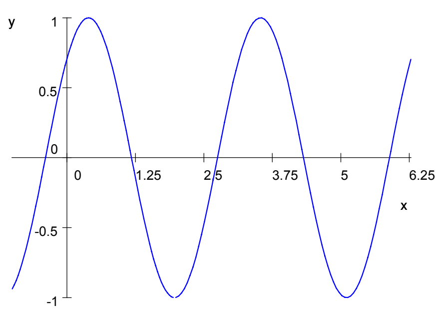
FAZNI POMAK faza Fazni pomak funkcije je definirana kao pomak pocetne tocke funkcije u odnosu na ishodiste. Taj pomak moze biti pozitivan ili negativan a izrazen je u radijanima ili stupnjevima.

Fazni pomak se racuna $faza = -\frac{c}{b}$

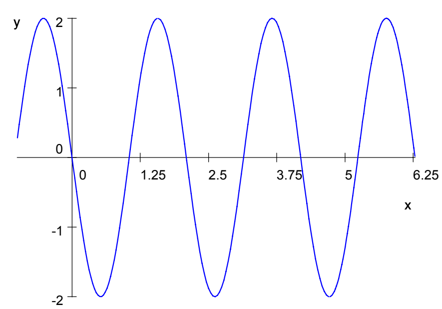
Promatrajmo funkcije: $y = a\sin(bx + c) = \sin\left(2x + \frac{\pi}{4}\right) \qquad faza = -\frac{c}{b} = -\frac{\frac{\pi}{4}}{2} = -\frac{\pi}{8}$

$y = a\sin(bx + c) = 2\sin(3x - \pi) \qquad faza = -\frac{c}{b} = -\frac{-\pi}{3} = \frac{\pi}{3}$

$y = \sin\left(2x + \frac{\pi}{4}\right)$



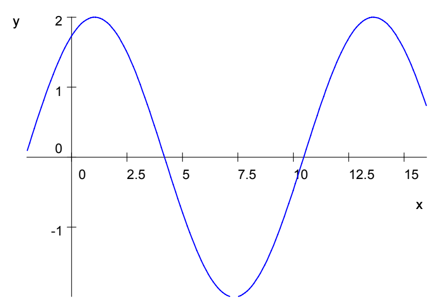
$y = -2\sin(3x - \pi)$



Za zadanu funkciju $y = a\cos(bx + c) = 2\cos\left(\frac{x}{2} - \frac{\pi}{6}\right)$ odredi amplitudu, periodu i fazni pomak.

Amplituda iznosi: $|a| = |2|$ Perioda iznosi: $P = \frac{2\pi}{b} = \frac{2\pi}{\frac{1}{2}} = 4\pi \qquad faza = -\frac{c}{b} = -\left(\frac{-\frac{\pi}{6}}{\frac{1}{2}}\right) = \frac{\pi}{3}$

$$y = 2\cos\left(\frac{x}{2} - \frac{\pi}{6}\right)$$



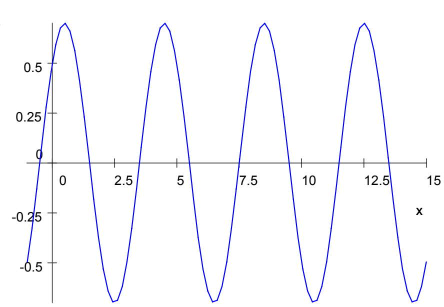
Vodeni val ima oblik funkcije $y = a\sin(bx + c) = 0.7\sin\left(\frac{\pi}{2}x + \frac{\pi}{4}\right)$

Odredi amplitudu, periodu i fazni pomak izrazenu u metrima.

Amplituda iznosi: $|a| = 0.7$ m   Perioda iznosi: $P = \frac{2\pi}{b} = \frac{2\pi}{\frac{\pi}{2}} = 4$ m

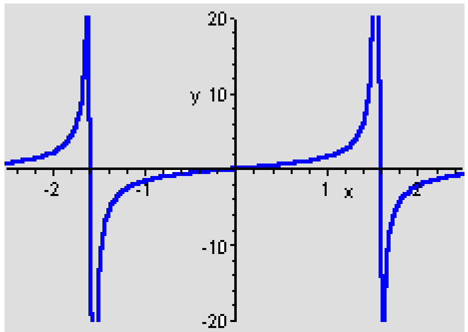
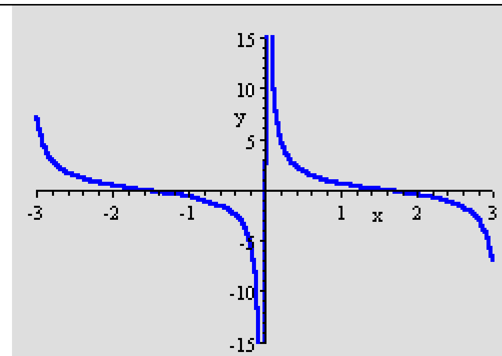
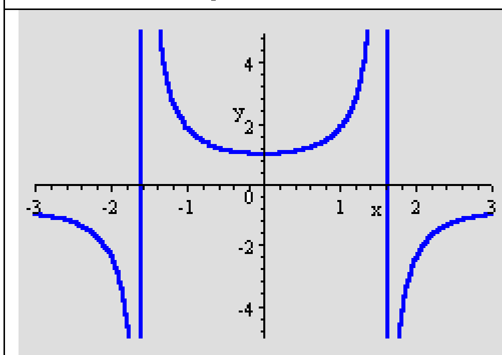
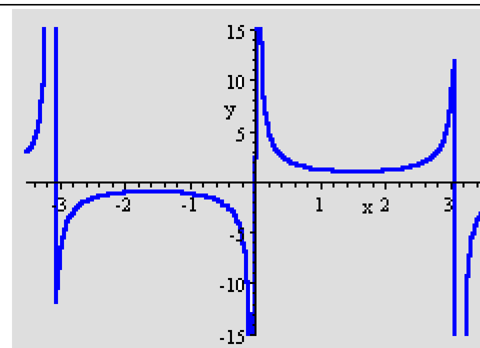
$$faza = -\frac{c}{b} = -\left(\frac{-\frac{\pi}{4}}{\frac{\pi}{2}}\right) = -\frac{1}{2} = -0.5 \text{ m}$$

$$y = 0.7\sin\left(\frac{\pi}{2}x + \frac{\pi}{4}\right)$$



Funkcije   y = tanx; y = cotx; y = secx; y = cscx

Ove trigonometrijske funkcije imaju periodu π, sto znaci da se funkcijska vrijednost ponavlja svakih pola kruga (vidi jedinicnu kruznicu).

$y = \tan x$	$y = \cot x$
	
$y = \sec x$	$y = \csc x$
	

63. Izraz $3\sin x + 4\cos x$ preuredi u oblik $a\cos(x-\varphi)$ i potom izracunaj maksimalnu i minimalnu vrijednost izraza $3\sin x + 4\cos x = a\cos(x-\varphi)$ u intervalu $0 \le x \le 2\pi$.

$a\cos(x-\varphi) = a(\cos x\cos\varphi + \sin x\sin\varphi)$

$$3\sin x + 4\cos x \Rightarrow \begin{Bmatrix} a\cos\varphi = 4 \to \cos\varphi = \dfrac{4}{a} \\ a\sin\varphi = 3 \to \sin\varphi = \dfrac{3}{a} \end{Bmatrix}$$

$$\cos^2\varphi + \sin^2\varphi = 1 \Rightarrow \left(\frac{4}{a}\right)^2 + \left(\frac{3}{a}\right)^2 = 1 \Rightarrow a^2 = 25 \Rightarrow a = \pm 5$$

$$\begin{Bmatrix} a = 5 \quad \cos\varphi = \dfrac{4}{5} \to \varphi = \cos^{-1}\dfrac{4}{5} = 0.6435 \\ a = -5 \quad \cos\varphi = -\dfrac{4}{5} \to \varphi = \cos^{-1}\left(-\dfrac{4}{5}\right) = 3.7851 \end{Bmatrix}$$

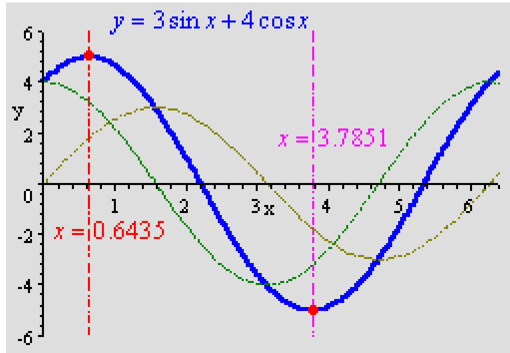
Nase jednadzbe inaju slijedeci izgled: $\begin{Bmatrix} 3\sin x + 4\cos x = 5\cos(x - 0.6435) \\ 3\sin x + 4\cos x = 5\cos(x - 3.7851) \end{Bmatrix}$

Maksimalna vrijednost:

$5\cos(x-0.6435)$ je za: $x-0.6435=0 \Rightarrow x=0.6435$ i $\max: 5\cos 0 = 5$

Minimalna vrijednost:

$5\cos(x-0.6435)$ je za: $x-0.6435=\pi \Rightarrow x=3.7851$ $\min: 5\cos\pi = -5$



64. Izraz $5\sin 3x + 12\sin 3x$ preuredi u oblik $a\cos(3x-\varphi)$

i potom izracunaj maksimalnu i minimalnu vrijednost izraza u intervalu $0 \le x \le 2\pi$.

$a\cos(3x-\varphi) = a(\cos 3x\cos\varphi + \sin 3x\sin\varphi) \equiv 5\sin 3x + 12\sin 3x$

$$\left\{\begin{matrix} a\cos\varphi = 5 \to \cos\varphi = \dfrac{5}{a} \\ a\sin\varphi = 12 \to \sin\varphi = \dfrac{12}{a} \end{matrix} \Rightarrow \cos^2\varphi + \sin^2\varphi = 1 \Rightarrow \left(\frac{5}{a}\right)^2 + \left(\frac{12}{a}\right)^2 = 1\right\}$$

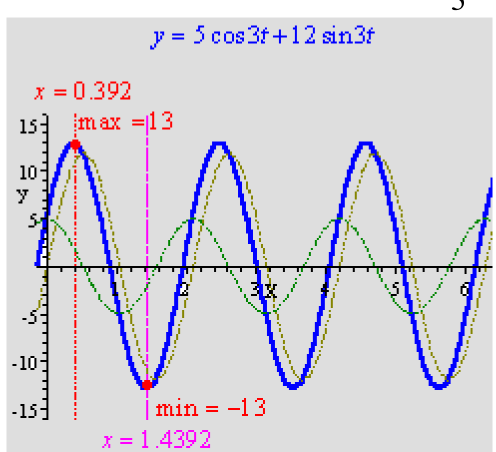
$\cos^2\varphi + \sin^2\varphi = 1 \Rightarrow 25 + 144 = a^2 \Rightarrow a = \pm 13$

$\cos\varphi = \dfrac{5}{13} \to \varphi = \cos^{-1}\dfrac{5}{13} = 1.176$

Jednadzba izgledaju ovako: $\underline{a\cos(3x-\varphi) = 13\cos(3x-1.176)}$

Max: $13\cos(3x-1.176)$ je za:$3x-1.176=1 \Rightarrow x=\dfrac{1.176}{3}=0.392$ $\max = 13$

Min: $13\cos(3x-1.176)$ je za: $3x-1.176=\pi \Rightarrow x=\dfrac{\pi+1.176}{3}=1.4392$ $\min=-13$



66. Izraz $\sin x - \cos x$ preuredi u oblik $a\sin(x-\varphi)$
i potom izracunaj maksimalnu i minimalnu vrijednost izraza
u intervalu $0 \le x \le 2\pi$.

$a\sin(x-\varphi) = a(\sin x\cos\varphi - \cos\varphi\sin x) \equiv \sin x - \cos x$

$$\left\{\begin{array}{l} a\cos\varphi = 1 \to \cos\varphi = \dfrac{1}{a} \\ a\sin\varphi = 1 \to \sin\varphi = \dfrac{1}{a} \end{array} \Rightarrow \cos^2\varphi + \sin^2\varphi = 1 \Rightarrow \left(\dfrac{1}{a}\right)^2 + \left(\dfrac{1}{a}\right)^2 = 1\right\}$$

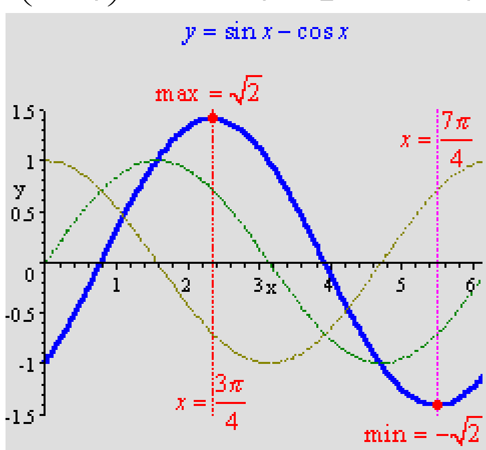
$\cos^2\varphi + \sin^2\varphi = 1 \Rightarrow a^2 \Rightarrow 2 \Rightarrow a = \pm\sqrt{2}$

$a\sin(x-\varphi) = 1 \Rightarrow \sin(x-\varphi) = \dfrac{1}{\sqrt{2}} \to \varphi = \sin^{-1}\dfrac{1}{\sqrt{2}} \Rightarrow \underline{\varphi = \dfrac{\pi}{4}}$

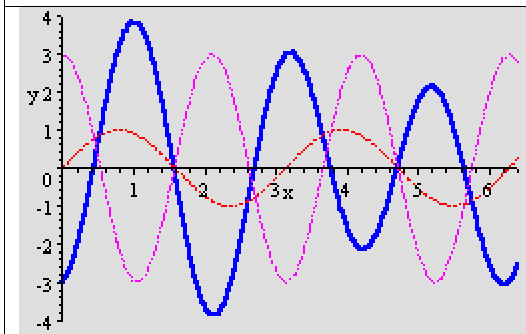
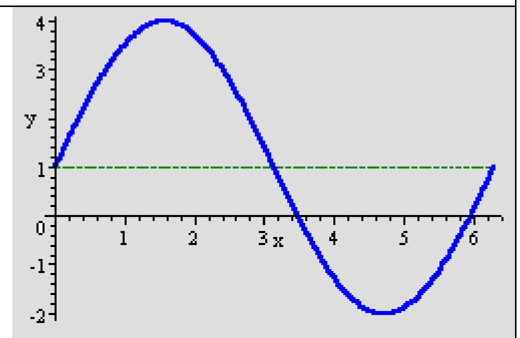
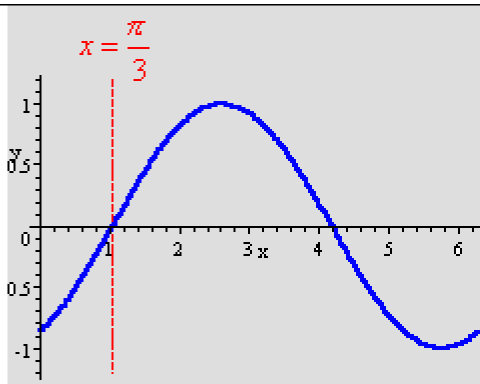
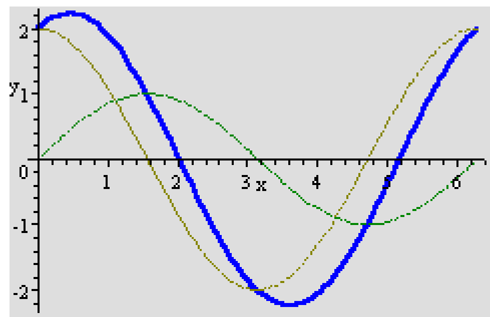
Jednadzba izgledaju ovako: $\underline{a\sin(x-\varphi) = \sqrt{2}\sin\left(x - \dfrac{\pi}{4}\right)}$

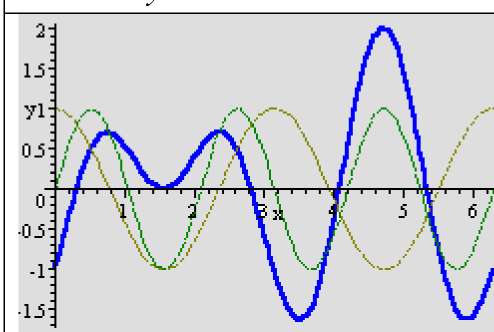
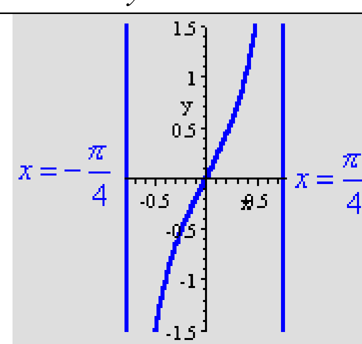
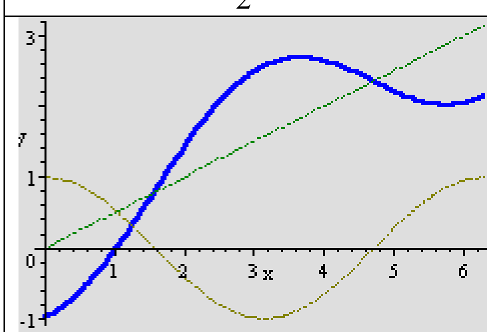
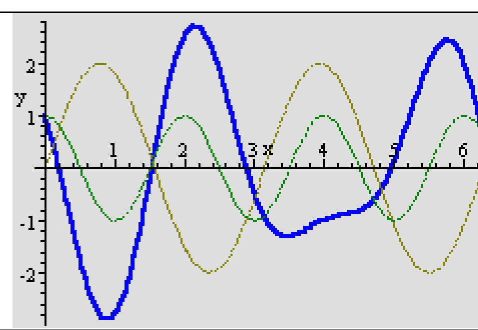
Maksimum: $\sqrt{2}\sin\left(x-\dfrac{\pi}{4}\right)$ je za: $x - \dfrac{\pi}{4} = \dfrac{\pi}{2} \Rightarrow x = \dfrac{3\pi}{4}$ max $= \sqrt{2}$

Minimum: $-\sqrt{2}\sin\left(x-\dfrac{\pi}{4}\right)$ je za: $x - \dfrac{\pi}{4} = \dfrac{3\pi}{2} \Rightarrow x = \dfrac{7\pi}{4}$ min $= -\sqrt{2}$

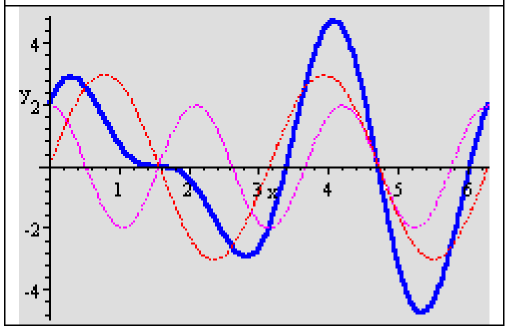


U nastavku su prikazani grafovi za razlicite kombinacije krivulja. Radi lakseg razumijevanja, svaka krivulja je prikazana drugacijom bojom. Eventualni pomak u fazi, se moze vidjeti na mjestu $y = 0$. Rezultirajuca funkcija nacrtana je plavom bojom.

$y = \sin 2x - 3\cos 3x$	$y = 3\sin x + 1$
	
$y = \sin\left(x - \frac{\pi}{3}\right)$	$y = \sin x + 2\cos x$
	

$y = \sin 3x - \cos 2x$	$y = \tan 2x$
	
$y = \frac{x}{2} - \cos x$	$y = \cos \pi x - 2\sin 2x$
	

$y = 3\sin 2x + 2\cos 3x$



15.7 Inverzne trigonometrijske funkcije

Za trigonometrijsku funkciju $y = \sin x$ kazemo da je y sinus luka koji zatvara kut x.
Inverzna funkcija toj funkciji je $x = \arcsin y$, Arcus sinus y.
To znaci, da je x luk kome je sinus jednak y.

Trigonometrijska funkcija $\quad y = \sin x$

Inverzna funkcija $\quad x = \arcsin y \Rightarrow$ obicno se pise, $y = \arcsin x$

75. $y = \arctan x \quad \Rightarrow y$ je kut ciji je tangens x
76. $y = arc\cot 3x \quad \Rightarrow y$ je kut ciji je cotangens $3x$
77. $y = 2\arcsin x \quad \Rightarrow y$ je dvaput kut ciji je sinus x
78. $\arccos\left(\frac{1}{2}\right) \quad \Rightarrow$ koji luk ima cosinus $\frac{1}{2} \Rightarrow \alpha = \frac{\pi}{3}\left(60^\circ\right)$
79. $\arcsin 0 \quad \Rightarrow$ koji luk ima sinus $0 \Rightarrow \alpha = 0\left(0^\circ\right)$
80. $\arctan\left(-\sqrt{3}\right) \quad \Rightarrow$ koji luk ima je tangens $-\sqrt{3} \Rightarrow \alpha = -\frac{\pi}{3}\left(-60^\circ\right)$
81. $\arctan\left(\frac{\sqrt{3}}{3}\right) \Rightarrow$ koji luk ima tangens $\frac{\sqrt{3}}{3} \Rightarrow \frac{\pi}{6}\left(30^\circ\right)$, jer je $\tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$
82. $\arcsin\left(-\frac{\sqrt{2}}{2}\right) \Rightarrow$ koji luk ima tangens $-\frac{\sqrt{2}}{2} \Rightarrow -\frac{\pi}{4}\left(-45^\circ\right)$, jer je $\tan\left(-\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$
83. $\text{arccsc}\sqrt{2} \Rightarrow$ koji luk ima kosecans $\sqrt{2} \quad \Rightarrow \frac{\pi}{4}\left(45^\circ\right)$, jer je $\csc = \frac{1}{\sin x} = \frac{1}{\sin\frac{\pi}{4}} = \sqrt{2}$
84. $\arcsin\left(-\frac{\sqrt{3}}{2}\right) \Rightarrow$ koji luk ima sinus $-\frac{\sqrt{3}}{2} \Rightarrow -\frac{\pi}{3}\left(-60^\circ\right)$, jer je $\sin = -\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$
85. $\cos\left[\arctan\left(-1\right)\right] \Rightarrow$ koji luk ima tangens(-1) $\Rightarrow \left(-\frac{\pi}{4}\right)\left(-45^\circ\right)$,

koliko iznosi $\cos\left(-\frac{\pi}{4}\right) \Rightarrow \frac{\sqrt{2}}{2}$

86. $\cos(2\arcsin 1) \Rightarrow$ koji luk ima je sinus $1 \Rightarrow \frac{\pi}{2}(90^\circ)$, koliko iznosi $\cos\left(\frac{\pi}{2}\right) \Rightarrow -1$

87. Rijesi zadanu jednadzbu po x:

$y = \sin 3x \Rightarrow 3x = \arcsin y \qquad x = \frac{\arcsin y}{3}$

88. $y = \arctan\left(\frac{x}{4}\right) \Rightarrow \frac{x}{4} = \arctan y \qquad x = 4\tan y$

89. $y = 1 + \sec 3x \Rightarrow \sec 3x = y - 1 \Rightarrow 3x = arc\sec(y-1) \Rightarrow x = \frac{arc\sec(y-1)}{3}$

90. $1 - y = \arccos(1-x) \Rightarrow (1-x) = \cos(1-y) \Rightarrow x = 1 - \cos(1-y)$

91. Rijesi pomocu arcus funkcije, po t: $y = A\cos 2(\omega t + \varphi)$

$\frac{y}{A} = 2\cos(\omega t + \varphi) \Rightarrow 2(\omega t + \varphi) = \arccos\frac{y}{A} \Rightarrow 2\omega t + 2\varphi = \arccos\frac{y}{A}$

$t = \frac{1}{2\omega}\arccos\frac{y}{A} - \frac{2\varphi}{2\omega} \Rightarrow \underline{t = \frac{1}{2\omega}\arccos\frac{y}{A} - \frac{\varphi}{\omega}}$

92. Rijesi po t: $i = I_{\max}\left[\sin(\omega t + \alpha)\cos\varphi + \cos(\omega t + \alpha)\sin\varphi\right]$

$i = I_{\max}\left[\sin(\omega t + \alpha + \varphi)\right] \Rightarrow \frac{i}{I_{\max}} = \sin(\omega t + \alpha + \varphi)$

$\omega t + \alpha + \varphi = \arcsin\left(\frac{i}{I_{\max}}\right) \Rightarrow \underline{t = \frac{1}{\omega}\left[\arcsin\left(\frac{i}{I_{\max}}\right) - \alpha - \varphi\right]}$

93. Rijesi na nacin koji znas: $\sec x + \tan x = \underline{\frac{\cos x}{1 - \sin x}}$

$\sec x + \tan x = \frac{1}{\cos x} + \frac{\sin x}{\cos x} = \frac{1 + \sin x}{\cos x} \cdot \left(\frac{1 - \sin x}{1 - \sin x}\right) = \frac{(1 - \sin^2 x)}{\cos x(1 - \sin x)} =$

$= \frac{\cos^2 x}{\cos x(1 - \sin x)} = \underline{\frac{\cos x}{1 - \sin x}}$

94. $3(\tan x - 2) = 1 + \tan x$

$3\tan x - 6 - 1 - \tan x = 0 \Rightarrow 2\tan x = 7 \Rightarrow \tan x = \frac{7}{2} = 3.5$

$x = 1.2925(74.055^\circ), 4.3441(254.049^\circ)$

95. $2\left(1-2\sin^2 x\right)=1$

$2-4\sin^2 x=1 \Rightarrow \sin^2 x=\frac{1}{4} \Rightarrow \sin x_{1,2}=\pm\frac{1}{2}$

$x=\frac{\pi}{6}\left(30^\circ\right),\frac{5\pi}{6}\left(150^\circ\right),\frac{7\pi}{6}\left(210^\circ\right),\frac{11\pi}{6}\left(330^\circ\right)$

96. $\cos^2 2x-1=0$

$\cos^2 2x=1 \Rightarrow \cos 2x_{1,2}=\pm 1 \Rightarrow 2x=0,\pi\left(180^\circ\right) \Rightarrow x=\frac{\pi}{2}\left(90^\circ\right),\frac{3\pi}{2}\left(270^\circ\right)$

97. $4\cos^2 x-3=0$

$\cos^2 x=\frac{3}{4} \Rightarrow \cos x=\pm\frac{\sqrt{3}}{2} \Rightarrow x=\frac{\pi}{6}\left(30^\circ\right),\frac{5\pi}{6}\left(150^\circ\right),\frac{7\pi}{6}\left(210^\circ\right),\frac{11\pi}{6}\left(330^\circ\right)$

98. $\cos 2x=\sin x$

$2\sin x\cos x=\sin x \Rightarrow \underline{\sin x\left(2\cos x-1\right)=0}$

$\sin x=0 \Rightarrow x=0,2\pi\left(360^\circ\right) \quad 2\cos x-1=0 \Rightarrow \cos x=\frac{1}{2} \Rightarrow x=\frac{\pi}{3}\left(60^\circ\right),\frac{5\pi}{3}\left(300^\circ\right)$

99. $\sin^2\frac{x}{2}-\cos x+1=0$

$\sin^2\frac{x}{2}-\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)+1=0 \Rightarrow 2\sin^2\frac{x}{2}-\cos^2\frac{x}{2}+1=0 \Rightarrow$

$\Rightarrow 2\sin^2\frac{x}{2}-\left(1-\sin^2\frac{x}{2}\right)+1=0 \Rightarrow 2\sin^2\frac{x}{2}-1+\sin^2\frac{x}{2}+1=0 \Rightarrow$

$\Rightarrow 3\sin^2\frac{x}{2}=0 \Rightarrow \frac{x}{2}=0$

100. Rijesi jednadzbu ako je $x=2\cos x$: $2\sin x=\sqrt{4-x^2}$

$2\sin x=\sqrt{4-\left(2\cos x\right)^2}=\sqrt{4-4\cos x^2}=2\sqrt{1-\cos^2 x}=\underline{2\sin x}$

101. Rijesi jednadzbu ako je $x=2\sec x$: $\sqrt{x^2-4}=2\tan x$

$\sqrt{\left(2\sec x\right)^2-4}=\sqrt{4\sec^2 x-4}=2\sqrt{\sec^2 x-1}=\underline{2\tan x}$

102. Rijesi jednadzbu ako je $x=\tan x$: $\frac{x}{\sqrt{1+x^2}}=\sin x$

$$\frac{x}{\sqrt{1+x^2}}=\frac{\tan x}{\sqrt{1+(\tan x)^2}}=\frac{\tan x}{\sqrt{1+\tan^2 x}}=\frac{\tan x}{\sec x}=\frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}}=\frac{\sin x\cos x}{\cos x}=\underline{\sin x}$$

15.8 Sinusov i kosinusov poucak

Sinusov poucak:
Omjer izmedju stranice trokuta i sinusa kuta suprotnog toj stranici,
je konstantan.

$$\frac{a}{\sin\alpha}=\frac{b}{\sin\beta}=\frac{c}{\sin\gamma}$$

Dodajmo jos slijedece: Zbroj stranica $2s=a+b+c$

Povrsina trokuta $P=\sqrt{s(s-a)(s-b)(s-c)}$

Upisana kruznica $\rho=\frac{P}{s}$

Opisana kruznica $r=\frac{P}{2\sin\alpha}=\frac{abc}{4P}$

Kosinusov poucak:
Kvadrat nad stranicom trokuta jednak je zbroju kvadrata drugih dviju stranica, umanjenog za dvostruki produkt tih dviju stranica i kosinusa kuta izmedju tih stranica.

$$a^2 = b^2+c^2-2bc\cos(\measuredangle b,c)$$

Prilikom rjesavanja zadataka, posebno treba voditi racuna o kutu izmedju stranica, kada je kut veci od $\varphi>90^o$. Tada funkcija $\cos\varphi$ mijenja vrijednost i predznak pa se dvostruki produkt zbroji kvadratima drugih dviju stranica.

103. Dva promatraca medjusobno udaljena 7,450 m, promatraju helikopter istocno od njih, pod kutem: prvi promatrac α=32°, drugi promatrac β_v=44°.
Odredi udaljenost helikoptera od prvog promatraca i visinu helikoptera.
α=32°, unutarnji kut drugog promatraca iznosi $\beta=180-\beta_v=180-44=136^\circ$
Kut na vrhu, gdje je helikopter iznosi $\gamma=180-(\alpha+\beta)=180-(32^\circ+130^\circ)=12^\circ$
Iz sinusovog poucka: $\frac{\mathrm{a}}{\sin\alpha}=\frac{\mathrm{b}}{\sin\beta}=\frac{\mathrm{c}}{\sin\gamma}\Rightarrow b=\frac{7540\sin 136^\circ}{\sin 12^\circ}=25{,}192\text{ m}$
Visina helikoptera iznosi: $h=b\sin 32^\circ=25{,}192\sin 32^\circ=13{,}192\text{ m}$

104. Rijesi trokut: $a = 45.7,\ \alpha = 65^\circ, \beta = 49^\circ$

$\gamma = 180 - (\alpha + \beta) = 180 - (65^\circ + 49^\circ) = 66^\circ$

Iz sinusovog poucka: $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \Rightarrow \frac{45.7}{\sin 65^\circ} = \frac{b}{\sin 49^\circ} = \frac{c}{\sin 66^\circ}$

$b = \frac{45.7 \sin 49^\circ}{\sin 65^\circ} = 38.055$ $\qquad c = \frac{45.7 \sin 66^\circ}{\sin 65^\circ} = 46.06$

105. Stol u obliku peterokuta ima dijagonalu duzine 1.3 m. Kolika je duzuna stranice.

Kut pri vrhu peterokuta iznosi $\alpha = \frac{360^\circ}{5} = 72^\circ$

Iz sinusovog poucka: $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \Rightarrow \frac{d}{\sin 72^\circ} = \frac{a}{\sin\frac{72^\circ}{2}} \Rightarrow a = 0.803$ m

106. Stup je usidren sa dva uzeta. Sila u desnom je 850 kp. Kut koji uzad cine na vrhu stupa je $\alpha = 105^\circ$ a kut desnog uzeta prema zemlji iznosi $\beta = 35.7^\circ$.
Odredi silu u lijevom uzetu.

Treci kut iznosi $\gamma = 180 - (\alpha + \beta) = 180 - (105^\circ + 35.7^\circ) = 37.8^\circ$

$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \Rightarrow \frac{F}{\sin 37.8^\circ} = \frac{850}{\sin 35.7^\circ} \Rightarrow F = 892.775$ kp

107. Rijesi trokut: $a = 0.1762,\ c = 0.5034,\ \beta = 129.20^\circ$

Iz kosinusovog poucka: $b^2 = a^2 + c^2 - 2ac\cos\beta \Rightarrow b^2 = a^2 + c^2 - 2ac\cos 129.20^\circ$

$b^2 = 0.1762^2 + 0.5034^2 - 2ac\cos 129.20^\circ = 0.3965 \Rightarrow b = 0.6297$

Iz sinusovog poucka: $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \Rightarrow \frac{0.1762}{\sin\alpha} = \frac{0.6297}{\sin 129.20^\circ} = \frac{0.5034}{\sin\gamma}$

$\sin\alpha = \frac{0.1762 \sin 129.20^\circ}{0.6297} = 0.2168 \Rightarrow \alpha = 12.52^\circ$

$\sin\gamma = \frac{0.5034 \sin 129.20^\circ}{0.6297} = 0.6196 \Rightarrow \gamma = 38.28^\circ$

08. Rijesi trokut: $a = 39.53,\ b = 45.22,\ c = 67.15$

Iz kosinusovog poucka: $a^2 = b^2 + c^2 - 2bc\cos\alpha \Rightarrow \cos\alpha = \frac{b^2 + c^2 - a^2}{2bc}$

$\cos\alpha = \frac{67.15^2 + 45.22^2 - 39.53^2}{2 \cdot 67.15 \cdot 45.22} = 0.82188 \Rightarrow \alpha = 34.72°$

Iz sinusovog poucka: $\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} \Rightarrow \sin\beta = \frac{b\sin\alpha}{a} = \frac{45.22 \sin 34.72°}{39.53} = 0.65155$

$\beta = 40.65° \quad \gamma = 180 - (\alpha + \beta) = 180 - (34.72° + 40.65°) = 104.63° \Rightarrow \gamma = 38.28°$

109. Cjevovod je zbog prirodne prepreke mijenjao pravac. Prvi dio je dug 3,756 km a drugi 4,675 km. Kut skretanja trase iznosi γ =168.85°. Koliko je povecana duzina cjevovoda zbog te prepreke.

$c^2 = a^2 + b^2 - 2ab\cos\gamma = 3,756^2 + 4,675^2 - 2 \cdot 3,765 \cdot 4,675 \cos 168.85$

$c^2 = 70,418,872.25$

Duzina zamisljene trase iznosi: $c = 8,391.59$ km

Duzina trase iznosi: $3,756 + 4,675 = 8,431$ km

Razlika u duzini iznosi: $\triangle l = 8,431 - 8,391.59 = 39.41$ km

110. Rijecni brod putuje brzinom 11.5 km/h ali zbog strujanja vode, ta je brzina 12.7 km/h u odnosu na obalu. Koja je brzina vode, ako brod putuje pod kutem γ =23.6°.

$c^2 = a^2 + b^2 - 2ab\cos\gamma = 11.5^2 + 12.7^2 - 2 \cdot 11.5 \cdot 12.7 \cos 23.6° \Rightarrow c^2 = 25.87$

$c = 5.0862 \Rightarrow$ Brzina vode iznosi: 5.0862 km/h

15.9 Trigonometrijske funkcije u parametarskom i polarnom obliku

Trigonometrijske funkcije zadane u parametarskom obliku

Ako se tocka (x, y) moze zadati u ovisnosti o trecoj promjenjivoj t, tada se jednadzbe $x = f(t)$ i $y = f(t)$ zovu parametarske jednadzbe a parametar je t.

111. Izrazi parametarski zadanu jednadzbu $x = 2\cos t, y = 3\sin t$ u pravokutnom koordinatnom sistemu:

$$\left.\begin{array}{ll} x = 2\cos t & \Rightarrow \cos t = \dfrac{x}{2} \\ y = 3\sin t & \Rightarrow \sin t = \dfrac{y}{3} \end{array}\right\} \Rightarrow \cos^2 t + \sin^2 t \qquad = 1 \Rightarrow \left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = \frac{x^2}{4} + \frac{y^2}{9} = 1$$

Rjesenje predstavlja jednadzbu elipse.

112. Izrazi parametarski zadanu jednadzbu $x = 2 + t, y = 2 + 3t$ u pravokutnom koordinatnom sistemu:

$$\left.\begin{array}{ll} x = 2 + t & \Rightarrow t = x - 2 \\ y = 2 + 3t & \Rightarrow t = \dfrac{y-2}{3} \end{array}\right\} \Rightarrow x - 2 = \frac{y-2}{3} \Big/ \cdot 3 \Rightarrow 3x - 6 = y - 2 \Rightarrow y = 3x - 4$$

Rjesenje predstavlja jednadzbu pravca

113. Izrazi parametarski zadanu jednadzbu $x = t^2 - 2, y = t + 1$ u pravokutnom koordinatnom sistemu:

$$\left.\begin{aligned} x = t^2 - 2 \Rightarrow t^2 = x + 2 \\ y = t + 1 \Rightarrow t = y - 1 \end{aligned}\right\} \Rightarrow x + 2 = (y-1)^2 \Rightarrow x + 2 = y^2 - 2y + 1 \Rightarrow x = y^2 - 2y - 1$$

Rjesenje predstavlja jednadzbu parabole

114. Izrazi parametarski zadanu jednadzbu $x = e^t, y = e^{-t}$ u pravokutnom koordinatnom sistemu:

$$\left.\begin{aligned} x = e^t \\ y = e^{-t} = \frac{1}{e^t} \Rightarrow e^t = \frac{1}{y} \end{aligned}\right\} \Rightarrow x = \frac{1}{y} \Rightarrow \quad xy = 1$$

Rjesenje predstavlja jednadzbu hiperbole

115. Izrazi parametarski zadanu jednadzbu $x = 3\cos\varphi, y = 3\sin\varphi$ u pravokutnom koordinatnom sistemu:

$$\left.\begin{aligned} x = 3\cos\varphi \Rightarrow \cos\varphi = \frac{x}{3} \\ y = 3\sin\varphi \Rightarrow \sin\varphi = \frac{y}{3} \end{aligned}\right\} \Rightarrow \cos^2\varphi + \sin^2\varphi = 1 \Rightarrow \left(\frac{x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$

Rjesenje predstavlja jednadzbu kruznice

116. Izrazi parametarski zadanu jednadzbu $x = 4 + 3\tan\varphi,\ y = -1 + 2\sec\varphi$ u pravokutnom koordinatnom sistemu:

$$\left.\begin{array}{l} x = 4 + 3\tan\varphi \Rightarrow \tan\varphi = \dfrac{x-4}{3} \\ y = -1 + 2\sec\varphi \Rightarrow \sec\varphi = \dfrac{y+1}{2} \end{array}\right\} \Rightarrow \tan^2\varphi + 1 = \sec^2\varphi \Rightarrow \left(\frac{x-4}{3}\right)^2 + 1 =$$

$$= \left(\frac{y+1}{2}\right)^2 \Rightarrow \frac{(x-4)^2}{9} - \frac{(y+1)^2}{4} = -1$$

Rjesenje predstavlja jednadzbu hiperbole

117. Izrazi parametarski zadanu jednadzbu $x = 2\tan\varphi,\ y = \cot\varphi$ u pravokutnom koordinatnom sistemu:

$$\left.\begin{array}{l} x = 2\tan\varphi \Rightarrow \tan\varphi = \dfrac{x}{2} \\ y = \cot\varphi \Rightarrow \tan\varphi = \dfrac{1}{\cot\varphi} = \dfrac{1}{y} \end{array}\right\} \Rightarrow \frac{x}{2} = \frac{1}{y} \Rightarrow xy = 2$$

Rjesenje predstavlja jednadzbu hiperbole

Trigonometrijske funkcije zadane u polarnim koordinatama

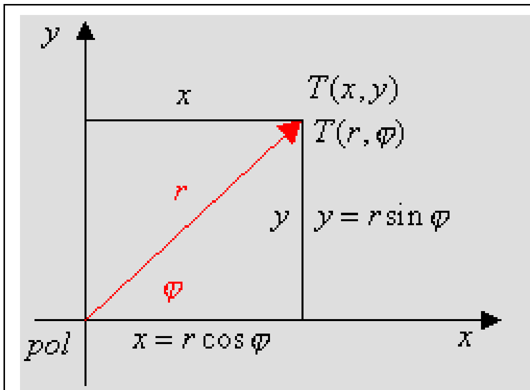
Polarni koordinatni sistem ima dvije koordinate sa kojima je odredjen polozaj tocke u ravnini.

$r -$ udaljenost tocke od ishodsta ili $r -$ radijvektor. Ishodiste se zove i pol polarnog koordinatnog sistema.

$\varphi -$ kut rotacije, kut izmedju nultog polozaja r (pozitivni dio x osi)

Na slici je prikazan odnos velicina u pravokutnom i polarnom koordinatnom sustavu.

Za pretvaranje jednog sistema u drugi koristimo slijedece relacije:



$$x = r\cos\varphi$$
$$y = r\sin\varphi$$
$$r^2 = x^2 + y^2$$
$$\varphi = Arc\tan\frac{y}{x}$$

118. Odredi pravokutne koordinate tocke zadane u polarnom obliku.

$T\left(4,240^\circ\right) \Rightarrow r = 4,\ \ \varphi = 240^\circ$

$x = r\cos\varphi\ = 4\cos 240^\circ\ = 4\left(-\cos 60^\circ\right) = 4\left(-\frac{1}{2}\right) = -2$

$y = r\sin\varphi\ = 4\sin 240^\circ\ = 4\left(-\sin 60^\circ\right) = 4\left(-\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$

$T\left(4,240^\circ\right) \Leftrightarrow T\left(-2,-2\sqrt{3}\right)$

119. Pretvori pravokutne koordinate tocke $T\left(0,2\right)$ u polarne.

$T\left(0,2\right) \Rightarrow x = 0, y = 2$

$r^2\ = x^2 + y^2\ = 0^2 + 2^2\ = 4 \Rightarrow r = \pm 2 \Rightarrow \varphi\ = Arc\tan\frac{y}{x} = Arc\tan\frac{2}{0} = \infty \Rightarrow \varphi\ = 90^\circ$

$T\left(0,2\right) \Leftrightarrow T\left(2,90^\circ\right), T\left(-2,270^\circ\right)$

120. Pretvori pravokutne koordinate tocke $T\left(-\sqrt{3},1\right)$ u polarne.

$T\left(-\sqrt{3},1\right) \Rightarrow x = -\sqrt{3}, y = 1$

$r^2\ = x^2 + y^2\ = \left(-\sqrt{3}\right)^2 + 1^2\ = 3 + 1 = 4 \Rightarrow r = \pm 2 \Rightarrow$

$\varphi\ = Arc\tan\left(-\frac{1}{\sqrt{3}}\right) = Arc\tan\left(-\frac{\sqrt{3}}{3}\right) = 150^\circ\ \Rightarrow \varphi\ = 150^\circ$

$T\left(-\sqrt{3},1\right) \Leftrightarrow T\left(2,150^\circ\right), T\left(-2,330^\circ\right)$

121. Pretvori $\left(1 + i\sqrt{3}\right)$ u polarne koordinate:

$x + iy = 1 + i\sqrt{3} \Rightarrow x = 1, y = \sqrt{3} \Rightarrow r^2\ = x^2 + y^2\ = 1 + \left(\sqrt{3}\right)^2\ = 4$

$\varphi\ = Arc\tan\frac{y}{x} = Arc\tan\frac{\sqrt{3}}{1} = 60^\circ$

$x + iy = 1 + i\sqrt{3} \Leftrightarrow r\left(x\cos\varphi + iy\sin\right) = \pm 2\left(\cos 60^\circ\ + i\sin 60^\circ\right) = 1 + i\sqrt{3}$

$T\left(2,60^\circ\right), T\left(-2,240^\circ\right)$

122. Pretvori u pravokutne koordinate:

$r\sin\varphi\ = -2 \Rightarrow y = -2$

$r\cos\varphi\ = 3 \Rightarrow x = 3 \Rightarrow$ Rjesenje je: $T\left(3,-2\right)$

123. Pretvori u pravokutne koordinate: $2\sqrt{6}\left(\cos 120^\circ + i\sin 120^\circ\right)$

$$x = r\cos\varphi = 2\sqrt{6}\cos 120^\circ = 2\sqrt{6}\left(-\sin 30^\circ\right) = -2\sqrt{6}\,\frac{1}{2} = -\sqrt{6}$$

$$y = r\sin\varphi = 2\sqrt{6}\sin 120^\circ = 2\sqrt{6}\cos 30^\circ = 2\sqrt{6}\,\frac{\sqrt{3}}{2} = \sqrt{18} = 3\sqrt{2}$$

$$2\sqrt{6}\left(\cos 120^\circ + i\sin 120^\circ\right) = \left(-\sqrt{6} + i3\sqrt{2}\right)$$