Unit One Physics

1.1 - Basic Physics

SI units

- All units of measurement in physics are based on **7 SI Units:**

	measurement	**Base SI unit**
length	metre	m
time	second	s
mass	kilogram	kg
current	amperes	A
temperature	kelvin	K
amount	moles	mol
light intensity	candella	cd

Each unit of measurement in physics can be expressed in terms of its base SI units.

All other units of measurement are *derived units*, meaning they are a combination of many base SI units.

1 - Velocity = displacement / time

- Displacement is measured in m
- Time is measured in s

Thus speed expressed as base units = m / s = ms^{-1}.

2 - Density = mass / volume

- Mass is measured in kg
- Volume is measured in m^3

Mass is measured in kg, volume is measured in m^3, thus density = kg / m^3 = kgm^{-3}.

3 - Acceleration = velocity / time

- Velocity is measured in ms^{-1}
- Time is measured in s

Thus, acceleration = ms^{-1} / s = ms^{-2}.

4 - Force = mass * acceleration

- Mass is measured in kg
- Acceleration is measured in ms^{-2}.

Thus, force = kg * ms^{-2} = kgms-2.

5 - Work = Force * Distance

- Force is measured in $kgms^{-2}$
- Distance is measured in m

Thus, work = $kgms^{-2}$ * m = kgm^2s^{-2}.

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Often, rather than expressing an equation in terms of its SI units, we simplify the SI units into a derived unit.

For example, rather than expressing force as $kgms^{-2}$, we simplify it to the Newton - N.

Rather than expressing work as kgm^2s^{-2}, we simplify it to the Joule - J.

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In a **homogenous equation**, the SI units on both sides of the equation are equal.

When proving that an equation is homogeneous, we take the known SI units on one side of the equation, and try to determine the SI units on the other side of the equation. If they are equal, the equations are homogeneous.

Any non-unit quantities in the equation, e.g. 1/2, 4…, are ignored.

If an equation contains a power, we have to raise the base SI units by that power.

E.g. if $v = ms^{-1}$, $v^2 = m^2s^{-2}$

If radius, r, is measured in m, $r^3 = m^3$.

All equations must be homogeneous before they can be considered to be correct. However, if an equation is homogeneous, it does not confirm an equation is correct, it is simply the first step before processes are carried out later on to confirm that it is correct.

Examples of proving homogeneity:

Prove that the equation Kinetic Energy = 1/2 mv^2 is homogeneous

We previously showed that energy, which is measured in joules, is kgm^2s^{-2}.

We need to prove that the equation $1/2mv^2$ gives these units.

m = kg
$v = ms^{-1} \Rightarrow v^2 = m^2s^{-2}$.
1/2 is ignored.

$1/2 * m * v^2 = kgm^2s^{-2}$

The units of kinetic energy are kgm^2s^{-2}, and this is equal to the units of the joule, hence the equation is homogeneous.

Prove that the equation change in momentum = Ft is homogeneous

Momentum is equal to the mass * velocity = $kgms^{-1}$. These are the units on the LHS.

On the right hand side, Ft can be expressed as $kgms^{-2} * s = kgms^{-1}$.

Thus LHS = RHS, so the equation is homogeneous.

Prove that the equation E = hf is homogeneous

Energy = work done = kgm^2s^{-2}.

h is Planck's constant, and is measured in Js.

However, J is not a base SI unit - it needs to be broken down into these units.

$J = kgm^2s^{-2}$, thus $h = Js = kgm^2s^{-2} * s = kgm^2s^{-1}$.

Frequency is a measure of the number of cycles per second. *Per second* can be expressed as 1 / s, or s^{-1}.

Thus $hf = kgm^2s^{-1} * s^{-1} = kgm^2s^{-2}$.

Hence, the LHS = RHS, so the equation is homogeneous.

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3. A large boulder of mass m lies in a riverbed. It can be rolled over by the water in the river flowing over it at speed v. Which of the following equations could relate the mass of the boulder to the speed of the river, v, its density ρ and the gravitational field strength, g? k is a constant with no units.

A. $m = \frac{k\rho v}{g}$ B. $m = \frac{k\rho v^2}{g^3}$ C. $m = \frac{k\rho v^6}{g^3}$ D. $m = \frac{k\rho g}{v^3}$

For this question, we need to determine which equation is homogeneous. In other words, we need to prove that the RHS has SI units of kg.

- k has no units, so we can ignore it.
- g is gravitational acceleration; acceleration is measured in ms^{-2}.
- v is velocity, measured in ms^{-1}.
- p is density, measured in kgm^{-3}

Equation A:

RHS = $(kgm^{-3} * ms^{-1}) / ms^{-2} = kgm^{-2}s^{-1} / ms^{-2} = kgm^{-3}s$

The RHS does not equal kg, so the equation is not homogeneous.

Equation B:

RHS = $(kgm^{-3}(ms^{-1})^2) / (ms^{-2})^3 = kgm^{-3}m^2s^{-2} / m^3s^{-6} = kgm^{-1}s^{-2} / m^3s^{-6} = kgm^{-4}s^4$.

Again, the units on the RHS are not kg, thus the equation cannot be homogeneous.

Equation C:

RHS = $(kgm^{-3}(ms^{-1})^6) / (ms^{-2})^3 = kgm^{-3}m^6s^{-6} / m^3s^{-6} = kg$

In this equation, RHS = LHS, so it is likely to be homogeneous.

Equation D:

RHS = $kgm^{-3}ms^{-2} / (ms^{-1})^2 = kgm^{-2}s^{-2} / m^2s^{-2} = kgm^{-4}$

This equation does not equal the LHS, so equation C must be correct.

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If an equation has two separate terms, e.g. V = u + at, all of the terms need to have the same SI units for the equation to be homogeneous.

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Prove the equation $v^2 = u^2 + 2as$ is homogeneous.

LHS = $(ms^{-1})^2 = m^2s^{-2}$

RHS = $m^2s^{-2} + ms^{-2}m = m^2s^{-2} + m^2s^{-2}$

All of the terms in the equation have the same combined SI units, so the equation is homogeneous.

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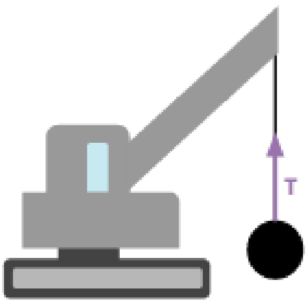
Concept of force

A force is a push or pull exerted on a body. All bodies that are in contact exert a force on each other.

There are many types of force, e.g. :

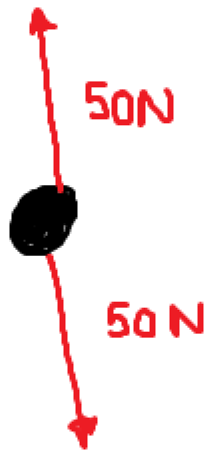
Weight	The force acting on an object due to a gravitational field.
Tension	A force exerted by a rope, cable or wire on a body that is attached to the rope / cable / wire. 
Driving force	The force exerted by a motor in the direction of travel.
Normal Reaction	The ground exerts a force perpendicular (hence *normal*) to the ground on any body resting on it. This is the normal reaction force.

force	
Friction	The resistance to motion that a body experiences when it moves across another surface.

Newton’s first law of motion

If there is an overall, or **net**, force, it will cause a body to accelerate.

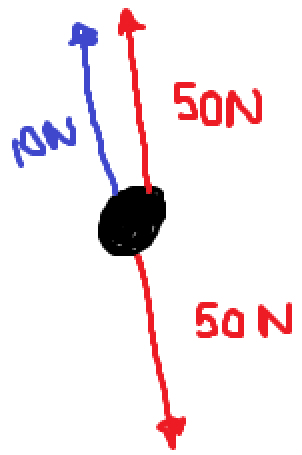
However, if there is no overall force, a body will not accelerate - it will remain at rest or at a constant velocity, e.g.:



In the above example, there is 50N of force acting upwards, and 50N of force acting downwards. These forces therefore cancel out, leading to no overall force.

Because there is no overall force exerted on this body, there is no acceleration. This means that if the body is at rest, it must remain at rest, or if the body is moving at a constant velocity, since there is no acceleration in any direction, it will remain at this constant velocity.

However, now suppose that another 10N force is applied to the previous body in the upwards direction:



There is now a net force of 10N in the upwards direction. A net force causes an acceleration, so the body will begin to accelerate in the upwards direction.

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These are the ideas presented in Newton's First Law: *a body at rest will remain at rest or a body in uniform motion will remain in uniform motion unless an external force is applied.*

An **external force** is a force that is exerted by one body on another body (e.g. one ball colliding with another ball).

There are also **internal forces** - forces that occur within a body (e.g. the force of gravity holding a body together). Internal forces cannot cause a change in motion.

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Newton's First Law is often described as the **law of inertia**. Inertia is a resistance to a change in motion.

Suppose a stationary body has an external force exerted upon it. This external force will cause an acceleration, but there will not be an instant change in motion if the body has mass. For example, if the force causes an acceleration of $0.4ms^{-2}$, it will take 5s to reach a speed of $2ms^{-1}$. The body has a resistance to a change in motion - or *inertia*.

The more massive a body, the greater its inertia. If a 50N force is exerted on a 1000kg body, the acceleration created will be 10x lower than the acceleration created when a 50N force is exerted on a 100kg body (as described by $a = F/m$). The 1000kg body will therefore have a greater resistance to a change in motion, i.e. a greater inertia.

Pushing at a wall will not cause it to accelerate greatly due to its high mass and thus its high inertia; exerting the same force on a football, on the other hand, will cause a much greater acceleration due to its lower inertia.

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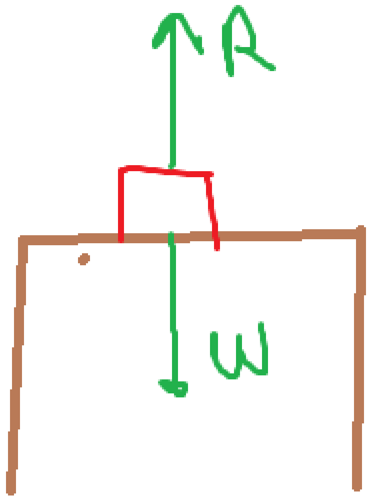
If a body is at rest or in uniform motion, it is said to be in **equilibrium**. An equilibrium can only be disturbed by an external force.

Example 1



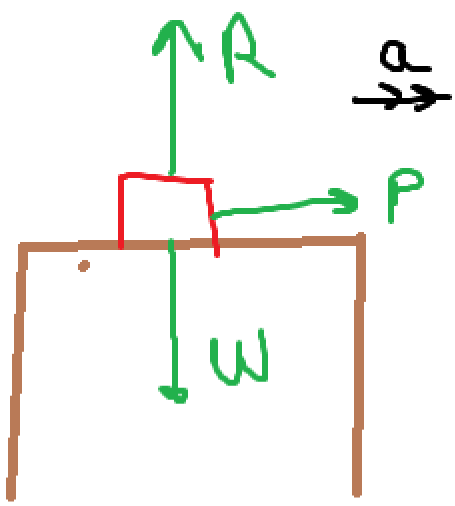
The body in the diagram above has a weight force, W, acting downwards, and a reaction force, R, acting upwards.

The reaction force is exerted on any body in contact with another surface.

When the body is at rest, R = W. For example, if the weight of the body, W, is 50N, the reaction force, R, must also be 50N. In this state, it is at **equilibrium**.

The only way to disturb this equilibrium is to apply an external force.

If a pushing force, P, is applied to the right, the body will now begin to accelerate to the right:

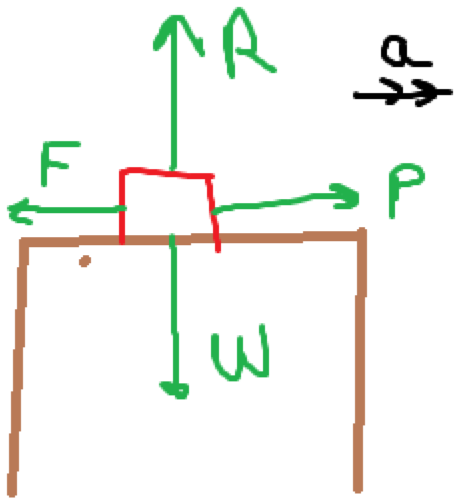


Now that the body is moving across a surface, if the surface is rough, a new force, **friction**, will emerge. Friction opposes the direction of travel.

As long as the force P is being applied to the body (e.g. somebody pushing the body), it will continue to accelerate (if the person stops exerting the push, the force goes away, and the acceleration stops). As the velocity increases due to the acceleration created by P, the friction increases.

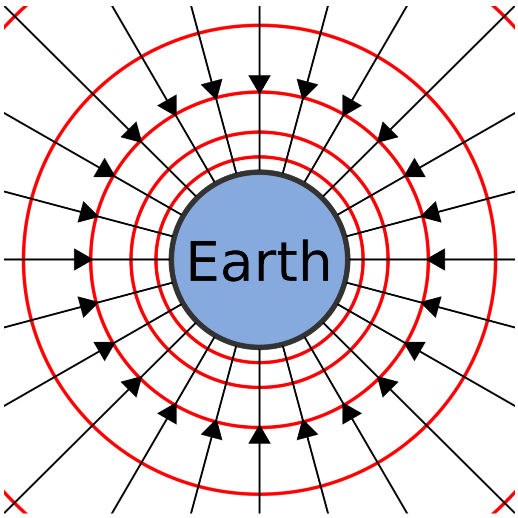
A force causes an acceleration - so the friction force is causing an acceleration in the opposite direction to the direction of the travel. When the friction force increases to the point where it is equal to P, the acceleration opposing the acceleration in the direction of travel will equal the acceleration created by P.

This means that there will be no overall acceleration, so the body enters into uniform motion, and is back into a state of equilibrium:



The forces in the horizontal plane, F and P, are equal, and the forces in the vertical plane, R and W, are equal. The body therefore remains in a state of equilibrium, with no motion in the vertical plane, and uniform motion in the horizontal plane.

Example 2



Now let us consider a body falling downwards in a gravitational field.

All bodies create a gravitational field around them. This gravitational field leads to the force of gravity: the pull exerted on a body towards the centre of a gravitational field.

On Earth, bodies fall towards the centre of Earth's gravitational field. The force acting towards Earth's core is gravity, and the force that gravity creates is known as the **weight** force.

Earth's gravitational field occurs throughout the Universe, meaning that all bodies throughout the Universe are pulled towards Earth; but the strength of this pull just weakens at greater distances from Earth. Earth's gravitational effects are only truly felt within close proximity to Earth (i.e. the solar system), but most prominently on Earth itself.

When a body is close to Earth's surface, we generally say that Earth's gravitational field causes the body to accelerate at 9.81ms^{-2} towards the centre of Earth.

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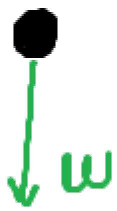
When a body is falling towards the centre of Earth (e.g. a skydiver or a book falling off a shelf), it will accelerate towards Earth at $9.81ms^{-2}$. The force that creates this acceleration is the *weight* of the body.



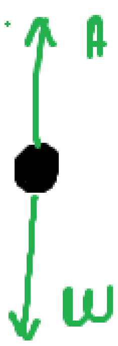
However, the net acceleration towards the centre of Earth is not $9.81ms^{-2}$ due to **air resistance**.

As the body accelerates downwards, it encounters air particles, which exert a resistive force called air resistance. This is a form of friction.

As the body is initially accelerating downwards, its velocity is increasing. This leads to more contacts with air particles per second, and thus the air resistance increases.

The air resistance is a force and it consequently causes acceleration in the upwards direction. So there are two accelerations acting on the body, an upwards acceleration and a downwards acceleration, but the net acceleration is initially downwards. As the air resistance increases, the upwards acceleration increases, and eventually the acceleration in both directions is equal, i.e. the air resistance is equal to the weight of the body.

At this point, the body enters into equilibrium; it is moving at a constant velocity - called the **terminal velocity**.



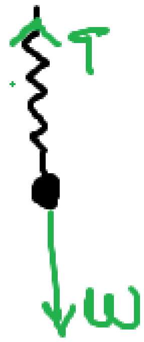
The only way for this equilibrium to be disturbed is for an external force to be exerted (e.g., a gust of wind).

Example 3

Now let us consider a system where tension and weight are involved.

Take an object that is attached to a spring, and the system is in equilibrium.

Acting downwards is the weight force. The force resisting this weight force, and allowing the system to be in equilibrium, is the **tension** force exerted by the spring:



For the system to be in equilibrium, T must equal W.

Now let us consider when the body was initially attached to the spring in its unstretched state. When the body is released, it will initially accelerate downwards due to the weight force being higher than the tension (so there is a net force in the downwards direction).

However, as the string stretches further out, its tension increases. This tension force opposes the downwards weight force, decreasing the net downwards acceleration. Eventually, the tension increases to a point where it is equal to the weight, so the equilibrium is obtained (in actuality, the spring will be in **simple harmonic motion,** oscillating about a mean position).

Newton's Second Law and the concept of a Newton

Newton's Second law allows us to quantify some of the ideas described by Newton's first Law.

The unit of force is the **newton**.

1 newton is defined as the force necessary to provide a mass of 1kg with an acceleration of $1ms^{-2}$.

- Suppose a 2N force is exerted on a 1kg body. This will cause the body to accelerate at $2ms^{-2}$.

- Suppose a 5kg body is accelerated to $1ms^{-2}$ by a force. The force required to achieve this is 5N.

- If a 10N force causes an acceleration of $5ms^{-2}$, the mass of this body must be 2kg.

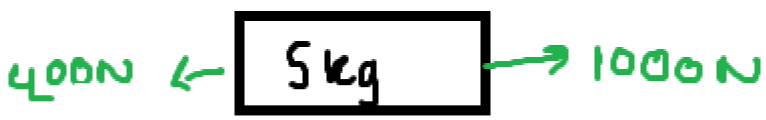
We can therefore say that Force = mass * acceleration => **F=ma**. This is Newton's second law of motion.

- Suppose you have a 24kg body accelerating at $2.5ms^{-2}$. The force exerted to cause this acceleration must be 24 * 2.5 = 60N.
- Suppose a 102kg body has a force of 200N exerted upon it. This will cause an acceleration of 200 / 102 = $1.96ms^{-2}$.

Net force:

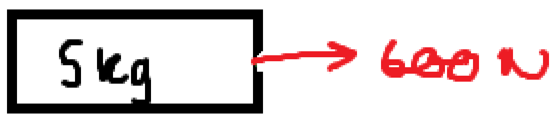
A key concept with Newton's second law calculations is the **resultant**, or **net**, force.

Consider the following diagram:



The 5kg body above has a 1000N force exerted on it to the right, and a 400N force exerted on it to the left.

The 400N force has the effect of opposing the 1000N force. This means that there is a *net* or *resultant* force of 1000 - 400 = 600N to the right.

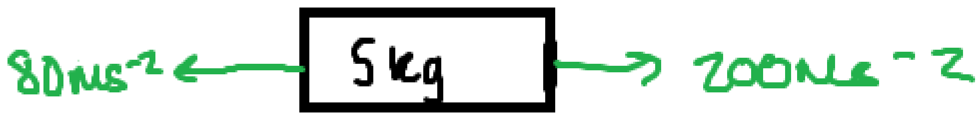


The acceleration produced when a 600N force acts on a 5kg body is **600 / 5 = $120ms^{-2}$**. This body therefore accelerates to the right at $120ms^{-2}$.

A body always accelerates in the direction of the net force.

Another way to consider this situation is to consider the accelerations created by each force.

- The 1000N force creates an acceleration of 1000 / 5 = 200ms^{-2}.
- The 400N force creates an acceleration of 400 / 5 = 80ms^{-2}.



The two accelerations oppose each other to create an overall acceleration of 200 - 80 = 120ms^{-2} to the right.

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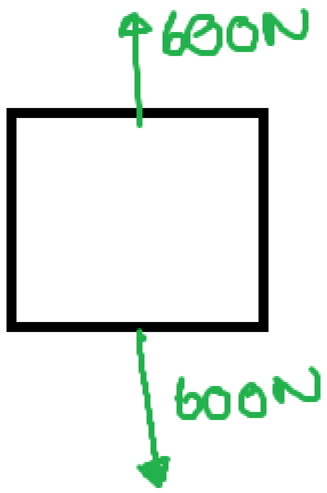
When a body is in equilibrium, the forces exerted in a single plane cancel each other out, e.g.:



The net force in this situation is 0N. The force is not favoured in a given direction, so there is no acceleration. This is why the body exists in an equilibrium.

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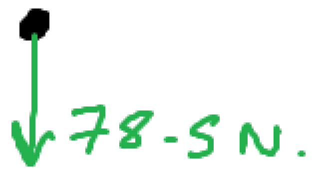
The first application of $F = ma$ that we will consider is **weight**. Weight is a force that all bodies in a gravitational field experience. Weight is exerted towards the centre of the gravitational field.

Earth's gravitational field causes an acceleration of $9.81ms^{-2}$, or $9.81\ Nkg^{-1}$. This means that Earth's gravitational field will exert a 9.81N force on a body with a mass of 1kg; or a 2(9.81) = 19.62N force on a body of mass 2kg. This force is the weight.

The gravitational field must exert a greater force on bodies with a higher mass to cause them to accelerate at $9.81ms^{-2}$.

Using F=ma, the force required to accelerate a 5.2kg body at $9.81ms^{-2}$ must be 5.2(9.81) = 51.0N => 51N is the weight of the body.

A body with a mass of 8.0kg must have a weight of 8.0(9.81) = 78.5N.



We can therefore say that **Weight = mass * acceleration due to gravity**

The acceleration due to gravity is referred to as the **gravitational field strength, g**. This value can vary from body to body (e.g. on the Moon, $g = 1.6ms^{-2}$).

Thus, **Weight = mg**, where **m** is the mass, and **g** is the gravitational field strength.

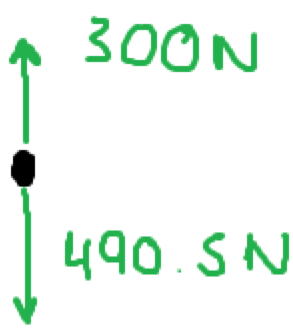
Applying Newton's Second Law to falling bodies

Now let us consider the weight of a body falling in Earth's gravitational field, and how this is opposed by the air resistance:

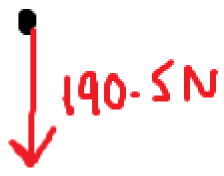
Suppose we have a body of mass 50kg falling through the sky, with an air resistance of 300N.

The weight of the body is 50(9.81) = 490.5N.

There is thus a force of 490.5N acting downwards and a force of 300N acting upwards:



The 300N force has the effect of opposing the 490.5N force. The net (overall) force felt by the body must be 490.5 - 300 = 190.5N in the downwards direction:



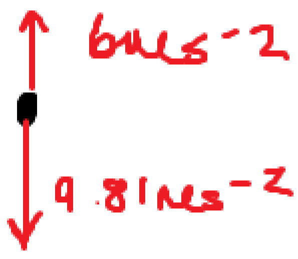
If a 50kg body experiences a force of 190.5N, its acceleration at this instant is: $a = F / m = 190.5 / 50 = 3.81ms^{-2}$.

So although Earth's gravitational field is accelerating the body at $9.81ms^{-2}$, there is an opposing acceleration created by air resistance, so the net acceleration felt is actually lower.

Another way to think of this is to think of the acceleration created by gravity and the acceleration created by air resistance.

The air resistance is 300N, and it is acting on a 50kg body, so the acceleration caused by the air resistance is: $300 / 50 = 6ms^{-2}$.

The acceleration created by gravity is $9.81ms^{-2}$.

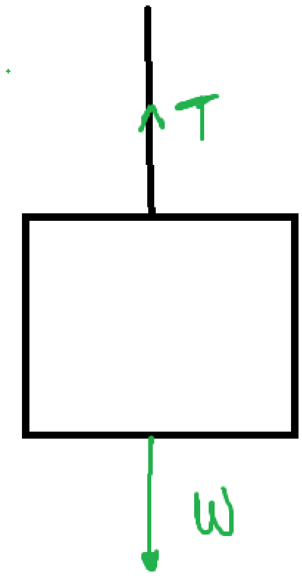


The net acceleration is therefore $9.81 - 6 = 3.81ms^{-2}$.

Considering tension

Now let us consider a lift. In the cable of a lift, there is a tension force.

In a lift, there are two opposing forces: the tension in the cable, and the weight of the lift:

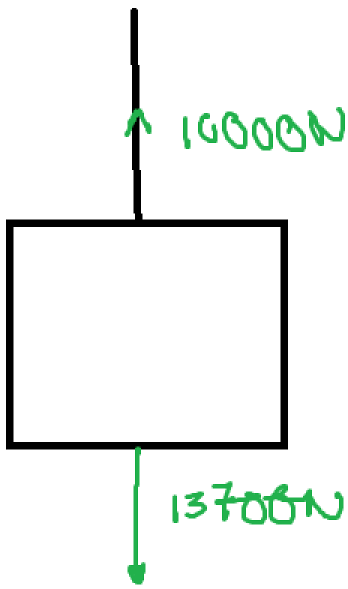


Suppose that the lift is accelerating upwards. The tension in the cable must be greater than the weight. If the lift is accelerating downwards, the weight of the lift must be greater than the tension in the cable.

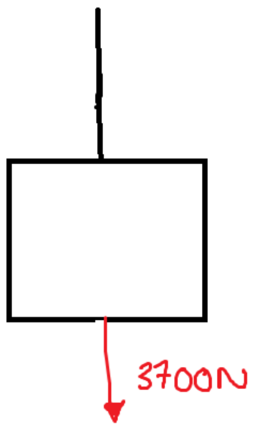
Suppose the tension in a lift cable is 10kN (10000N), and the mass of the lift itself is 1200kg, and the mass of the passengers in the lift is 200kg. **What is the acceleration of the lift?**

The total mass of the lift is 1200 + 200 = 1400kg. This gives a weight of 1400(9.81) = 13700N.

The weight is therefore greater than the tension, so the lift must be accelerating downwards:



The net force is therefore 13700 - 10000 = 3700N.



When a 1400kg body has a force of 3700N exerted on it, according to Newton's second Law, its acceleration must be: 3700 / 1400 = 2.64ms^{-2}.

Another way to view this is that the acceleration caused by the tension is 10000 / 1400 = 7.14ms^{-2}.

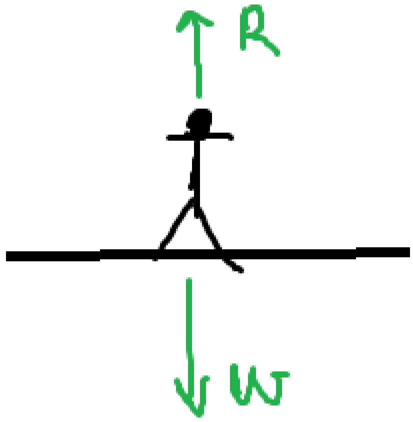
Thus, since the acceleration caused by weight is 9.81ms^{-2}, the overall acceleration is 9.81 - 7.14 = 2.67ms^{-2} downwards *(this is not the same as the value calculated previously due to the rounding of the weight figure to 3SF).*

Considering the reaction force

In the previous example, the people in the lift have a reaction force exerted upon them.

A surface exerts an upwards force called the *normal reaction force* on any bodies that lie on it. The normal reaction force is always perpendicular to the surface.

The reaction force in the case of the lift example is the upwards force exerted by the floor of the lift on the people, R:



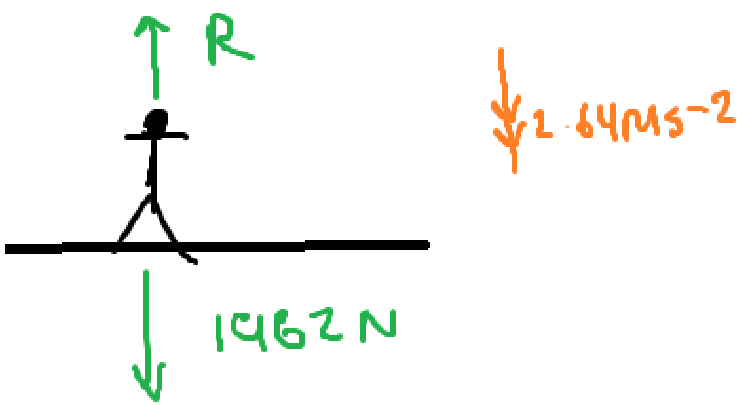
The reaction force does not always have to equal the weight; if a body is accelerating in the vertical plane, the reaction force and the weight force will be different. The reaction force only equals the weight when the body is in equilibrium in the vertical plane.

- When a body is accelerating upwards, the reaction force must be greater than the weight, because the force exerted by the ground is causing the body to accelerate upwards.
- When a body is accelerating downwards, the weight must be greater than the reaction force.

Let us find the reaction force exerted on the people in the lift (*referring to previous example*). Newton's second law can only be applied to a single body, so we have to consider the people as a separate system to the lift.

The mass of the people in the lift is 200kg; we will model this as one person with a weight of 200(9.81) = 1962N.

In the example used previously, the lift is accelerating downwards, so the weight must be greater than the reaction force. The acceleration of the lift was found to be 2.64ms^{-2} in the downwards direction. Since the people are standing in the lift, the acceleration of the people must also be 2.64ms^{-2}:



The mass of the people is 200kg. The force required to cause a downwards acceleration of $2.64ms^{-2}$ is therefore: 200 * 2.64 = 528N.

Thus, the net force in the downwards direction must be 528N.

We can express the net force as (1962 - R). Thus, 1962 - R = 528 => R = 1962 - 528 = 1434N.

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Example problem: A lift has a mass of 500kg, and there is a 50kg person in the lift. The lift is accelerating upwards, with a tension in its cable of 10000N. **Find the reaction force exerted on the person in the lift**.

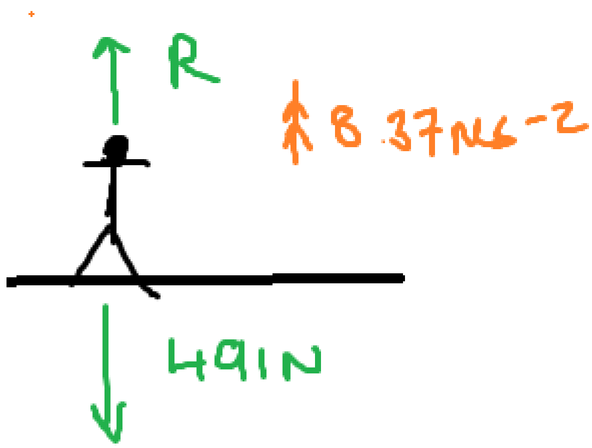
We can start by finding the acceleration of the lift.

Weight of lift = 9.81(500+50) = 5396N.

Thus net force = 10000 - 5396 = 4604N.

Acceleration = 4604 / 550 = $8.37ms^{-2}$.

R must be greater than the weight of the person since the lift is accelerating upwards.



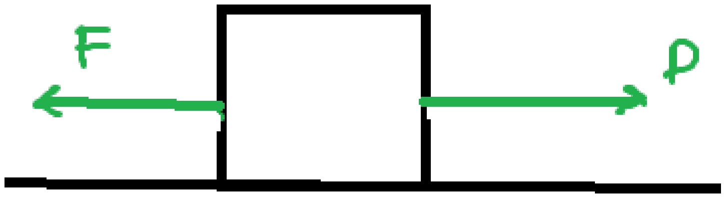
Net force = R - (50*9.81) = R - 491

Net force = m * a => R-491 = 50 * 8.37

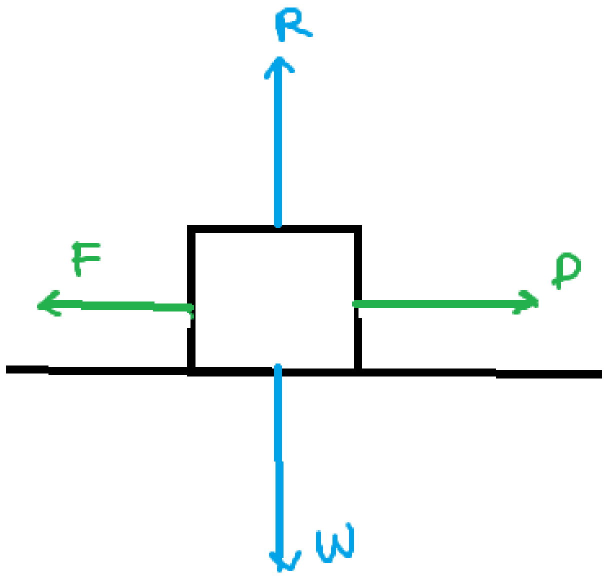
R = (50*8.37) + 491 = 906.5N.

Considering a car accelerating on a road

A car has two main forces exerted upon it in the horizontal plane: friction (F) and the **driving force** (D):



The car also has forces acting on it in the vertical plane: the weight of the car and the reaction force:

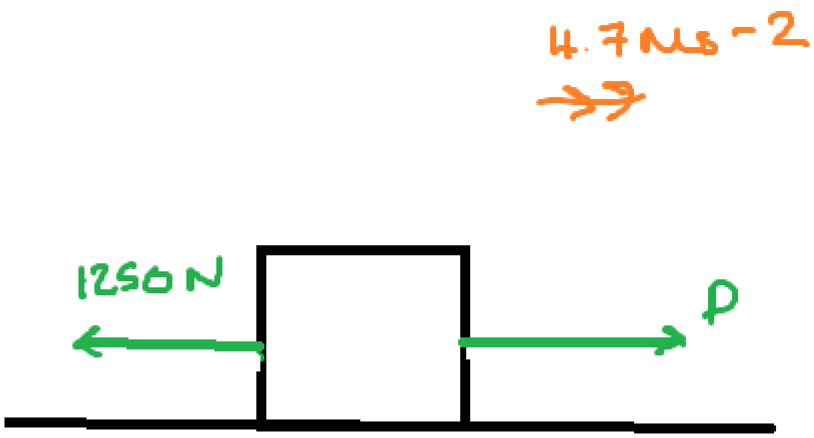


However, the car is not moving in the vertical plane, meaning that the weight and the reaction force must be equal. We therefore do not have to consider these forces.

Suppose a car is accelerating. The driving force must be greater than the friction force for this acceleration to occur.

-

Take a 1150kg car accelerating at $4.7ms^{-2}$ in the direction of the driving force. If the friction acting on the car is 1250N, **what must the driving force be?**



If a 1150kg body is accelerating at 4.7ms^{-2}, the force required to cause this acceleration is:

F = m * a = 1150 * 4.7 = 5405N.

This means that there must be a *net* force of 5405N.

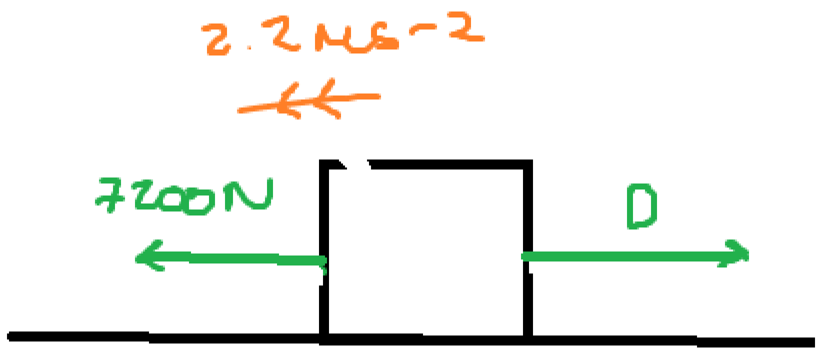
The net force can be expressed as D - 1250.

Thus, D - 1250 = 5405 => D = 5405 + 1250 = 6655N.

-

Now suppose the car is *decelerating*. Now, the friction force must be greater than the driving force.

Suppose the friction force is 7200N and the deceleration is 2.2ms^{-2}. **What is the driving force?**



We know that the net force is 7200 - D.

The net force is equal to: m * a = 1150 * 2.2

Thus, 7200 - D = 1150 * 2.2

D = (7200) - (1150 * 2.2) = 4670N.

Key points on Newton's First and Second Laws of motion:

- A force is a push or pull exerted on an object.
- A force causes acceleration.
- For a body to accelerate in a certain direction, there has to be a net force in that direction.
- If there is no net force in a certain direction, the body is in equilibrium.
- The unit of force is the newton; 1N is the force required to accelerate a 1kg body at $1ms^{-2}$. A 2N force would produce an acceleration of $2ms^{-2}$ for the same body. Thus, **F=ma**.
- To find the acceleration of a body in a certain plane, we find the net force acting on it by adding the forces in this plane, and divide it by the mass of the body.
- All bodies on Earth experience a gravitational acceleration of **g** (where $\mathbf{g=9.81ms^{-2}}$). Since **F=ma** and **a=g**, the gravitational force that is exerted is **F=mg**. This is the weight of a body.
- All bodies that lie on a surface have a normal reaction force exerted on them by the surface. This reaction force can be greater than or less than the weight.

Newton’s Third Law of motion

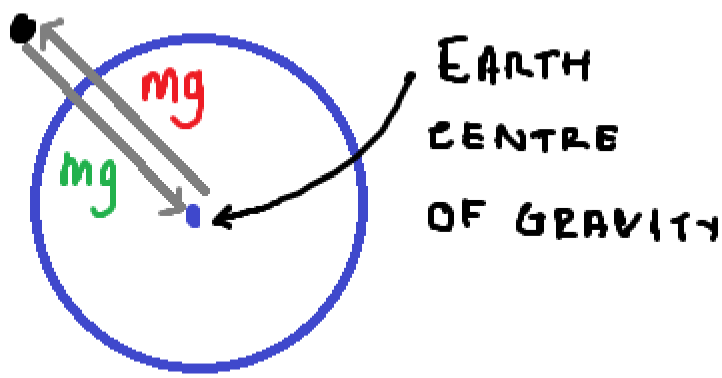
Newton’s third law of motion states that if body A exerts a force on body B, body B exerts an equal and opposite force on body A *of the same type*.

The force exerted on body B by body A is the **action force** and the force exerted by body B on body A is the **reaction force**.

-

A key example of this is weight.

Earth’s gravitational field exerts a gravitational pull on a body, equal to **mg**. According to Newton’s third law, if Earth exerts a weight force on the body, the body must exert an equal and opposite force on the Earth:



So all bodies on Earth (e.g. cars) are pulled towards the Earth due to weight, but Earth is pulled equally towards the body due to Newton’s third law.

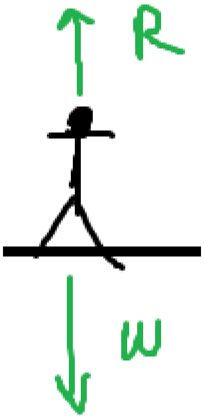
Other examples:

- *If a tennis ball hits a tennis racket, the ball exerts an action force on the racket, and the racket exerts an equal and opposite reaction force on the ball*
- *If a swimmer kicks off of a wall, the swimmer exerts a force on the wall, and the wall exerts an equal and opposite force on the swimmer, allowing the swimmer to glide off*

It is important to note that the forces in a Newton’s third law pair must be:

- Acting on different bodies
- Of the same type

For example:



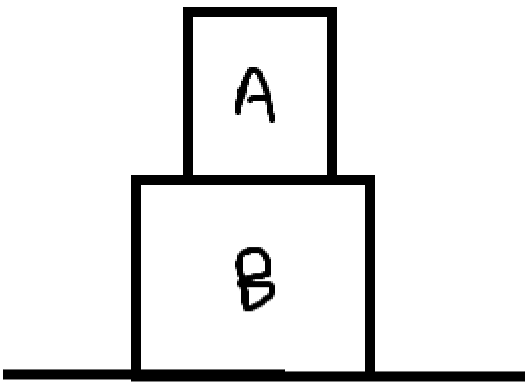
The person has a normal reaction force acting upwards and a weight force acting downwards.

This is not an example of a N3L pair because the forces act on the same body, and they are not of the same type. The weight force is caused by gravitational attraction to Earth, whereas the normal reaction force is caused by electrostatic repulsion between the floor and the person.

A N3L pair in this case would be the electrostatic repulsion exerted by the person and the floor, and the resulting reaction force caused by the electrostatic repulsion exerted by the floor on the person. Another N3L pair would be the gravity exerted by Earth on the person, and the resulting gravitational force exerted by the person on Earth.

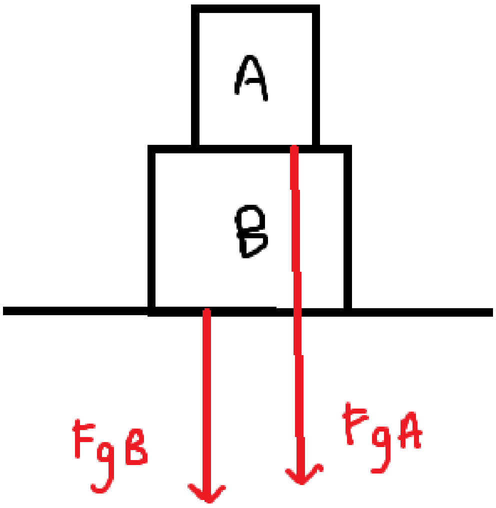
Example 1: two boxes in a lift

Let us consider two boxes in a lift, box A and box B. Box A is smaller than box B and lies on top of box B:



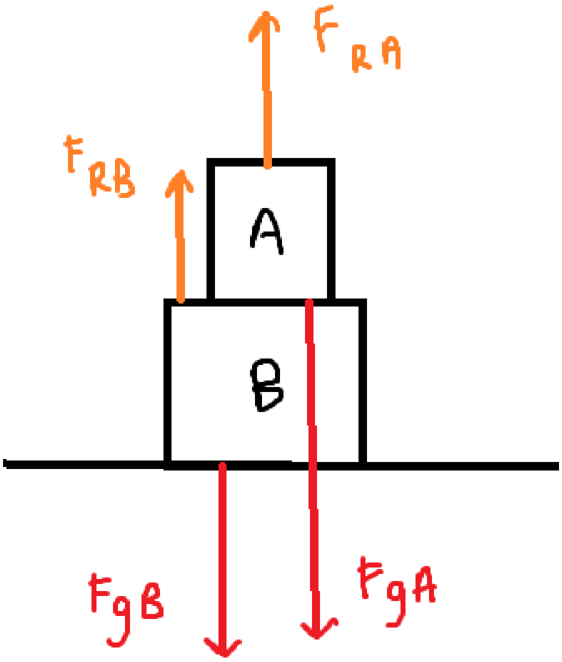
There are many forces involved in this system.

Firstly, both box A and box B have a gravitational force acting downwards. We will call these forces F_{gA} and F_{gB}:



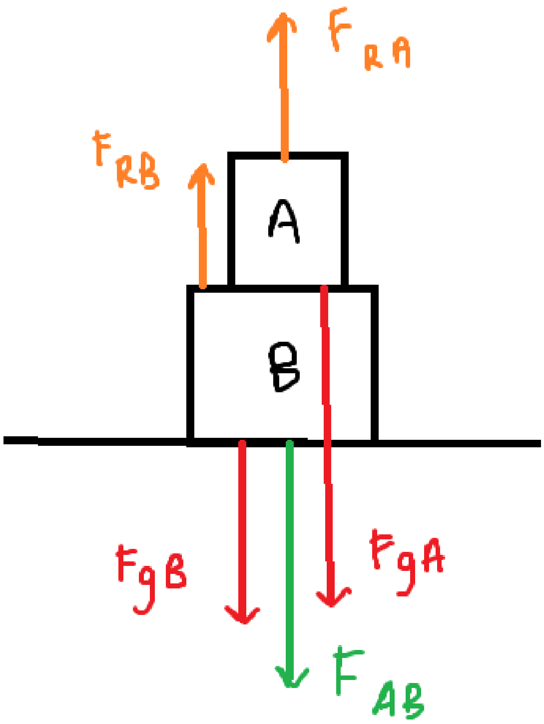
The floor of the lift exerts a normal reaction force on box B. We will call this F_{RB}.

Force B also exerts a normal reaction force on box A. We will call this F_{RA}.



Newton's Third Law states that if body 1 exerts a force on body 2, body 2 exerts an equal and opposite force on body 1.

Since box B exerts a normal reaction force, F_{RA}, on box A, box A must exert an equal and opposite reaction force on box B, F_{AB}:

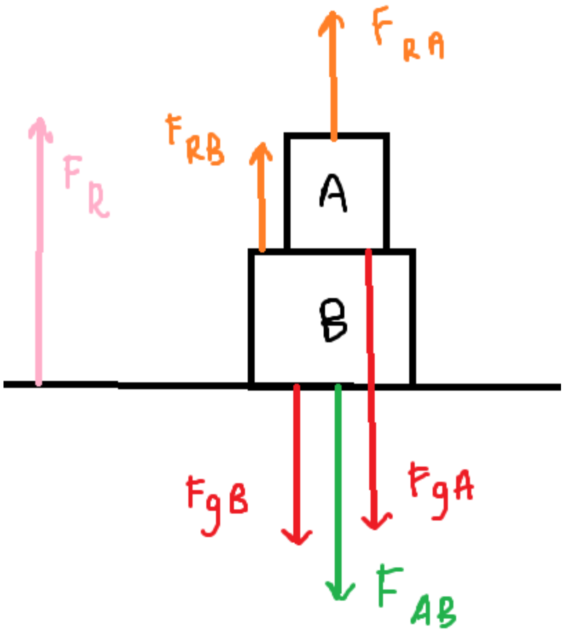


Now let us consider how these forces compare if the lift is accelerating upwards. We know that when a body that lies on a surface accelerates upwards, the normal reaction force must be greater than the weight.

In the case of A, this means $F_{RA} > F_{gA}$, and in the case of B, this means $F_{RB} > F_{gB}$. Additionally, the force of reaction exerted by the floor on B must be greater than the overall forces acting downwards on B; thus, $F_{RB} > (F_{gB} + F_{AB})$.

We can also say that the reaction force B exerts on A, F_{RA}, is equal to the reaction force that A exerts on B, F_{AB}, since it is a Newton's third law pair.

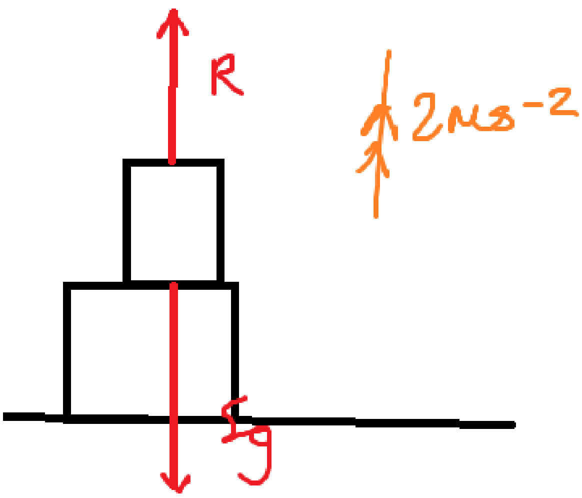
Another point to consider is that the ground exerts a reaction force on B, F_{RB}. According to N3L, box B must exert an equal and opposite reaction force on the ground. We will call this F_R:



Now let us consider how these ideas can be applied to a problem.

Two boxes, A and B, of mass 5kg and 10kg respectively are in a lift. Box A lies on top of box B. The lift is accelerating upwards at 2ms^{-2}.

a) ***Find the reaction force exerted by A on B.***



If we consider only box A, box A is accelerating upwards at 2ms^{-2}.

It has the reaction force, R, exerted by box B acting upwards, and its weight acting downwards.

Since weight = mg, rather than expressing weight as its actual value, 5 * 9.81, it is often easier to express it as 5g before we go into calculations.

We know that R must be greater than 5g. The resultant force is R - 5g. This resultant force is causing an upwards acceleration of 2ms^{-2}. The force required to cause a 5kg body to accelerate at 2ms^{-2} is m * a = 5 * 2.

We can therefore say: R - 5g = 5 * 2

Therefore R = (5*2) + (5g) = 10 + (5 * 9.81) = 59.1N.

As mentioned previously, if box B exerts a reaction force of 59.1N on box A, box A must exert an equal and opposite reaction force on box B of 59.1N according to N3L.

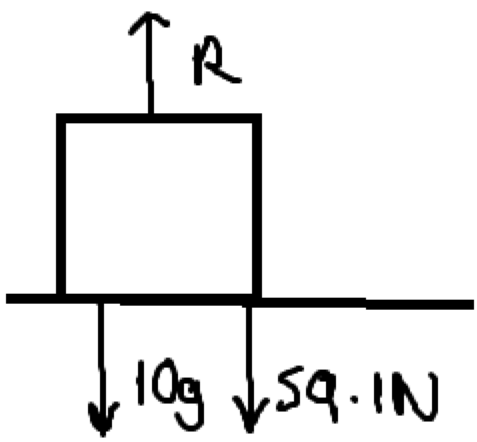
Hence, the reaction force exerted by box A on box B is **59.1N**.

b) Find the reaction force exerted on the lift of the floor on box B.

Force B has three forces acting on it:

- The weight force acting downwards, 10g.
- The upwards reaction force from the lift.
- The downwards reaction force exerted by box A.

The reaction force exerted by box A is equal and **opposite** to the reaction force exerted by box B, so it acts in the downwards direction.



The overall downwards force is 10g + 59.1 = (10*9.81) + 59.1 = 157.2N.

R must be greater than the downwards force.

The net force can be expressed as R - 157.2.

The net force is causing a body of mass 10kg to accelerate at $2ms^{-2}$, so F_{net} = 10 * 2 = 20N.

Thus R - 157.2 = 20

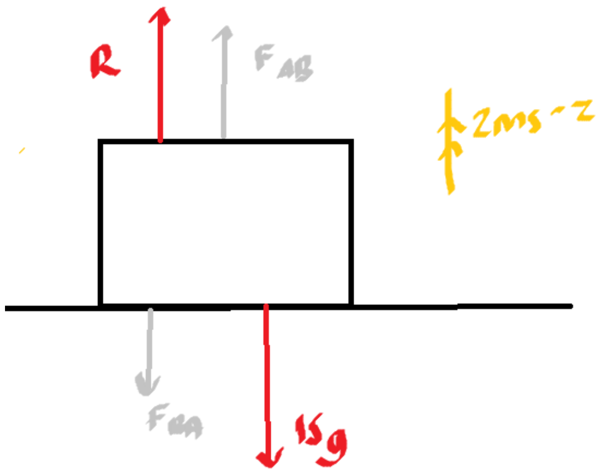
R = 177.2N.

-

Another way to derive the same value of **R** is to model the 5 and 10kg boxes as a single box. We consider the two boxes as **connected particles**. When we have two connected particles, they can be modelled as one single particle.

When we model the boxes as a single box, we get a box with a combined mass of **15kg**. There are four forces exerted on the box:

- The weight force of **15g**
- The reaction force exerted by A on B
- The reaction force exerted by B on A
- The reaction force exerted by the floor on B



The key point to consider when modelling two connected particles as a single particle is that the forces exerted between the particles (**internal forces**) cancel out. In this case, because the reaction forces are equal and opposite, in our overall model, they cancel out.

We only have to consider **R** and the weight force as a result.

Thus, to find R, we consider that the system is accelerating at **2ms^{-2}**:

R - 15g = 2 * 15 => **R = 30 + 15g = 177.2N** => the same answer as shown above.

The problem is designed so that the internal forces have no effect when we model the particles as a single particle.

-

Example:

A child of mass m kg, is sitting over the shoulders of his father whose mass is 80 kg. The father, with the child over his shoulders, is standing in a lift of mass 800 kg. When the tension in the cable of the lift is 9050 N, the lift and its occupants is accelerating, with the shoulders of the father exerting a force of 250 N on the child.

Determine the value of m.

We will let the acceleration of the lift be **a**.

The child has two forces acting on it; the reaction force from the father (250N) and the weight force, **mg**.

Since the lift is accelerating upwards, we can show: **250 - mg = ma**.

Now we will form a second equation, because we currently have two unknowns.

As a whole, the lift has a mass of **800 + 80 + m**. The weight of the lift is therefore **(880 + m)g**.

The resultant force acting on the lift is **mass * acceleration = (880+m)a**.

Since the lift is accelerating upwards, we can say: **9050 - (880+m)g = (880+m)a**.

Expanding the brackets gives: **9050 - 880g - mg = 880a + ma**.

We previously showed that **250 - mg = ma**, so if we substitute this for **ma** in the above equation, we get:

9050 - 880g - mg = 880a + 250 - mg.

Now, **mg** cancels to give:

9050 - 880g = 880a + 250

Thus, **a = (9050-880g-250)/880 = 0.19ms^{-2}**.

We can substitute this into the original equation to find **m**:

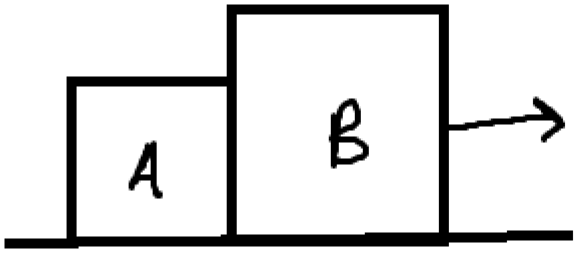
250 - mg = 0.19m

250 = m(0.19+g)

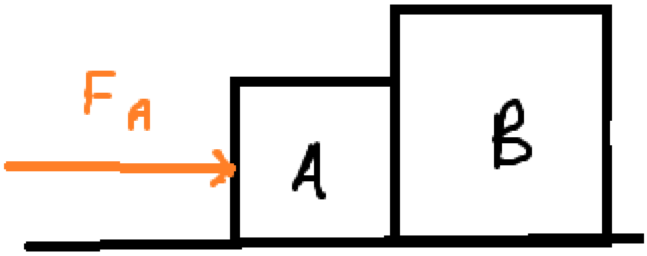
m = 25kg.

Example 2: one box pushing another box

Suppose a box, A, pushes another box B, along a rough surface:

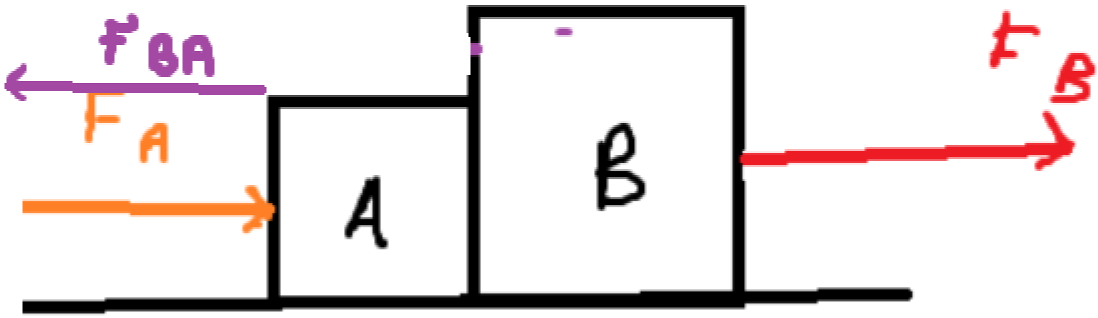


For this to happen, a force must be being exerted on box A. We will call this force F_A:



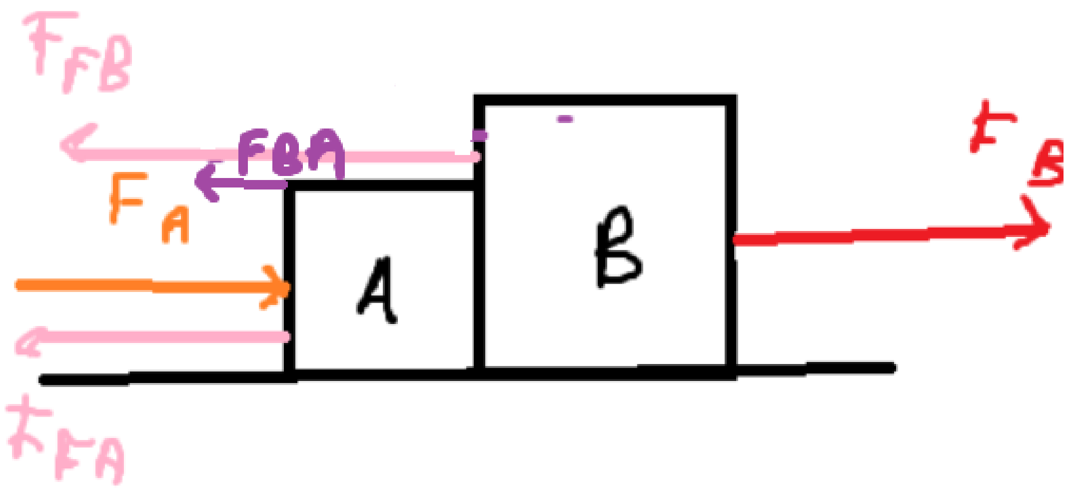
F_A will cause box A to exert a force on box B. We will call this F_B. F_B is *not equal to* F_A since box A exerts a separate force on box B than the force exerted on box A.

If A exerts a force on B (F_B), B must exert an equal and opposite force on A. We will call this force F_{BA}:



$$F_{BA} = F_B$$

Finally, since A and B are moving on a rough surface, they both experience friction. We will call the friction on A F_{FA} and the friction on B F_{FB}:



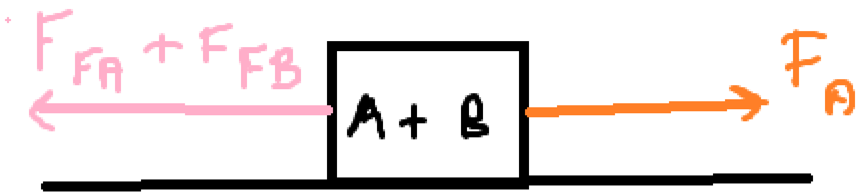
An important aspect of answering N3L questions is the ability to model two bodies as one body.

Because A and B are accelerating at the same rate, we can model them as a single body.

When we model them as a single body, we do not need to consider the forces that occur between them (these are ***internal forces***). This is because the internal forces will cancel out.

This means that we can ignore the force exerted by A on B, F_B, and the force exerted by B on A, F_{BA}.

The only forces that we consider are the **external forces**, which are the force on A, F_A, and the two frictional forces, F_{FA} and F_{FB}, which add up to a single force:



This is useful because we can consider these forces to determine the acceleration of the system. Once we know the acceleration, we can look individually at A and B to determine the force between them.

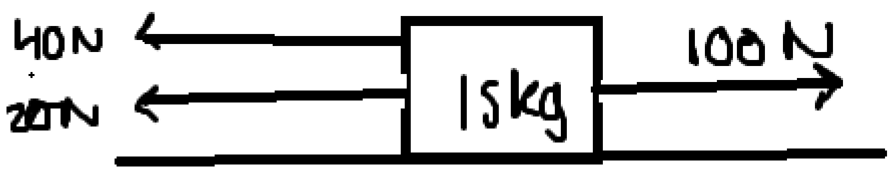
Example question: *A 10kg box has a 100N force exerted on it. The 10kg box is in contact with a 5kg box, causing the 5kg box to be pushed forwards. There is a frictional force of 40N acting against the 10kg box, and a frictional force of 20N acting against the 5kg box.*

a) Determine the acceleration of the system.

A key aspect of N3L problems is modelling two interacting bodies as a single body. When we do this, we only consider the external forces - not the internal forces.

In the case of the two boxes, the external forces are the 100N force, and the two frictional forces. The boxes exert forces on each other, as described above, but these forces are internal - so they don't influence the overall motion of the system.

We start by modelling A and B as a single box with a mass of 15kg. This has a force of 100N acting forwards, and the two frictional forces of 40N and 20N acting backwards (as shown in the example above):



The net force acting on the body is therefore 100 - 60 = 40N.

When a 40N force acts on a 15kg body, it causes an acceleration of F / m = 40 / 15 = 2.67ms^{-2}.

b) Determine the contact force that one box exerts on the other

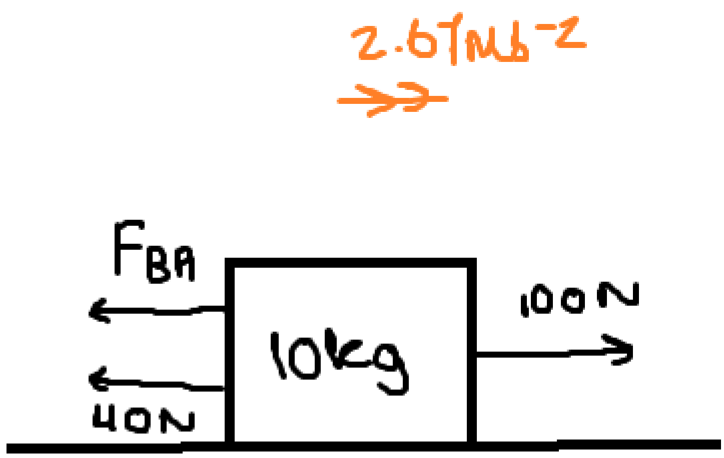
The 10kg box exerts a contact force on the 5kg box, causing the 5kg box to exert an equal and opposite contact force on the 10kg box.

To find this contact force, we consider the 10kg box and the 5kg box individually. Because the force exerted by the 10kg box on the 5kg box is equal and opposite to the force exerted by the 5kg box on the 10kg box, if we consider both individually, we should find that the contact force is the same in both situations.

Considering 10kg box:

The 10kg box has three forces exerted on it:

- The 100N force
- The 40N frictional force
- The equal and opposite contact force exerted by B (let this be F_{BA}).



The box is accelerating at 2.67ms^{-2}, and it has a mass of 10kg, so the net force required to produce this acceleration is 10 * 2.67 = 26.7N.

The overall backwards force is F_{BA} + 40.

The net force is therefore 100 - (F_{BA}+40) = 100 - F_{BA} - 40

This is equal to the net force of 26.7N, so 100 - F_{BA} - 40 = 26.7

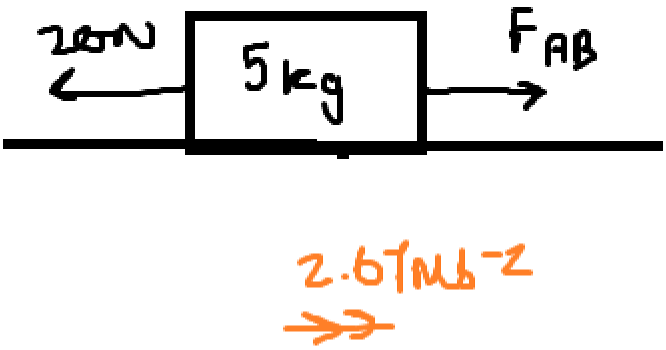
Thus F_{BA} = 100 - 40 - 26.7 = 33.3N.

This is the contact force exerted by the 5kg box on the 10kg box.

Considering the 5kg box:

The contact force exerted by the 10kg box on the 5kg box must be the same as the 33.3N force exerted by the 5kg box on the 10kg box according to N3L. The forces acting on the 5kg box in the horizontal plane are:

- The 20N frictional force
- The contact force from the 10kg box (let this be F_{AB}).



If a 5kg box is accelerating at 2.67ms^{-2}, there must be a net force of 5 * 2.67 = 13.35N acting on the box in the direction of F_{AB}.

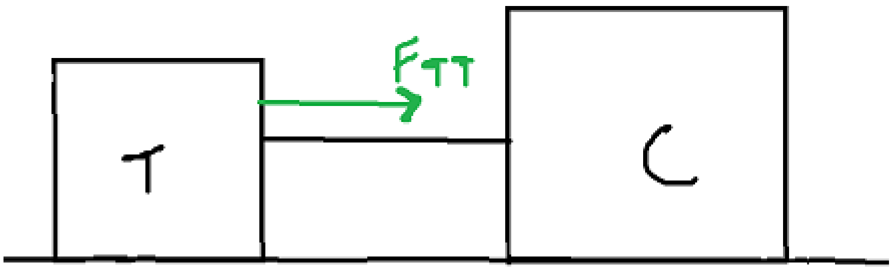
Thus, F_{AB} - 20 = 13.35 => F_{AB} = 33.35N (would be 33.3N if acceleration wasn't rounded).

We have thus proven that the contact force exerted by the 10kg box on the 5kg box is equal and opposite to the contact force exerted by the 5kg box on the 10kg box.

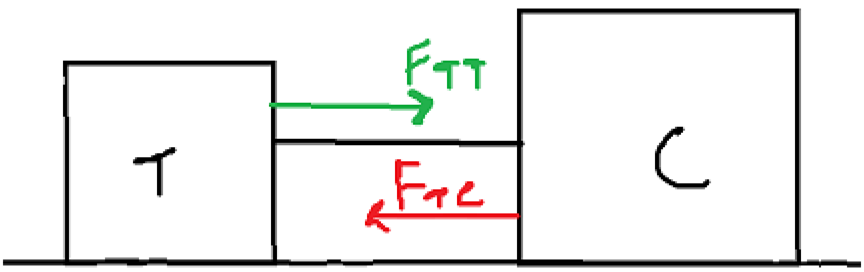
Example 3: a car towing a trailer

Suppose we have a car, C, that is accelerating whilst pulling a trailer, T, along.

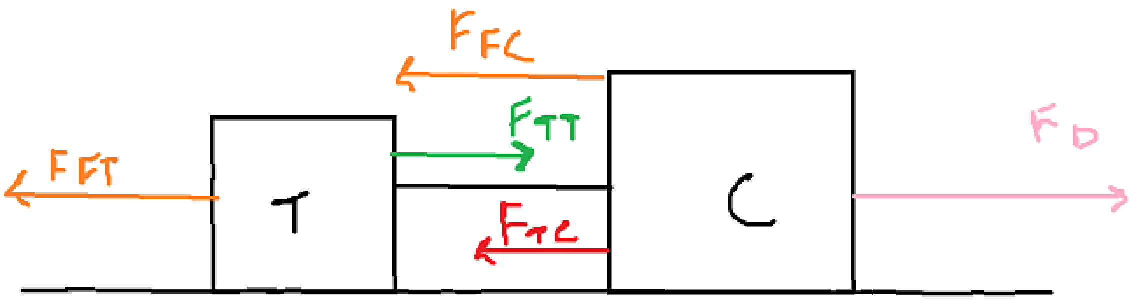
There is a tension force in the bar connecting the car to the trailer that pulls the trailer along (let this force be F_{TT}):



Following Newton's third law, if the car exerts a tension force on the trailer, the trailer must exert an equal and opposite tension force on the car (let this be F_{TC}):



If the road is rough, there will be a friction force acting against T and C (let this be F_{FT} and F_{FC} respectively). The car must also have a driving force (let this be F_D).



Like in the previous example, we can model the trailer and car as one body, with the tension forces F_{TT} and F_{TC} not existing in the overall model because they are internal and cancel each other out.

We can then consider each body individually to find F_{TT} and F_{TC}. These forces should have an equal magnitude since they are equal and opposite.

Example problem: *A 1000kg car tows a 200kg trailer. The car has a driving force of 4kN, causing the system to accelerate forwards, and a resistance to motion due to friction of 500N. The trailer has a friction of 200N.*

a) ***Find the tension in the tow bar.***

First, we can model both the car and trailer as a single body with mass 1200kg, a driving force of 4kN, and a total resistance of 500 + 200 = 700N:

The net force is 4000 - 700 = 3300N.

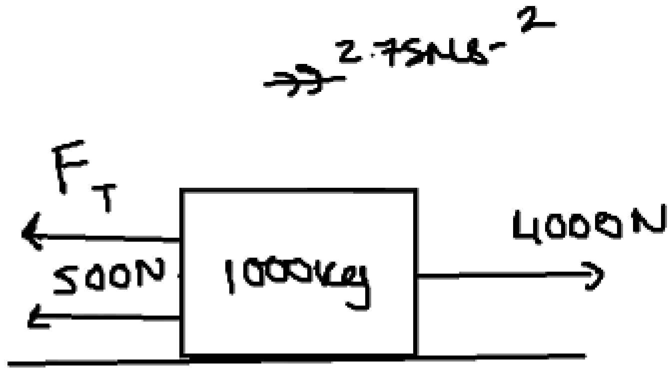
The acceleration of a 1200kg body experiencing a force of 3300N is: 3300 / 1200 = $2.75ms^{-2}$.

We can now consider either the trailer or the car to determine the tension force.

Car:

The car has three forces acting on it in the horizontal plane:

- A driving force of 4000N
- A friction of 500N
- An unknown tension force exerted by the trailer



We can say that: 4000 - (F_T + 500) = net force.

Net force = mass * acceleration = 1000 * 2.75

4000 - (F_T + 500) = 1000 * 2.75

4000 - F_T - 500 = 2750

F_T = 4000 - 500 - 2750 = 750N.

Trailer:

We can prove that this force is equal for the trailer. The forces acting on the trailer are:

- Friction of 200N
- Tension force acting in the direction of travel, F_T

Net force = F_T - 200
Net force = m * a = 200 * 2.75 = 550N

F_T - 200 = 550 => F_T = 750N.

The 200kg mass of the trailer consists of a 150kg mass of the trailer itself, and a 50kg box inside of the trailer.

b) **Find the reaction force exerted by the road on the 150kg trailer structure**.

This problem is similar to the situation where there is one box on top of another box in a lift.

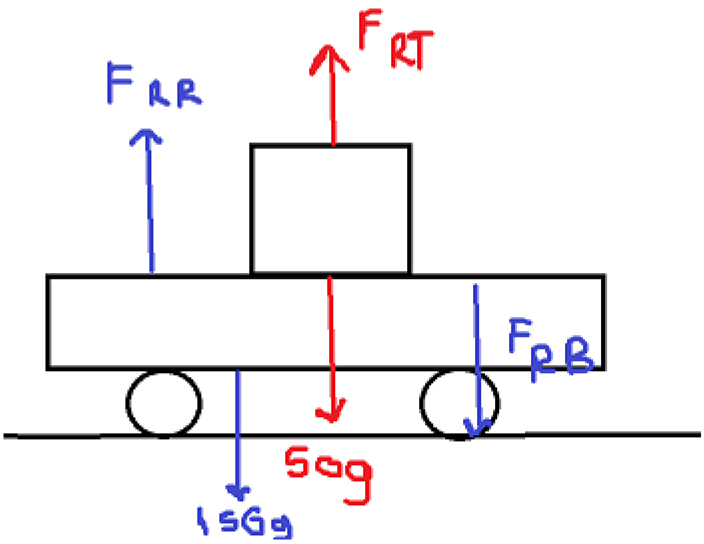
The 50kg box has two forces acting on it in the horizontal plane:

- The downwards weights force (50g)
- The upwards normal reaction force exerted by the trailer (F_{RT})

The 150kg trailer has three forces acting on it in the horizontal plane:

- The downwards weight force (150g)
- The normal reaction force from the road (F_{RR})
- The equal and opposite reaction force from the box (the trailer exerts the force F_{RT} on the box, so the box exerts an equal and opposite force in the downwards direction) - let this be F_{RB}.

This can be considered in the following diagram:



If we first consider the box on the trailer, since the box is not moving in the vertical plane, we can say that the reaction force and the weight force must be equal to each other (it is in *equilibrium*).

Thus, F_{RT} = 50g = 50 * 9.81 = 490.5N.

The magnitude of the reaction force exerted by the trailer on the box is equal to the magnitude of the reaction force exerted by the box on the trailer.

Thus, F_{RB} = 490.5N.

If we now consider the trailer structure, it has two downwards forces acting on it, F_{RB} and 150g, and one upwards force, F_{RR}. Since the trailer is not moving vertically, these forces must be equal.

F_{RR} = 150g + F_{RB} = (150 * 9.81) + 490.5 = 1960N.

Hence, the reaction force exerted by the road on the trailer is 1960N.

-

Another way to consider this is to consider the trailer and the box as one particle. The trailer and the box have a combined mass of 200kg, so there is a weight force of **200g** acting downwards. There is also the reaction force exerted by the floor and the trailer structure. We do not have to consider the internal reaction forces between the box and the trailer as they cancel out. Thus, the total downwards force, **200g**, is equal to the reaction force, which makes the reaction force **200g**, which is **1960N**.

-

Now suppose the car begins to decelerate, and the driving force of the car is now replaced with a braking force opposing the direction of motion (F_B).

Now, rather than the trailer being pulled along by the car by a tension force, the trailer begins to be compressed towards the car. This causes the trailer to exert a force on the car: a **thrust** force (let this be F_{ThC}). This force occurs in the forward direction.

Following Newton's third law, if the trailer exerts a forward force on the car, the car responds by exerting an equal and opposite force on the trailer (let this be F_{ThT}).

The frictional forces (F_{FC} and F_{FT}) still exist.

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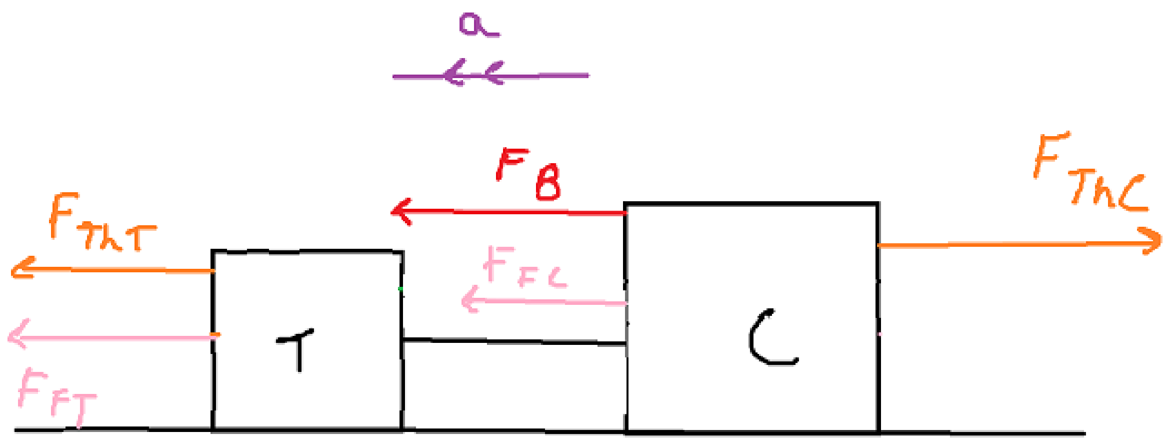


Example problem: A car of mass 800 kg pulls a trailer of mass 200 kg along a straight horizontal road using a light towbar which is parallel to the road. The horizontal resistances to motion of the car and the trailer have magnitudes 400 N and 200 N respectively. The car is moving along the road when the driver sees a hazard ahead. He reduces the force produced by the engine to zero and applies the brakes. The brakes produce a force on the car of magnitude F newtons and the car and trailer decelerate. Given that the resistances to motion are unchanged and the magnitude of the thrust in the towbar is 100 N, **find the value of F**.

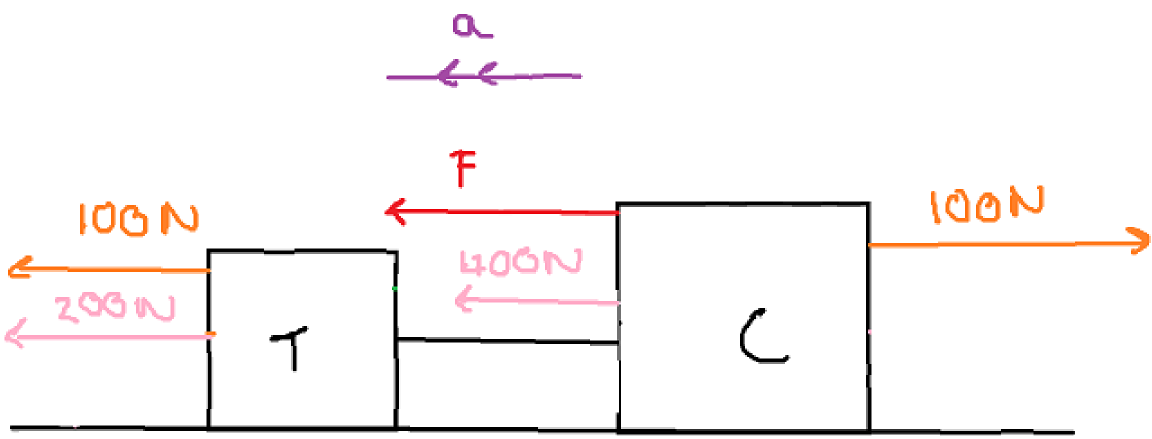
First let us consider all of the forces acting.

The car has three forces acting on it:

- The braking force, F, in the backwards direction.
- The thrust force of 100N in the forward direction.
- A frictional force of 400N acting in the backward direction.

The trailer has two forces acting on it:

- The equal and opposite thrust force from the car, acting in the backwards direction.
- A frictional force of 200N acting in the backwards direction.



We need to determine the value of F.

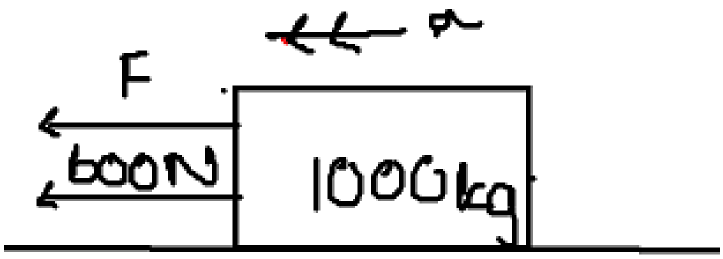
First, let us consider the entire system.

When we consider both the trailer and the car (total mass 1000kg), we ignore the forces acting between the trailer and the car (the internal forces), which are the thrust forces.

We therefore have three forces in the entire system:

- The backwards driving force, F.
- The two frictional forces acting on the car and trailer (total magnitude of 600N).

There are no forces in the forward direction.



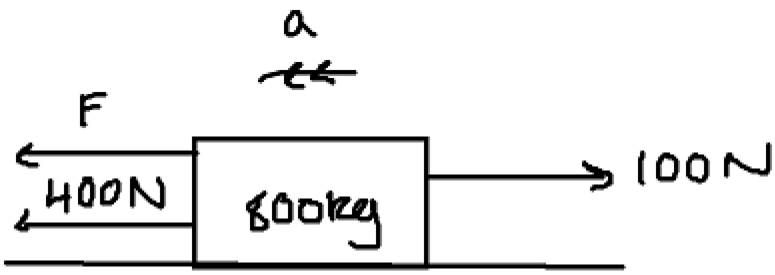
The overall force acting on the system is F + 600.

This is equal to the mass of the system multiplied by the acceleration, 1000 * a.

$$F + 600 = 1000a$$

This does not allow us to solve for any values, but we can say that $F = 1000a - 600$.

Now we can consider the forces acting on the car only:



The net force is: $(F+400) - 100$

The net force is equal to the mass * acceleration = $800a$.

Thus, $F+400-100 = 800a \Rightarrow F = 800a - 300$.

We previously found that $F = 1000a - 600$.

Thus, $1000a - 600 = 800a - 300$

$200a = 300 \Rightarrow a = 1.5ms^{-2}$.

Thus, we can find F.

$F = 1000(1.5) - 600 = 900N$.

///

Example problem: *A person of mass 60kg pulls a 40kg crate with a rope with a force of 100N. The person and the crate accelerate at 0.1ms^{-2} as a result of this pulling action.* ***Find the force of friction between the floor and the person and the floor and the crate****.*

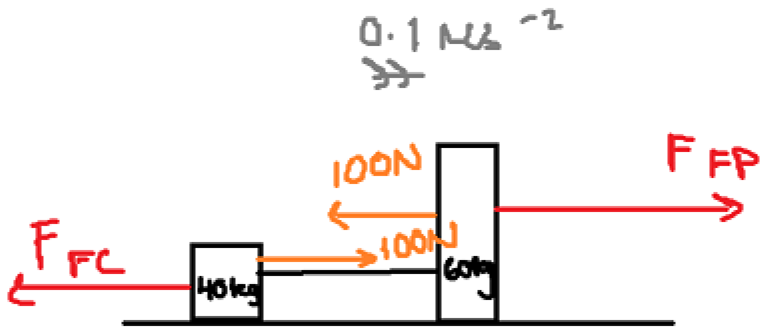
This is similar to a car tugging a trailer - there is a tension force of 100N pulling the crate along. The crate must exert an equal and opposite tension force of 100N on the person.

There is a friction force between the crate and the ground in an opposite direction to the direction of motion (F_{FC}).

For the person to be accelerating, there must be a force that is greater than the tension exerted by the crate that is in the direction of travel. The force causing the system to accelerate is a frictional force (F_{FP}).

Previously, the frictional force has always opposed the direction of travel, but friction does not always have to do this. Sometimes, friction can occur in the direction of travel. The key idea of friction is that it opposes slippage.

The crate is slipping towards the person, so the friction opposes this slippage. The person is being pulled by the crate due to the tension force, so the friction opposes this slippage.



Considering the crate only:

The forces acting on the crate are the 100N tension force and the frictional force, F_{FC}. The crate has a mass of 40kg and is accelerating at 0.1ms^{-2}.

Thus, 100 - F_{FC} = 40 * 0.1 => F_{FC} = 100 - (40*0.1) = 96N.

Considering the person only:

The net force acting on the person is F_{FP} - 100, and the person is accelerating at 0.1ms^{-2}.

Thus, $F_{FP} - 100 = 0.1 * 60 \Rightarrow F_{FP} = (0.1 * 60) + 100 = 106N$.

-

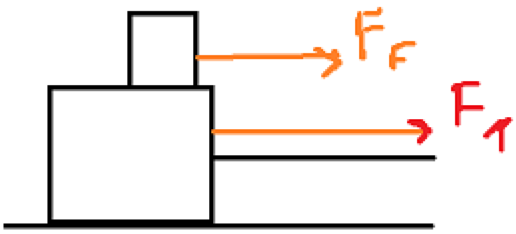
Example 4: sliding box on another box

Suppose you have a box being pulled along by a rope on a smooth surface (so no friction). The box must be being pulled by the tension force in the rope (F_T).

Because the small box is not being pulled by the rope, it has no direct force being exerted upon it.

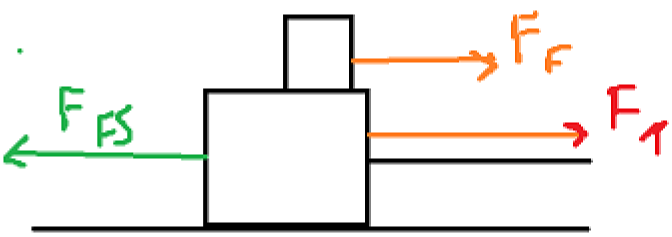
If the large box was frictionless, because there is no force being exerted on the small box, it would quickly slide off the large box when it begins to accelerate. If the large box is rough, on the other hand, the box will not slide off instantly, as there will be a resistive force - friction (F_F) - resisting the change in motion.

The friction will act in the direction of travel and will cause the small box to have a small acceleration in the direction of travel:



However, the acceleration of the small box due to friction cannot be greater than the acceleration of the large box due to the tension force. This means that the large box will be accelerating at a higher rate than the small box, so the small box will begin to slide backwards along the large box.

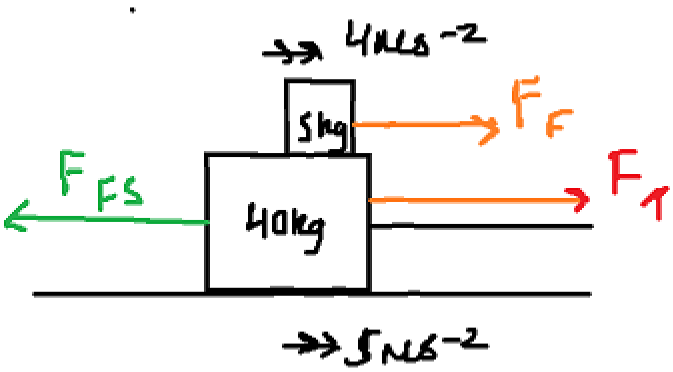
If the large box exerts friction on the small box, the small box must exert an equal and opposite friction on the large box (let this be F_{FS}):



Example problem:

A 40kg box is pulled by a rope along a frictionless surface, creating an acceleration of 5ms^{-2}. On top of the 40kg box is a 5kg box that experiences friction with the 40kg box as it accelerates. This friction causes the 5kg box to accelerate at 4ms^{-2} in the direction of the large box.

Find the tension in the rope.



The 5kg box is accelerating at 4ms^{-2}, and there is only one force, the friction, causing this acceleration.

The friction is therefore: F = 4 * 5 = 20N.

The 5kg box exerts an equal and opposite friction on the 40kg box, so the 40kg box has a resistance to motion of 20N.

Thus, the net force acting on the 40kg box is F_T - 20. This net force causes an acceleration of 5ms^{-2}, so F = 5 * 40 = 200N.

Thus, $F_T - 20 = 200N \Rightarrow F_T = 220N$.

/

Summary of Newton's Third Law and connected particles

- If body A exerts a force on body B, body B exerts an equal and opposite force on body A of the same type.

- This law only applies when the forces act on separate bodies, and the equal and opposite forces are of the same type.
- When two particles are connected and in-motion (or stationary in some instances), there will be equal and opposite forces exchanged between the particles.
- We can consider the connected particles as one overall particle. In such an instance, we do not include the internal forces between the particles; as these are equal and opposite and thus cancel out.

Vectors

Vector Introduction

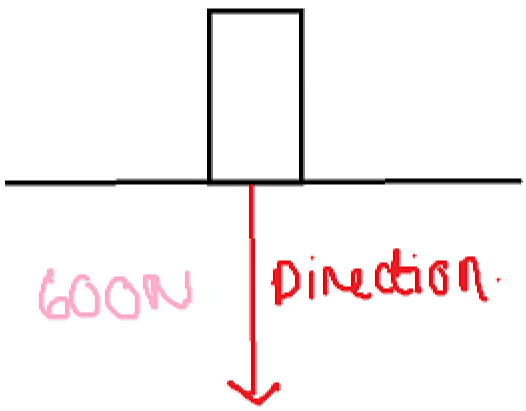
A quantity in physics can either be **scalar** or **vector**.

- A scalar quantity *only has magnitude*
- A vector quantity has magnitude *and* direction.

An example of a vector quantity is force. All forces have a magnitude and they occur in a certain direction.

For example, suppose a person has a weight of 600N.

The magnitude of the force is 600N, and the direction in which the force acts is towards the centre of the gravitational field in which the person is located.



An example of a scalar quantity is mass. Mass only has magnitude, it does not occur in any particular direction. The *vector* quantity of mass is weight.

Another example of a scalar quantity is speed. Speed does not occur in any direction; a body simply has a speed of xms^{-1}. The vector quantity of speed is *velocity*. Velocity is the speed in a particular direction.

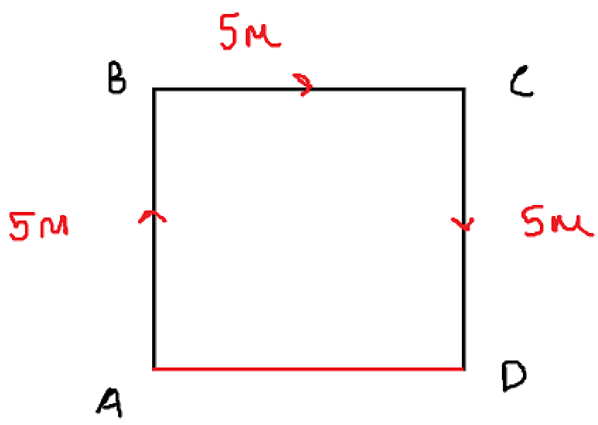
Scalar	Corresponding Vector quantity
-	Force
Speed	Velocity
-	Acceleration
Mass	Weight
Distance	Displacement
-	Momentum
Energy	-

Comparing distance to displacement

Distance is a scalar quantity; a body travels a certain distance, but this is not in any particular direction.

The vector quantity of distance is **displacement**.

Displacement is defined as the distance of a body from the origin.

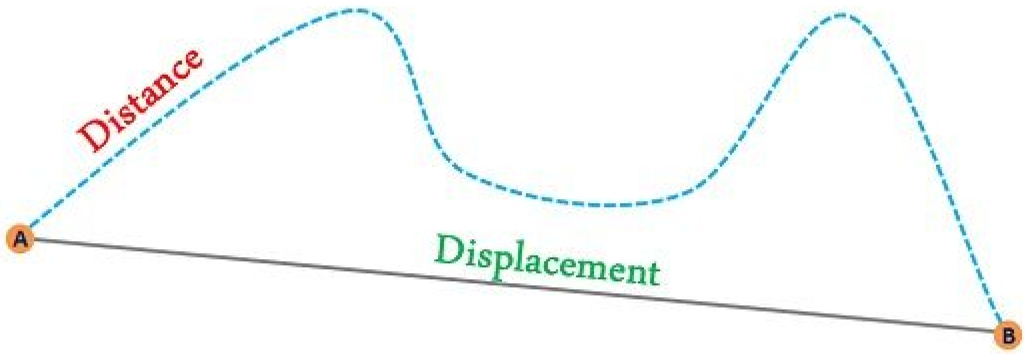


Suppose a particle travels from A to B to C to D. The total distance that the particle travels is 15m.

The origin of the particle is point A. When it is at point D, the particle is 5m from the origin. This means that the **displacement** of the particle is 5m.

So the particle travels 15m, but is only *displaced* by 5m from the origin.

The displacement is simply the length of a straight line from the origin to the current position of a particle.



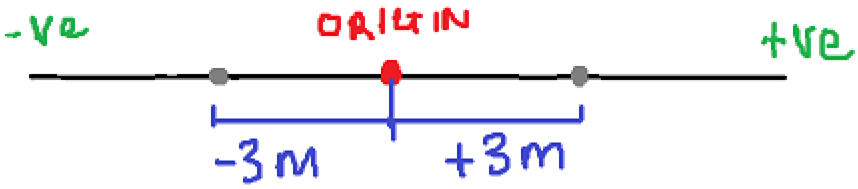
To represent vectors mathematically, we have to use positive and negative values.

When considering vectors, we always define a ‘positive direction’ and a ‘negative direction’.

For example, suppose a particle is confined to a line in the horizontal plane, and it can move along this line in any direction. We need to define a positive direction, and a negative direction. We will define any position right to the origin as positive, and any position to the left of the origin as negative:

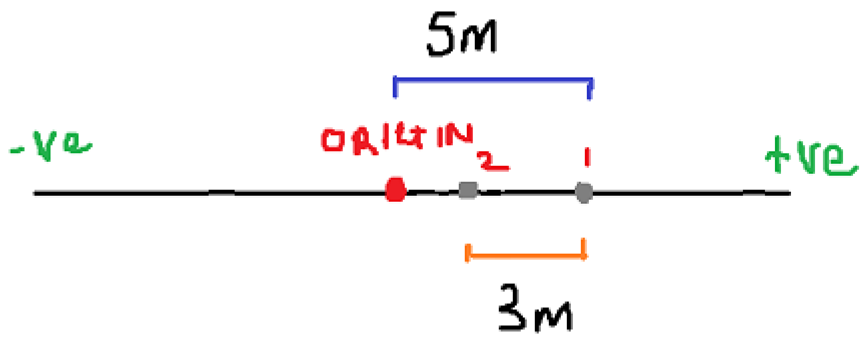
If the particle moves 3m to the right of the origin, its displacement is +3m.

If the particle moves 3m to the left of the origin, its displacement is -3m.



Suppose the particle is subject to two displacements, a +5m displacement, followed by moving 3m to the left.

The particle will move 5m away from the origin, and then to a distance of 2m from the origin. Its final displacement is therefore **2m**.



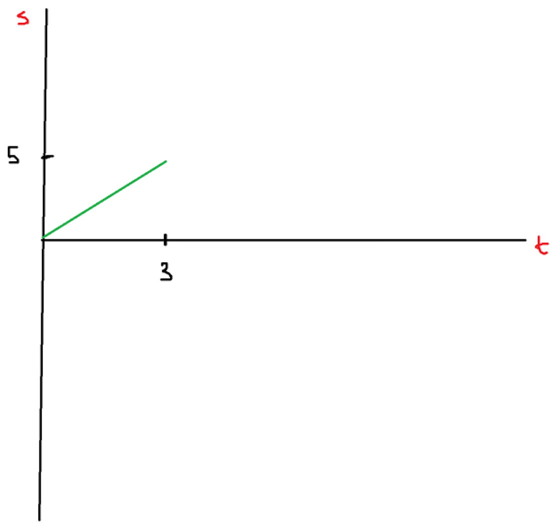
The displacement of a particle over time can be represented in a displacement-time graph.

Suppose a particle takes the following path, starting from the origin:

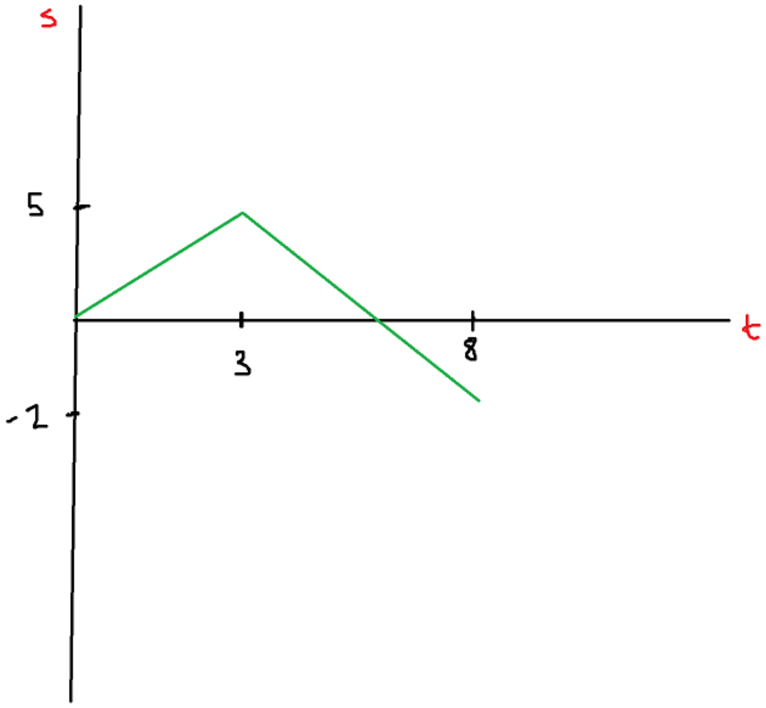
- Moves 5m to the right of the origin over 3s.
- Moves 7m to the left of this position over 5s.
- Moves 6m to the left of this position over 2s.
- Moves 15m to the right of this position over 5s.

We will define the right as the positive displacement, and the left as the negative displacement.

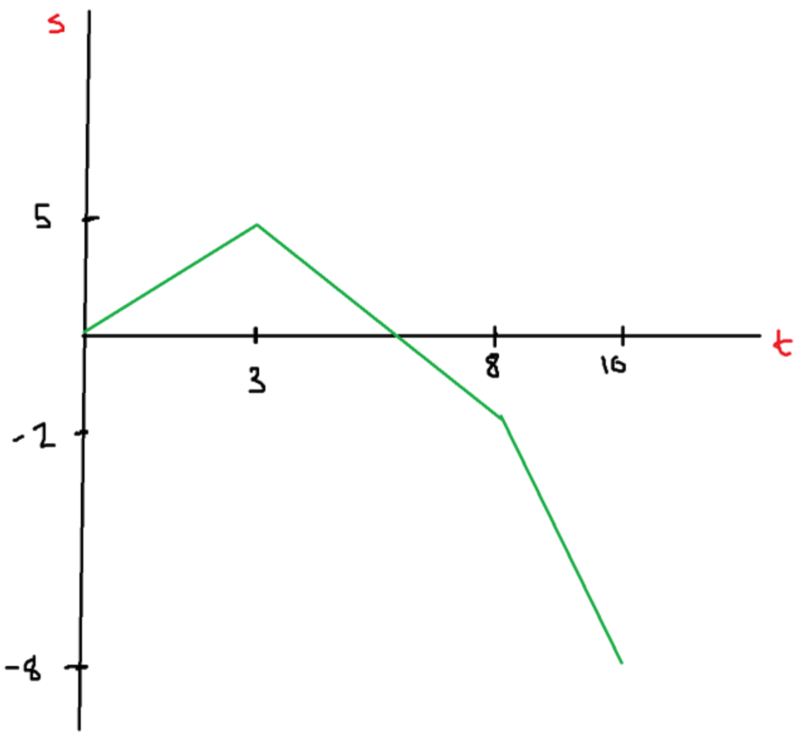
The particle starts by moving to a displacement of 5m over 3s:



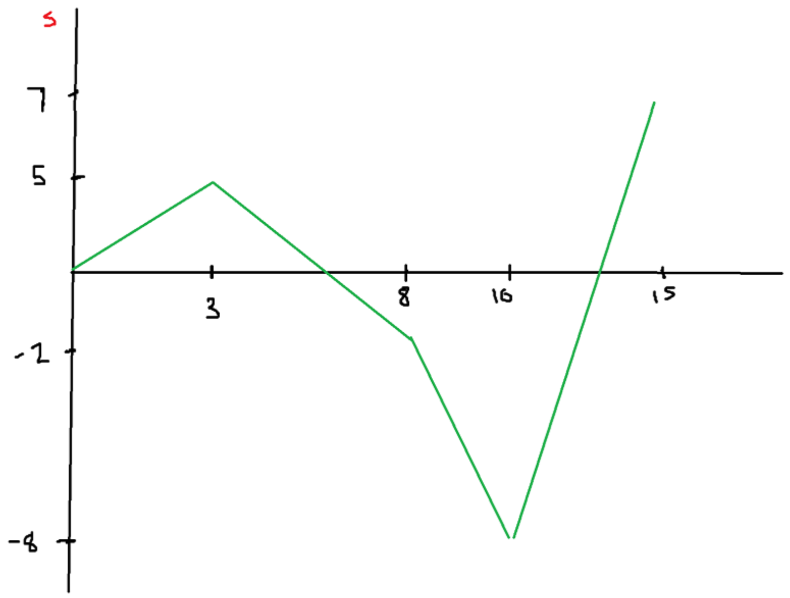
The particle then moves 7m to the left over 5s. Its final position will be 2m to the left of the origin, which is a displacement of -2m:



Next, the particle moves a further 6m to the left over 2s. The particle therefore moves from a displacement of -2m to a displacement of -8m:



Finally, it moves 15m to the right over 5s. Its displacement will therefore change from -8m to (-8+15) = 7m.



So after the 15s of motion, the displacement of the particle is 7m from the origin. However, the particle has travelled a total distance of: 5 + 7 + 6 + 15 = 33m.

As we will see soon, **velocity = displacement / time**, so the **average velocity** during the journey is the final displacement, 6m, over 15s, 6 / 15 = 0.4ms^{-1}.

Speed = distance / time, so the **average speed** of the journey is 33 / 15 = 2.2ms^{-1}.

-

Comparing speed to velocity

Speed considers how far an object travels (not in any given direction) over time. For example, if an object travels 50m in 20s, its speed must be 50 / 20 = **2.5ms^{-1}**.

Velocity, on the other hand, does consider the direction of motion. Velocity is equal to the displacement of a particle over the time. Since displacement is a vector, velocity must be a vector.

For example, if a particle moves 5m away from the origin in the positive direction 10s, its velocity is 5 / 10 = **0.5ms^{-1}**.

-

Speed and velocity are only the same when a particle moves in a single direction for the entire journey. If a particle changes its direction, its velocity becomes negative, which means that speed and velocity are no longer the same.

Suppose a particle moves 5m from the origin over 3s. It then moves 4m back towards the origin in a further 2s.

The total **distance** travelled by the particle is **9m**. This is over a time of **5s**, so the average speed for the journey is **9/5 = 1.8ms^{-1}**.

However, the total **displacement** for the journey is only **1m**, because the particle ends up 1m from the origin. This means that the average velocity for the journey is **1 / 5 = 0.2ms^{-1}** => which shows that speed and velocity are not the same if a journey involves a change in the direction of motion.

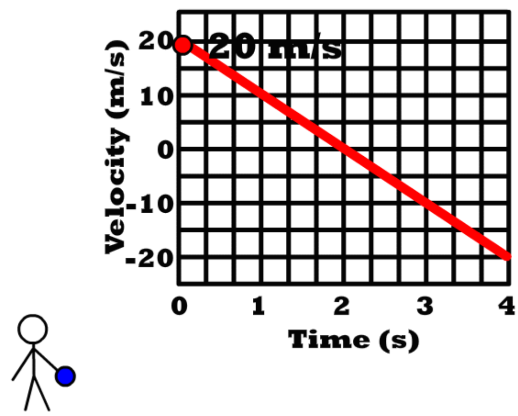
-

When considering velocity, we have to again consider a positive and negative direction due to velocity being a vector quantity.

Suppose a ball is thrown upwards, and we define upwards as positive, and downwards as negative.

When the ball is travelling upwards, its velocity is positive, because its displacement is increasing over time.

When the ball is travelling downwards, its velocity is negative, because its displacement is now decreasing over time (becoming more negative) - as it is returning to the origin, and thus its distance from the origin is decreasing.



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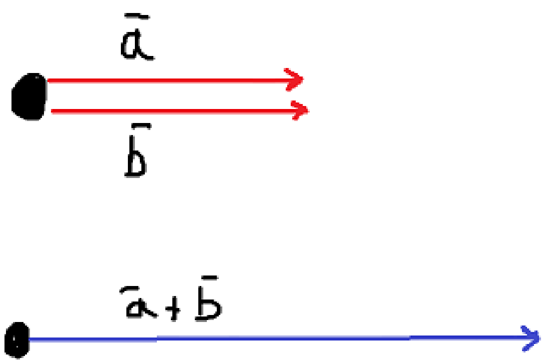
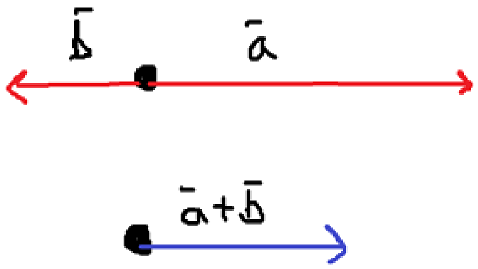
A particle travelling in the positive direction thus has a positive velocity, and a particle travelling in the negative direction has a negative velocity.

Vector addition in one dimension

When two vectors act in the same direction on the same particle, they add up to a single larger vector:



Similarly, when two vectors oppose each other, they add up into a single vector:



*In the example above, **b** would have a negative value that opposes the direction of **a**.*

Thus, when you have two vectors, **a** and **b**, the sum of the vectors is: **a** + **b**.

For example, suppose a particle is subject to two velocities in the horizontal plane: $6ms^{-1}$ to the right, and $2ms^{-1}$ to the left.

If we define right as positive and left as negative, the particle is subject to two velocities:

- $v_1 = 6ms^{-1}$
- $v_2 = -2ms^{-1}$

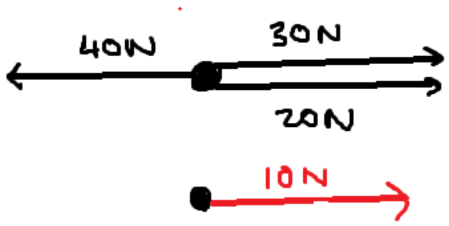
The overall velocity is $v_1 + v_2 = 6 - 2 = 4ms^{-1}$.

Because this overall velocity is positive, the particle is moving to the right with a velocity of $4ms^{-1}$.

We have been utilising the principle of vector addition in force and Newton's laws calculations.

For example, suppose a body has two forces, F_1 and F_2 to the right, and a force F_3 to the left, acting upon it:

F_1 = 30N
F_2 = 20N
F_3 = -40N



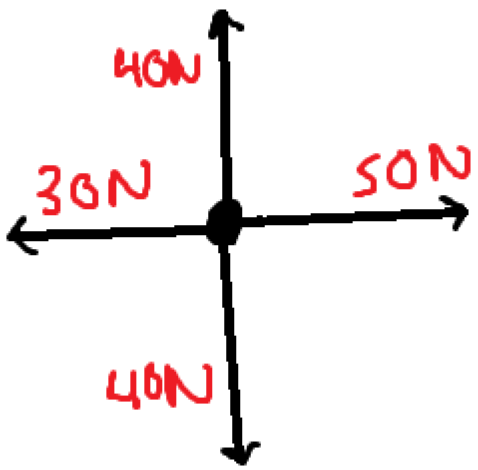
The net force is 30 + 20 - 40 = 10N. This force is positive, so it acts to the right, the direction we defined as positive.

Similarly, suppose a particle has an acceleration of $4ms^{-2}$ to the right and $2ms^{-2}$ to the left. The overall acceleration is 4 - 2 = $2ms^{-2}$ to the right.

Addition of vectors in two dimensions

In the previous Newton's Laws calculations, we never carried out calculations in both the horizontal and vertical planes: these planes were considered separately.

In vector calculations, the horizontal and vertical planes are considered as independent. Vectors can only be added together if they are **coplanar** - meaning *in the same plane*. We can never add a horizontal force to a vertical force; only a horizontal force to a horizontal force and a vertical force to a vertical force.



In the example on the left,

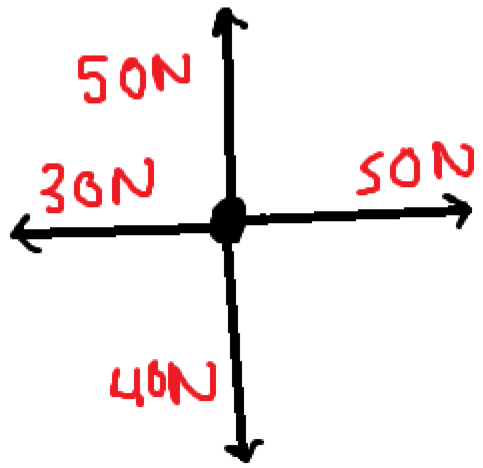
Net force in the horizontal plane = 50 - 30 = 20N

Net force in the vertical plane = 20 - 20 = 0N

This means that the particle will accelerate in the horizontal plane, but not in the vertical plane.

The horizontal and vertical planes are considered independently of each other.

Now consider this situation:



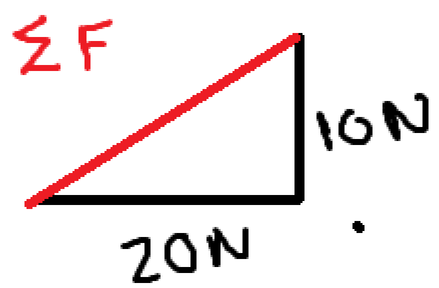
Resultant force in vertical plane = 50 - 40 = 10N upwards
Resultant force in horizontal plane = 50 - 30 = 20N to the right

This body therefore has 2 forces acting on it in different planes:

<u>So in what direction will the particle move?</u>

The particle is being subject to two forces, a 20N force to the right, and a 10N force upwards. If the particle had a mass of 1kg, this would mean that it is accelerating $20ms^{-2}$ to the right and $10ms^{-2}$ upwards. These forces will combine into a single, or resultant, force.

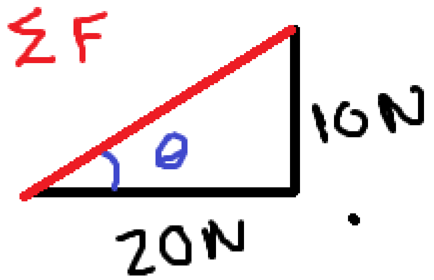
To find the resultant force acting on the particle, we use the Pythagorean theory:



The resultant force, ∑F, is given by ∑F = $\sqrt{(20^2 + 10^2)}$ = 22.3N.

So, the 20N force to the right and the 10N force upwards combine to give an overall force of 22.3N.

To find the direction of travel of the particle, we consider trigonometry.



$$\tan\theta = O/A = 10/20 = 0.5$$

$$\therefore \theta = \tan^{-1}(0.5) = 26.6°$$

So the particle is subject to a 22.3N force at an angle of 26.6° to the horizontal.

If it has a mass of 5kg, its overall acceleration is therefore 22.3 / 5 = 4.46ms^{-2} at 26.6° to the horizontal.

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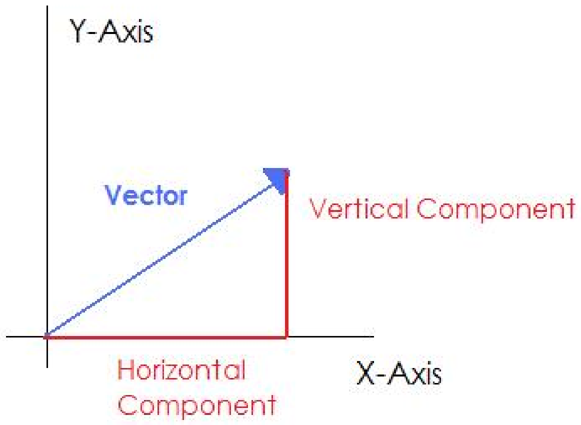
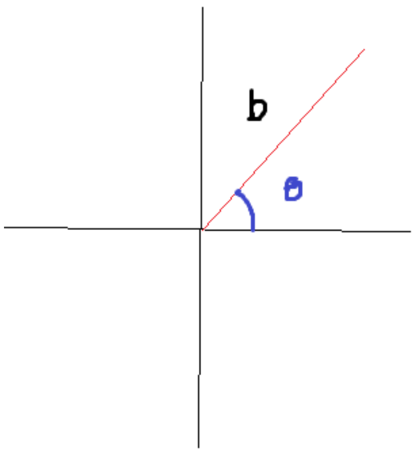
We will now generalise the above principles to all vector calculations.

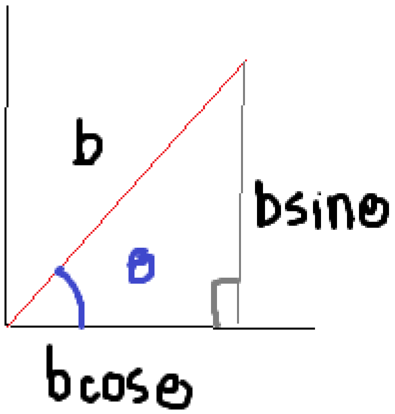
All vectors that act at an angle to the horizontal plane can be **resolved** into horizontal and vertical **components**.



Suppose we have a vector, **b**, acting at an angle of Θ:



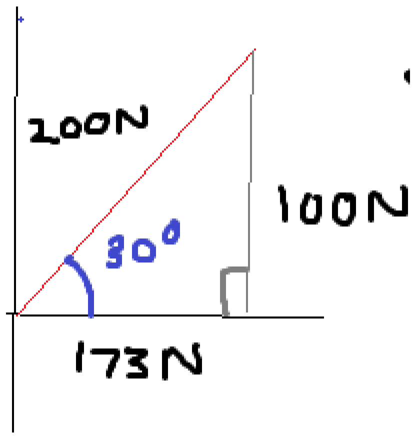
If we consider **b** as a right-angled triangle, using trigonometry:



- The **horizontal component** of b is equal to **bcosΘ**
- The **vertical component** of b is equal to **bsinΘ**

For example, suppose you have a force of 200N acting at 30° to the horizontal:

The horizontal component of the force is 200cos30 = 173N
The vertical component of the force is 200sin30 = 100N.

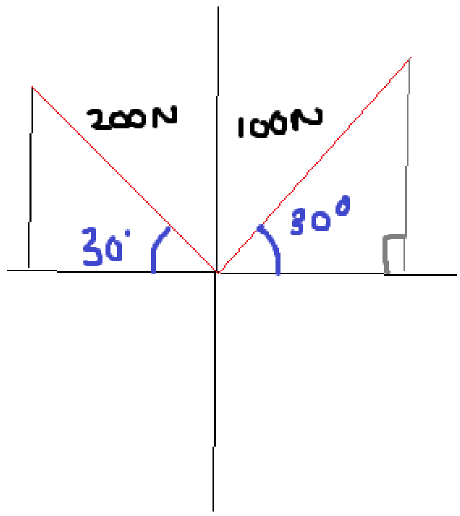


These components would combine as shown previously to a 200N force ($\sqrt{(100^2+173^2)}$ = 200).

/

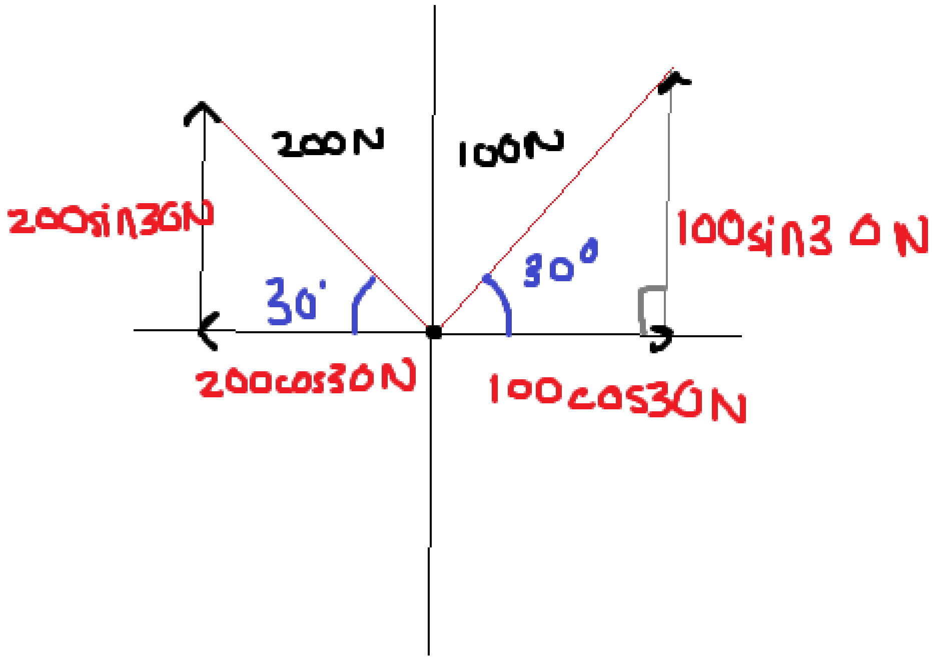
Considering the force in terms of its components is useful as we can only add the components of a vector; never the overall magnitude of it.

- Suppose you have a force of magnitude 100N acting at a bearing of 060, and a force of magnitude 200N acting at a bearing of 300, both acting on the same body, and we want to determine the **resultant force** acting on the body, and the direction of this force:



In this current layout, there is no way to determine the overall force that is acting on the body, because you can only add forces together if they are *coplanar*.

We therefore have to resolve the forces into their horizontal and vertical components:



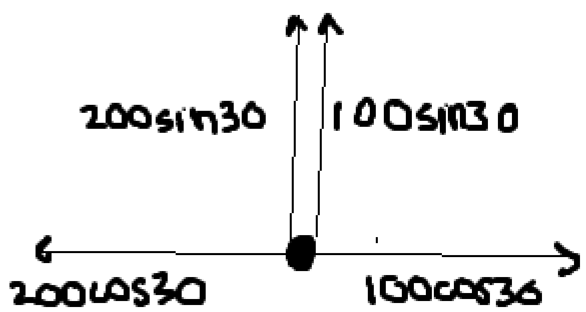
So, in the vertical plane, the particle has two forces acting on it:

- A force of 200sin30N
- A force of 100sin30N

In the horizontal plane, the particle has two forces acting on it:

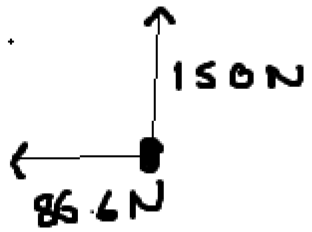
- A force of 100cos30N
- A force of 200cos30N

Now we will consider this in a free-body diagram:

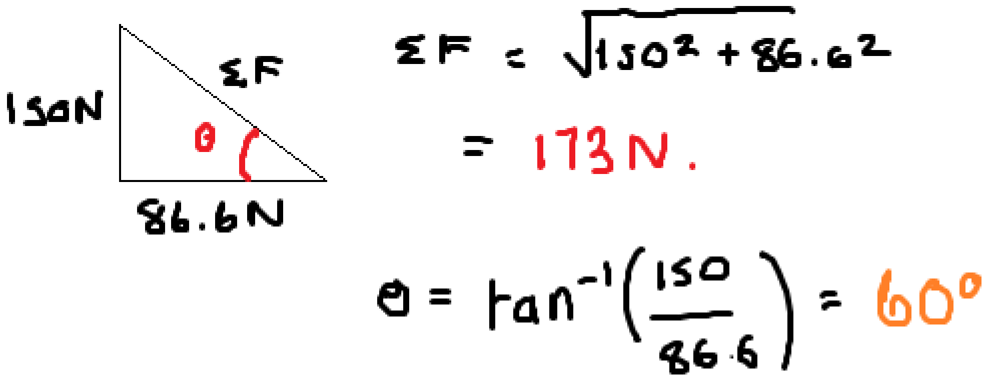


This enables us to add together the forces in the horizontal and vertical planes:

- Total force in horizontal plane = 200cos30 - 100cos30 = 86.6N to the left
- Total force in vertical plane = 200sin30 + 100sin30 = 150N upwards



From this, we can find the resultant force using Pythagoras' theorem, and the direction of the force using trigonometry:



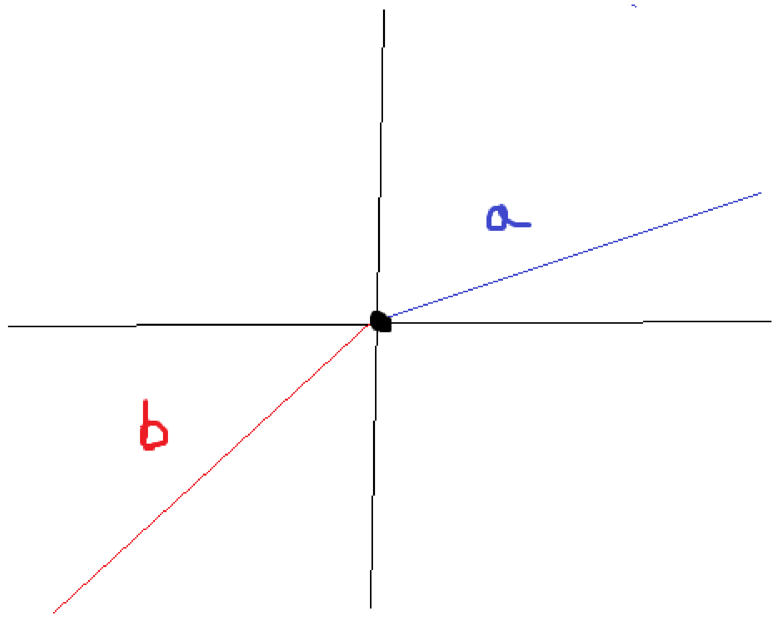
So, the body has an overall force of 173N exerted on it, which will cause it to accelerate at a bearing of 330°.

Suppose the body has mass 10kg - its acceleration must be 173 / 10 = 17.3ms^{-2} at a bearing of 330°.

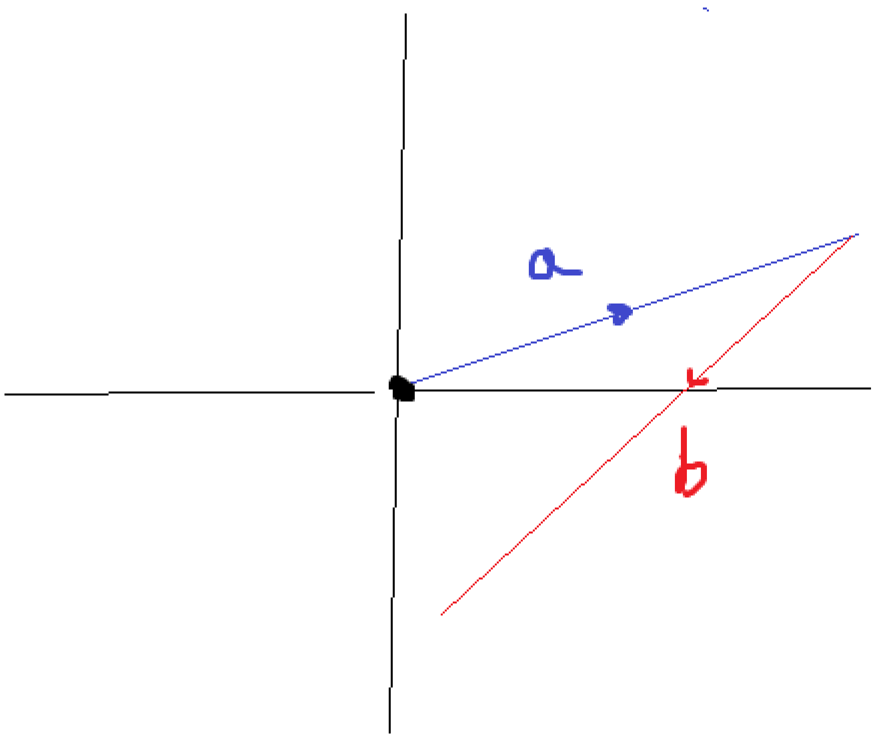
Adding force vectors from head-to-tail

Another way to consider force vectors is the addition of vectors from head-to-tail.

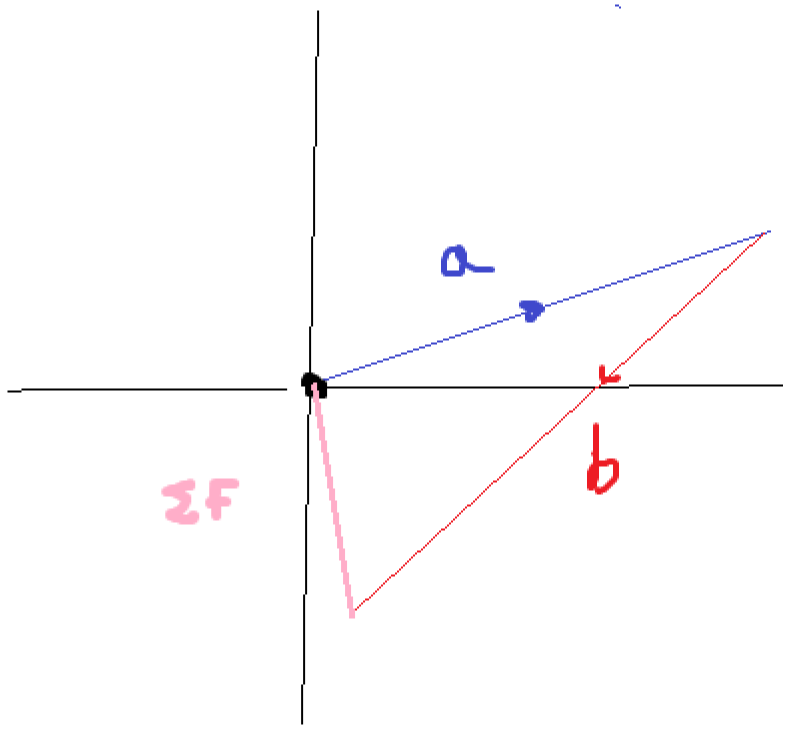
Suppose a body has two force vectors acting on it, **a** and **b**, where **a** occurs at 30° to the horizontal plane, and **b** occurs at a bearing of 225°:



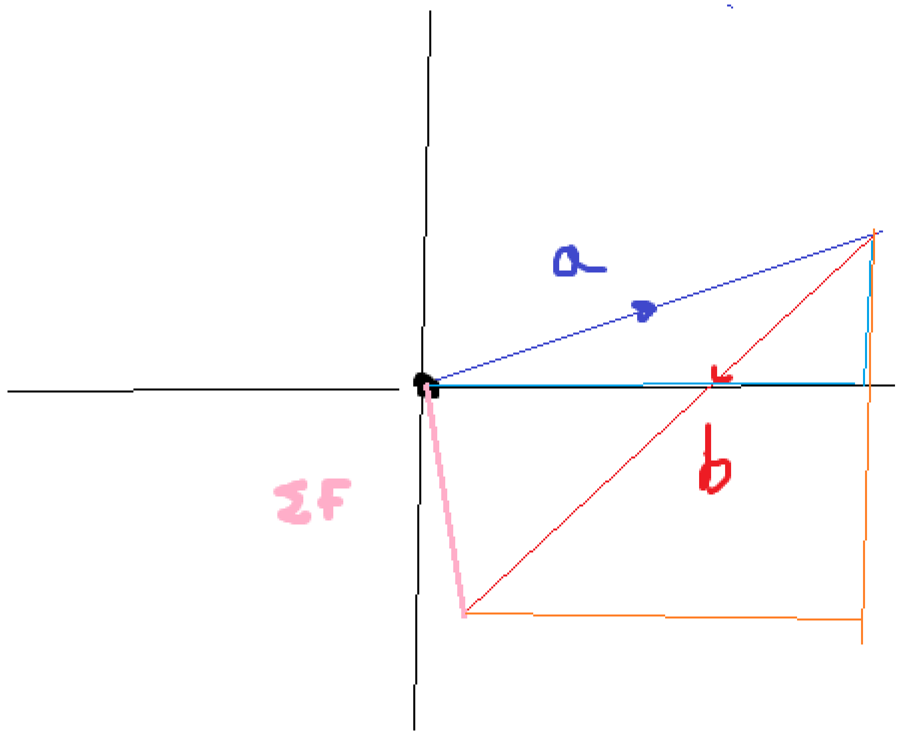
One way to consider the resultant force would be to add **a** and **b** from head-to-tail:



The resultant force would be the line from the start of **a** to the end of **b**:



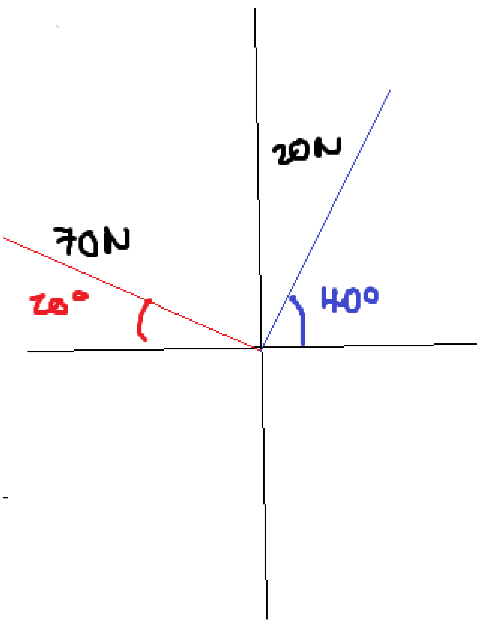
This visualisation can be explained by considering the components of **a** and **b**:



By adding the horizontal components together and the vertical components together, then finding the magnitude of the horizontal and vertical components, the resultant force is found.

E.g. *A particle is subject to two forces:*

F_1 = 30N at a 050º bearing
F_2 = 70N at a 290º bearing



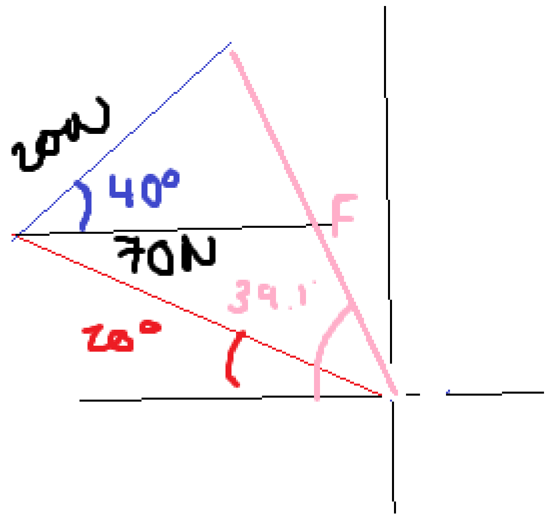
Find the resultant force

As shown previously:
The total force in the horizontal plane is: 70cos20 - 20cos40 = 45.3N to the left.
The total force in the vertical plane is: 70sin20 + 20sin40 = 36.8N upwards.

This gives a resultant force of $\sqrt{(36.8^2+45.3^2)}$ = 58.4N.

With a direction of: $\tan^{-1}(36.8 / 45.3)$ = 39.1° (bearing: 309.1°)

This can be visualised by adding the force vectors from head to tail:



Considering equilibria

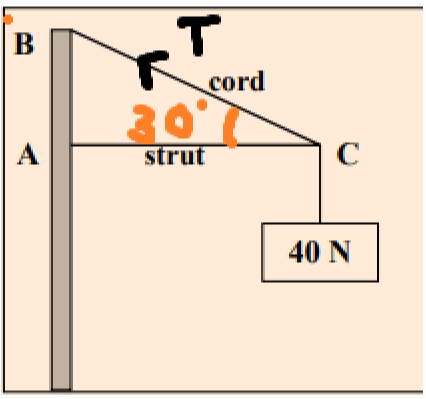
We previously established with Newton's First Law that if a body is at rest or in uniform motion, the resultant force acting on it must be 0 - as it is not accelerating in any direction (and a resultant force causes acceleration).

If a body is in equilibrium in the horizontal plane (i.e. it is not accelerating horizontally), the horizontal components acting on the body must add to 0.

Similarly, if a body is in equilibrium in the vertical plane, the vertical components acting on the body must add to 0.

This is useful in many vector questions.

Example 1: tension in a crane



A crane has a cord that is used to haul bodies upwards.

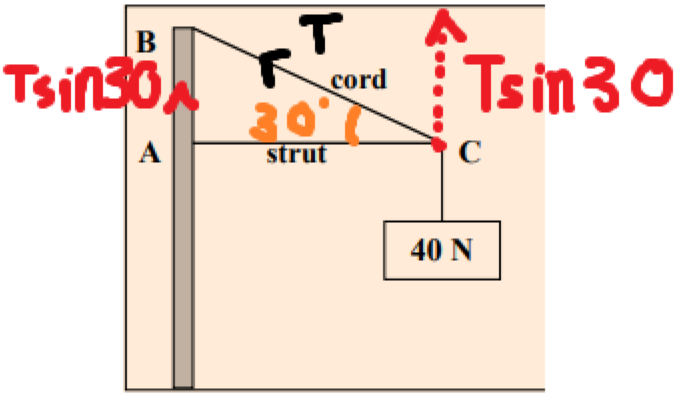
The force that is hauling the body upwards is tension; like the tension in a lift cable.

This is indicated by **T** on the diagram.

Suppose the 40N body being lifted by the crane is stationary.

This does not mean that the tension in the cord indicated on the diagram is equal to the weight, because the tension in the cord occurs at an angle.

Instead, we have to consider the vertical component of the tension, Tsin30:



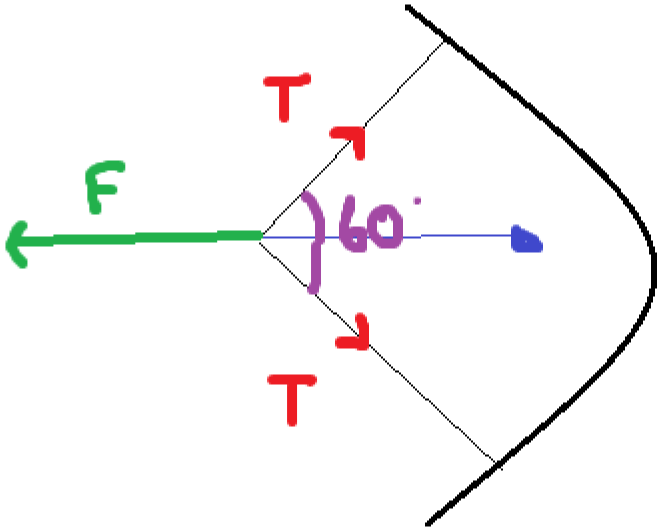
The weight force and the vertical component of the tension must be in equilibrium for the body to be in equilibrium.

Thus, Tsin30 = 40, which tells us **T = 40 / sin 30 = 80N**.

Example 2: tension in bow string

In a drawn bow string, there are three forces:

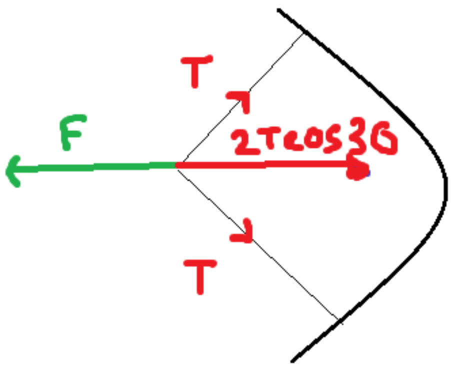
- Tension in both sides of the string (T)
- The backwards pulling force (F)



Suppose the bow string is in equilibrium. We cannot say that the tension forces in the string, T, are equal to the backwards pulling force, F - because T and F are not coplanar.

Instead, since we can only add coplanar forces, we say the *horizontal components* of the tension are equal to the backwards pulling force.

The horizontal component of the tension force is Tcos30. However, because there are two tension forces, the overall horizontal tension force is 2Tcos30.



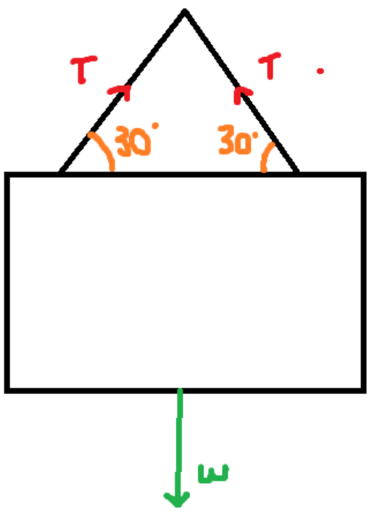
Thus, when the bow is in equilibrium, F = 2Tcos30.

So if F = 800N, 800 = 2Tcos30 => T = 800 / 2cos30 = 462N.

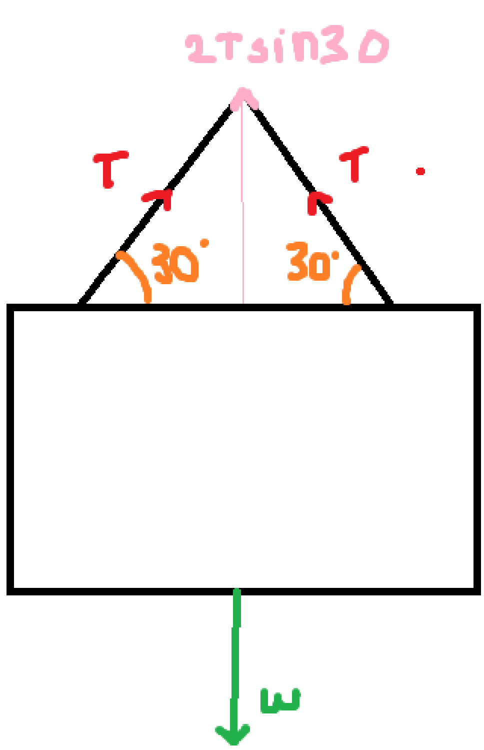
Example 3: tension in picture frame

A similar example to tension in the bow is tension in a picture frame.

If a picture frame is held in the position shown below, there are again two tension forces acting on the body, as well as a weight force acting downwards:



The vertical components of the tension forces must equal the weight force for an equilibrium to exist:



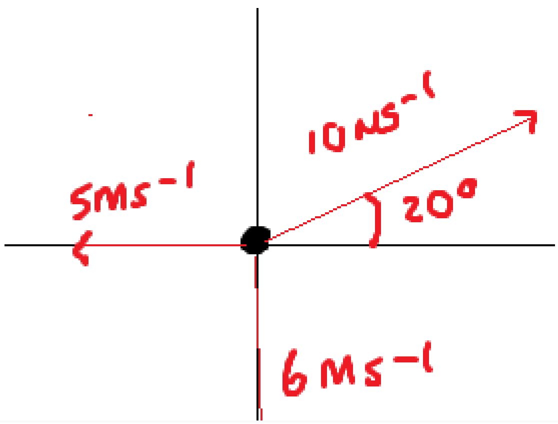
2Tsin30 = W

-

The same principles apply to other vectors such as velocity vectors.

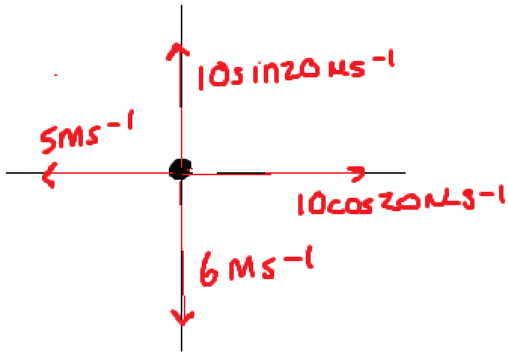
Suppose a particle is subject to 3 velocities:

- A $10ms^{-1}$ velocity at a 070° bearing.
- A $5ms^{-1}$ velocity to the west.
- A $6ms^{-1}$ velocity to the south.

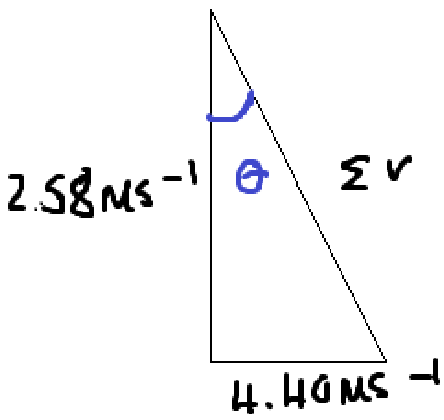


We cannot currently add the $10ms^{-1}$ velocity to the other velocities since we can only add vectors in the same plane.

We therefore have to resolve the vector into its components, 10cos20, and 10sin20.



The resultant velocity in the horizontal plane is: 10cos20 - 5 = 4.40ms^{-1} to the right
The resultant velocity in the vertical plane is: 6 - 10sin20 = 2.58ms^{-1} downwards



$$\Sigma V = \sqrt{4.40^2 + 2.58^2} = 5.1 ms^{-1}$$

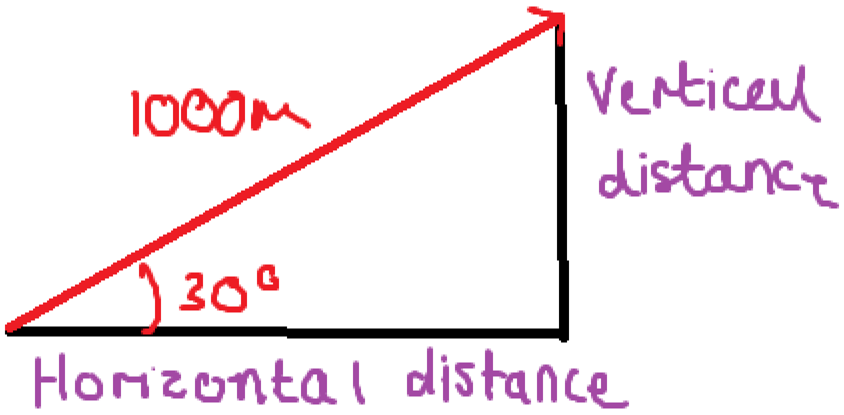
$$\theta = \tan^{-1}\left(\frac{4.40}{2.58}\right) = 59.6°$$

So the particle travels at 5.1ms^{-1} at a bearing of 120.4°.

- Multiple distances at an angle
- Summarise
- Example questions

We can also apply the principles of vectors in two dimensions to displacement problems.

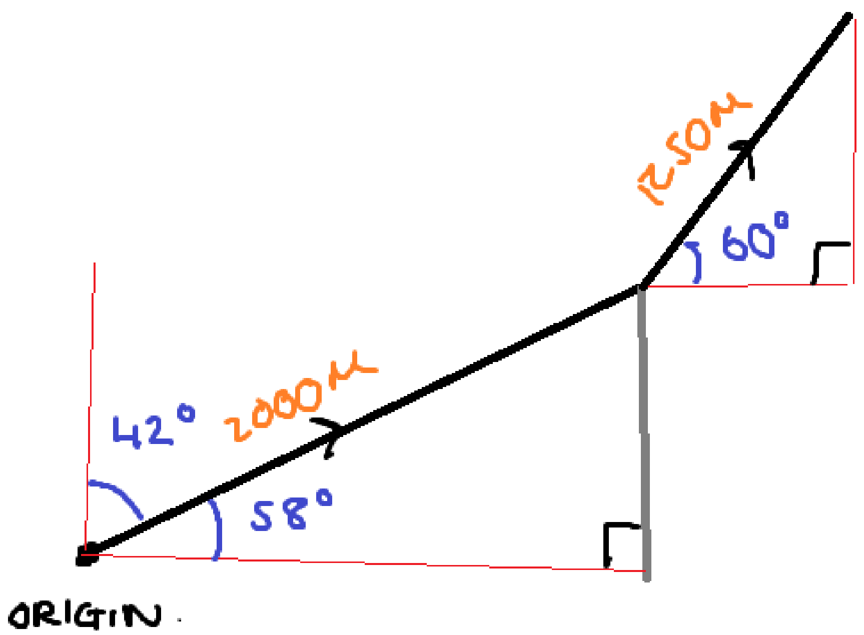
Example 1: A particle is projected at 30° to the horizontal, and travels 1000m in this direction. *How far does it travel horizontally and vertically?*



Distance travelled in horizontal plane = 1000cos30 = 866m
Distance travelled in vertical plane = 1000sin30 = 500m

Example 2: A particle is projected at an angle of 42° to the vertical plane, and travels a distance of 2000m in a straight line in this direction. The particle then glances off the top of a wall, so that it travels at 60° to the horizontal for 1250m in a straight line.

a) **Calculate the distance the particle travels in the horizontal plane and in the vertical plane**.



In the initial projection, the particle travels a horizontal distance of 2000cos58 = 1060m.

After glancing off the wall, the particle travels a horizontal distance of 1250cos60 = 625m.

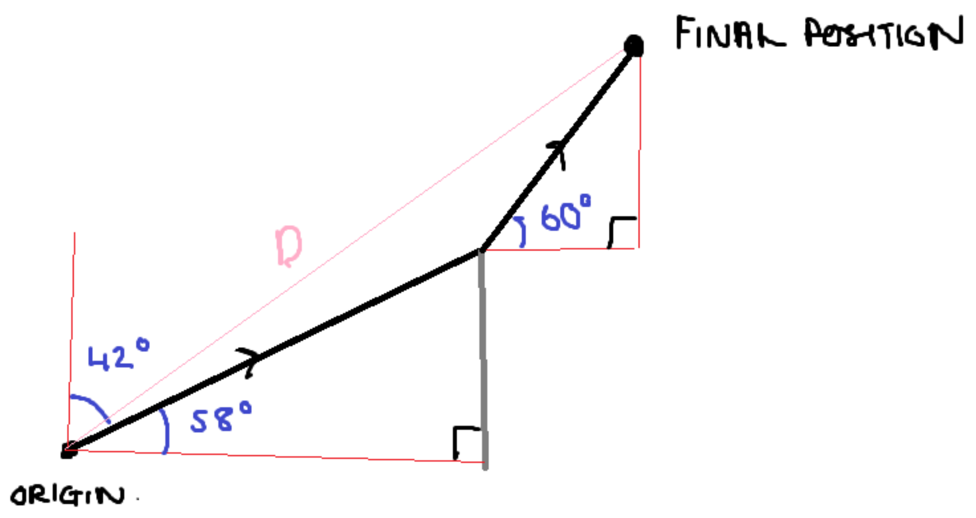
Thus, total horizontal distance travelled **= 1060 + 625 = 1685m**

-

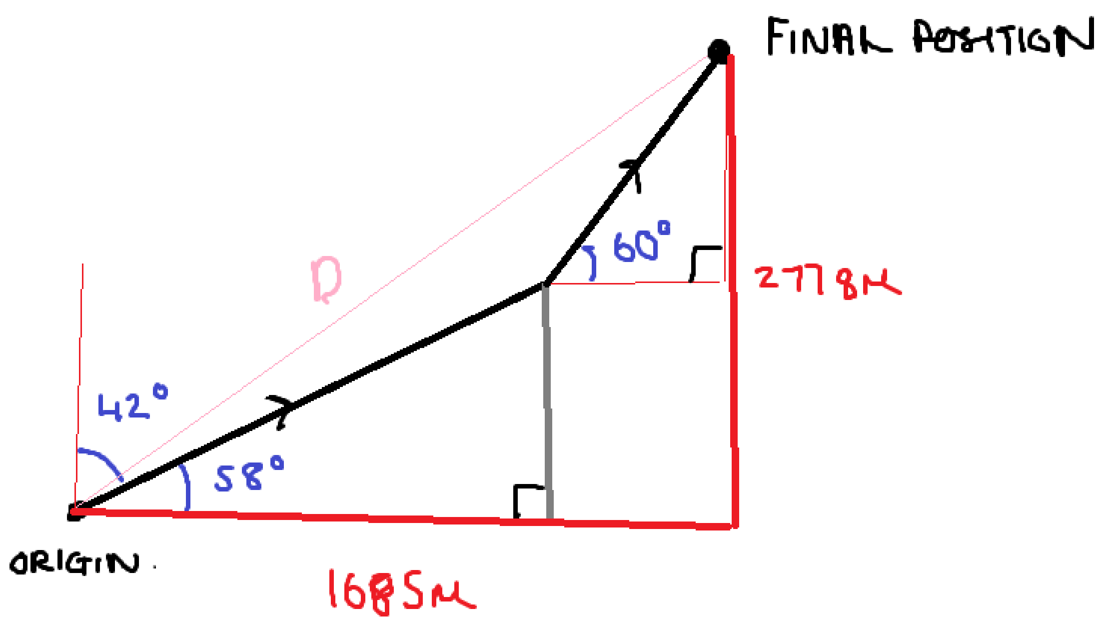
The total vertical distance travelled would be **(2000sin58 + 1250sin60) = 2778m**.

b) Use these values to calculate the final distance of the particle from the origin.

The final distance of the particle from the origin is the length of a line from the origin to the particle:

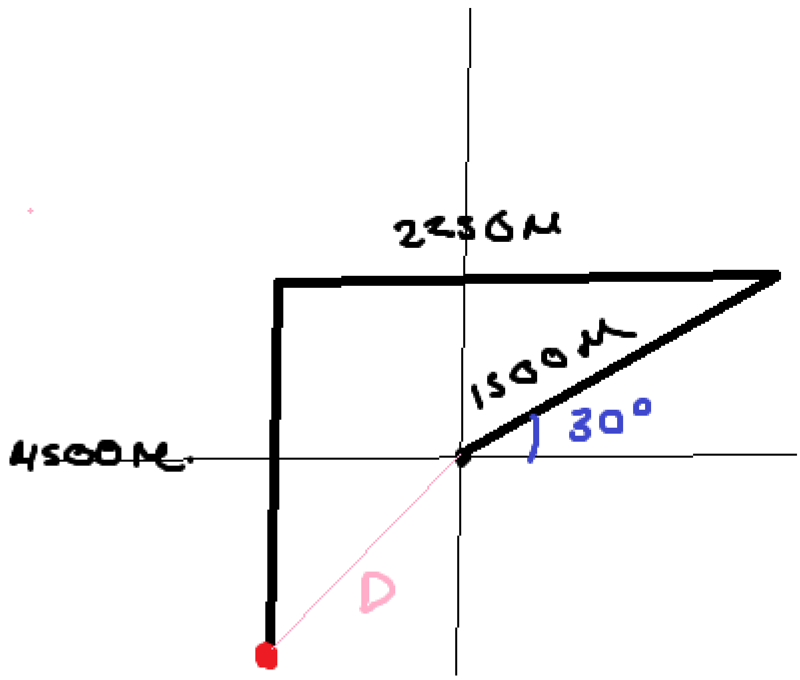


This distance can be found by considering the total horizontal distance travelled and the total vertical distance travelled:



Thus, D = $\sqrt{(1685^2+2778^2)}$ = 3250m.

Example 3: A particle travels 1500m from the origin at an angle of 30°. After this, it changes its angle of motion to a bearing of 270° from its current position, and travels 2250m in this direction. It then travels 4500m directly downwards. **Find its final distance from the origin**.



Consider the distances travelled in the vertical plane, taking upwards as positive and downwards as negative:

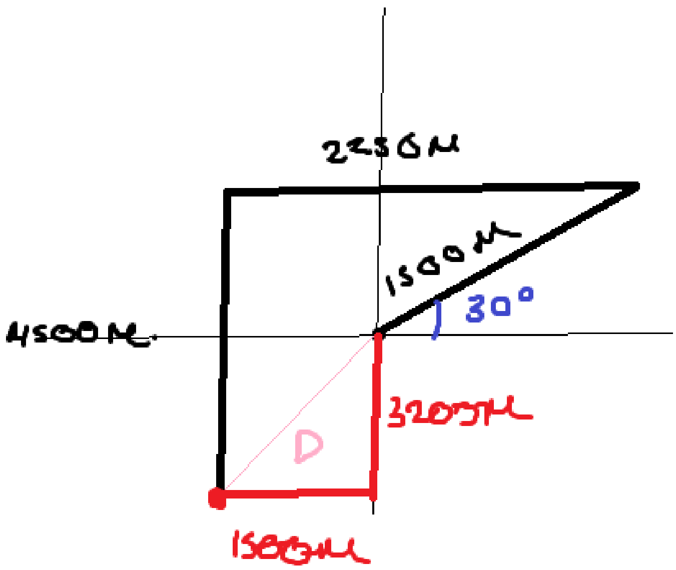
Stage of motion	Distance travelled
1	1500cos30m
2	0m
3	-4500m
Total	-3200m

Now consider the total horizontal distance travelled, taking the right as positive and the left as negative.

Stage of motion	Distance travelled
1	1500sin30m
2	-2250m
3	0
Total	-1500m

So the particle travels 3200m to the left of the origin, and 1500m downwards.

This gives a total distance from the origin of $\sqrt{(3200^2 + 1500^2)} = 3530$m

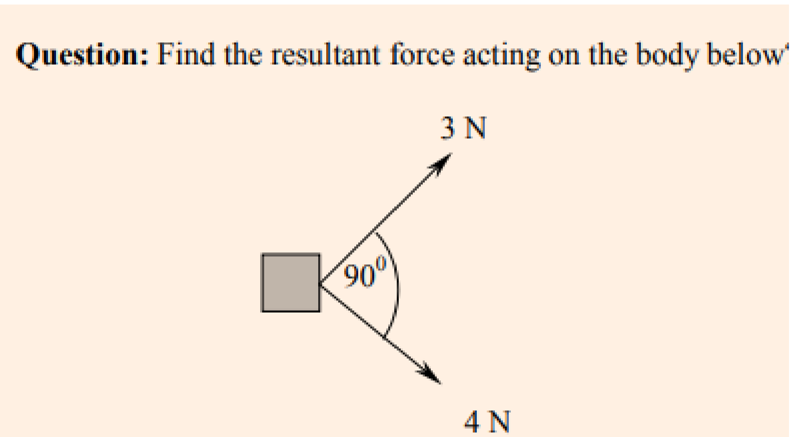


Vector summary:

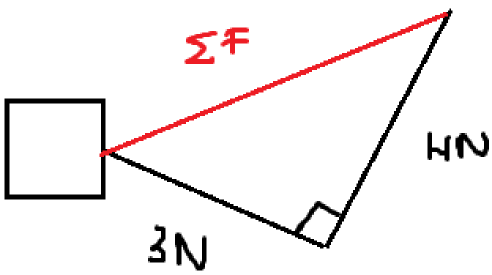
- A vector has magnitude and direction
- All vectors that occur at an angle can be split into component vectors
- It is not possible to add vectors that are at an angle; you can only add the components of the vectors.
- Vectors can only be added if they are coplanar.

Vector Problems

1



If we add the two force vectors from head-to-tail, it gives the following triangle:



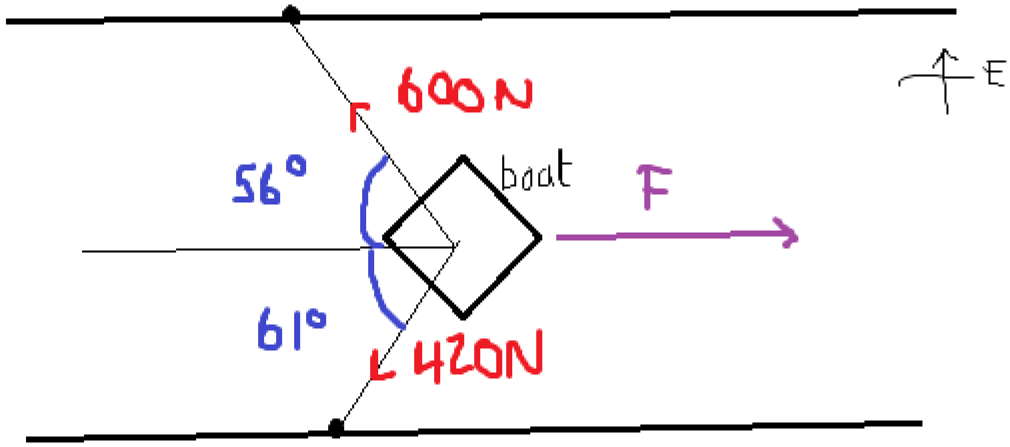
This gives a resultant force of $\sqrt{(3^2+4^2)} = 5N$.

2

A sailing boat is in a river canal, and is held stationary in the horizontal plane by two ropes and the force of the water.

The two ropes are attached to the river bank, and occur at the angles shown in the diagram below. The tension in the ropes is also indicated.

The river is flowing to the east, and exerts a force of FN on the boat.



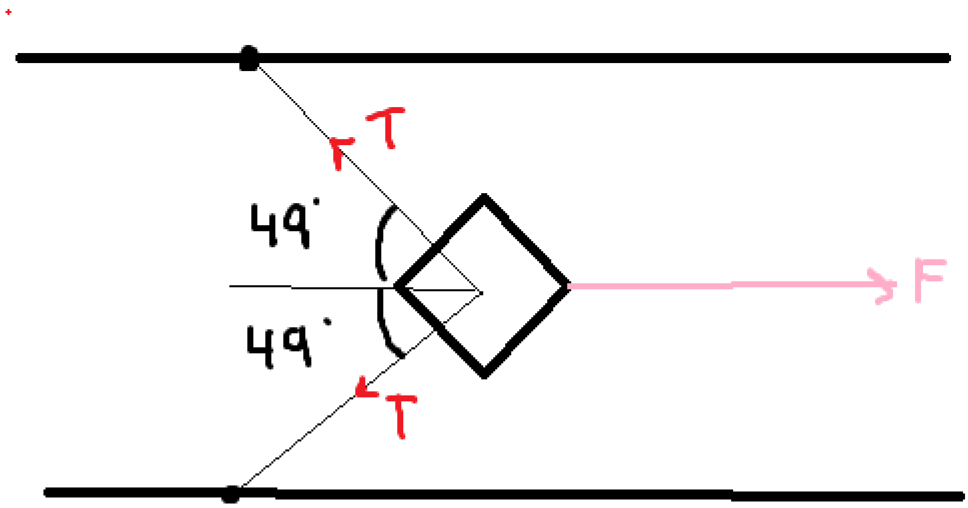
Find the force that the river must exert on the boat for it to be stationary.

The horizontal components of the tension forces must equal the force exerted by the river for an equilibrium to exist.

Thus, 600cos56 + 420cos61 = F => F = 539N.

--

The force of the river increases to 1070N during a flooding event, and the ropes stretch out as shown below. The ropes now have an equal tension.



If the tension in one of the ropes exceeds 800N, it will snap.

Determine by calculation if the ropes will snap.

If F = 1000N, the total backwards component of tension is also equal to 1000N.

The total horizontal tension is equal to 2Tcos49.

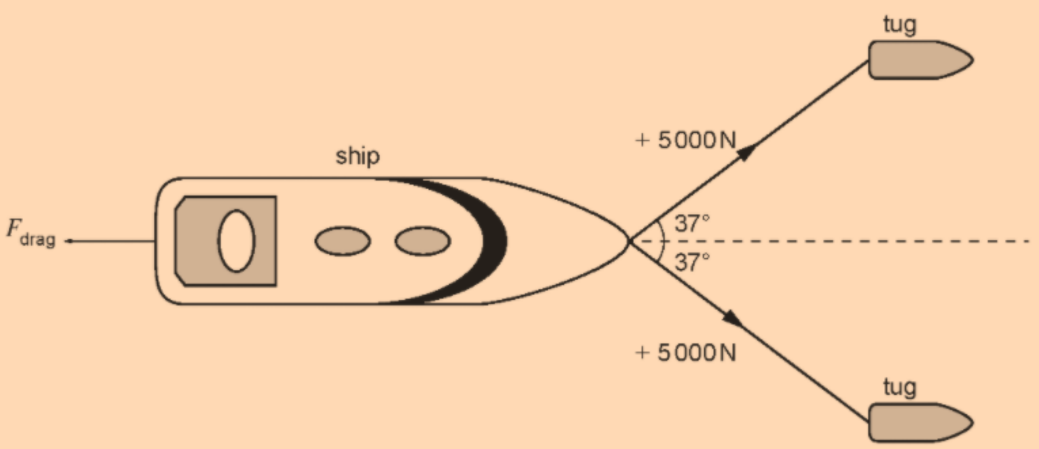
Thus 2Tcos49 = 1000 => T = 762N.

The tension in each rope is 762N.

762 < 800, so the ropes will not snap.

3

(c) A ship is being pulled along by cables attached to two tugs as shown. F_{drag} represents the total drag force that acts on the ship at the instant shown.



(i) Show clearly that the magnitude of the resultant of the forces applied **by the tugs** is approximately 8000N. [2]

The total horizontal component of tension is: 2(5000cos37) = 7990N, which is approximately 8000N.

(ii) Given that $\sum F = +2000\,\text{N}$ for the situation shown above, determine the value of F_{drag}. [1]

The resultant force is 2000N.

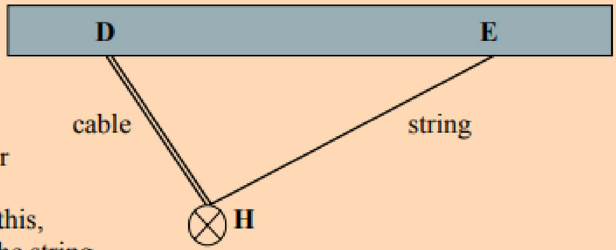
The resultant force in this situation is: 8000 - F_{drag}.

Thus, 8000 - F_{drag} = 2000

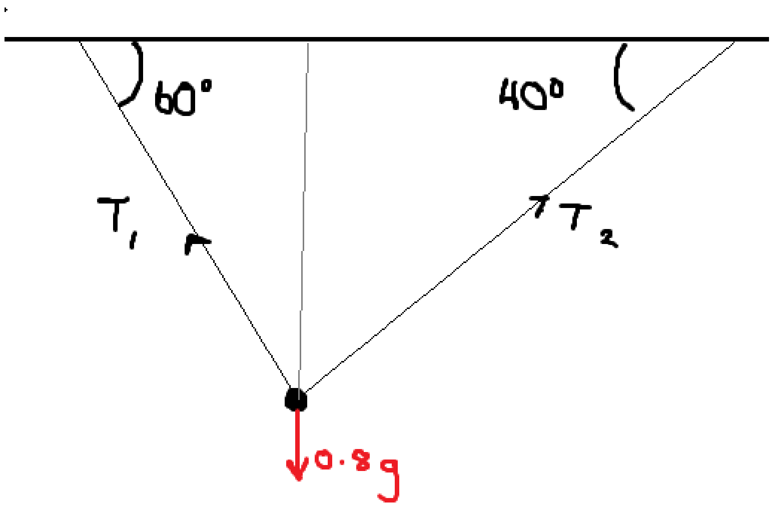
F_{drag} = 6000N.

4

Q4
An electric lamp of mass 0,8 kg is attached by an electric cable to the ceiling at point D. To position it directly over her desk, Thembi has pulled the lamp to one side by a string tied to the lamp holder H and fixed to point E on the ceiling. Thembi is curious as to whether the tension in the electric cable is affected when the lamp is pulled aside. To investigate this, she measures the angles which the cable and the string make with the ceiling and finds them to be 60^0 and 40^0 respectively as shown in the diagram at the right.



Find the tension in the cable and the tension in the string.



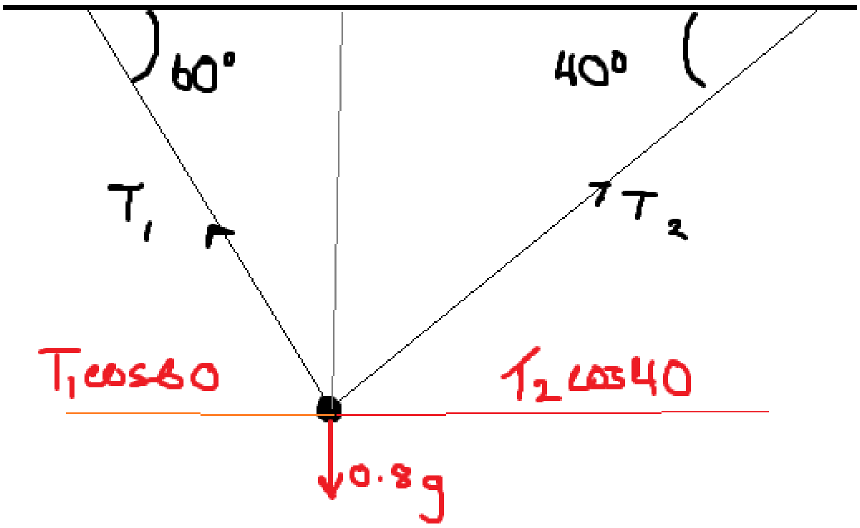
Horizontal component of T_1 = T_1sin60
Horizontal component of T_2 = T_2sin40

These horizontal components must add up to the weight for an equilibrium:

T_1sin60 + T_2sin40 = 0.8g.

The only way we can solve this is by finding another equation to express T_1 and T_2.

Since the lamp is not moving horizontally, the horizontal components of T_1 and T_2 must be equal to each other.



$T_1\cos60 = T_2\cos40$

Thus, $T_1 = T_2\cos40 / \cos 60 = 1.53T_2$.

We can now substitute this into the previous equation:

$1.53T_2\sin60 + T_2\sin40 = 0.8g$

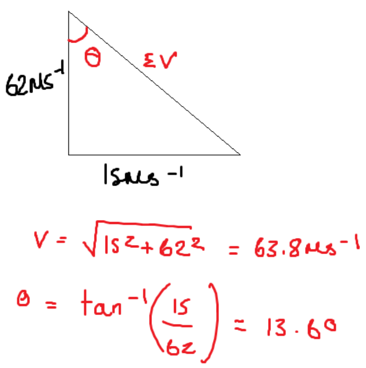
$T_2\ (1.53\sin60 + \sin40) = 0.8g$

$T_2 = 3.99N$.

Thus, $T_1 = 1.53(3.99) = 6.10N$.

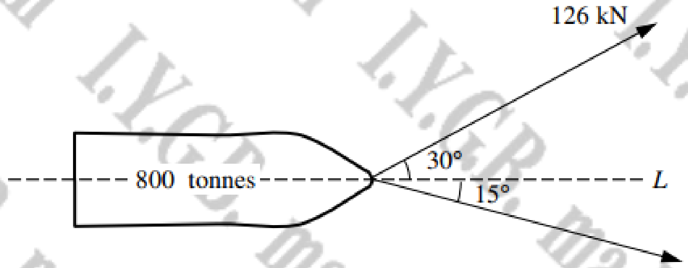
5

An airplane must land with a vertical velocity of 62ms^{-1}. If there is a crosswind of 15ms^{-1}, **find the angle and resultant velocity with which the airplane must land.**



The airplane must land with a resultant velocity of 63.8ms^{-1} at an angle of (90-13.6) = 76.4° below the horizontal.

6



The figure above shows a ship, of mass 800 tonnes, being pulled by two tugs using horizontal cables, so that the ship is moving in a straight line L.

One tug exerts a force of 126 kN, at an angle of 30° to L.

The other tug exerts a force at an angle of 15° to L.

The ship is accelerating along L at 0.05 ms^{-2}.

Determine the magnitude of the resistance opposing the motion of the ship.

We will let the force exerted at 15° be **F** and the resistive force be **R**.

To find **F**, we consider that the ship is not moving vertically - it is only moving horizontally. This means that the forces in the vertical plane must be equal.

The forces in the vertical plane are the vertical components of the **126kN** force and the force **F**, which are **126sin30** and **Fsin15** respectively (the resistive force does not act at an angle, so it has no vertical component).

Thus, **126sin30 = Fsin15** => **F = 243.4kN**.

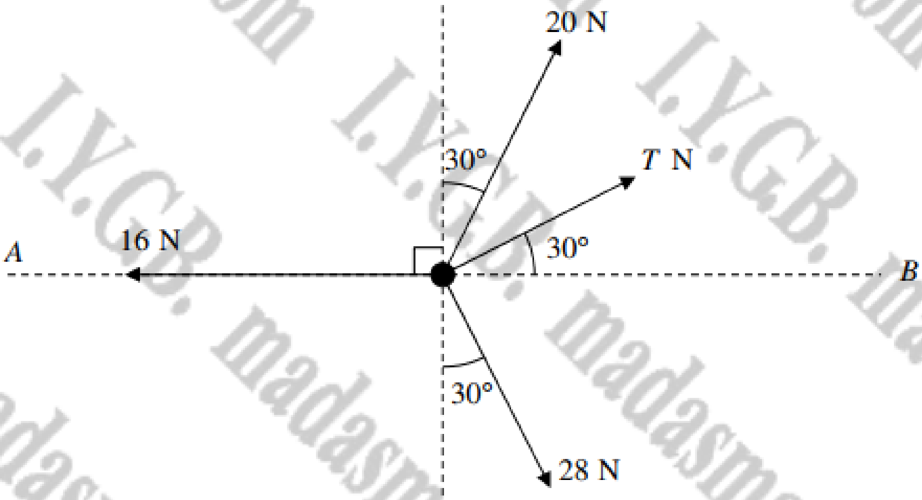
Now we can consider the forces that are acting in the horizontal plane:

- The horizontal components of the tugging forces, **126cos30** and **243.4cos15**.
- The resistive force acting against the tugging forces.

The net force acting on the boat is equal to **126cos30 + 243.4cos15 - R = 344.2 - R**

This is equal to **ma**, so **(344.2×10^3) - R = (800×10^3) * 0.05** => **R = (344.2×10^3) - ($800 \times 10^3 \times 0.05$) = 304000N = 304kN**.

7



A particle of mass 80 kg is accelerating in the direction AB.

Four **horizontal** forces of different magnitudes are acting on the particle. The magnitude and direction of these four forces, together with other important information is shown in the figure above.

Find the acceleration of the particle.

First, we will consider the overall horizontal force exerted on the particle:

F_H = Tcos30 + 20cos60 + 28cos60 - 16 = Tcos30 + 8

And the overall vertical force:

F_V = 20sin60 + Tsin30 - 28sin60 = 1/2T - 6.93.

Because the particle is accelerating in the direction AB, there must be no net force in the vertical plane, because the acceleration would be at a slight angle otherwise.

Thus, **1/2T - 6.93 = 0** => **T = 13.9N**.

The overall horizontal force is therefore **13.9cos30 + 8 = 20.04N**.

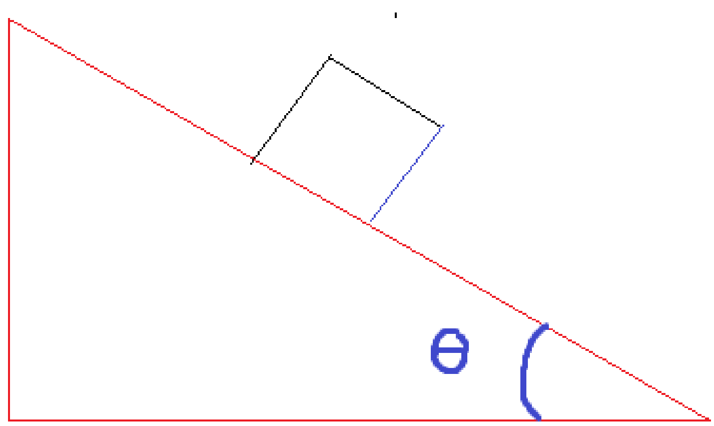
Thus, **20.04 = 80 * a** => **a = 0.25ms^{-2}**.

Forces on a slope

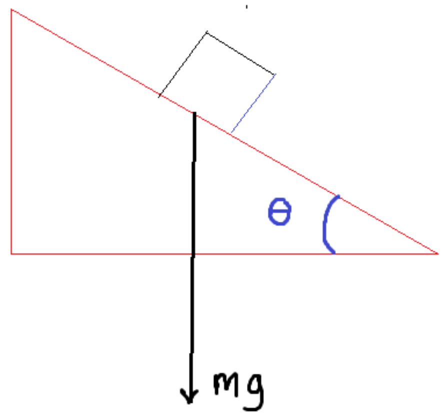
We often have to consider how bodies behave on slopes.

When a body is on a flat, horizontal plane, the weight force has no influence over the motion of the body in the horizontal plane, because it acts perpendicular to the horizontal (and the forces in the horizontal and vertical plane are independent of each other).

Take a block of mass **m** on a slope:

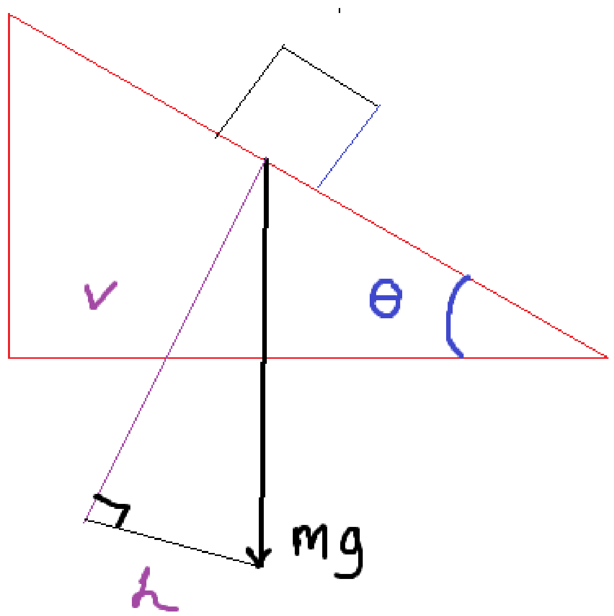


The weight of the block, **mg**, acts directly downwards:

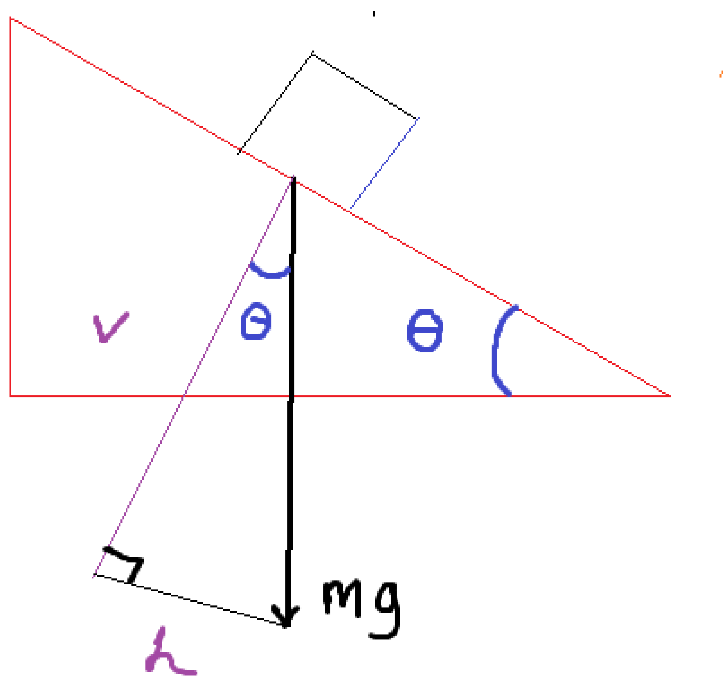


However, this weight is not useful in calculations; we instead have to consider the components of the weight.

Consider the components of the weight acting perpendicular to the slope (**v**) and parallel to the slope (**h**):

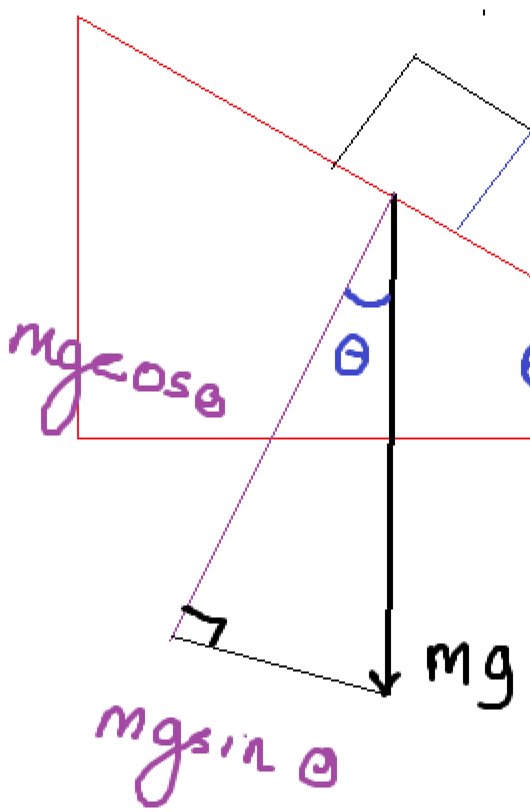


We can prove with simple geometry that the angle between **v** and mg is equal to the angle of the slope, Θ:

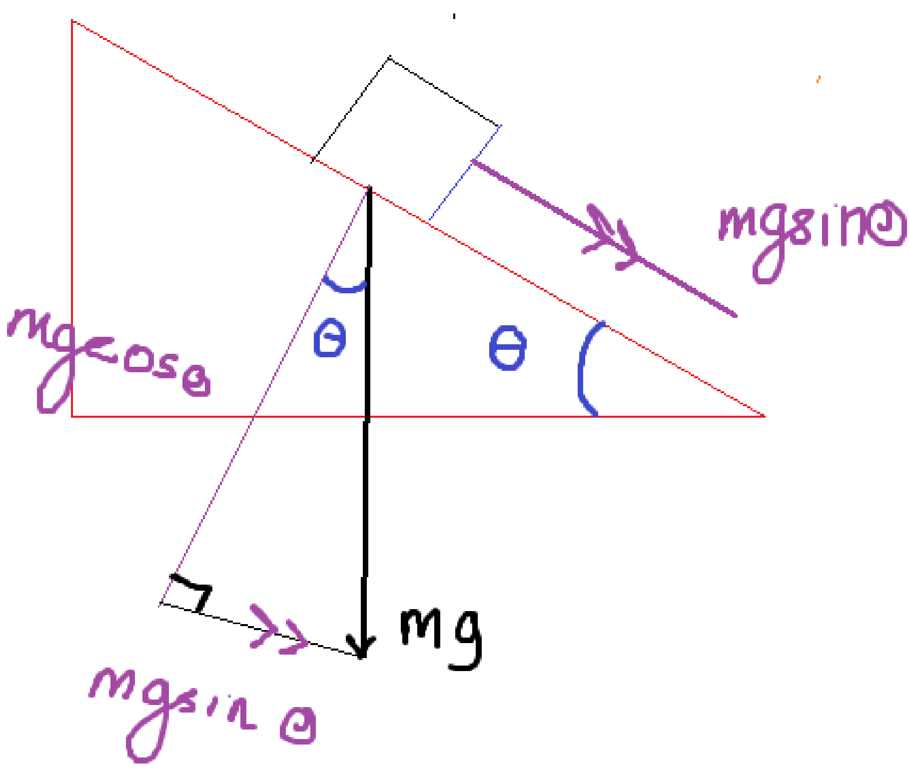


Hence, using trigonometry:

v = mgcosΘ
h = mgsinΘ



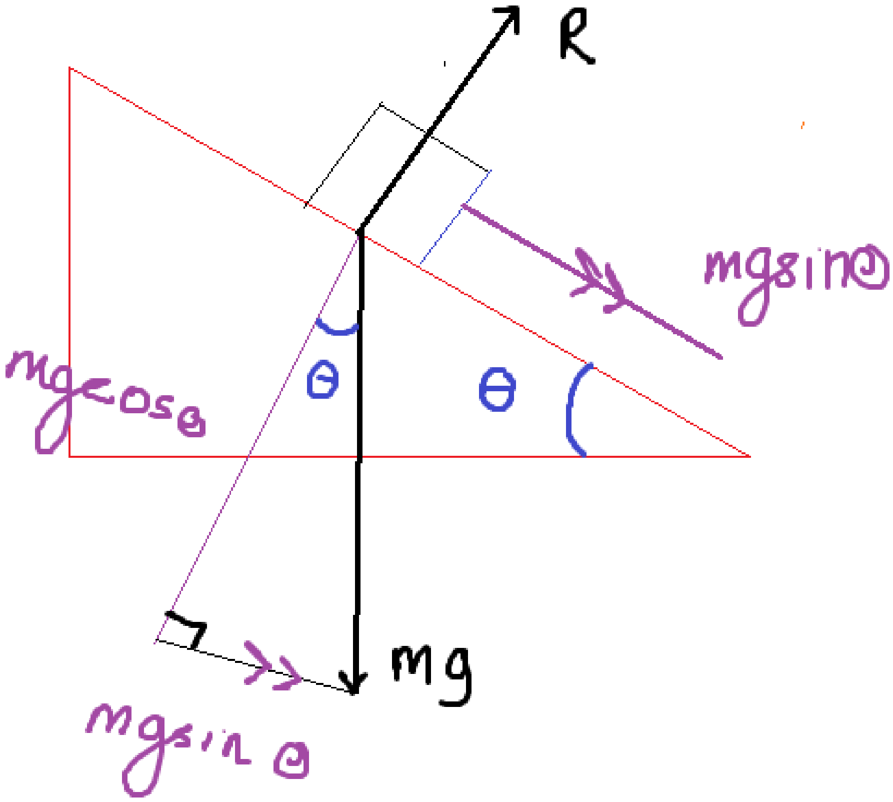
The horizontal component of the weight acts directly down the slope:



This is very useful as the horizontal component is coplanar with the slope, so we can essentially consider it as a flat plane once we know the horizontal component.

We can also consider the contact forces with the slope.

The normal contact force, R, acts perpendicular to the slope:



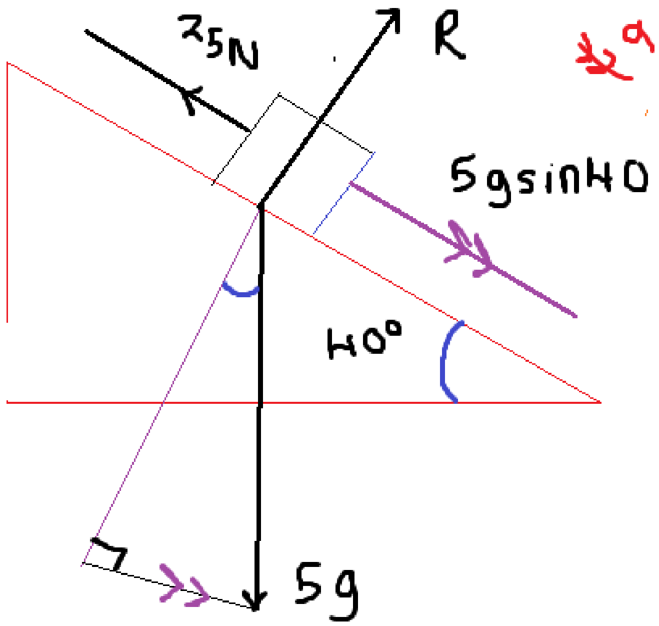
From this, we can see that the normal reaction force is coplanar with **v** (as **v** is also perpendicular to the slope) If the body is not moving in this plane, **mgcosΘ = R**.

Example 1

A 5kg block is left on a slope inclined at 40° to the horizontal. The particle begins to slide down the slope, and a friction force of 25N opposes this motion.

Find the acceleration of the particle.

By finding the horizontal component of the particle's weight, since this is coplanar with the friction force, we can determine the acceleration of the block.



Horizontal component = 5gsin40.

Net force = 5gsin40 - 25.

Net force = m * a = 5 * a

5gsin40 - 25 = 5a

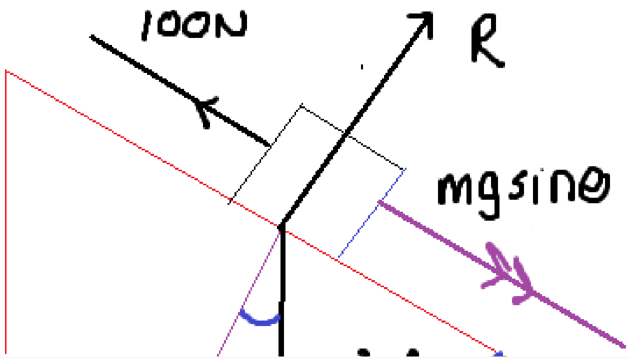
a = (5gsin40-25) / 5 = 1.31ms^{-2}.

-

Example 2

A block of mass m is positioned stationary on a slope with an incline of 30°. The block is attached to a rope that moves up the slope. The rope has a tension of 100N.

Find the normal reaction force between the block and the slope.



If the block is stationary, the tension force in the rope and the horizontal component of the weight must be equal.

Thus, mgsin30 = 100.

This enables us to find m.

m = 100 / gsin30 = 20.4kg.

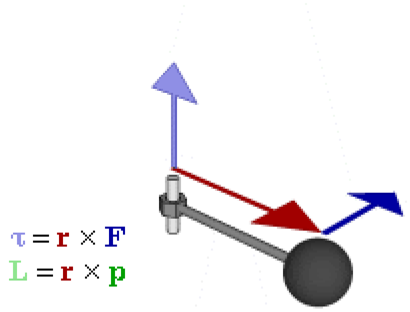
The normal reaction force is equal to the vertical component of the weight.

Vertical component of weight = 20.4gcos30 = 173N.

Thus, normal reaction force = 173N.

Moments

If a body is attached to a pivot point, and a force is applied to the pivot point, the body will rotate around the pivot point:

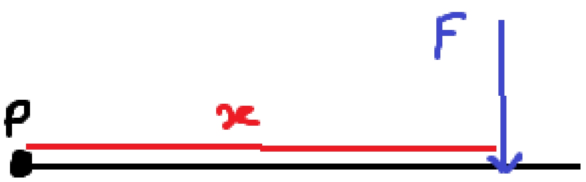


The greater the force, the greater the rotation; the greater the distance at which the force is applied, the greater the rotation.

A **moment** is the turning effect of a force.

- Moment = force * perpendicular distance to force (units = N * m = Nm)

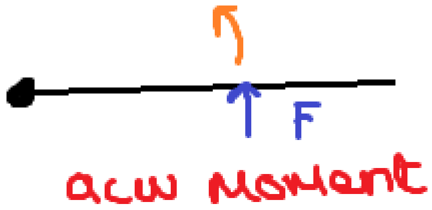
So at a greater force and a greater perpendicular distance, there is a greater turning effect.

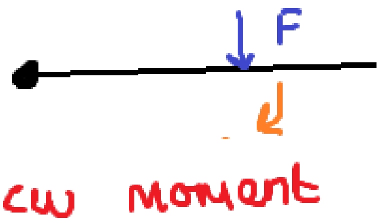


- A force can cause clockwise rotation or anticlockwise rotation.

A clockwise rotation creates a *clockwise moment*.

An anticlockwise rotation creates an *anticlockwise moment.*



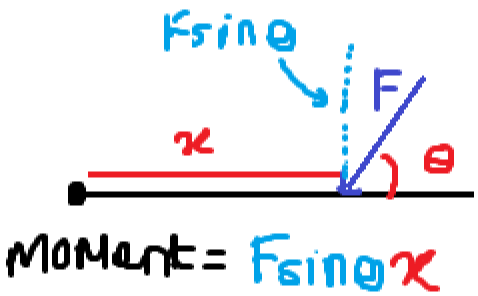


- We previously showed that the net force acting on a body must be 0 for it to be in equilibrium.
- Another condition for equilibrium is the **net moments are equal**.

For an equilibrium to occur, **total clockwise moment = total anticlockwise moment**

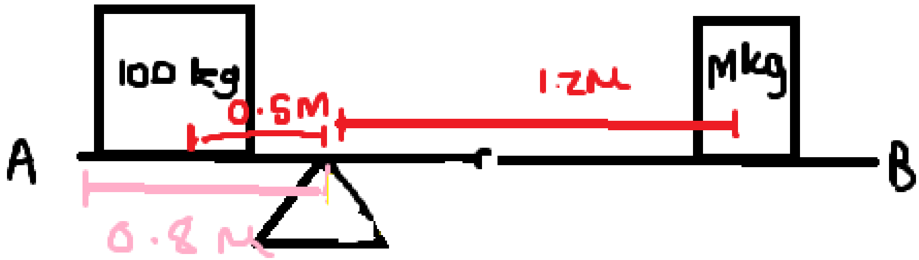
This is the principle of moments

- If a force acts at an angle, since moment = force * **perpendicular distance** to force, we need to find the component of the force that is perpendicular to the distance to the force.



Example 1

A uniform steel rod, AB, of length 3m and mass 60kg is bound to a pivot, as shown in the diagram below. The pivot is 0.8m from A. On the rod, there is a mass of 100kg 0.5m from the pivot. There is also a mass of **m**kg at a distance of 1.2m from the rod.



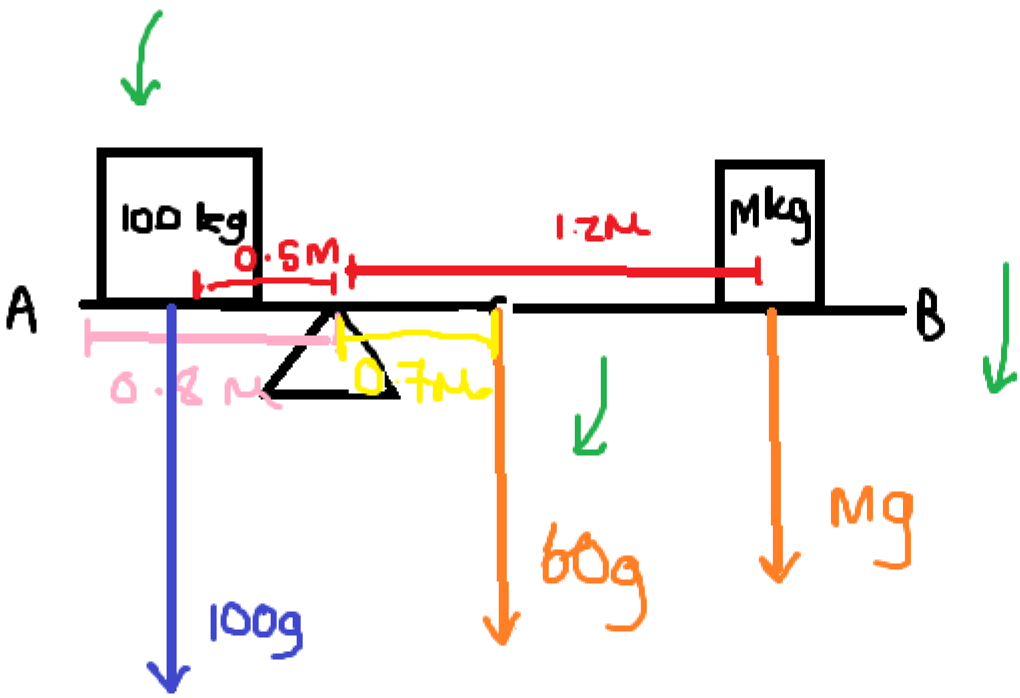
Given that the beam is in equilibrium, a) **determine the value of m**.

If the beam is in equilibrium, the clockwise moments must equal the anticlockwise moments.

There are three forces acting on the beam:

- A 5g force
- A mg force
- The weight of the rod, 60g

The weight of the rod acts at the centre of the rod, since it has uniform density. The rod is



3m long, so the weight must be 1.5m along the rod. This is 0.7m from the pivot.

The weight of the rod and the weight **mg** cause the rod to rotate clockwise, as indicated in the diagram. The 100g weight causes the rod to rotate anticlockwise.

According to the principle of moments, the total clockwise moment must equal the total anticlockwise moment.

Clockwise moments:

Moment of centre of gravity = 60g * 0.7 (force * perpendicular distance from pivot)
= 412Nm

Moment of mkg box = mg * 1.2
= 11.77m Nm

Total clockwise moment = 412 + 11.77m

Anticlockwise moments:

Moment of 100kg mass = 100g * 0.5 = 490.5Nm.

-

Total clockwise moment = total anticlockwise moment

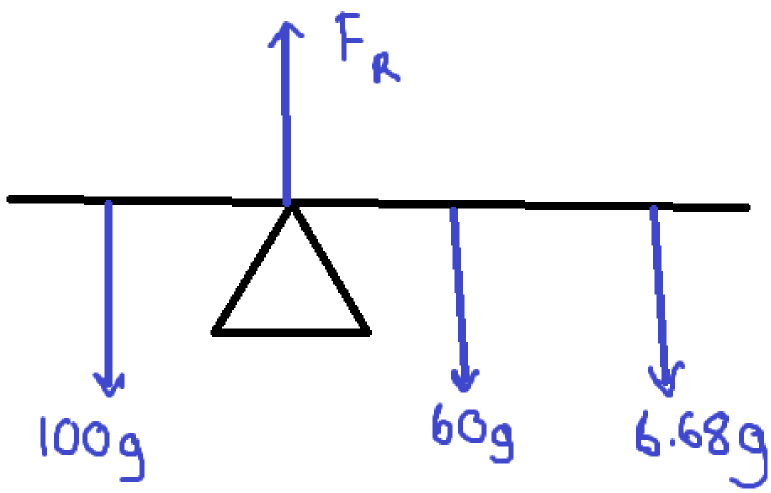
Thus, 412 + 11.77m = 490.5

m = (490.5 - 412) / 11.77 = 6.68kg.

So, a mass of 6.68kg a distance of 1.2m from the pivot allows it to be in equilibrium.

b) **Determine the value of the force exerted by the pivot.**

The pivot exerts an upwards contact force on the beam (F_R):



We previously said that for a body to be in equilibrium, the net force must be 0 and the net moments must be 0.

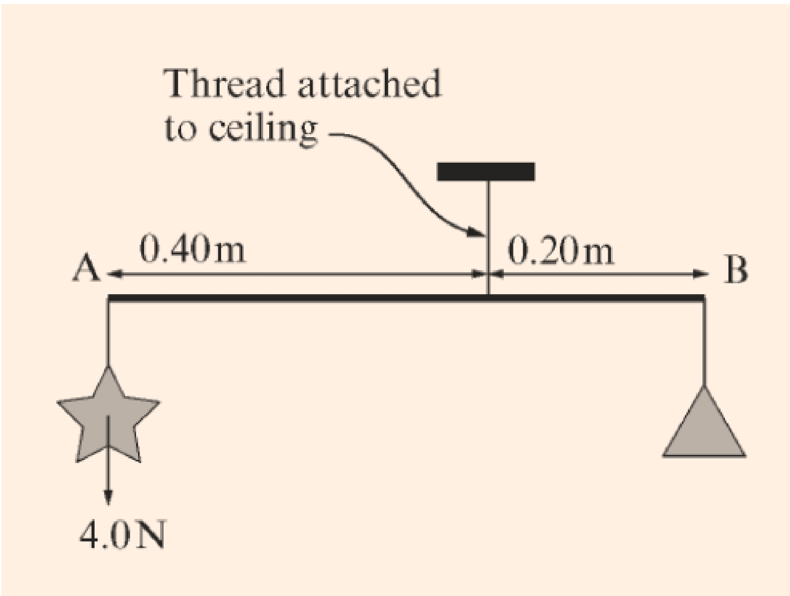
If the net force is 0, the total downwards forces must equal the total upwards force.

Hence, $F_R = 100g + 60g + 6.68g = g(100+60+6.68) = 1640N$.

-

Example 2

(b) (i) A simple toy mobile, consisting of a star and a triangle is shown hanging freely. Assume the rod AB is weightless. Calculate the weight of the triangle. [2]



Let the mass of the triangle be **m**.

The total clockwise moments must equal the total anticlockwise moments for equilibrium.

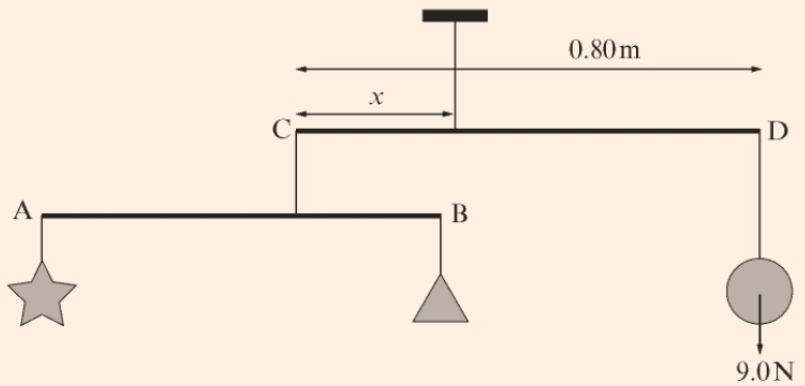
Thus, 0.40 * 4 = 0.20 * mg => mg = 1.6 / 0.20 = 8N.

(ii) Hence calculate the tension in the thread attached to the ceiling. [1]

The thread exerts an upwards tension force. The total upwards force must equal the total downwards force.

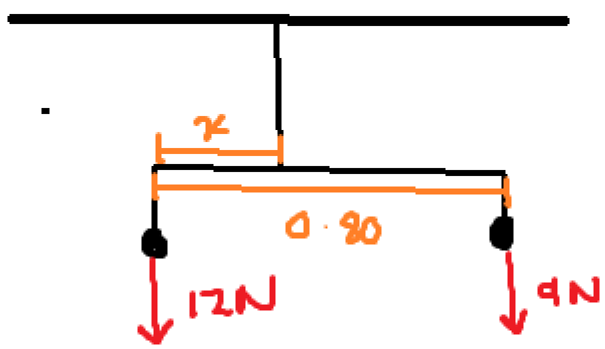
Hence, T = 4 + 8 = 12N.

(c) The mobile is now attached to another weightless rod CD of length 0.80 m, making a more complex mobile as shown. A sphere of weight 9.0 N is attached to D and the mobile is hung freely from the ceiling.



(i) The distance x can be adjusted to allow the rod CD to hang horizontally. Calculate x. [3]

We can model A and B as a single mass. The weight of A and B would therefore be 12N.



The pivot in this instance is the point where the ceiling contacts the point CD.

The distance from the pivot to the 12N force is x, and the distance from the pivot to the 9N force is 0.80 - x.

Thus, 12x = 9(0.80-x)

12x = 7.20 - 9x.

21x = 7.20 => x = 0.34m.

Example 3

Now we will consider when there are two pivot points.

When there are two pivot points, we can consider either pivot as the 'main pivot' in our calculations.

A steel girder AB, of mass 200 kg and length 12 m, rests horizontally in equilibrium on two smooth supports at C and at D, where AC = 2 m and DB = 2 m. A man of mass 80 kg stands on the girder at the point P, where AP = 4 m, as shown in Figure 1.

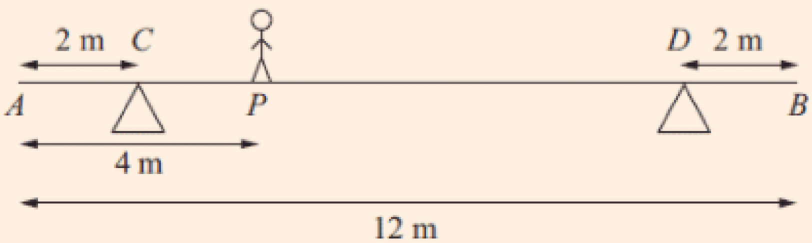


Figure 1

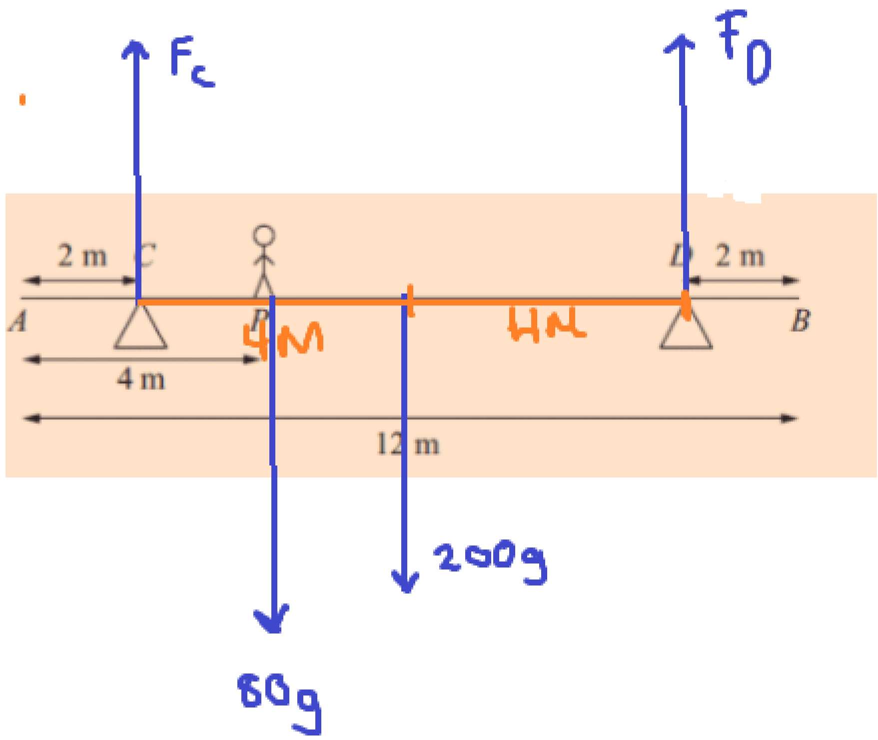
The man is modelled as a particle and the girder is modelled as a uniform rod.

(a) Find the magnitude of the reaction on the girder at the support at C.

(3)

The rod is uniform, so its centre of gravity must be 6m from A. This means that the centre of gravity is 4m from C, and 4m from D.

The two pivots exert upwards contact forces (we will call these F_c and F_D). The weight of the man, 80g, acts downwards.

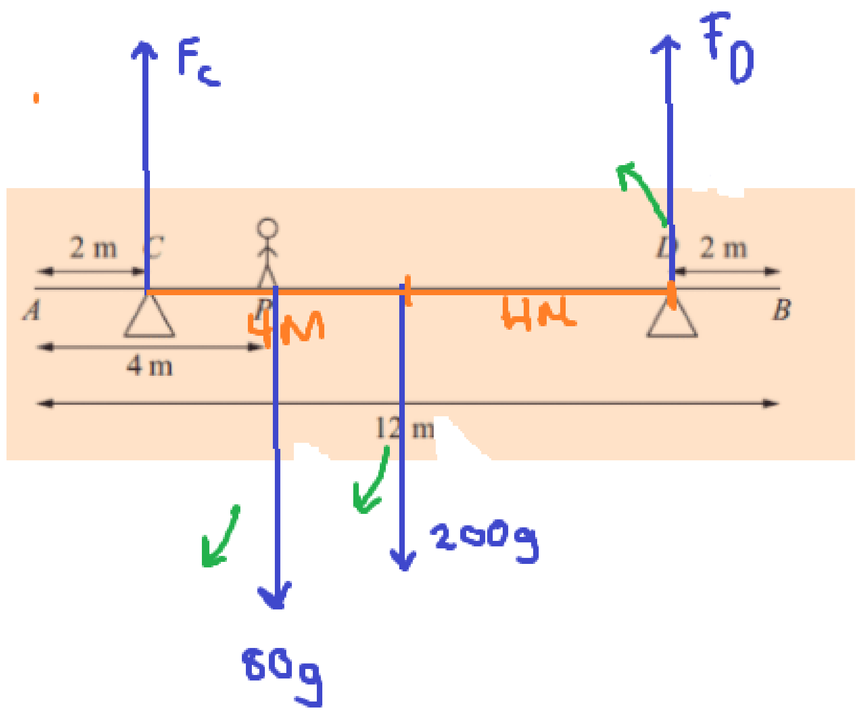


When taking moments, we can consider either pivot as the pivot point.

Suppose we consider C as the pivot point. When we do this, we can ignore the force exerted by C (because moment = force * perpendicular distance, and perpendicular distance = 0m for F_c).

The forces that create moments relative to C are:

- The 80g force of the person (clockwise moment)
- The 200g force of the beam (clockwise moment)
- The force exerted by pivot D, F_D (anticlockwise moment).



So, if we take moments about C:

- Total clockwise moment = (80g * 2) + (200g * 4) = 9472N
- Total anticlockwise moment = (F_D * 8), 8 because pivot D is 8m from pivot C.

Thus, 9472 = $8F_D$ => F_D = 1180 N (3SF).

We need to find the support force exerted by C.

We must therefore use the fact that the net force must be 0 if the beam is in equilibrium, so:

80g + 200g = F_c + 1180 => F_c = 280g - 1180 = 1570N.

-

We can also take moments about D instead of C, and prove that F_c is the same:

Taking moments about D:

Total clockwise moment = $8F_c$
Total anticlockwise moment = (80g * 6) + (200g * 4) = 12560N.

$8F_c$ = 12560
F_c = 1570N.
--

The support at D is now moved to the point X on the girder, where $XB = x$ metres. The man remains on the girder at P, as shown in Figure 2.

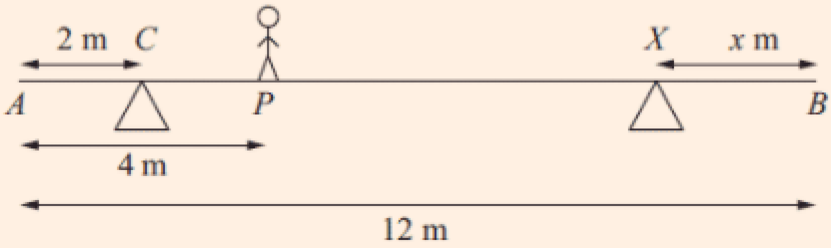


Figure 2

Given that the magnitudes of the reactions at the two supports are now equal and that the girder again rests horizontally in equilibrium, find

(b) the magnitude of the reaction at the support at X,
(2)

(c) the value of x.
(4)

b) The total downwards force is 200g + 80g = 2750N.

The force exerted by the pivots is equal. Let the force exerted by each pivot be F.

We can say that the total force exerted by the pivots (total upwards force) must equal the total downwards force.

Thus, 2F = 2750 => F = 1375N.

c)

To answer this question, we can determine the distance of point X from pivot C by taking moments.

This distance must be: 12 - 2 - x = 10 - x.

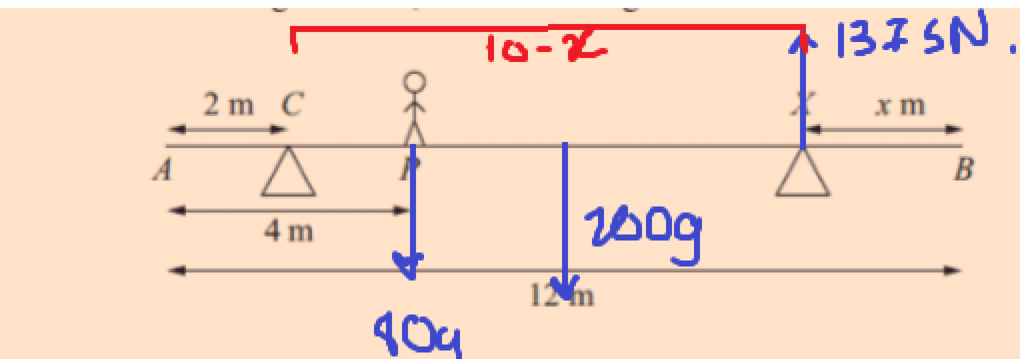


Figure 2

(80g * 2) + (200g * 4) = 1375(10-x)

9422 = 13750 - 1375x

1375x = 4328

x = 3.2m.

Example 4

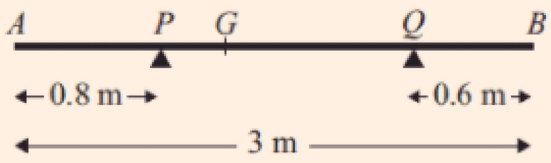


Figure 1

A non-uniform rod *AB* has length 3 m and mass 4.5 kg. The rod rests in equilibrium, in a horizontal position, on two smooth supports at *P* and at *Q*, where *AP* = 0.8 m and *QB* = 0.6 m, as shown in Figure 1. The centre of mass of the rod is at *G*. Given that the magnitude of the reaction of the support at *P* on the rod is twice the magnitude of the reaction of the support at *Q* on the rod, find

(a) the magnitude of the reaction of the support at *Q* on the rod, **(3)**

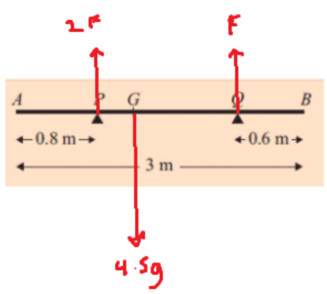
(b) the distance *AG*. **(4)**

a)

The rod has three forces acting on it:

- The 4.5g weight at G
- The two reaction forces at P and Q.

If we say the reaction force at Q is F, the reaction force at P must be 2F.



Since the beam is in equilibrium, the total downwards force = the total upwards force.

$3F = 4.5g$
$F = 1.5g = 14.7N$.

b)

Because the rod has a non-uniform density, we cannot assume that its centre of mass is at the centre of the rod; instead, it is at G.

Firstly, $AG = AP + PG$, so $AG = 0.8 + PG$, $PG = AG - 0.8$.

To find the distance AG, we need to take moments about either of the pivots.

We will take moments about P.

-

The distance from P to Q must be 1.6m.

The distance from P to G is (AG - 0.8)m.

Thus, $1.6(14.7) = 4.5g(AG-0.8)$

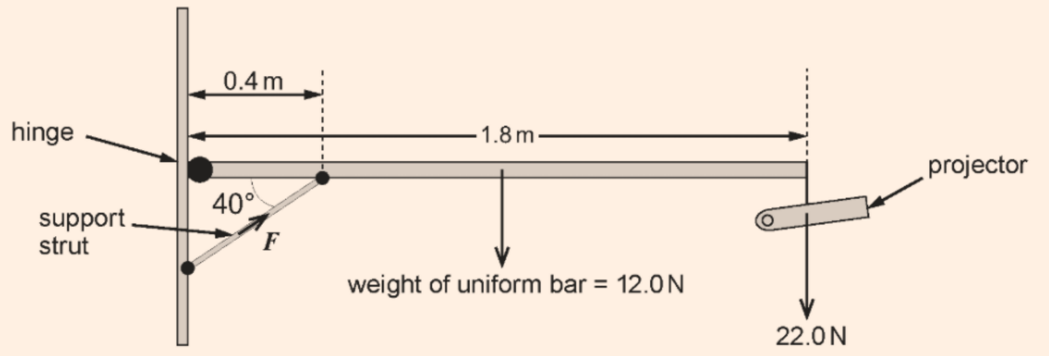
$23.5 = 4.5gAG - 4.5g(0.8)$

$AG = (23.5 + 4.5g(0.8)) / 4.5g = 1.33m$.

Example 5:

Now we will consider an example where a force acts at an angle:

(b) A classroom projector is set up as shown.

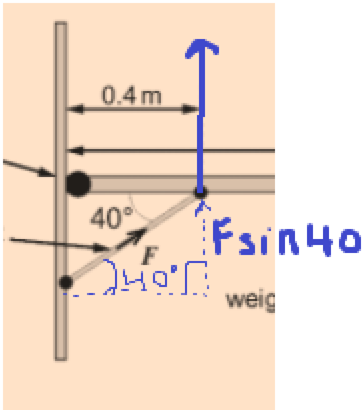


(i) By taking moments about the hinge, show that the force, F, exerted by the support strut on the uniform bar is approximately 200 N. [3]

The force **F** is exerted at a 40° angle.

However, we can only take moments if the force exerted is perpendicular to the distance to that force.

We therefore have to use the vertical component of the force, **Fsin40:**



There are three forces that we need to consider:

- The vertical component of the force exerted by the support struct, Fsin40 (creates an anticlockwise moment)
- The weight force of the bar (creates a clockwise moment)
- The weight force of the projector (creates a clockwise moment).

Total clockwise moment = (0.9 * 12.0) + (1.8 * 22) = 50.4N.

Total anticlockwise moment = 0.4 * Fsin40

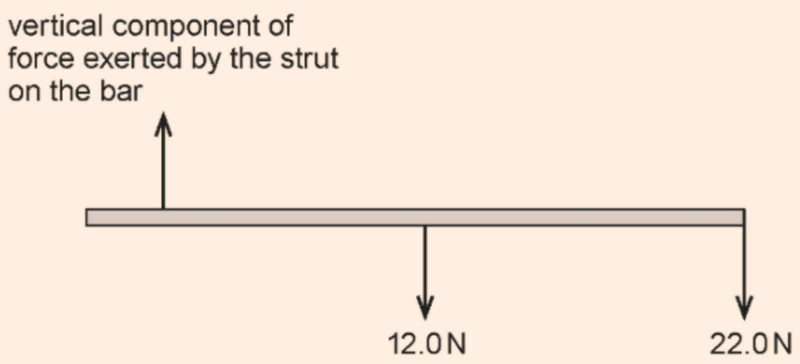
Thus, 0.4Fsin40 = 50.4

F = 50.4 / 0.4(sin40) = 196N.

This is approximately 200N.

-

(ii) The free body diagram below shows **some of the vertical forces** acting on the uniform bar.



(I) Calculate the value of the **vertical component** of the force exerted by the strut on the bar. [1]

(III) Calculate the size of the vertical force on the bar due to the hinge. [1]

I) The vertical component of the force exerted by the strut is Fsin40 = 196sin40 = 126N.

II)

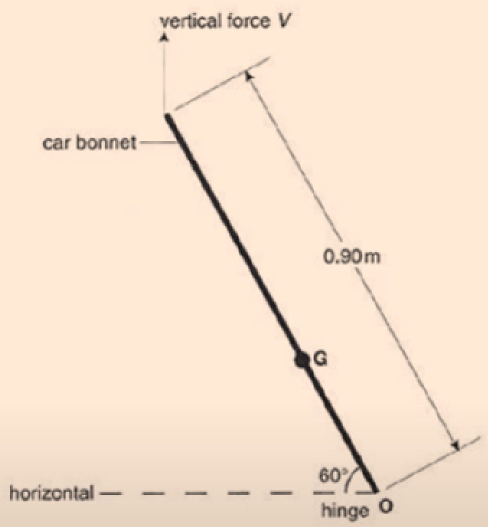
The hinge must exert a downwards force, because the bar is in equilibrium, and the downwards force of 12 and 22N alone are not great enough to resist the vertical force on the bar.

Thus, F + 12 + 22 = 126 => F = 92N.

Example 6

The diagram below shows the open bonnet of a car. The bonnet is held open at an angle of 60° to the horizontal by a vertical force V applied at one end of the bonnet *(shown on the diagram)*. The bonnet is 0.90 m long, has a weight of 25 N and its centre of gravity G is 0.35 m from the hinge at O.

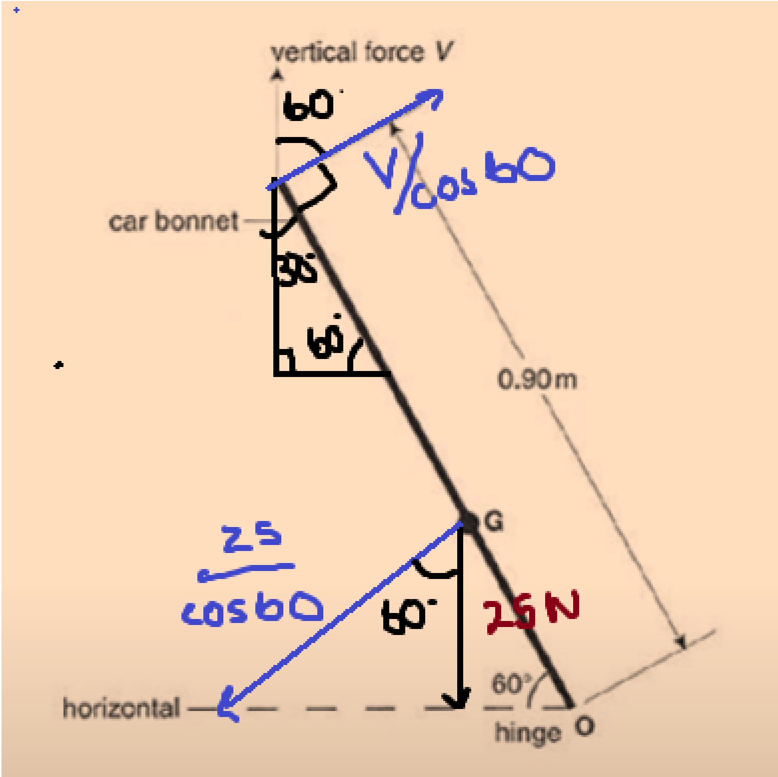
Centre of gravity is the point we model the weight force coming from.



a) Show that the force V is 9.7N.

We need to find the component of weight and the component of V that are perpendicular to the distance.

Using geometry, we can prove that the forces acting perpendicular to the bonnet are **V / cos60** and **25/cos60**:



Now taking moments about the hinge O:

The clockwise moment is the moment created by force V, and the anticlockwise moment is the force created by gravity.

(25/cos60) * 0.35 = (V/cos60) * 0.9

17.5 = 0.9V / cos60

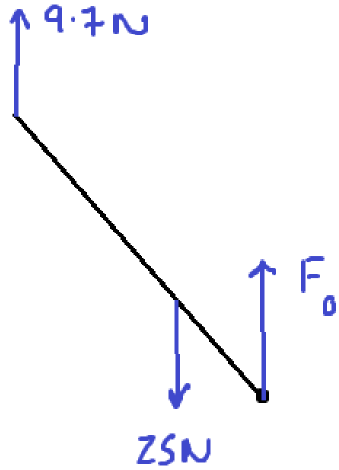
V = 17.5cos60 / 0.9 = 9.7N.

b) Find the magnitude of the force exerted at O

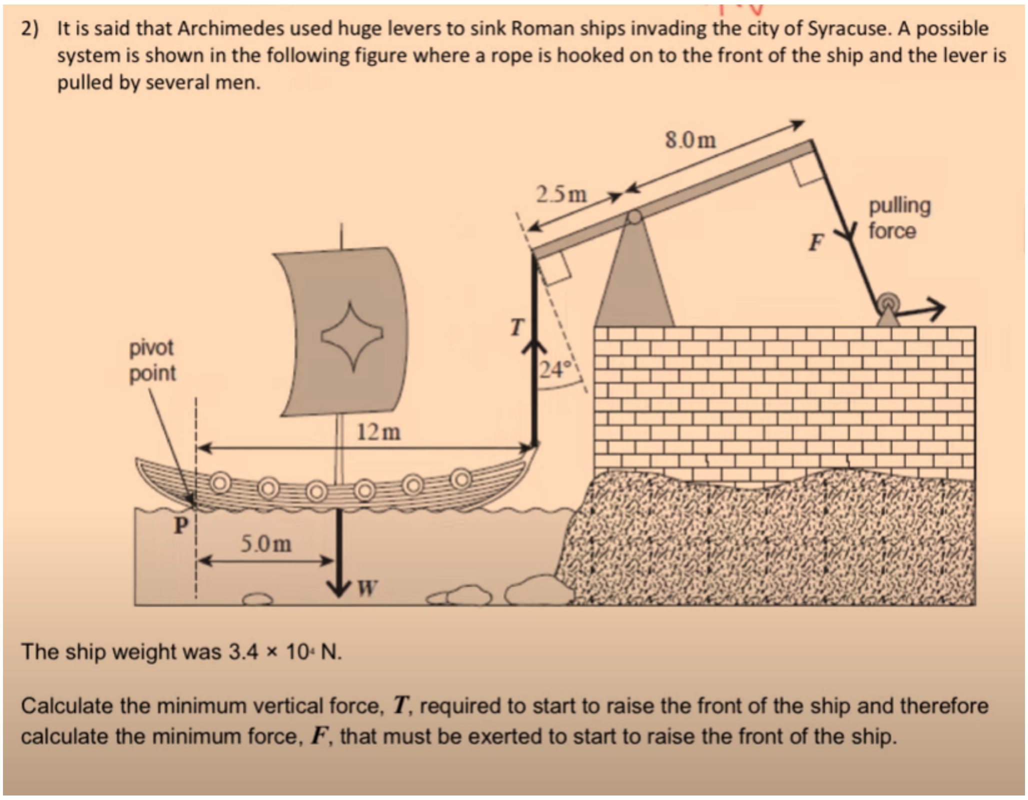
There is a normal reaction force at O.

To find this normal reaction force, we consider that the body is in equilibrium. Thus, the total upwards forces must equal the total downwards forces.

$F_O + 9.7 = 25 \Rightarrow F_O = 15.3$N.



Example 7



2) It is said that Archimedes used huge levers to sink Roman ships invading the city of Syracuse. A possible system is shown in the following figure where a rope is hooked on to the front of the ship and the lever is pulled by several men.

The ship weight was 3.4×10^4 N.

Calculate the minimum vertical force, ***T***, required to start to raise the front of the ship and therefore calculate the minimum force, ***F***, that must be exerted to start to raise the front of the ship.

The pivot point, P, is where the ship will be lifted up from.

We can first take moments about P to determine the value of T.

-

Clockwise moment = $5 * 3.4*10^4 = 1.7*10^5$N

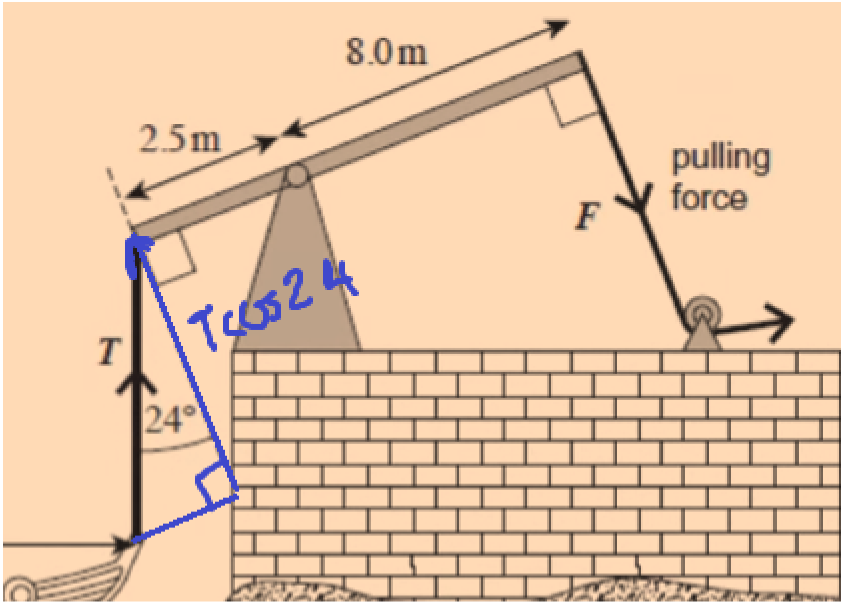
Anticlockwise moment = 12 * T

$12T = 1.7*10^5 \Rightarrow T = 14200$N.

-

Now, we consider a separate system, where we have the tension force, T, and the pulling force, F, being exerted on the lever.

We need to consider the component of tension that is perpendicular to the pivot of the lever:



Taking moments about the pivot of the lever:

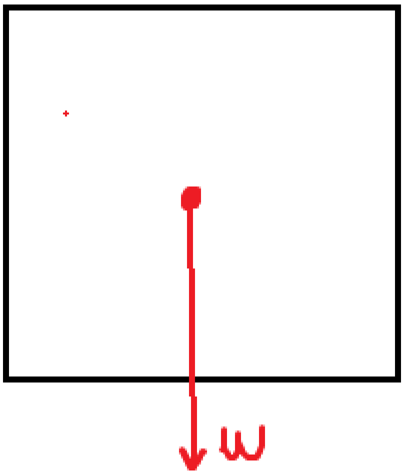
8F = 2.5T/cos24

F = (2.5Tcos24) / 8 = (2.5(14200)cos24) / 8 = **4050N**

Centre of gravity

The centre of gravity is the point about which all of a body's weight appears to act around.

For example, consider a square with uniform density:

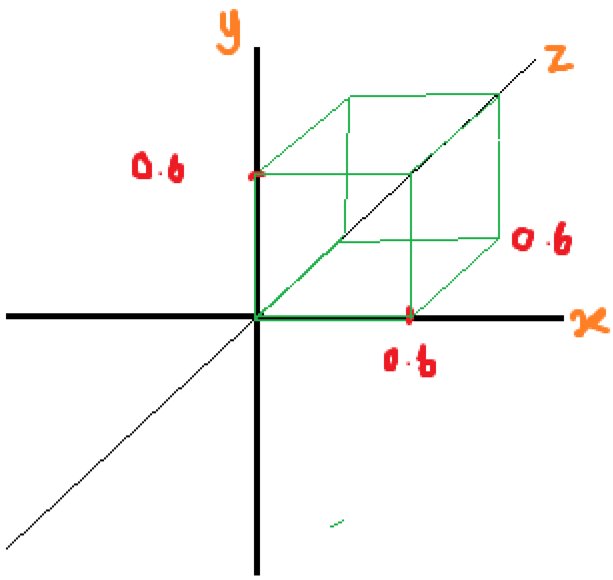


Because the square has the same density throughout, its centre of gravity is at the centre of the square.

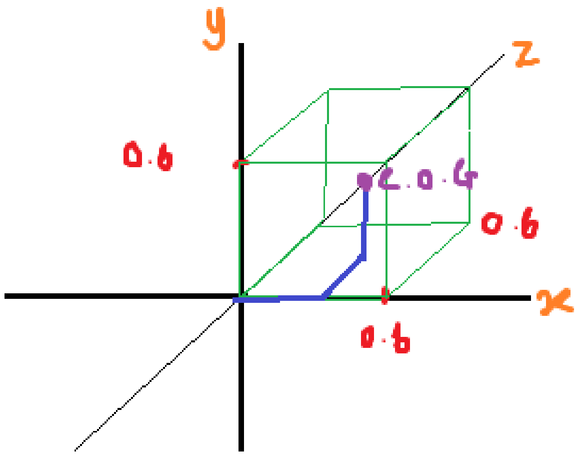
The gravity of the square will act about this point.

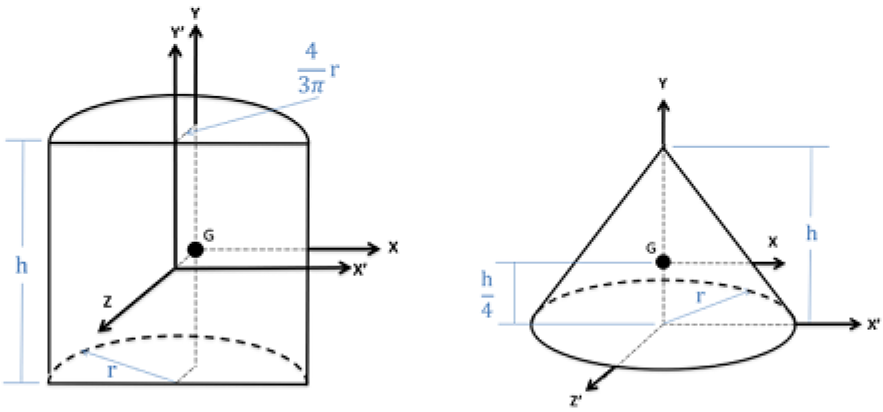
-

Consider a cube with uniform density and dimensions x=0.6, y=0.6 and z=0.6:



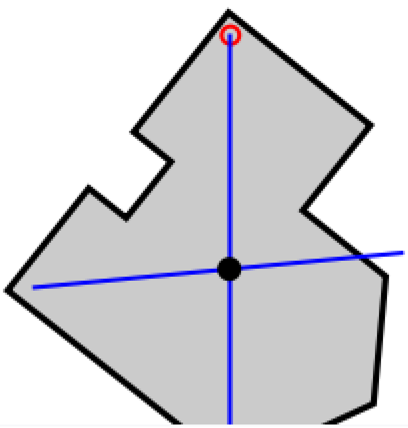
The centre of gravity of the cube must be half of each of the x, y and z values, i.e. (0.3,0.3,0.3):





However, if a body has a non-uniform density, we cannot assume that the centre of gravity is at the centre of the body; it will instead be in a region away from the centre where the most mass is concentrated.

To determine the centre of gravity of a non-uniform body, we can hang it from a pivot, at multiple positions, and draw a line directly downwards representing the direction of the weight:

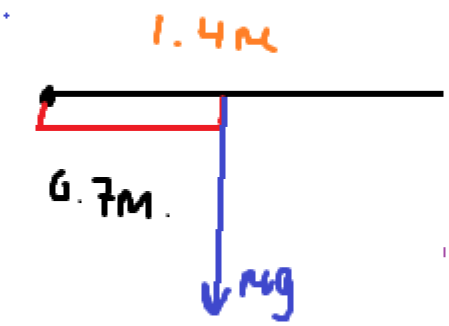


The point where these lines converge is the centre of gravity.

-

Now let us consider how the centre of gravity is applied to different situations.

Take a beam of length 1.4m and uniform density attached to a pivot:



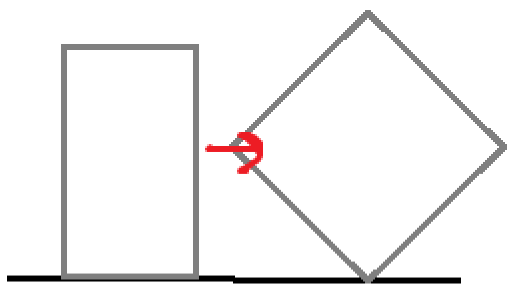
Since the beam is uniform and has a length of 1.4m, its centre of gravity is 0.7m from the pivot.

The centre of gravity is the point where all of a body's weight appears to act. Thus, the weight of this body acts 0.7m from the pivot. This will create a turning effect (moment) that rotates the beam in a clockwise direction about the pivot.

-

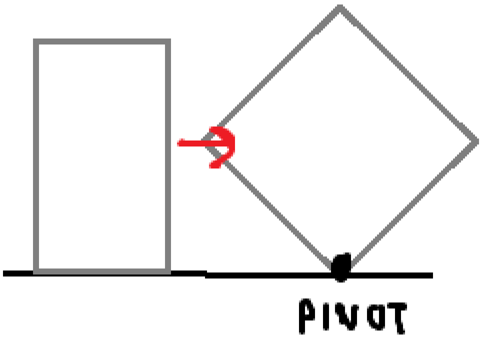
Another situation where the centre of gravity can be applied is in toppling problems.

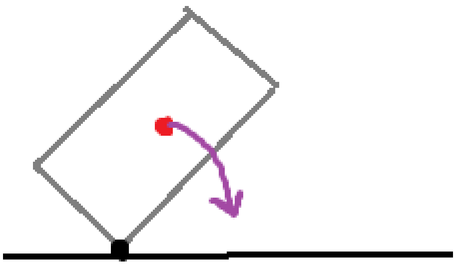
Take a body that is tilting:



The point about which the body is tilting is a pivot point:

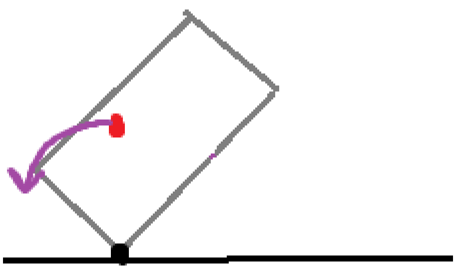
If the centre of gravity of the body is outside of this pivot point, a clockwise moment is created that causes the body to topple. If the centre of gravity is inside the pivot point, an anticlockwise moment iscreates that causes the body to return to its original position:



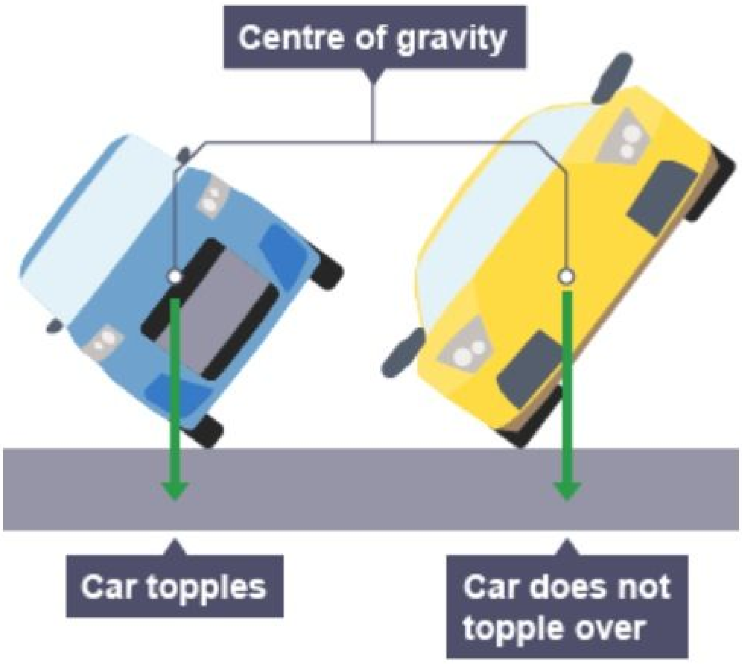


In the top diagram, the centre of gravity is outside of the pivot, so a moment is created that causes the body to topple.

In the lower diagram, the centre of gravity is inside of the pivot point, so a moment is created that causes the body to return to its original position.



To maximise the stability of a body, the centre of gravity should be as low as possible. This is because the angle at which toppling occurs is higher for a higher centre of gravity.



The wider car will not topple over because it has a lower centre of gravity, but the narrow car will

1.2 - Kinematics

Describing motion

A particle in motion can be described by 3 main units of measurement:

- Distance (scalar) / Displacement (vector)
- Speed (scalar) / Velocity (vector)
- Acceleration

Suppose a body travels a distance of 6m over 3s. It must have an average speed of $2ms^{-1}$, so we can say that:

Speed = distance / time

If the displacement of a body is 12m over 4s, its average velocity must be $3ms^{-1}$, so we can say that:

Velocity = displacement / time

It is important to use the term *average* here, because the body could change its speed / velocity, and still travel the same distance / displacement.

E.g If a body changes its displacement by 6m over 10s, we say its average velocity is $0.6ms^{-1}$.

However, it could equally travel a distance of 6m over 10s by accelerating from rest to a velocity of $1.2ms^{-1}$ (we will see how this is calculated later).

Thus, when using speed = distance / time, or velocity = displacement / time, we are assuming that the particle is moving at a constant speed / velocity, or we are calculating its *average* speed / velocity.

-

Acceleration is a change in a particle's velocity over time.

Consider a particle with a constant acceleration. If it has a constant acceleration, its velocity is increasing at a constant rate.

Suppose the particle starts off with a velocity of $6ms^{-1}$, and accelerates to $12ms^{-1}$ over 2s.

The particle must be gaining velocity at a rate of $3ms^{-1}$ per second; so its acceleration is $3ms^{-2}$.

Acceleration = change in velocity / time

$$a = (v-u) / t$$

a = acceleration, v = final velocity, u = initial velocity, t = time

If a particle starts with a velocity of $8ms^{-1}$, and accelerates to $15.6ms^{-1}$ over 4s, its acceleration is:

$a = (15.6-8) / 4 = 1.9ms^{-2}$

If a particle starts with a velocity of $9ms^{-1}$, and has an acceleration of $3ms^{-2}$, its velocity after 4s will be:

$v = 9 + 3(4) = 21ms^{-1}$. (the particle is gaining $3ms^{-1}$ per second, so in 4s, it gains $12ms^{-1}$).

-

Motion-time graphs

The movement of a particle over time can be represented in a motion-time graph.

If we are plotting a scalar quantity (distance or speed) over time, we do not need to consider any negative values.

However, if we are plotting a vector quantity (displacement, velocity or acceleration) over time, we do need to consider negative values; as a vector has both direction and magnitude.

Distance and displacement-time graphs

- A distance-time graph shows the distance a particle travels over a given time.

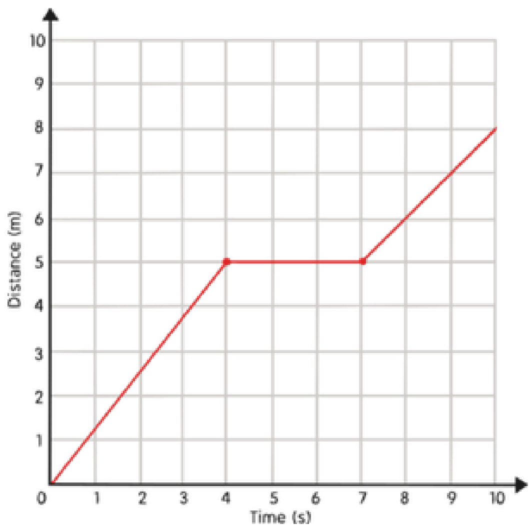
From a distance-time graph, there is no way to determine the direction in-which a particle is travelling; the graph simply shows how far a particle moves over a given time.

- A displacement-time graph shows the distance of a particle *from the origin* over a given time.

When constructing a displacement-time graph, we need to set a positive direction and a negative direction.

If a particle has a positive displacement, it is on one side of the origin; while if it has a negative displacement, it must be on the other side of the origin.

Distance-time graphs



This distance-time graph on the left can be split into three sections:

1 - The particle moves 5m over 4s

2 - The particle does not move for 3s

3 - The particle moves 3m over 3s

For each stage of the motion, we do not know in what direction the particle is moving; simply that it is moving a given distance over a given time.

We can calculate the speed for each stage of the motion:

1 - In the first stage, the particle moves a distance of 5m over 4s, so its speed is 5/4 = 1.25ms^{-1}.

2 - In the second stage, the particle is not moving, so its speed is 0ms^{-1}.

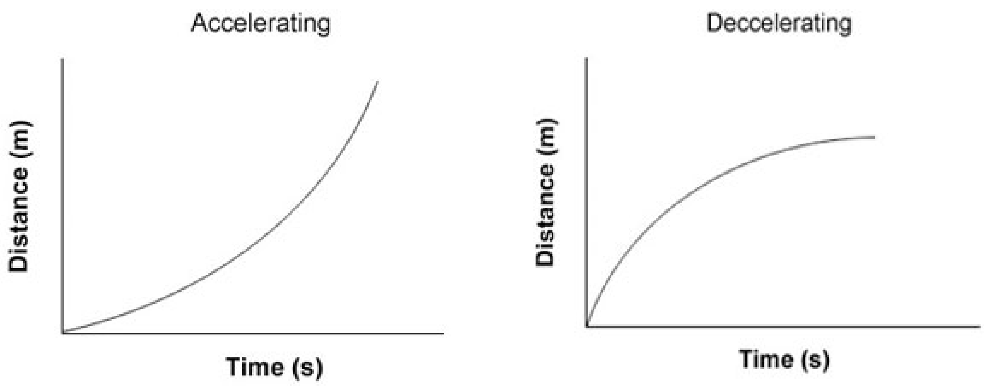
3 - In the third stage, the particle moves 3m over 3s, so its speed is 1ms^{-1}.

The speed of a particle is also the gradient of a distance-time graph, since gradient = distance / time = speed.

If the gradient of the graph is 0 (as seen in stage 2), the speed of the particle is 0ms^{-1}.

If the gradient of the graph is constant, the particle has a constant speed. However, if the graph is curved, it implies that the particle's speed is changing (i.e. it is accelerating):

The



gradient of the graph is different at different points in time; so the particle's speed must be changing.

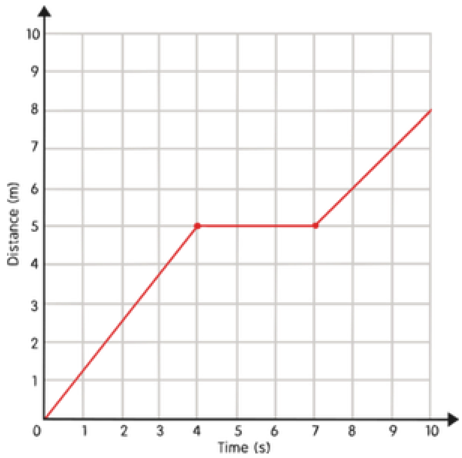
To find the speed at a given point in time from a distance-time graph, we can use two methods:

- Draw a tangent to the point in time and find the gradient of the tangent
- Differentiate the equation of the curve, and find dd/dt at that point in time

-

To find the **average speed** for a particular journey, we divide the entire distance travelled by the time.

Consider the first graph:



This graph shows how a particle moves a total of 8m over 10s, so its average speed is **8 / 10 = 0.8ms^{-1}**.

Suppose a particle travels a distance of 1500m over 300s in a straight line, and this journey is split into two stages:

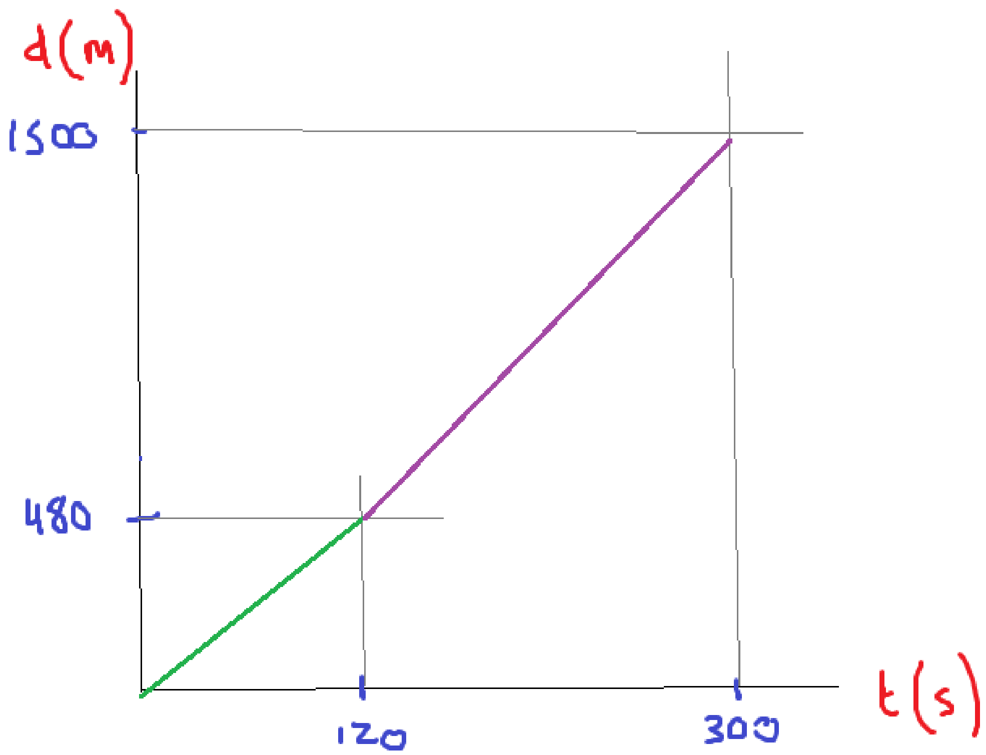
- The particle moves at 4ms^{-1} for 120s
- The particle moves at xms^{-1} for the remaining 180s.

How would we draw the distance-time graph for this journey?

In the first stage of the motion, the particle travels a distance of: 4 * 120 = 480m.

This means that in the second stage, the particle travels 1500 - 480 = 1020m. Its speed, x, for the second stage is therefore 1020 / 180 = 5.67ms^{-1}.

We would expect the gradient for the second stage to be higher since the particle is travelling at a higher speed.

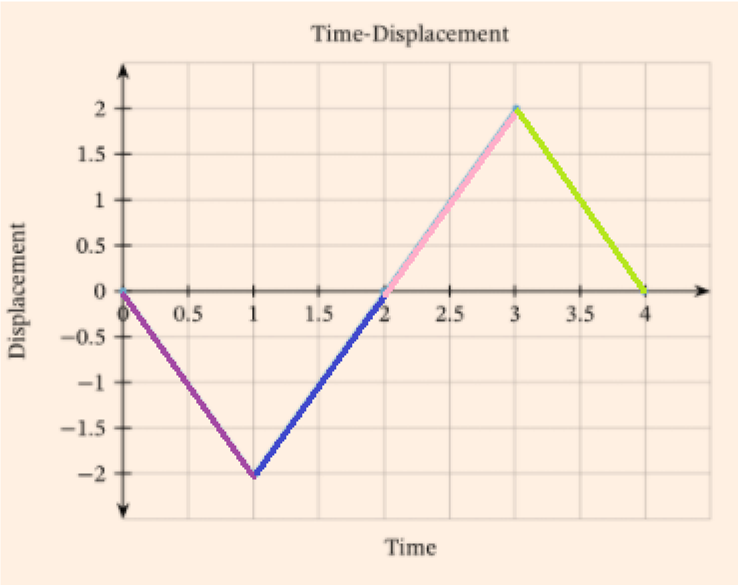


The average speed for the journey is 1500 / 300 = 5ms^{-1}.

-

Displacement-time graphs

In the displacement-time graph on the left, there are 4 stages of motion.



We will define ‘left’ as negative displacement, and ‘right’ as positive displacement.

1 - The particle first moves 2m to the left of the origin over 1s.

2 - The particle then returns to the origin over another 1s.

3 - The particle moves 2m to the right of the origin over 1s.

4 - The particle returns to the origin over 1s.

We can also determine the *velocity* (not speed) for each stage of the motion.

1 - The particle moves to a displacement of -2m over 1s, so its velocity = -2 / 1 = -2ms^{-1}.

2 - The particle gains a displacement of +2m over 1s, so its velocity = 2 / 1 = 2ms^{-1}.

3 - This velocity of 2ms^{-1} continues into stage 3.

4 - The particle’s displacement decreases by 2m over 1s, so its velocity = -2 / 1 = -2ms^{-1}.

- If the gradient of a displacement-time graph is positive - the particle’s velocity is positive (i.e. it is moving in the positive direction).

- If the gradient of a displacement-time graph is negative - the particle’s velocity is negative (i.e. it is moving in the negative direction).

The **average velocity** for the entire journey is the final displacement over the total time taken to reach that displacement.

In the above graph, the particle finishes at the origin, so its total displacement (distance from the origin) is 0m. This means its average velocity is 0ms^{-1}.

Whenever a particle starts at the origin and finishes at the origin, its average velocity is $0ms^{-1}$.

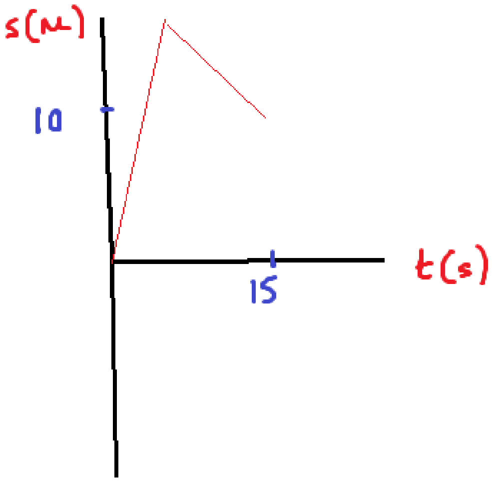
-

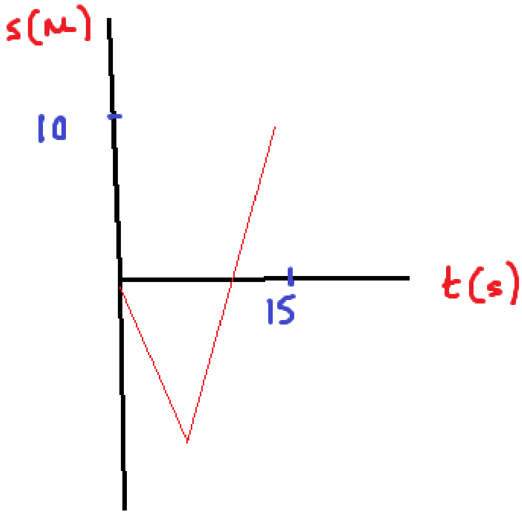
From the average velocity of a particle, it is only possible to determine the particle's final position.

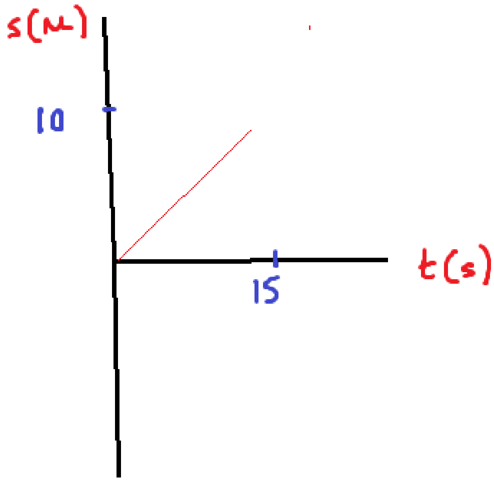
It is not possible to determine how the particle reached this position.

- For example, suppose a particle has a final displacement of 10m, and it took 15s to reach this displacement.

There are infinite possibilities as to how the particle can move over this displacement over this time:



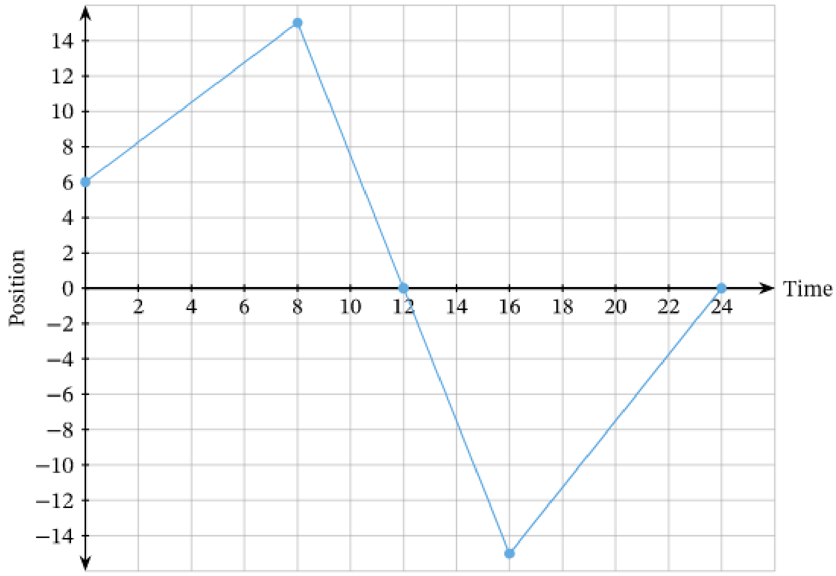




In each instance above, the particle has a total displacement of 10m over 15s, and so it has an average velocity of 2/3ms^{-1}.

-

From a displacement-time graph, it is also possible to determine the average speed.



Assume position is in metres and time is in seconds.

In the graph above, the particle has 4 stages to its motion:

- It moves 8m from the original position over 8s. (velocity = 8 / 8 = 1ms^{-1})
- It moves 15m in the opposite direction over 4s. (velocity = -15 / 4 = -3.75ms^{-1})
- It moves a further 15m in the opposite direction to the original direction over 4s. (velocity = -15 / 4 = -3.75ms^{-1})
- It moves 15m in the original direction over 8s. (velocity = 15 / 8 = 1.875ms^{-1})

The average *speed* is the total distance travelled over the time.

- The total distance travelled is 9 + 15 + 15 + 15 = 54m.
- The total time is 24s.
- Thus, average speed = 54 / 24 = 2.25ms^{-1}.

The particle begins at a position of 6m, and ends at a position of 0m, so its displacement is -6m.

This means its *average velocity* is -6 / 24 = -0.25ms^{-1}.

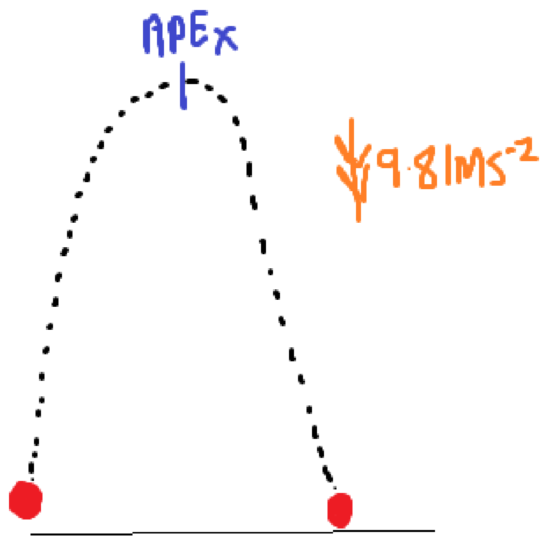
Now let us consider how different motions would be represented in a displacement-time graph.

Example 1: *throwing a ball directly upwards.*

If a ball is thrown directly upwards, it is pulled back towards Earth with an acceleration of $9.81ms^{-2}$ due to gravity.

It will start off with a certain velocity in the upwards direction, but since it is being accelerated downwards at $9.81ms^{-2}$, its velocity will decrease by $9.81ms^{-1}$ per second (ignoring air resistance, which would increase the rate of decline).

Eventually, the ball's velocity will be reduced to 0, and it will reach an apex (maximum height). After this point, the ball will start to be accelerated towards the ground at $9.81ms^{-2}$, so it will fall back downwards over the same time it took to reach its apex.

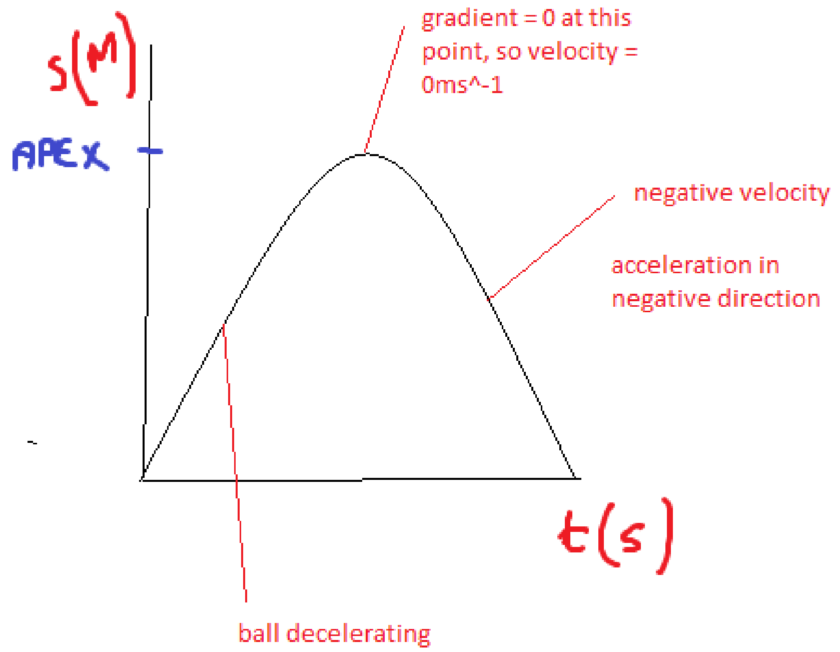


How would this be represented in a displacement-time graph?

- Because the ball has changing velocity, we know that the graph must be curved.

- The velocity of the ball decreases as it rises upwards, so the gradient of the curve must decrease over time in a parabola shape.

- As the ball falls downwards, it is now returning to its origin. This means that the displacement decreases over time (so the velocity is negative). The rate of change of displacement (i.e. the velocity) must be increasing since the ball is accelerating towards the ground (i.e. its negative velocity is increasing).

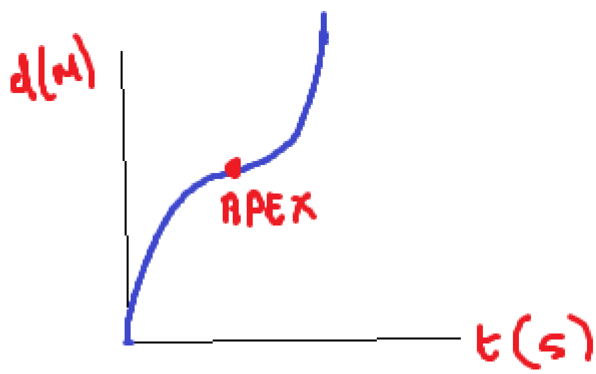


Now let us consider how we would draw the distance-time graph for throwing the ball directly upwards. Since direction is not specified in a distance-time graph, the distance travelled will continue to increase throughout the journey.

During the upwards motion, the speed of the ball decreases at a constant rate (of $9.81ms^{-2}$) due to gravitational deceleration, meaning the distance travelled will decrease at a decreasing rate (gradient decreasing).

During the downwards motion, the speed of the ball increases at a constant rate (of $9.81ms^{-2}$) due to gravitational acceleration. This leads to the distance travelled will increase at an increasing rate (gradient increasing).

This gives the following curve:



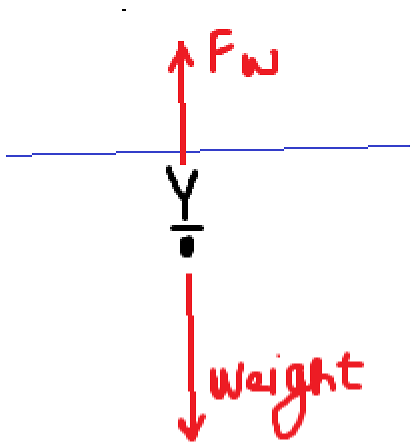
Example 2: jumping off diving board

If a person jumps off a diving board, they will accelerate downwards at 9.81ms^{-2} (ignoring air resistance).

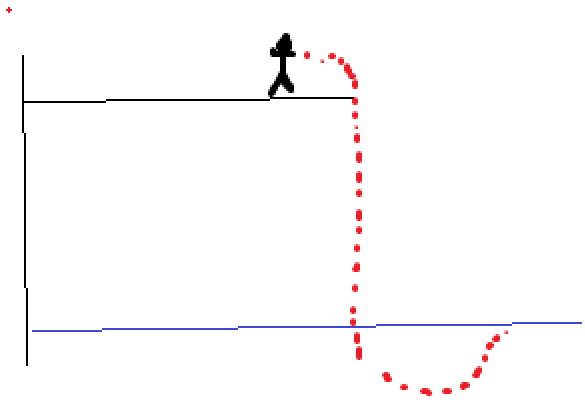
This will result in the velocity of the person increasing, so the rate of change of displacement will increase at an increasing rate.

When the person hits the water, their acceleration will suddenly decrease from 9.81ms^{-2} to a negative value due to the force exerted by the water on the person (this is an equal and opposite force to the force exerted by the person on the water).

The water exerts an upwards force that opposes gravity:

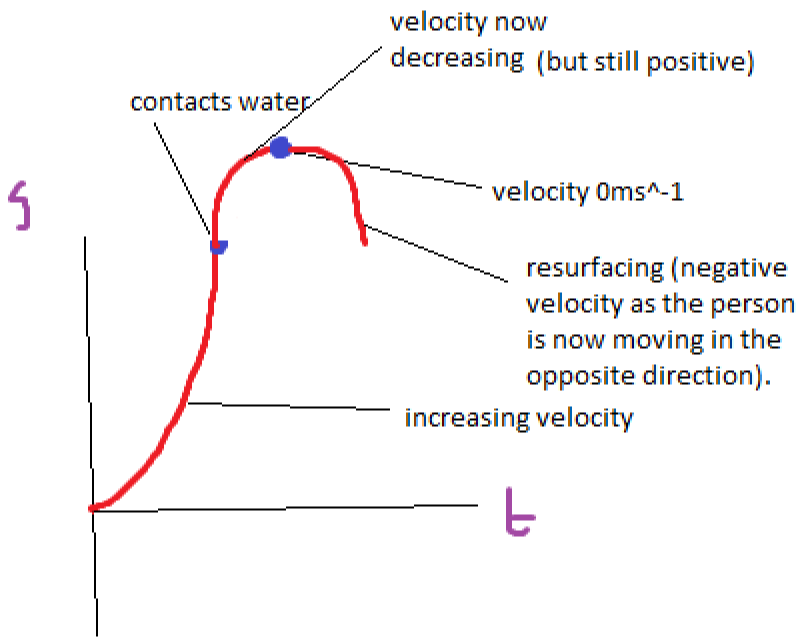


The net force is now in the upwards direction, so the person begins to decelerate. Their velocity eventually decreases to 0ms^{-1}, and the deceleration continues, causing the person to have a negative velocity and to therefore resurface.

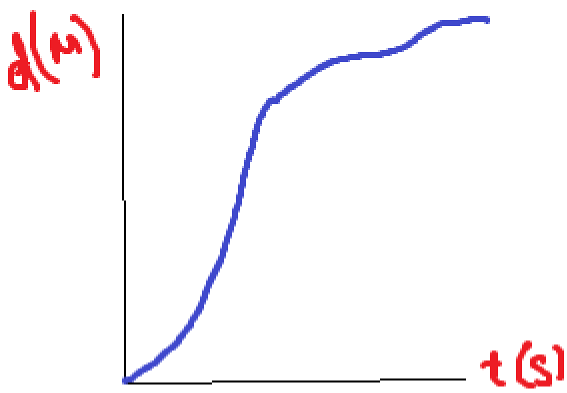


The diving board is the origin. We will take downwards as the positive direction.

- The displacement initially increases at an increasing rate due to gravitational acceleration
- Upon contacting the water, the displacement now begins to decrease at a decreasing rate due to water resistance.
- Eventually, the velocity will reduce to 0 when the diver reaches the deepest displacement underwater.
- The velocity continues to decrease and now becomes negative. The person resurfaces (i.e. moves back towards the origin (diving board)), so they have negative velocity.



The distance-time graph for this motion will look as follows:



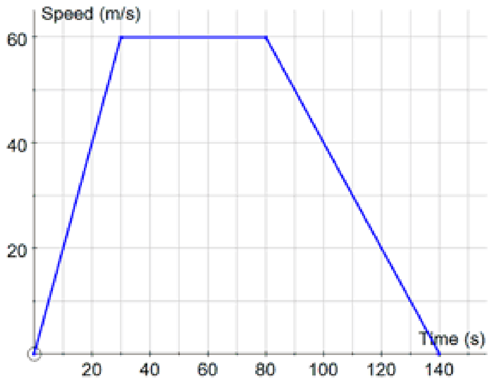
- The speed increases at an increasing rate during the dive, so the gradient of the distance-time graph increases.
- The speed then suddenly decreases upon hitting the water, so the distance covered over time decreases.
- The speed eventually decreases to $0ms^{-1}$ at the maximum depth below water, before increasing again as the diver ascends to the surface.

- We do not consider in what direction the diver is travelling.

Speed and velocity-time graphs

Speed-time graph

- A speed-time graph shows the speed of a particle over time.
- Speed is a scalar quantity; so it can never be less than $0ms^{-1}$, unlike velocity.
- It is impossible to tell in which direction a particle is travelling from a speed-time graph alone.
- If given information about the journey (e.g. throwing a ball directly upwards), it is possible to infer the direction.
- If the speed on a speed-time graph is increasing, the particle must be accelerating; if the speed on a speed-time graph is decreasing, the particle must be decelerating.
- Acceleration is the change in speed (or velocity) over time, so the gradient of a speed-time graph is the acceleration.
- The area under a speed-time graph tells us the distance travelled.



Consider the speed-time graph on the left.

The journey can be split into three sections:

1- The particle's speed increases from $0ms^{-1}$ to $60ms^{-1}$ over 30s.

2 - The particle maintains a constant speed of $60ms^{-1}$ for 50s.

3 - The particle's speed decreases from $60ms^{-1}$ to $0ms^{-1}$ over 60s.

From this graph, we can analyse multiple aspects of the journey:

- The acceleration
- The distance travelled

Acceleration:

In the first stage of the motion, the particle accelerates at a constant rate from $0ms^{-1}$ to $60ms^{-1}$ over 30s.

Thus, a = (60-0) / 30 = $2ms^{-2}$ (i.e. the particle is gaining $2ms^{-1}$ per second).

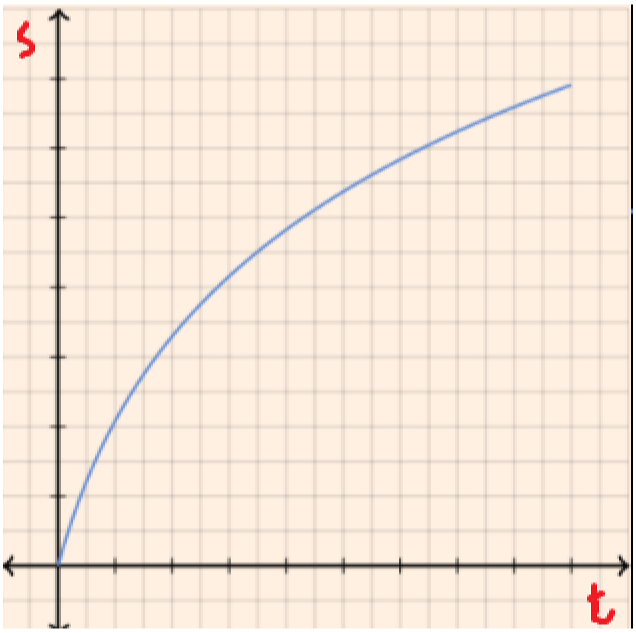
In the second stage, the particle remains at a constant velocity - so its acceleration is $0ms^{-2}$.

In the third stage of the motion, the particle decelerates from $60ms^{-1}$ to $0ms^{-1}$ in 60s, so its deceleration is 60 / 60 = $1ms^{-2}$ (i.e. its velocity is decreasing by $1ms^{-1}$ per second).

-

Acceleration is the *gradient* of a speed-time graph, since gradient = change in speed / time = acceleration.

If the gradient of a speed-time graph is constant, the acceleration is constant. If the gradient of a speed-time graph is not constant (i.e. curved), the acceleration must be increasing or decreasing.



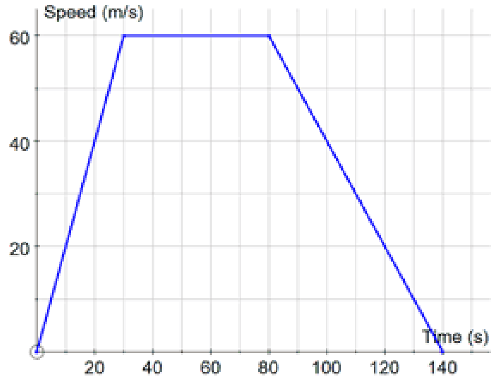
In the speed-time graph above, the gradient is decreasing over time, so the acceleration must be decreasing.

To find the acceleration from a curved speed-time graph, it is necessary to take a tangent to the curve and find the gradient of the tangent.

However, a more accurate method is to differentiate the equation of the speed-time graph and find the gradient at time t.

Distance travelled:

The distance travelled is the area under a speed-time graph.



In the first section of the motion, the area under the graph is: 1/2 * 60 * 30 = 900m.

So the particle travels 900m.

The total distance travelled is the area of the entire trapezium:

1/2 (50 + 140) * 60 = 2700m.

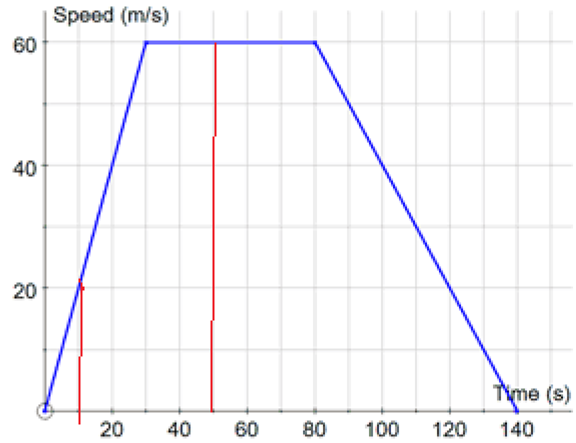
-

We previously said that the **average speed = total distance / time**.

In a distance-time graph, we could simply divide the final distance by the time.

In a speed-time graph, to find the total distance travelled, we find the area under the graph; and hence the average speed for a given time interval is the **area under the graph / time**.

In the above example, the total distance is 2700m, and it occurs over 140s; so the average speed is $2700 / 140 = 19.3ms^{-1}$.



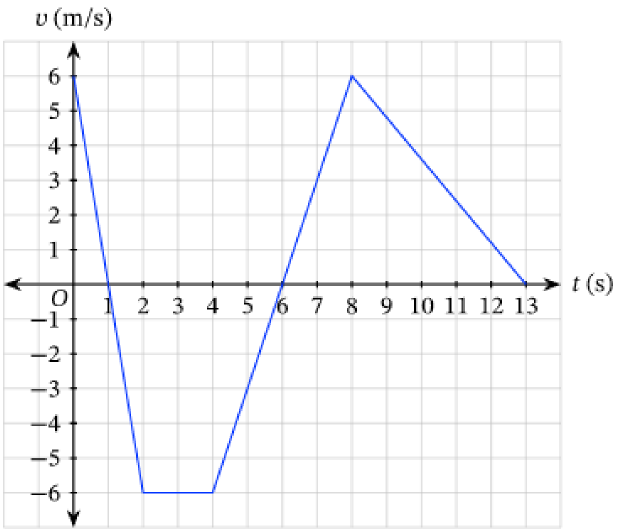
Suppose we wanted to find the average speed for a smaller time interval of between t=10 and t=50s.

We need to find the area under the area under the graph in this interval, and divide it by the time, 40s.

Area $_{\text{with limits of t=10 and t=50}}$ = (0.5 * (20 + 60) * 20) + (20 * 60)) = 2000m
Thus, average speed $_{\text{with limits of t=10 and t=50}}$ = 2000 / 40 = 50ms^{-1}.

Velocity-time graph

- A velocity-time graph shows the rate of change of a particle's displacement (i.e. velocity) over time.
- Velocity is a vector, and can therefore be positive or negative.
- If a particle is moving in the positive direction, its velocity is positive; if a particle is moving in the negative direction, its velocity is negative.
- The acceleration of a particle is the change in velocity over time, which is the gradient of a velocity-time graph.
- The total displacement in a given time interval is the area under a velocity-time graph.



Consider the graph on the left:

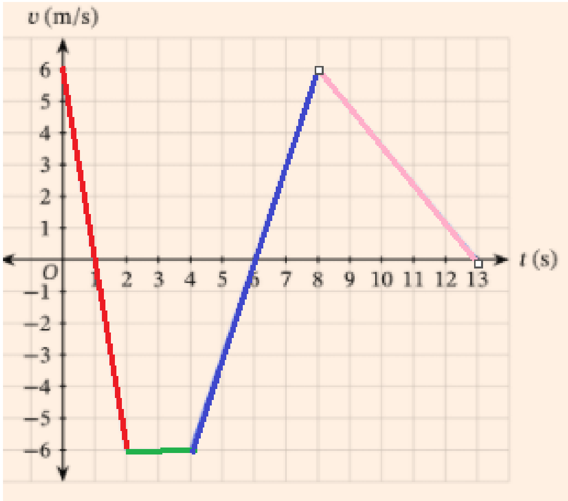
Whenever v>0, the particle is moving in the positive direction; whenever v<0, the particle is moving in the negative direction. If v=0, the particle is stationary.

- The particle starts with a velocity of 6ms^{-1}, so it is moving away from the origin at a rate of 6m every second.
- The particle then decelerates 0ms^{-1} over 1s.
- The particle keeps on decelerating and now starts to move in reverse to the original direction.
- The deceleration continues until the particle reaches a velocity of -6ms^{-1} over 1s.
- [At this point, the particle must be back at the starting position]

- The particle then moves at a constant velocity (no acceleration) of -6ms^{-1} over 2s, so it moves 12m in the opposite direction to the original direction.
- The particle then begins to accelerate, with its negative velocity increasing to become more positive.

- Between t = 4 and t = 6, the particle is slowing down in the negative direction, but is still moving in the negative direction.
- It then reaches a velocity of 0ms^{-1} at 6s, and begins to accelerate in the original direction up to a speed of 6ms^{-1} between t = 6 and t = 8.
- Between t = 8 and t = 13, the particle decelerates to 0ms^{-1}, still travelling in the positive direction.

We can determine the acceleration of the particle for each stage of the motion:



1 - Initial velocity = 6ms^{-1}, final velocity = -6ms^{-1}, over 2s. Thus, a = (-6 - 6) / 2 = -6ms^{-2} (the particle's velocity decreases by 6ms^{-1} per second).

2 - Velocity remains constant, so no acceleration.

3 - Initial velocity = -6ms^{-1}, final velocity = 6ms^{-1}, over a time of 4s, thus a = (6--6) / 4 = 3ms^{-2}.

4 - Initial velocity = 6ms^{-1}, final velocity = 0ms^{-1}, time = 5s, so a = (0-6) / 5 = 1.2ms^{-2}.

Acceleration is simply equal to the gradient of the velocity-time graph.

-

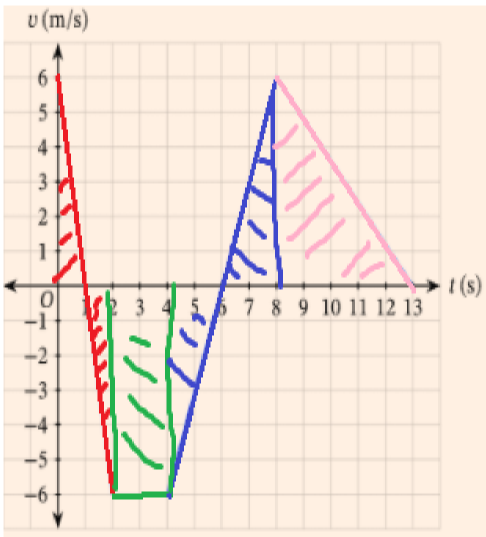
By finding the area under a velocity-time graph within a specific interval, we find the displacement during that interval.

<u>Why?</u>

We know that the total displacement in a given time interval is the average velocity for that time interval over time.

*Total displacement = average velocity * time*

If we consider the positive section of the red stage in the graph below, the average velocity of the particle is clearly 3ms^{-1} (0+6 / 2). Thus, from the above equation, the displacement is 3 * 1 = 3m.



This is also the same as the area under the graph in this time interval: 1/2 * 6 * 1 = 3m.

The area calculated must always be the area between the line and the time axis.

1 - Displacement = (1/2 * 6 * 1) + (1/2 * -6 * 1) = 0m

2 - Displacement = 2 * -6 = -12m.

3 - Displacement = (1/2 * 2 * -6) + (1/2 * 2 * -6) = 0m

4 - Displacement = 1/2 * 5 * 6 = 15m.

Thus, total displacement = 0 - 12 + 0 + 15 = 3m.

The particle finishes 3m from its original position. This was also its position at t=1.

We previously said that **average velocity = total displacement / time**, so the average velocity for this journey is 3 / 13 = 0.231ms^{-1}

-

When finding the total distance travelled from a velocity-time graph, we find the total area under the graph, but change all of the negative values to positive values, because distance does not consider the direction of motion.

For example, in stage 1, between t=0 and t=1, the particle's displacement is (1/2 * 6 * 1) = 3m. Between t=1 and t=2, the total displacement is (1/2 * -6 * 1) = -3m.

So the overall displacement for the first part of the motion is 0m. However, the particle has travelled an actual distance of 6m, travelling 3m from the origin, then 3m back towards the origin.

In stage 2, the displacement is -12m; this is a distance of +12m.

In stage 3, the displacement between t=4 and t=6 is (1/2 * -6 * 2) = -6m, and the displacement between t=6 and t=8 is +6m, which gives a total distance of 12m.

In stage 4, the displacement is 3m, so the distance travelled is 3m.

The total distance is thus 6 + 12 + 12 + 3 = 33m, giving an average speed of 33 / 13 = 2.54ms^{-1}.

-

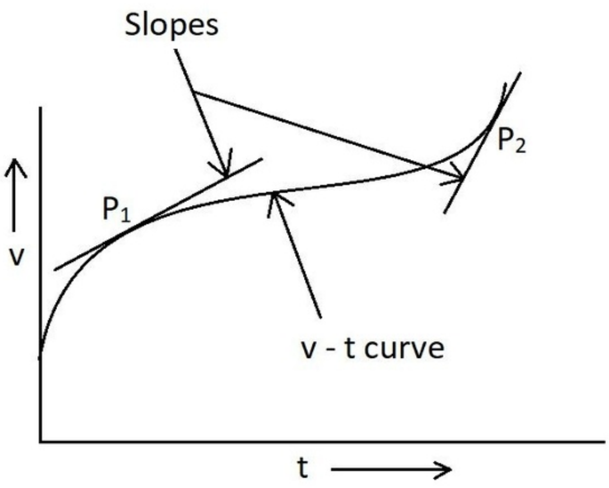
If a velocity-time graph has a constant gradient, the acceleration of the particle must be constant.

If a velocity-time graph has a variable gradient, there must be variable acceleration, because gradient = rate of change of velocity = acceleration, so if the gradient is varying, the acceleration must be varying.

In a situation where the acceleration is variable, to find the acceleration at time t, we can take two approaches:

- Draw a tangent to the curve at time t and find the gradient of the tangent
- Differentiate the equation of the curve and find the gradient at time t

To find the displacement of a particle when there is variable acceleration, it is necessary to integrate between the time intervals.



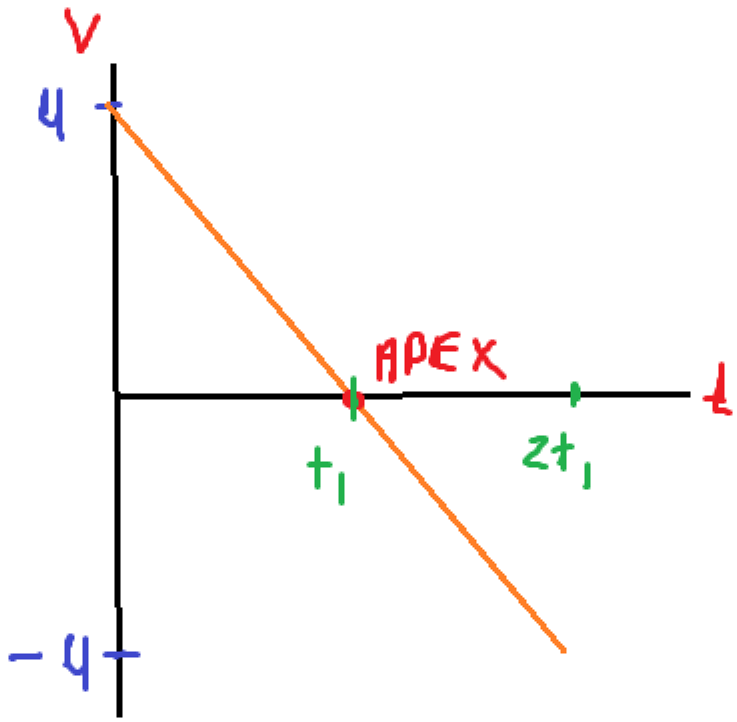
-

Now let us consider how a velocity-time graph can be applied to different situations.

Example 1: ball thrown vertically upwards

- When a ball is thrown vertically upwards, it must be thrown with some velocity, u.
- The ball is accelerated downwards by gravity at a rate of $9.81ms^{-2}$, ignoring gravity.
- This means that the velocity will decrease over time, at a constant rate due to the constant acceleration.
- The ball will reach its apex, where its velocity has been reduced to $0ms^{-1}$ by gravitational deceleration.
- The ball will then fall back towards the ground with an acceleration of $9.81ms^{-2}$, so its velocity will increase in the negative direction (to the original positive direction).
- The velocity at which the ball is thrown must be equal to the velocity at which the ball lands.

The velocity-time graph for this motion looks like this:



The velocity of the ball starts off at ums^{-1}. The ball is then decelerated to a velocity of $0ms^{-1}$ at its apex. The ball then begins to accelerate downwards at $9.81ms^{-2}$, with its velocity now being negative since it is moving in the opposite direction to the original direction (i.e. back towards the origin).

Notice how the gradient of the graph is constant, because the acceleration due to gravity is always $9.81ms^{-2}$ - it does not change. This of course ignores the effects of air resistance.

-

Suppose the ball is thrown upwards with velocity $25ms^{-1}$. Its velocity will decrease at a rate of $9.81ms^{-2}$.

We previously said that a = (v-u) / t.

If the velocity of the particle when it reaches its apex is $0ms^{-1}$, we can say:

-9.81 = (0 - 25) / t => t = -25 / -9.81 = 2.55s.

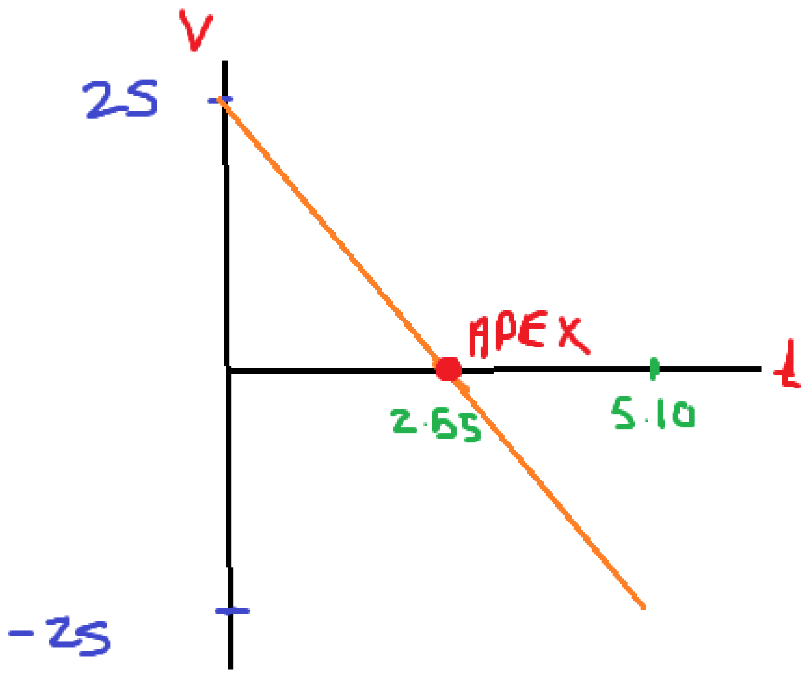
So the ball takes 2.55s to reach its maximum height.

The ball will then fall back to the ground with an acceleration of $9.81ms^{-2}$. If its initial velocity at the apex is $0ms^{-1}$, and the time taken to reach the apex is the time taken to return to the ground (if air resistance is ignored), we can say:

9.81 = (v - 0) / 2.55

v = 9.81 * 2.55 = $25ms^{-1}$ => so initial velocity = final velocity.

This gives us the following graph:



We can find the height the ball reaches by finding the area under the graph between t=0 and t=2.55.

Area = (1/2 * 25 * 2.55) = 31.9m.

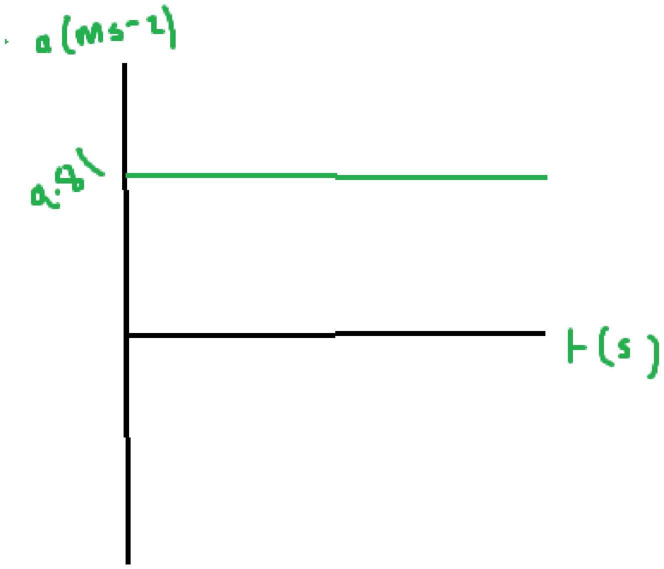
So the ball reaches a vertical height of 31.9m.

Acceleration-time graphs

An acceleration-time graph shows how the acceleration of a particle varies over time.

Suppose we are considering the motion of a particle in the vertical plane and the only force the particle is subject to is gravity.

Because the gravitational acceleration can be approximated to be a constant of 9.81ms^{-2} close to the surface of Earth, the acceleration of the particle will remain constant throughout its motion:



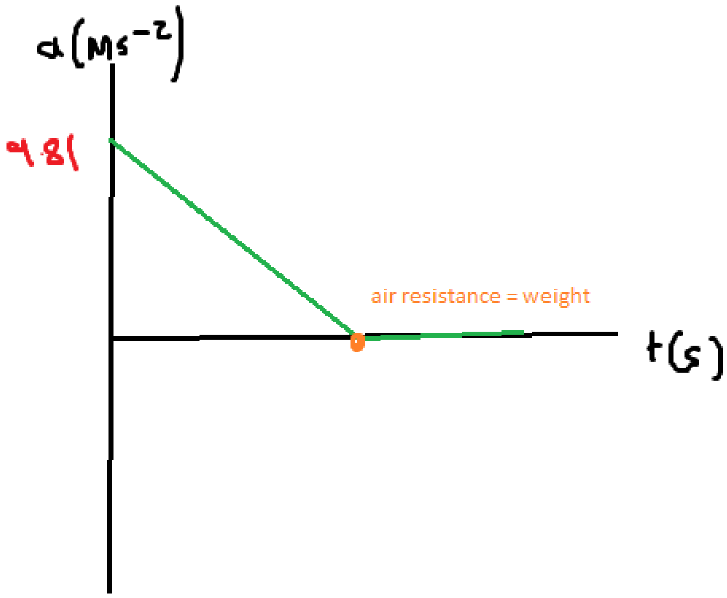
However, if we consider air resistance, the acceleration becomes variable.

Take a particle falling downwards close to Earth's surface that has been released from rest.

Initially, the air resistance exerted on the particle will be 0N, but as it geings to accelerate downwards due to gravity, its velocity increases, and therefore the number of collisions with air particles per second increases. This leads to the air resistance gradually increasing.

As the air resistance increases, the upwards acceleration of the particle increases. Eventually, the upwards acceleration equals the downwards acceleration, at which point, an equilibrium is established.

On an acceleration-time graph, this would look as follows:



When a=0ms^{-2}, the particle is at its terminal velocity.

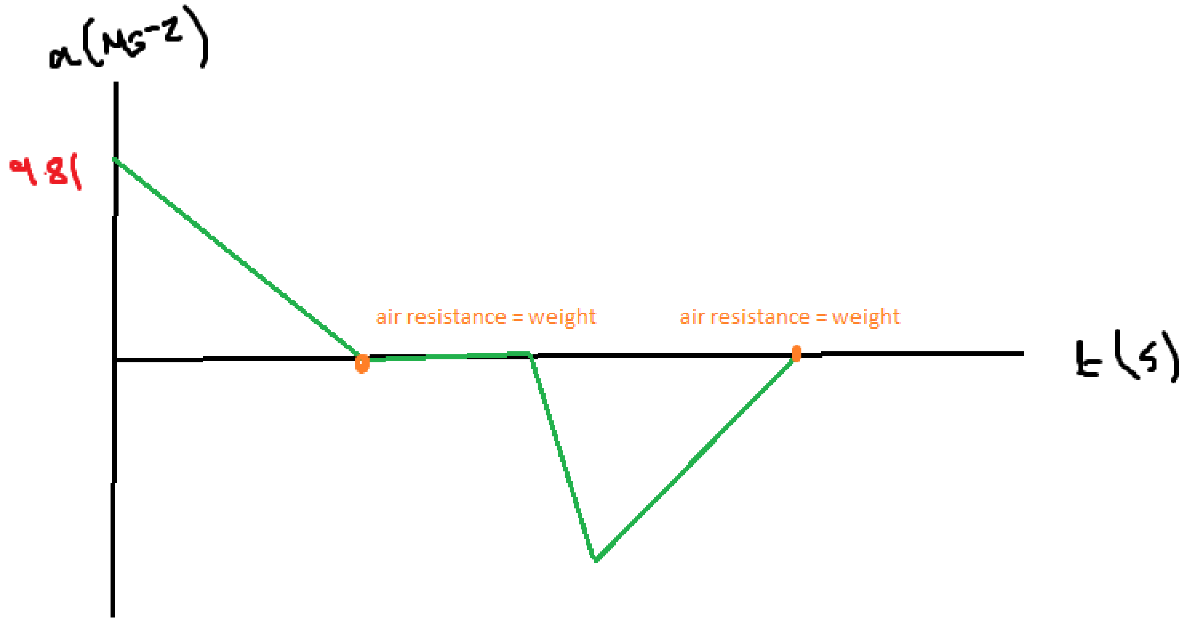
Now suppose that a few seconds after terminal velocity is reached, the particle deploys a parachute.

The parachute has a very high surface area, so it increases the number of contacts with air particles per second, leading to a higher air resistance.

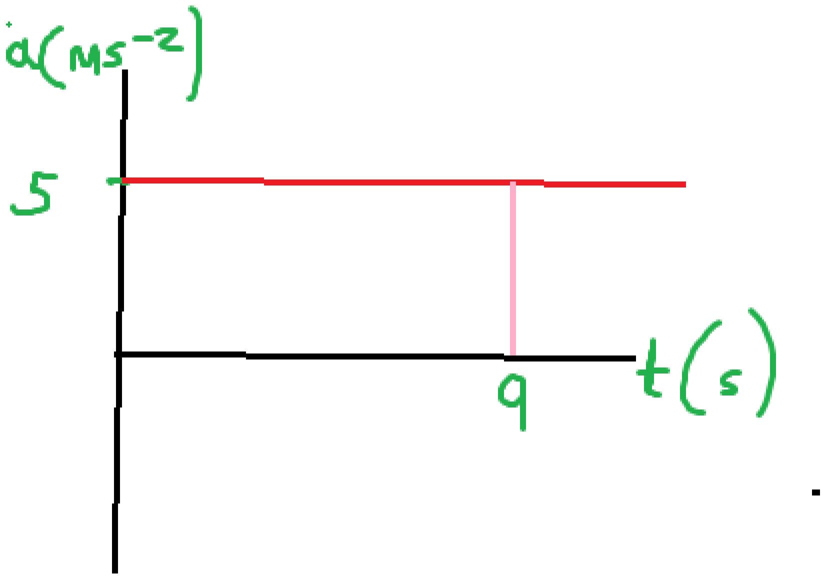
When the parachute is deployed, the air resistance becomes greater than the weight, so there is now a net upwards acceleration.

This upwards acceleration causes the particle to slow down in the downwards direction. As it slows down, however, the air resistance decreases. Eventually, the air resistance decreases until it equals the weight again, so the particle falls at terminal velocity.

This would be represented by a sudden decrease in acceleration (to a negative acceleration) when the parachute is deployed. The negative acceleration would then begin to increase (become more positive) until it again equals the gravitational acceleration:



From the area under an acceleration-time graph, we find the gain / loss in velocity.



Suppose a particle accelerates at a constant rate of $5ms^{-2}$ over 9s. The particle will gain a velocity of 5(9) = $45ms^{-1}$.

However, we cannot determine the velocity of the particle from this alone; we have to know the velocity at the start of the motion.

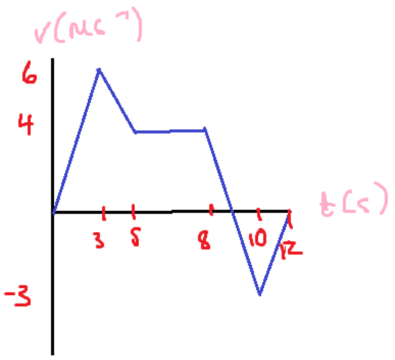
Suppose the particle starts at 20ms^{-1}. Its velocity after this motion will be 20+45 = 65ms^{-1}.

Mapping motion graphs onto each other

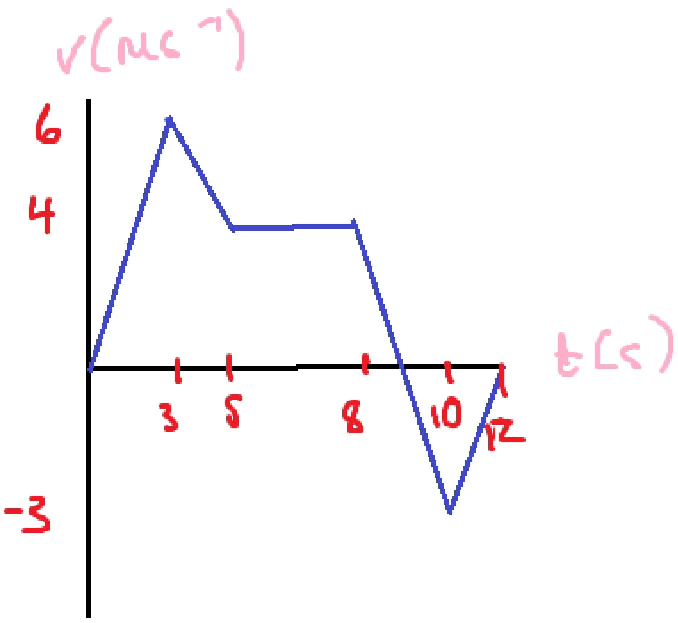
Now we will consider how we can take a motion-time graph (e.g. v-t, a-t, s-t), and determine what another type of motion-time graph would look like for the same motion.

Example 1:

<u>Take the following velocity-time graph:</u>



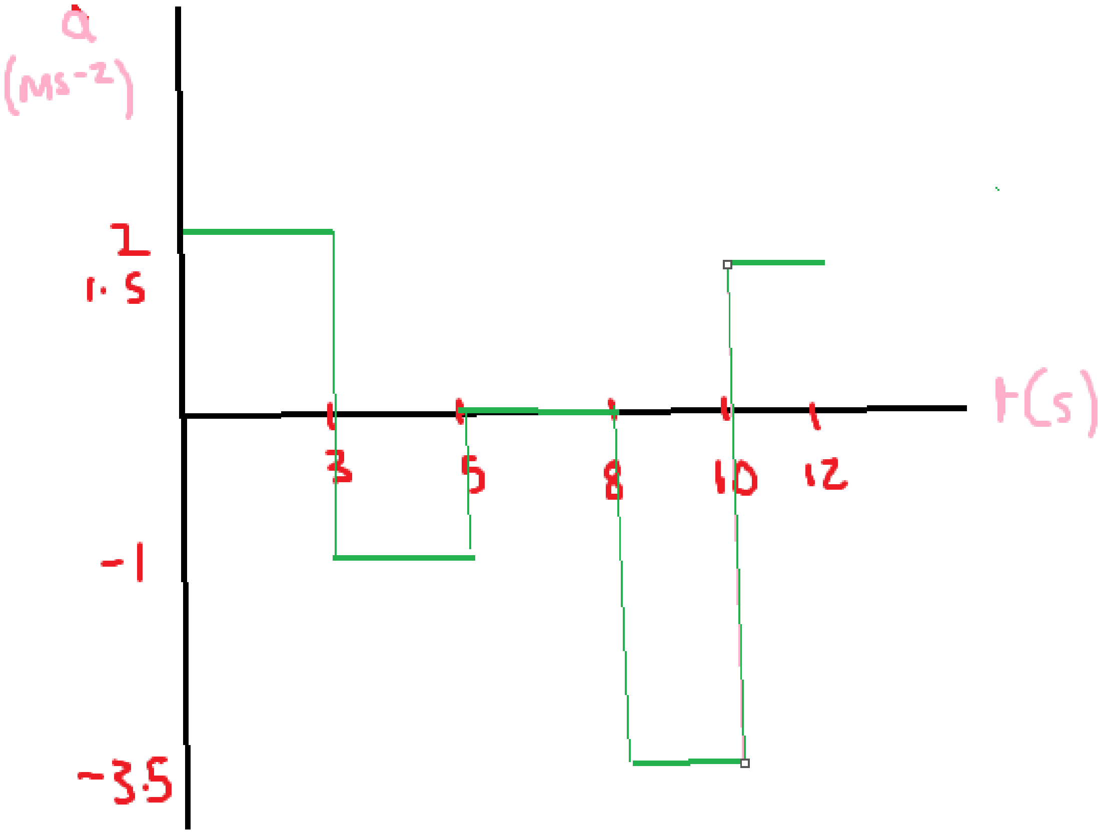
In this graph, the particle accelerates to 6ms^{-1} over 3s. This acceleration would lead to a curved displacement-time graph (displacement is increasing at an increasing rate). The particle then decelerates to 4ms^{-1}, but it is still moving in the positive direction - so its displacement is still increasing. The particle then retains a constant velocity of 4ms^{-1}; which will lead to a constant rate of change of displacement. The particle's velocity then decreases to 0ms^{-1}, and beyond this point, the particle continues to decelerate, until its velocity becomes negative. The particle is now moving in the opposite direction to the original direction. At t=10, the particle begins to accelerate (i.e. its velocity is increasing). This means that its velocity in the negative direction is decreasing, but it is still moving in the negative direction, so its displacement is decreasing. At t=12, has slowed down to rest.



Suppose we want to draw the acceleration-time graph for this motion.

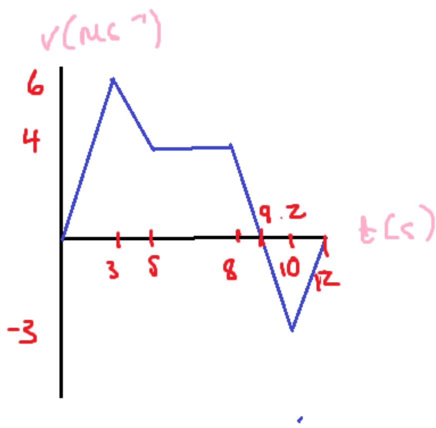
- In the first stage of motion, a = 6/3 = 2ms^{-2}
- In the second stage, a = (4-6) / 2 = -1ms^{-2}.
- In the third stage, a = 0 (no change in v)
- In the fourth stage, a = (-3-4) / 2 = -3.5ms^{-2}.
- In the fifth stage, a = (0--3) / 2 = 1.5ms^{-2}.

This gives the following a-t graph:



The accelerations change suddenly, which explains the sudden jumps. Instant changes in acceleration cannot occur in reality because all bodies have inertia; it will take some time for a new force that is being exerted to change the acceleration.

We could also plot the displacement-time graph for the journey:



Example 4

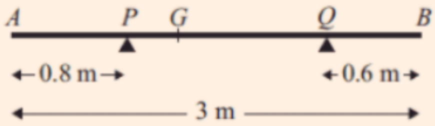


Figure 1

A non-uniform rod AB has length 3 m and mass 4.5 kg. The rod rests in equilibrium, in a horizontal position, on two smooth supports at P and at Q, where $AP = 0.8$ m and $QB = 0.6$ m, as shown in Figure 1. The centre of mass of the rod is at G. Given that the magnitude of the reaction of the support at P on the rod is twice the magnitude of the reaction of the support at Q on the rod, find

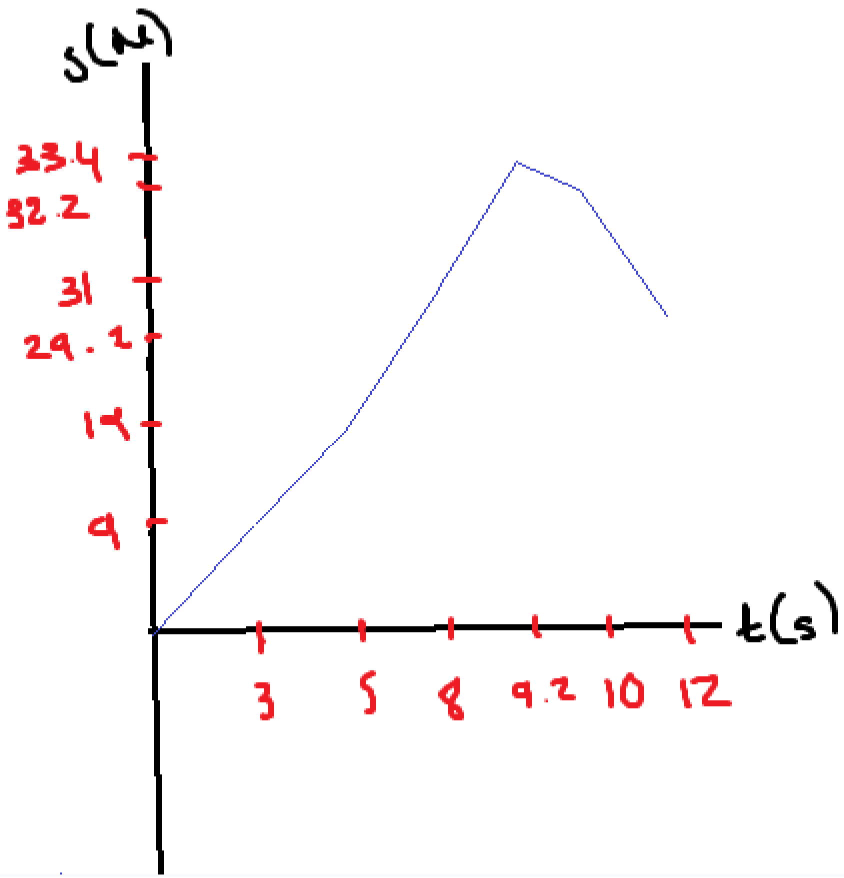
(a) the magnitude of the reaction of the support at Q on the rod,

(3)

(b) the distance AG.

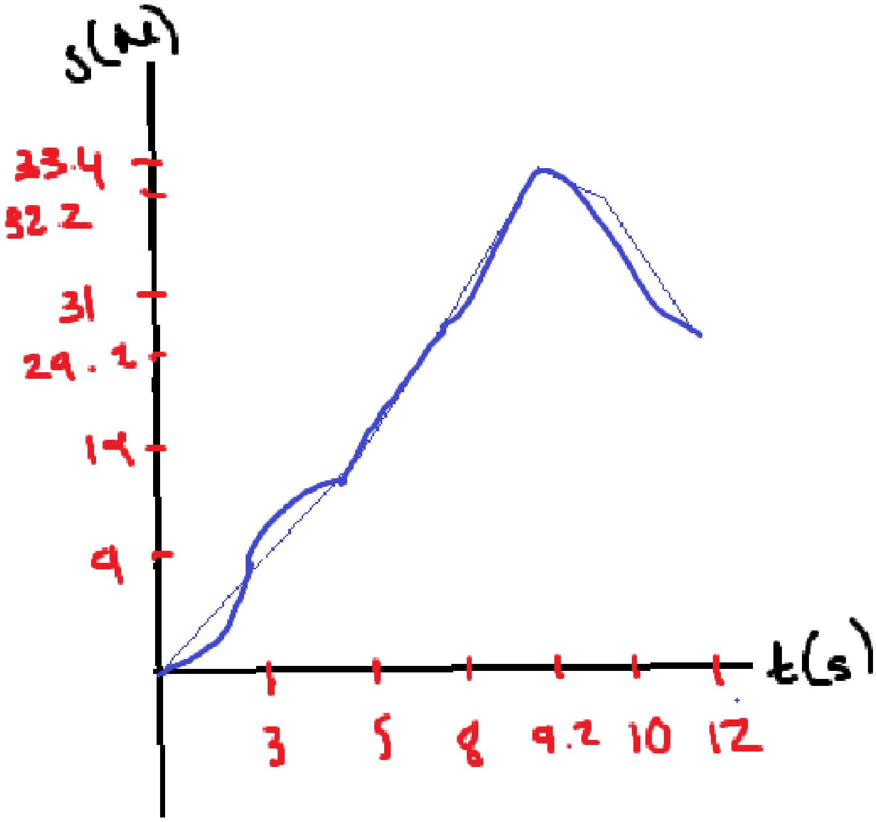
(4)

Stage 1: s = (1/2 * 6 * 3) = 9m
Stage 2: s = (1/2 * (4+6) * 2) = 10m (so the particle is now 19m from the origin)
Stage 3: s = 3 * 4 = 12m (now 31m from the origin)
Stage 4: s = (1/2 * 1.2 * 4) = 2.4m (now 33.4m from the origin)
Stage 5: s = (1.2 * 0.8 * -3) = -1.2m (now 32.2m from the origin)
Stage 6: s = (1/2 * 2 * 3) = -3m (now 29.2m from the origin)

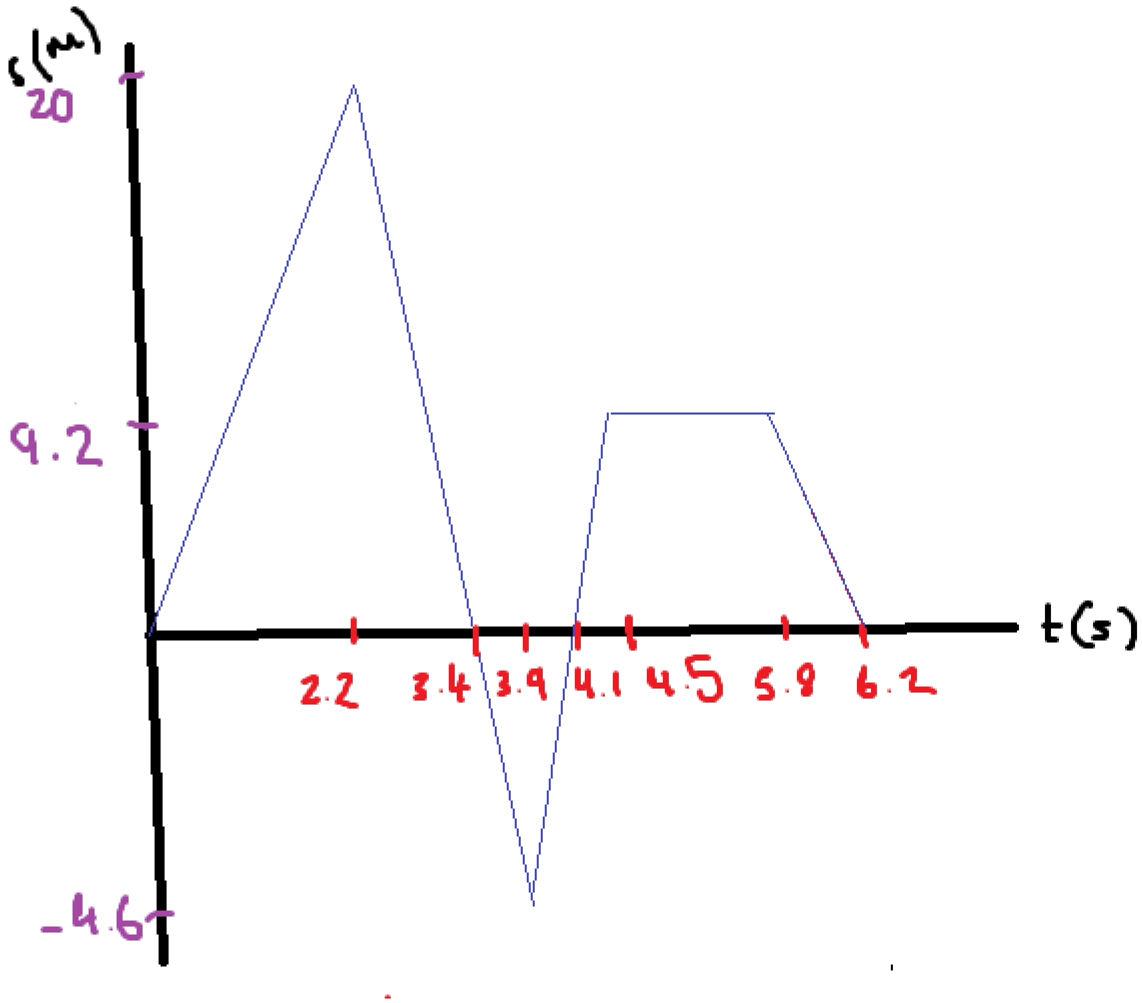


(graph would actually be curved due to changing velocity (i.e. changing gradient)

Example:



Example 2:



The above graph is a displacement-time graph.

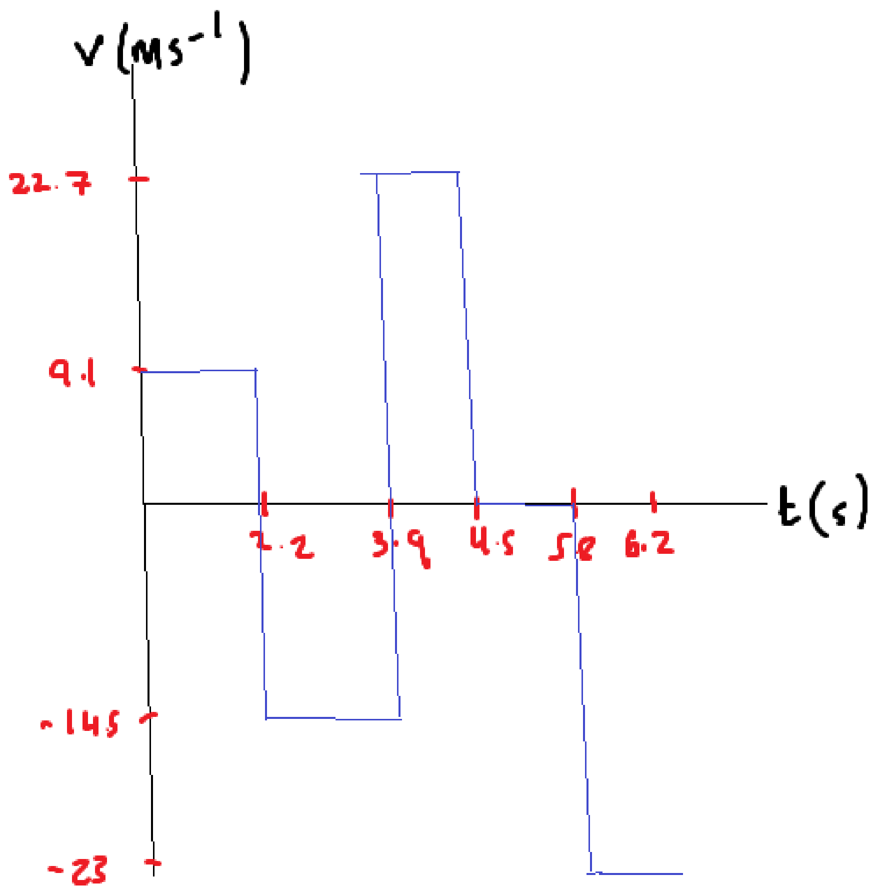
Because the gradient of the graph is not curved at any point, we know that the velocity is always some constant value - i.e. there is no acceleration. This is not a possible situation in reality since it takes time for a body to accelerate or decelerate to a new velocity.

The particle starts off by moving at a constant positive velocity for 2.2s, until it reaches a distance of 20m from the origin. It then begins to move in the opposite direction to the original direction, such that its displacement is decreasing. At t=3.4, the particle is back at its original position. Between t=3.4 and t=3.9, the particle continues to move in the negative direction, until it is 4.6m from the origin in the opposite direction to the original direction. The particle's velocity then becomes positive, so its displacement begins to increase. At t=4.1, it is back at the origin, and beyond this point, its displacement continues to increase. At t=4.5, the particle comes to rest at a distance of 9.2m from the origin. It remains in this state for 1.3s. Its velocity then becomes negative, and it returns to the origin over 0.4s.

By finding the gradient of the graph during each stage of the motion, we can determine the velocity of the particle.

Stage 1 (t=0 to t=2.2) - v = 20 / 2.2 = 9.1ms^{-1}.
Stage 2 (t=2.2 to t=3.9) - v = -24.6 / 1.7 = -14.5ms^{-1}
Stage 3 (t=3.9 to t=4.5) - v = 13.6 / 0.6 = 22.7ms^{-1}
Stage 4 (t=4.5 to t=5.8) - v = 0ms^{-1}.
Stage 5 (t=5.8 to t=6.2) - v = -9.2 / 0.4 = -23ms^{-1}

Since the changes in velocity are instantaneous, the graph for the motion looks as follows:



Because the velocity of the particle changes instantly, the particle has infinite acceleration, so drawing an acceleration-time graph would not be possible.

Summary of motion-time graphs:

- A displacement-time graph shows the distance of a particle from the origin over time.
- If the displacement of a particle is increasing, it must have a positive velocity.
- If the displacement of a particle is decreasing, it must have a negative velocity (moving in opposite direction to original direction).

- If the gradient of a displacement-time graph is constant, this means the displacement of the particle is increasing at a constant rate, so its velocity is constant.
- A curved displacement-time graph implies an increasing or decreasing rate of change of displacement, so the velocity of the particle must be increasing or decreasing (i.e. it is accelerating).
- To find the average velocity of a journey, divide the total displacement by the total time; to find the average speed from a displacement-time graph, consider the total distance travelled (so ignore negative displacements), and divide this by the time.

- In a velocity-time graph, if the velocity is positive, the particle is moving in the positive direction; if the velocity is negative, it is moving in the negative direction.
- Whenever a particle has positive velocity, it has increasing displacement over time; whenever the particle has negative velocity, it has decreasing displacement over time.
- The area under a velocity-time graph gives displacement.
- By dividing the total area by the total time, this must give the average velocity for the journey.
- By dividing the total area of the velocity-time graph by time, but changing any negative values to positive values (because distance does not consider direction), this gives the average speed.
- If the velocity of a particle is increasing or decreasing over time, it is accelerating or decelerating respectively.
- A curved velocity-time graph implies that the rate of change of velocity is changing, so the acceleration is variable rather than constant.

SUVAT equations of motion

Introduction

The **suvat equations** are equations that relate 5 variables:

- Displacement (s)
- Initial velocity (u)
- Final velocity (v)
- **Constant** acceleration (a)
- Time (t)

The suvat equations only work when acceleration is **constant**; if acceleration is variable, it is instead necessary to use calculus to model motion.

The equations are simply a shortcut to relating the displacement of a particle to its initial velocity, final velocity, acceleration and time. They can be simply derived.

-

Equation 1: ***v=u+at***

Suppose a particle starts off with a velocity of $6ms^{-1}$, and accelerates at $2ms^{-2}$ for 5s. What will its final velocity be?

The acceleration of $2ms^{-2}$ tells us that the particle is gaining $2ms^{-1}$ per second; so in 5s, it will gain $10ms^{-1}$, meaning that its final velocity is 6+10 = $16ms^{-1}$.

So, to find the final velocity, v, we take the initial velocity, u, and add this to the acceleration, a, multiplied by the time, t (this tells us the velocity gain).

Thus, **v = u + at** => this is the first suvat equation.

Notice how this re-arranges to our original acceleration equation:

$v = u + at$
$v - u = at$
$a = (v-u) / t$ => final velocity - initial velocity all divided by time

-

Suppose a particle starts off with velocity $6ms^{-1}$ and accelerates at $2ms^{-2}$ to a velocity of $9.2ms^{-1}$. *How long does this take?*

$t = (v-u) / a = (9.2 - 6) / 2 = 1.6s$

Suppose a particle accelerates at $3ms^{-2}$ to $5ms^{-1}$ over 9s. *What was its initial velocity?*

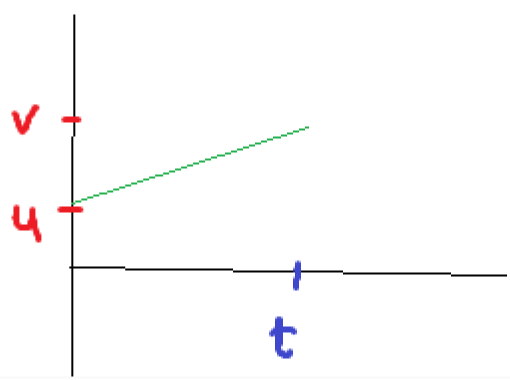
$u = v - at = 5 - (3 * 9) = -22ms^{-1}$.

Equation 2: *s = 1/2 (u+v)t*

We previously said that: average velocity = total displacement / time

Thus, total displacement = average velocity * time

Consider a single section of a particle's motion, where it accelerates from an initial velocity of **u** to a final velocity of **v** over a time **t**:



Since the acceleration is constant, the average velocity for this section of the motion is simply: (u+v) / 2 = 1/2 (u+v)

If we multiply this by time, we get **s = 1/2 (u+v) * t** => this is the second equation of motion

-

Suppose a particle accelerates from $5ms^{-1}$ to $7ms^{-1}$ over 10s; what is the distance travelled?

u = 5, v = 7, t = 10

s = 1/2 (5 + 7) * 10 = 65m.

-

Suppose a particle starts at a velocity ums^{-1}, and decelerates to $9ms^{-1}$ in 5s, travelling a distance of 120m. *What was its initial velocity?*

u = u, v = 9, t = 5, s = 120

120 = 1/2 (u + 9) * 5

48 = u + 9 => u = $38ms^{-1}$.

--

We now have two equations: v = u + at, and s = 1/2 (u+v) t.

v = u + at allows us to relate initial and final velocity to acceleration and time.

s = 1/2 (u+v) t allows us to relate initial and final velocity to displacement and time.

Suppose we were only given v, u and a, and we wanted to find s. Currently, using only one of the above equations would not allow us to relate these variables; we would instead have to use simultaneous equations to relate them.

It is therefore useful to create an equation that relates these variables by using our known equations.

The equation that relates v, u and a is $\mathbf{v^2 = u^2 + 2as}$.

Derivation:

From **v = u + at**, **t = (v-u) / a**.

This can be substituted into **s = 1/2(u+v)t** to derive **v^2=u^2+2as**:

$$t = \frac{v-u}{a} \quad \Big| \quad s = \tfrac{1}{2}(u+v)t$$

$$\therefore s = \tfrac{1}{2}(u+v)\left(\frac{v-u}{a}\right)$$

$$s = \tfrac{1}{2}\,\frac{v^2-u^2}{a}$$

$$2as = v^2-u^2$$

$$\boxed{v^2 = u^2 + 2as}$$

We now have an equation that allows us to relate v, u, a and s, without forming two simultaneous equations out of the first two equations.

Example question: A particle accelerates from $10ms^{-1}$ to $22ms^{-1}$ over 56m. **Find the particle's acceleration.**

$(v^2 - u^2) / 2s = a$

$a = (22^2 - 10^2) / 2(56) = 3.43ms^{-2}$.

Example question: A particle accelerates from $20ms^{-1}$ to vms^{-1} with an acceleration of $9ms^{-2}$. This acceleration occurs over 92m. **Find v.**

$v = \sqrt{(20^2 + 2(9)(92))} = 45.3 ms^{-1}$.

Currently, we also have no equation that relates s, u, t and a.

The equation that relates these variables is **s = ut + 1/2at²**.

We can derive this equation by substituting v = u + at into s = 1/2 (u + v) t.

$$v = u + at$$

$$s = \frac{1}{2}(u + v)t$$

$$s = \frac{1}{2}(u + u + at)t$$

$$s = \frac{1}{2}(2ut + at^2)$$

$$s = ut + \frac{1}{2}at^2$$

The equation can also be derived by integrating v = u+at with respect to t:

$$v = u + at$$

$$vt = ut + \frac{at^2}{2}$$

$$vt = s.$$

$$s = ut + \frac{1}{2}at^2.$$

Example question:

A particle starts from rest, and accelerates at $9.81ms^{-2}$ over 5s. **How far does it travel?**

$u = 0$, $a = 9.81$, $t = 5$

$s = (0 * 5) + (1/2 * 9.81 * 5^2) = 123m$.

Example question:

A particle accelerates from $6ms^{-1}$ over 10s. It travels a distance of 15m. **What is its acceleration?**

$15 = 6(10) + 1/2(a)(10^2)$
$15 = 60 + 50a$

$a = -0.9ms^{-2}$.

The particle must be decelerating to cover 15m in 10s starting from a velocity of $6ms^{-1}$.

-

The 4 main SUVAT equations are:

$$v = u + at$$
$$s = 1/2\,(u+v) * t$$
$$v^2 = u^2 + 2as$$
$$s = ut + 1/2at^2$$

With these equations, if we know any 3 variables from s, u, v, a or t, we can find the other 2 unknown variables.

Example: car in motion

Consider the following situation:

A man drives a car along a straight road. As he passes the point *A*, the car is travelling at a constant speed of $30\,ms^{-1}$. He continues at the speed of $30\,ms^{-1}$ for 5 minutes until he approaches a built-up area, when he applies a constant deceleration for 20 seconds until the car slows down to a speed of $16\,ms^{-1}$. On reaching the speed of $16\,ms^{-1}$, he sees his destination point *B* and applies a constant deceleration for 8 s until the car stops at *B*.

Suppose we want to find the distance between **A** and **B**.

There are two methods that we could apply here:

1 - Using SUVAT equations
2 - Using a velocity-time graph

Method 1:

To find the total distance between A and B, we can use suvat equations to determine the displacement in each stage of the motion.

Stage 1:

The car first travels at a constant speed of $30ms^{-1}$ for 5 minutes. The SUVAT equations only apply when a particle is accelerating, so they do not need to be used to calculate the distance travelled in this case.

If a car is travelling at $30ms^{-1}$ for a total of 300s, it will travel a distance of 30(300) = 9000m.

Stage 2:

In the second stage, the car slows from $30ms^{-1}$ to $16ms^{-1}$ over 20s.

To find the distance travelled here, we can use a SUVAT equation.

u = 30, v = 16, t = 20.

An equation relating s, u, v and t is **s = 1/2 (u+v) * t**

Thus, s = 1/2 (30+16) * 20 = 460m.

Stage 3:

In the final stage of the motion, the car decelerates from $16ms^{-1}$ to $0ms^{-1}$ in 8 seconds.

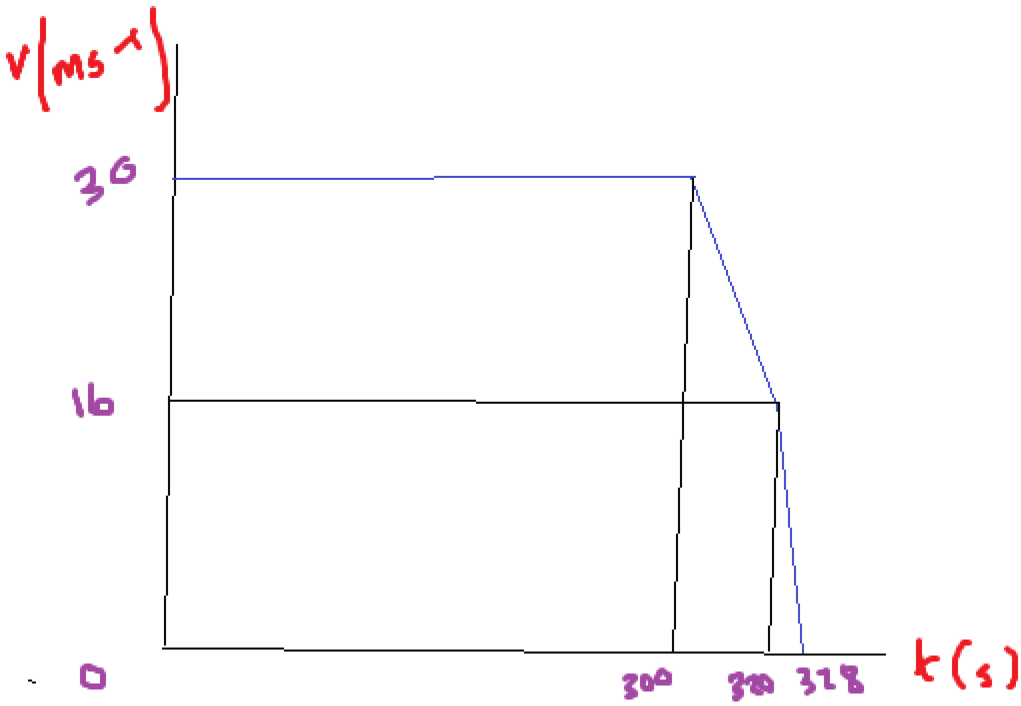
u = 16, v = 0, t = 8.

Thus, s = 1/2 (0+16) * 8 = 64m.

Thus, total distance travelled = 9000 + 460 + 64 = 9524m.

Method 2:

Consider the velocity-time graph for this motion:



The total distance travelled is the area under the graph:

Distance = (300*30) + (1/2 (30+16)*20) + (1/2 * 8 * 16) = 9524m.

Example: cyclist in motion

6. A cyclist is moving along a straight horizontal road and passes a point A. Five seconds later, at the instant when she is moving with speed 10 m s^{-1}, she passes the point B. She moves with constant acceleration from A to B.

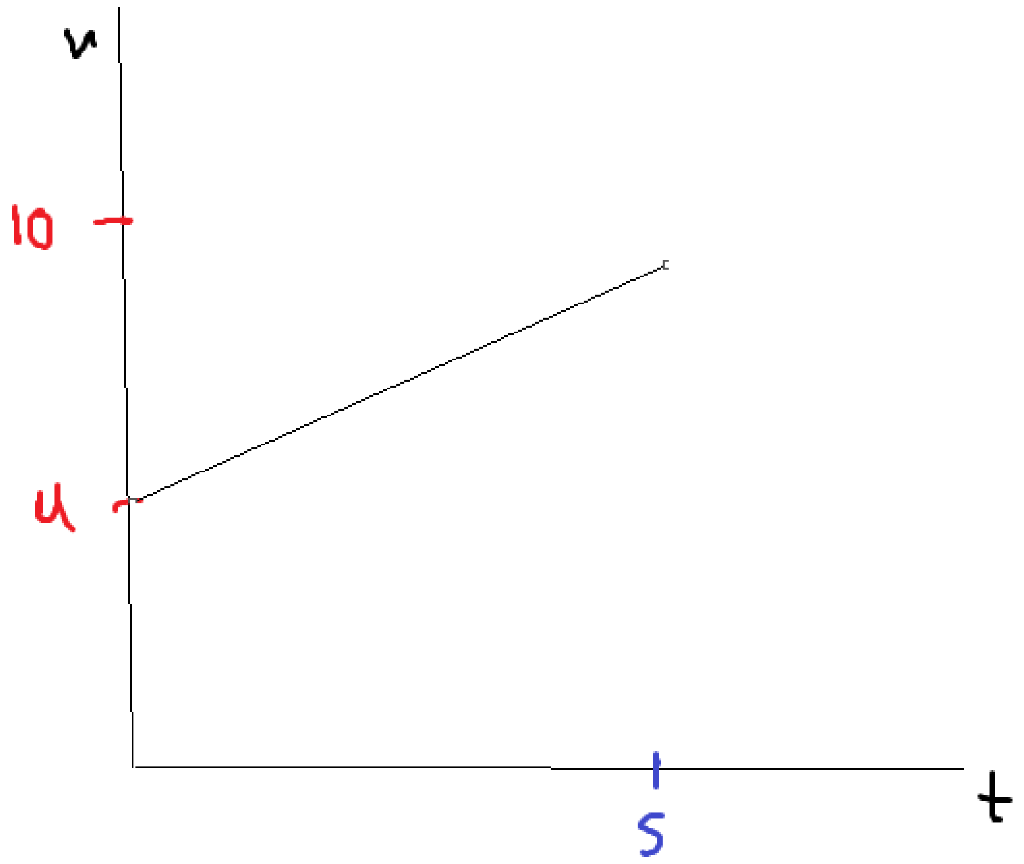
Given that AB = 40m, find

(a) the acceleration of the cyclist as she moves from A to B, **(4)**

(b) the time it takes her to travel from A to the midpoint of AB. **(5)**

a)

Consider the velocity-time graph for this motion:



The cyclist passes point A with an unknown velocity. They then accelerate to $10ms^{-1}$ over 5s, travelling a distance of 40m.

Consider the variables that we know: s = 40, v = 10, t = 5

We can first find the initial velocity of the cyclist using **s = 1/2(u+v)t**

40 = 1/2 (u + 10) * 5

16 = u + 10

u = $6ms^{-1}$.

So, the cyclist accelerates from $6ms^{-1}$ at A to $10ms^{-1}$ at B over 5s.

Thus, a = 10-6 / 5 = $0.8ms^{-2}$.

b)

The midpoint of AB must be 20m from A.

The cyclist starts with a velocity of $6ms^{-1}$, and is accelerating at a constant rate of $0.8ms^{-2}$ between A and B.

Thus, s = 20, u = 6, a = 0.8.

Using **$s = ut + 1/2at^2$:**

$20 = 6t + 1/2(0.8)t^2$

$20 = 6t + 0.4t^2$.

This can be solved using the quadratic formula:

$t = (-6 \pm \sqrt{(6^2 - (4 * 0.4 * -20))}) / (2 * 0.4)$ = 2.81s, -17.8s (cannot be negative).

Thus, t = **2.81s**.

Example: free fall

Let us consider how we can apply suvat equations to a particle accelerating in a gravitational field.

Whenever a particle is in free-fall on Earth, we assume that its acceleration is always $9.81ms^{-2}$, and we ignore the effects of air resistance.

In freefall problems, we often have particles being thrown vertically upwards. In these situations, the particle has an initial velocity at which it is thrown. Because of gravity, the velocity of the particle in the vertical direction reduces by **$9.81ms^{-1}$** each second. Eventually, the velocity of the particle is reduced to **$0ms^{-1}$**, at which point the particle is at its apex. Beyond this point, gravity will continue to accelerate the particle downwards until it hits the ground. If the initial direction is positive, when the particle moves downwards, its velocity is negative.

We can also have situations where a particle is simply released from a certain height. In these situations, the particle has an initial velocity of $0ms^{-1}$ in the vertical direction. The particle is accelerated towards the ground by gravity at **$9.81ms^{-1}$** per second.

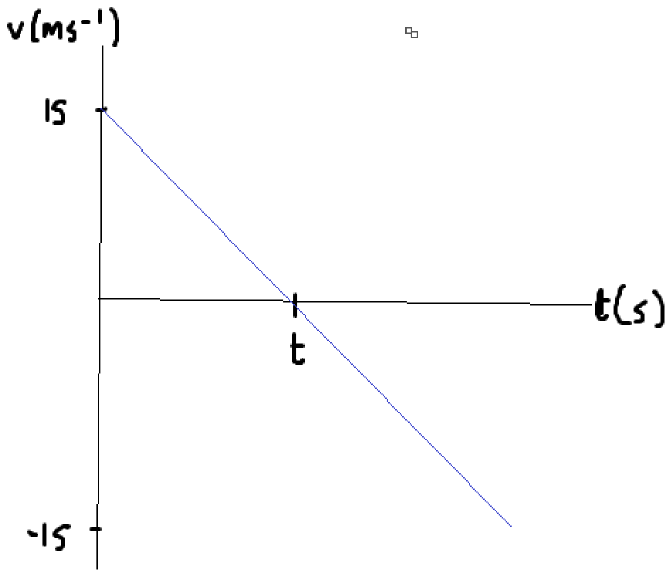
Because we assume the gravitational acceleration is constant and that there are no other external forces like air resistance acting on the falling particle, SUVAT can be used to model freefall.

-

A ball is thrown vertically upwards with a velocity of $15ms^{-1}$.

a) Calculate the maximum height reached by the ball

The ball will be decelerated by gravity at $9.81ms^{-2}$. When the ball is at its maximum height (**apex**), its velocity will be reduced to $0ms^{-1}$. It will then start accelerating back towards the ground:



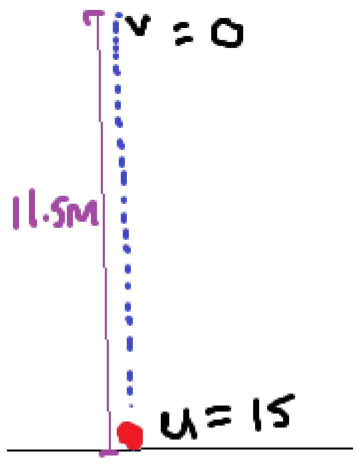
We need to find the time t taken for the ball to decelerate to $0ms^{-1}$.

u = 15, v = 0, a = -9.81.

What equation relates s, u, v and a?

$v^2 = u^2 + 2as$.

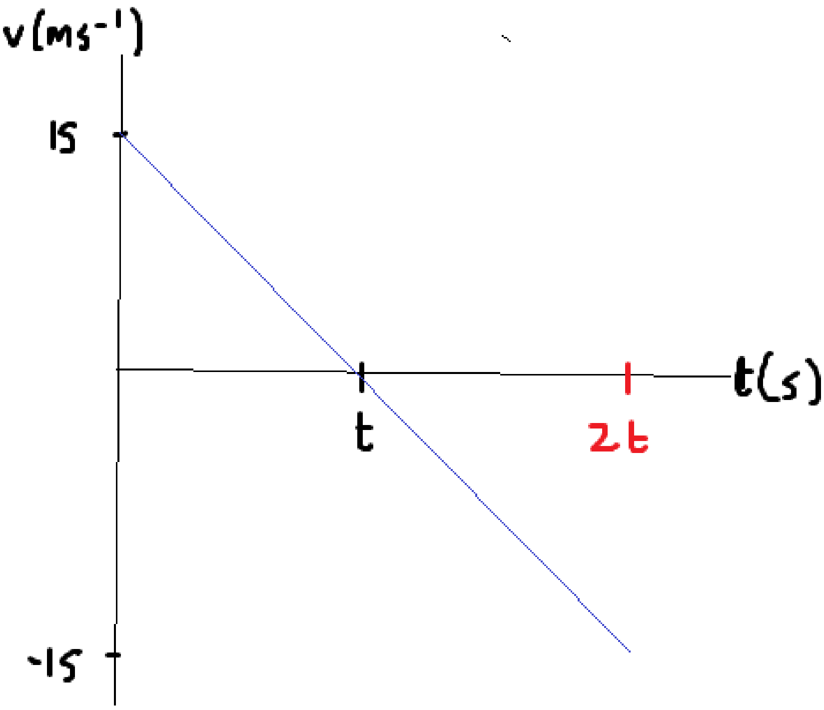
$s = (v^2 - u^2) / 2a = 0^2 - 15^2 / (2*-9.81) =$ **11.5m**.



b) Calculate the total time of flight of the ball

Because we can ignore the effects of air resistance, the total time taken to reach the apex is the total time taken to return to the ground.

We can therefore find the total time taken to reach the apex, and double this to find the total time of flight.



Considering the ball rising to its apex:

u = 15, v = 0, a = -9.81

The equation that relates u, v and a is **v = u+at**.

v = u+at
t = (v-u) / a = (0-15) / -9.81 = 1.53s.

The time taken to return to the ground is equal to this; thus total time of flight = 2t = 2(1.53) = **3.06s**.

-

We can also find this value by considering the initial and final velocity of the ball.

If we take upwards as positive, the ball is launched at $15ms^{-1}$. The ball must land at $-15ms^{-1}$. The velocity is now negative because the ball is travelling in the opposite direction.

The magnitude (not direction) of the initial and final velocities are equal because the ball travels the same distance from the ground to the apex as it does from the apex to the ground, and the acceleration is constant.

Thus, for the entire motion: **u = 15, v = -15, a = -9.81**

When the particle falls to the ground, it is being accelerated by gravity; so why is a = -9.81ms^{-2} for the entire journey, as this implies a constant deceleration?

We have to consider that the particle begins to move in the negative direction when it is returning to the ground, so although it is accelerating downwards, it is still decelerating relative to the initial direction.

A suvat equation that relates u, v, a and t is **v = u + at**

Thus, t = (v-u) / a = (-15-15) / -9.81 = **3.06s** => as calculated previously.

--

Example: free fall from a height

Now suppose that rather than throwing a ball vertically upwards from the ground, we throw it vertically upwards from a height, h.

A ball is thrown vertically upwards from a height of 15m at a velocity of 22ms^{-1}. Ignoring the effects of air resistance:

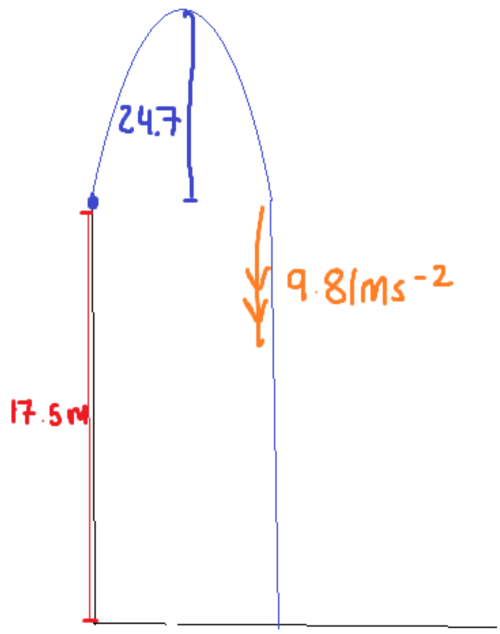
a) **Calculate the maximum height the ball reaches above the ground**

The ball will be decelerated at 9.81ms^{-2} by gravity when it is thrown upwards. It will reach its apex, at which point, its velocity is 0ms^{-1}.

We can say: u = 22, v = 0, a = -9.81.

Using $v^2 = u^2 + 2as$ => $s = (v^2 - u^2) / 2a = (0^2 - 22^2) / 2(-9.81) =$ 24.7m.

This is the height that the ball reaches from its original position. Thus, the overall height above the ground is: 15 + 24.7 = **39.7m**.



b) Calculate the total time of flight for the ball

In this instance, the time taken to reach the apex is not equal to the time taken to reach the ground, since the ball is thrown from a height of 15m.

We instead need to calculate the time taken in each stage of the motion.

-

Upwards motion:

For the upwards motion, $u = 22$, $v = 0$, $a = -9.81$

Thus, $t = (v-u) / a = (0-22) / -9.81 = 2.24$s.

Downwards motion:

If we consider the downwards motion, the ball starts from the apex, and descends a height of 39.7m, accelerating at $9.81ms^{-2}$.

At the apex, the ball has a velocity of $0ms^{-1}$, so:

$u = 0$, $s = 39.7$, $a = 9.81$.

Using $s = ut+1/2at^2$

Since $u=0$, we can say: $s = 1/2at^2$

Thus, $t = \sqrt{(2s/a)} = \sqrt{(2(39.7) / 9.81)} = 2.84$s.

-

Thus, **total time of flight = 2.24 + 2.84 = 5.08s**.

-

Another method for finding the total time of flight is to consider the entire motion.

If the particle starts from 17.5m above the ground, its displacement from this origin when it is on the ground is -17.5m, if downwards is taken as negative.

It has an initial velocity of $22ms^{-1}$, and it is accelerating at $-9.81ms^{-2}$.

Thus, $s = -17.5$, $u = 22$, $a = -9.81$.

$s = ut + 1/2at^2$

$-17.5 = 22t + 1/2(-9.81)t^2$

$-17.5 = 22t - 4.905t^2$

$4.905t^2 - 22t - 17.5 = 0$

Using the quadratic formula, **t = 5.17s** (this is a more accurate value; the previous value calculated of 5.08s was offsetted by rounding many of the figures calculated).

Combining SUVAT and force

Kinematics (SUVAT) and dynamics (force) concepts can be integrated with each other. The variable that relates force to kinematics is **acceleration**.

By considering the net force on a body of a known mass, we can find its acceleration. We can then use that acceleration in the context of SUVAT to determine the distance travelled, the time etc.

Example 1

A particle P is projected vertically upwards from a point A with speed u m s^{-1}. The point A is 17.5 m above horizontal ground. The particle P moves freely under gravity until it reaches the ground with speed 28 m s^{-1}.

(a) Show that $u = 21$

(3)

At time t seconds after projection, P is 19 m above A.

(b) Find the possible values of t.

(5)

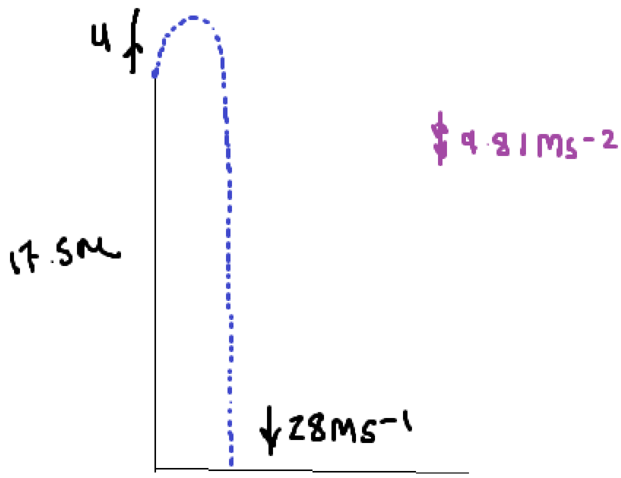
The ground is soft and, after P reaches the ground, P sinks vertically downwards into the ground before coming to rest. The mass of P is 4 kg and the ground is assumed to exert a constant resistive force of magnitude 5000 N on P.

(c) Find the vertical distance that P sinks into the ground before coming to rest.

(4)

a)

When the particle hits the ground, its displacement is now -17.5m, if we take upwards as positive.



We also know that it hits the ground with a velocity of 28ms^{-1}. Since the particle is travelling downwards when it hits the ground, if upwards is positive, the final velocity of the particle is -28ms^{-1}.

The particle is accelerating at -9.81ms^{-2} due to the downwards pull of gravity; this remains constant throughout the motion because although the particle is being accelerated by gravity when it descends, its velocity is becoming more negative - so in the upwards direction, it is still decelerating.

We can thus say that for the entire motion:

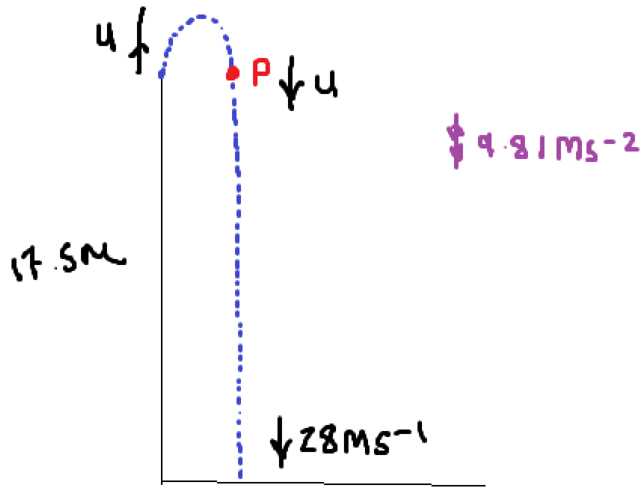
u = u
v = -28
a = -9.81
s = -17.5

Thus, using **$v^2 = u^2 + 2as \Rightarrow u = \sqrt{(v^2 - 2as)}$**

$u = \sqrt{((-28)^2 - 2(-9.81)(-17.5))} = 21ms^{-1}$.

-

Another way to consider this motion is that the magnitude of the velocity at the point marked **P** must be the same as the initial velocity.



This is because the particle travels the same height whilst decelerating to the apex as it does whilst accelerating to point P; and the acceleration is constant, so the velocity must be the same.

We can therefore just consider the downwards part of the motion from point P. If we take downwards as positive:

s = 17.5
u = u
v = 28
a = 9.81

Solving using $v^2 = u^2 + 2as$ gives $u = 21ms^{-1}$.

-

We can also consider this problem algebraically.

First, we can determine the maximum height the particle reaches above the ground in terms of u:

$s = h$

$u = u$

$v = 0$

$a = -g$

$$v^2 = u^2 + 2as$$

$$0^2 = u^2 + 2(-g)(h)$$

$$u^2 = 2gh$$

$$h = \frac{u^2}{2g}.$$

We can then consider the acceleration of the particle from the apex to the ground. If the apex is **u^2 / 2g** above the original position, the distance to the ground must be **$17.5 + u^2/2g$**.

$s = -\left(17.5 + \frac{u^2}{2g}\right)$

$u = 0$

$a = -g$

$v = -28$

$$v^2 = u^2 + 2as$$

$$(-28)^2 = 0^2 + 2(-g)\left(-17.5 - \frac{u^2}{2g}\right)$$

$$784 = -2g\left(-17.5 - \frac{u^2}{2g}\right)$$

$$784 = 35g + \frac{2gu^2}{2g}$$

$$784 = 35g + u^2$$

$$u = \sqrt{784 - 35g} = 21\,\text{ms}^{-1}$$

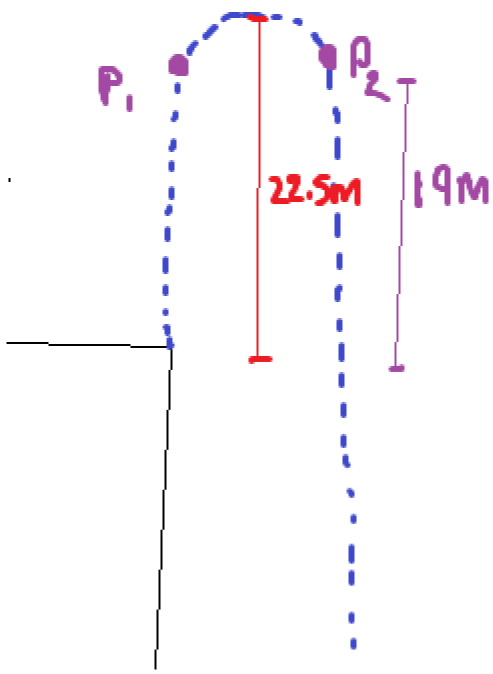
b)

Firstly, let us consider the maximum height obtained by the particle.

u = 21
v = 0
a = -9.81

$v^2 = u^2 + 2as \Rightarrow s = (v^2 - u^2) / 2a = (0^2 - 21^2) / 2(-9.81) = 22.5m$.

This means that the particle must be 19m above A at two points, at one point during its ascent, and one point during its descent (marked as P_1 and P_2):



When the particle is 19m above A, it must have a displacement of +19m.

Thus, we can say:

s = 19, u = 21, a = -9.81

Using $s = ut + 1/2at^2$

$19 = 21t + 1/2(-9.81)t^2$

$4.905t^2 - 21t + 19 = 0$

Using the quadratic formula, this gives us two values of t: t = 1.30s, t = 2.98s.

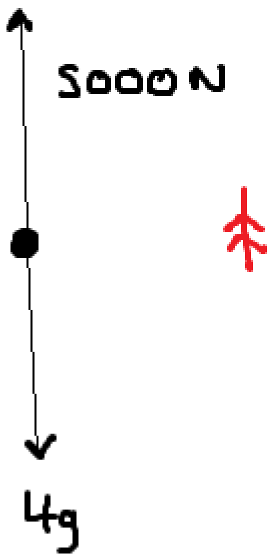
So the particle is 19m above A at 1.30s after projection and 2.98s after projection.

-

c)

In this question, we combine SUVAT and force concepts.

The particle has two forces acting on it when it hits the ground:



- A gravitational force of 4g (downwards)
- A resistive force of 5000N (upwards) => this force is resisting the movement of P into the ground, so it must act upwards.

Clearly, the resistive force is greater than the weight, so the particle will be subject to a deceleration.

$5000 - 4g = 4 * a$

$a = (5000 - 4(9.81)) / 4 = 1240ms^{-2}$.

The particle enters the ground with a velocity of $28ms^{-1}$, and it is instantly subject to this deceleration of $1240ms^{-2}$.

The deceleration will cause the velocity of the particle to reduce to $0ms^{-1}$.

We need to find the distance travelled for a particle decelerating at $1240ms^{-2}$ from $28ms^{-1}$ to $0ms^{-1}$ using SUVAT:

s = s
u = 28
v = 0
a = -1240

$v^2 = u^2 + 2as$

$s = (v^2 - u^2) / 2a = (0^2 - 28^2) / 2(-1240) = 0.316m$.

So the particle sinks for 0.316m before reaching a velocity of $0ms^{-1}$.

-

Example 2

A car of mass 800 kg is travelling on a horizontal road. It experiences a resistance to motion which is constant throughout the journey. The car accelerates from rest under a constant tractive force of 300 N exerted by its engine. After 50 seconds, the car reaches a speed of 15 ms^{-1}.

(a) Determine the magnitude of the acceleration of the car. [3]

(b) Calculate the magnitude of the constant resistance to motion. [3]

(c) When the car reaches the speed of 15 ms^{-1}, the engine is switched off and the car is brought to rest by a constant braking force. The total distance covered by the car for the **whole** journey is 500 m. Find the constant force exerted by the brakes. [7]

a)

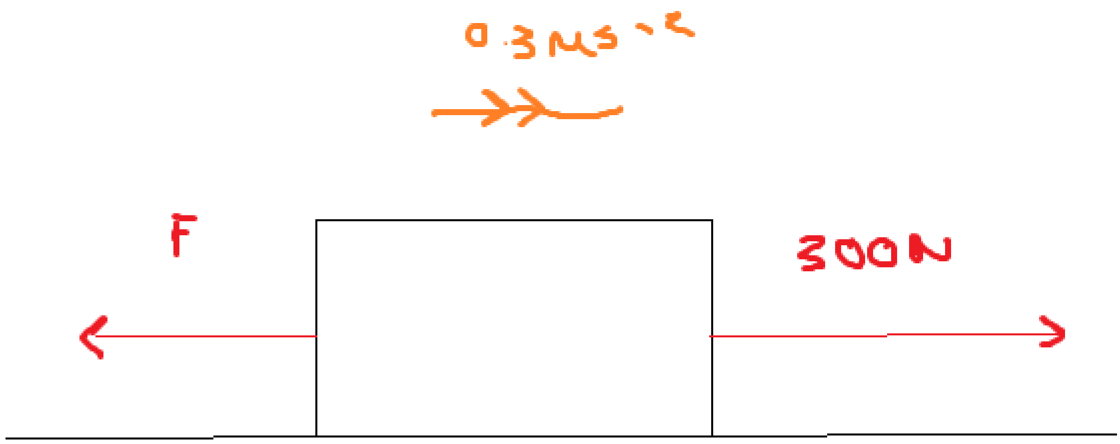
The car starts with a velocity of 0ms^{-1}, and accelerates to 15ms^{-1} over 50 seconds.

Thus, a = (15-0) / 50 = 0.3ms^{-2}.

b)

The car has two forces acting on it:

- The tractive force of 300N (in the direction of motion)
- A frictional force resisting the tractive force (let this be **F**)



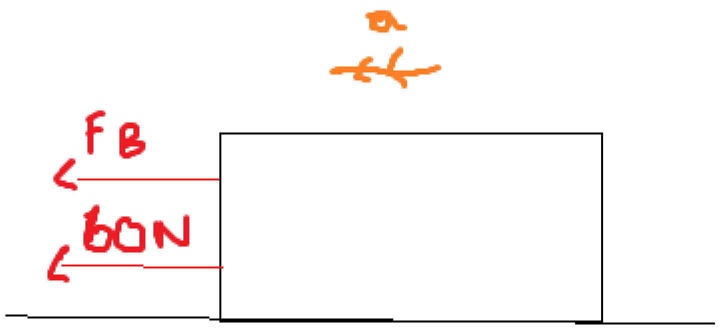
The net force acting on the car is **300 - F**.

If a body of mass 800kg is accelerating at 0.3ms^{-2}, it has a net force of **800 * 0.3 = 240N**.

Thus, **300 - F = 240 => F = 60N**.

c)

When the car is breaking, it now has two forces acting on it in the same direction: the frictional force of 60N (remains constant *throughout the journey*) and a new braking force (let this be F_B). This causes the car to decelerate:



We can therefore say: $F_B + 60 = 800 * a$

If we find the value of **a** using SUVAT, we can find the braking force.

Currently, we only know the initial velocity of the car, $15ms^{-1}$, and the final velocity: $0ms^{-1}$. We need to know a third variable to be able to calculate a.

The third variable we can find with the data given is the distance travelled during the deceleration.

-

During the first section of the journey (acceleration), we know:

u = 0, v = 15, t = 50

Thus, we can find the distance travelled during this section of the journey:

s = 1/2 (u+v) * t = 1/2 (15) * 50 = 375m.

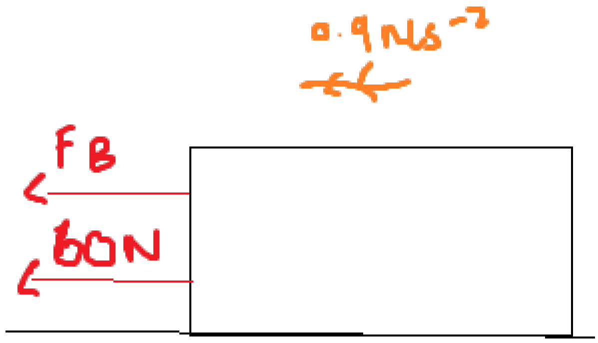
If the whole journey is 500m, the deceleration must therefore occur over a distance of 500-375 = 125m.

We now know: **u=15, v=0, s=125**.

Thus, using $\mathbf{v^2 = u^2 + 2as \Rightarrow a = (v^2-u^2) / 2s}$

$a = (0^2 - 15^2) / 2(125) = -0.9ms^{-2}$.

This means that the car must be accelerating at $0.9ms^{-2}$ in the direction of the braking force:



We previously derived: **$F_B + 60 = 800 * a$**

Thus, $F_B = (800 * 0.9) - 60 =$ **660N**.

-

Example 3:

A lift accelerates upwards at a constant rate with a constant tension of **15000N** in the lift cable. In a certain section of the lift's motion, it accelerates from $15ms^{-1}$ to $18ms^{-1}$ in 40s. Given that the lift itself has a mass of 500kg, and there are people inside the lift of total mass **m** kg, **find the reaction force exerted on the people by the floor of the lift**.

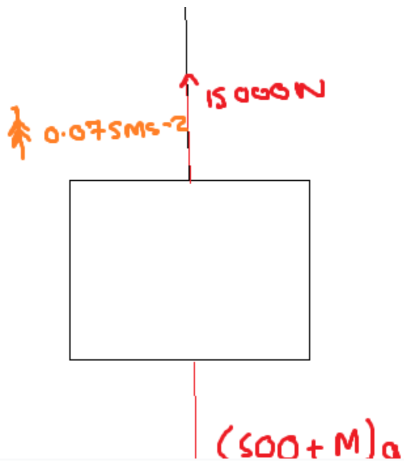
We can first use SUVAT to determine the acceleration of the lift.

u = 15, v = 18, t = 40

$a = (18-15) / 40 = 0.075ms^{-2}$

-

Consider a free body diagram for the lift:



*[The weight of the lift is its total mass, **500+m** multiplied by the gravitational field strength, g.]*

The resultant force acting on the lift is: **15000 - (500+m)g**

If a lift of mass **500+m** has an acceleration of 0.075ms^{-2}, the net force acting on it must be **(500+m) * 0.075**

Thus, **15000 - (500+m)g = (500+m) * 0.075**

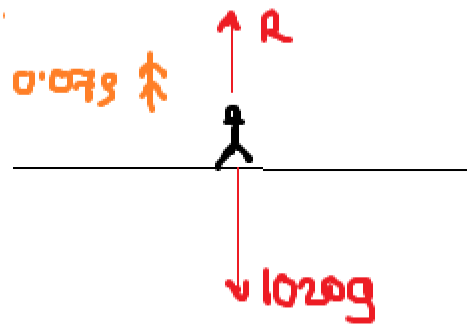
15000 - 500g - mg = 37.5 + 0.075m

15000 - 500g - 37.5 = 0.075m + mg

m(0.075 + g) = 15000 - 500g - 37.5

m = (15000-500g-37.5) / (0.075 + g) = 1020kg.

Thus, we can now model all of the people as a single body, and consider the forces acting on this body:



R - 1020g = 1020 * 0.075 => R = (1020 * 0.075) + 1020g = 10100N

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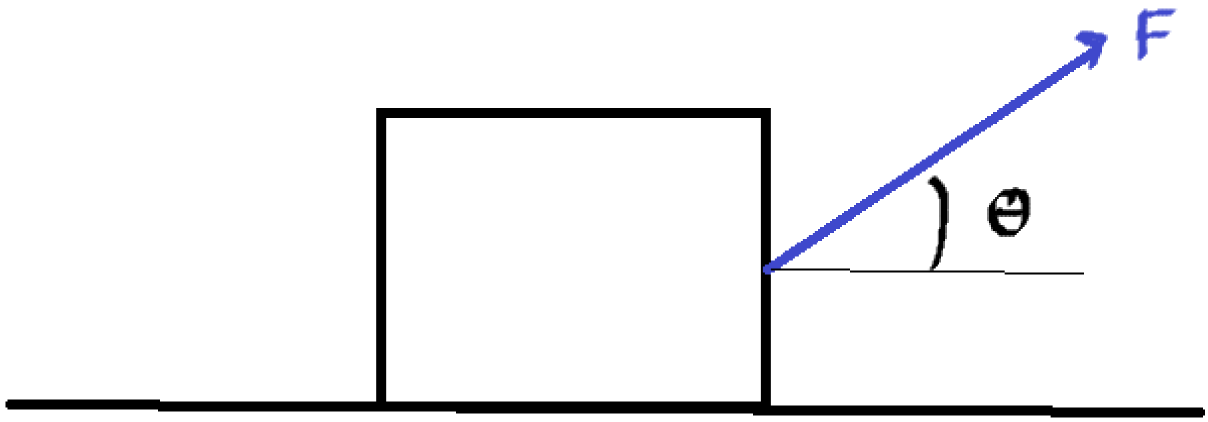
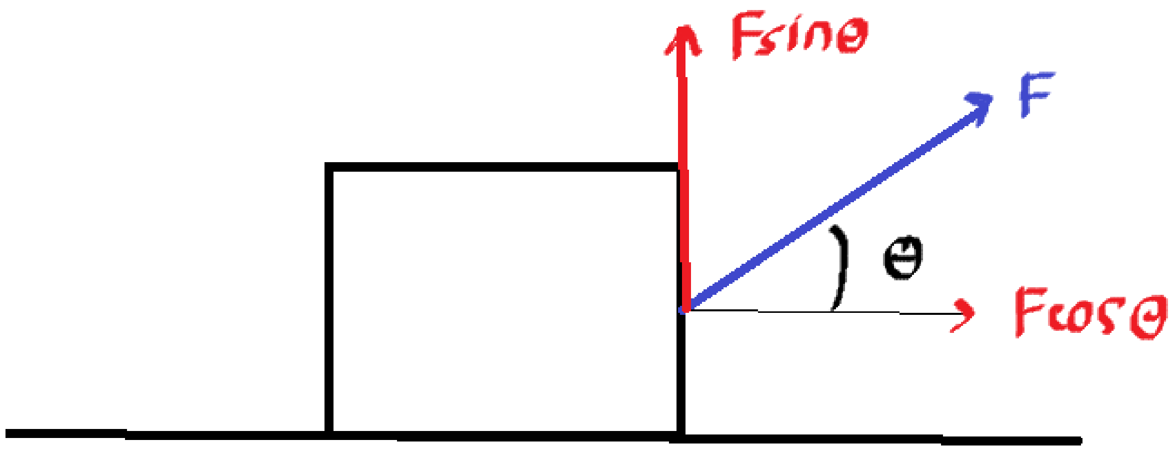
Example 4

A common type of problem involves a particle being pulled along by a force acting at an angle:



In these situations, we have to consider that the force has a component in the horizontal direction *and* the vertical direction:



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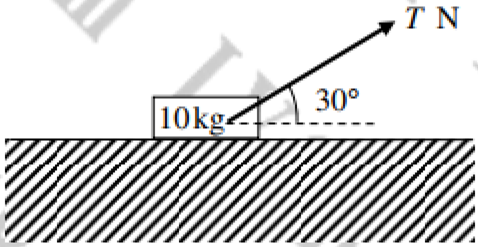


The figure above shows a small box of mass 10 kg, pulled by a rope inclined at 30° to the horizontal, along rough horizontal ground.

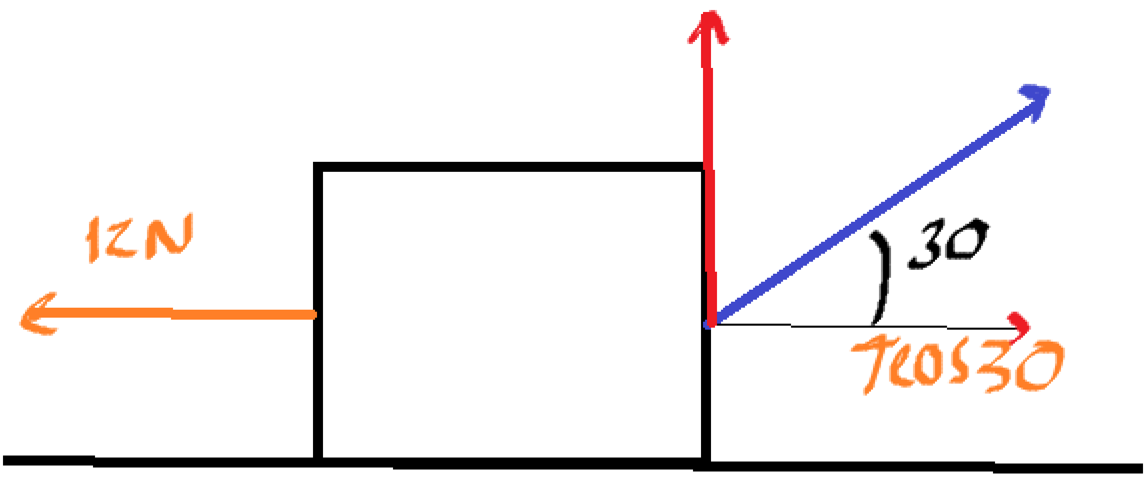
The tension in the rope is T N and the box is accelerating at 0.4 ms^{-2}.

The box is modelled as a particle experiencing a frictional force of 12 N and a normal reaction of R N.

Determine the value of T and the value of R.

Finding T:

To find T, we consider the forces acting on the particle in the horizontal plane:



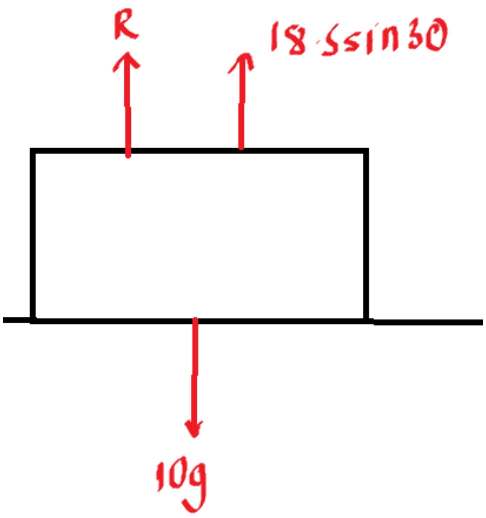
The particle is accelerating at **$0.4ms^{-2}$** in the horizontal plane, thus:

Tcos30 - 12 = 0.4(10) => **T = (4 + 12)/cos30 = 18.5N**.

Finding R:

We might initially assume that because the particle is not moving in the vertical plane, the reaction force is simply equal to the weight.

However, we cannot make this assumption, because the particle has another force acting in the vertical plane: the vertical component of **T** (18.5sin30).

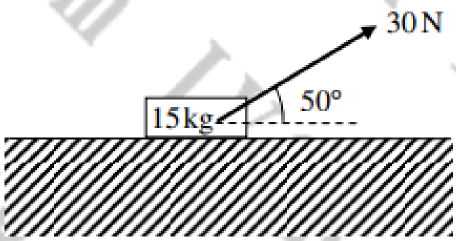


Because there is no acceleration in the vertical plane, we can say the total upwards forces equal the total downwards forces.

Thus,

R + 18.5sin30 = 10g => **R = 10g - 18.5sin30 = 88.9N**.

-



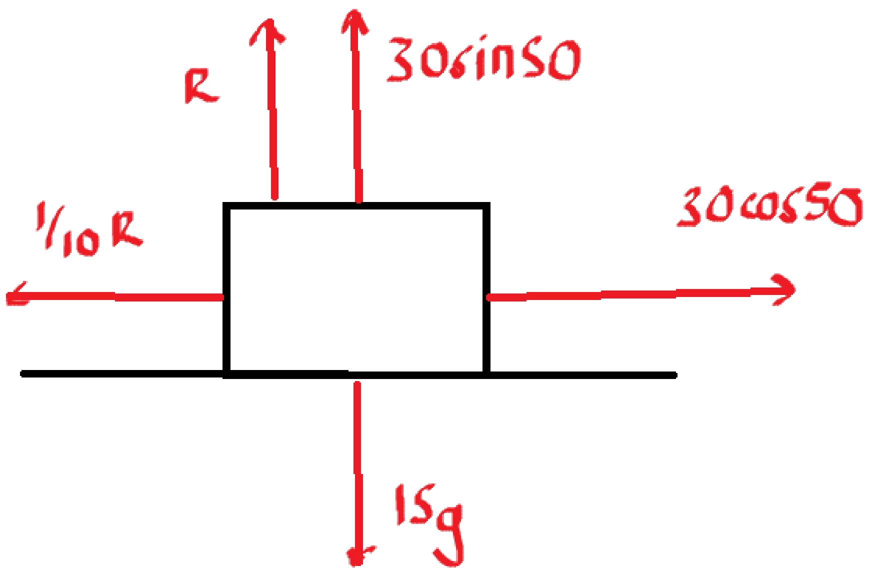
The figure above shows a small box of mass 15 kg, pulled by a rope inclined at 50° to the horizontal, along rough horizontal ground.

The tension in the rope is 30 N and the particle is accelerating at a ms^{-2}.

The box is modelled as a particle experiencing a normal reaction of R N and a constant frictional force of magnitude $\frac{1}{10}R$ N.

Determine the value of a.

We can first draw a diagram of the forces acting on the body (i.e. a free-body diagram). We will split the tension force into its horizontal and vertical components in the diagram:



The acceleration of the body in the vertical plane is 0ms^{-2}, because it is only moving horizontally. The upwards and downwards forces are therefore equal, which enables us to find R:

R + 30sin50 = 15g => **R = 15g - 30sin50 = 124.2N**.

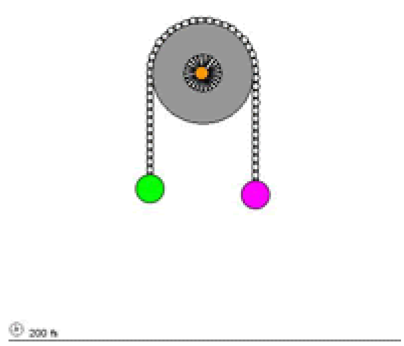
This means that the frictional force acting on the particle is **1/10(124.2) = 12.42N**.

Now, we can consider the acceleration of the particle in the horizontal plane. We only consider the horizontal forces because these are independent of the vertical plane:

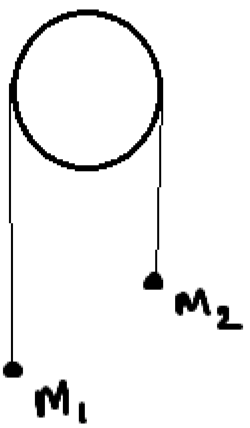
30cos50 - 12.42 = 15a => **a = 0.458ms^{-2}**.

Pulleys

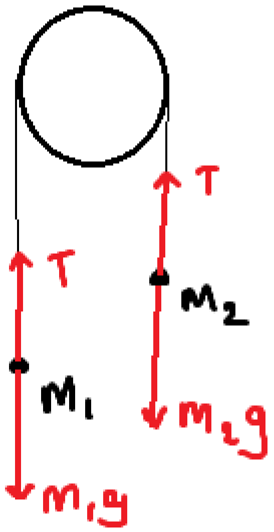
A pulley is a wheel with a grooved rim that a wire / rope can pass over. This can be used for pulling heavy bodies.



Consider a pulley with a string running over it, and two masses **m_1** and **m_2** are attached to the strings:



There is a force of tension that is exerted upwards on **m_1** and **m_2**, as well as the weights of **m_1** and **m_2** acting downwards:

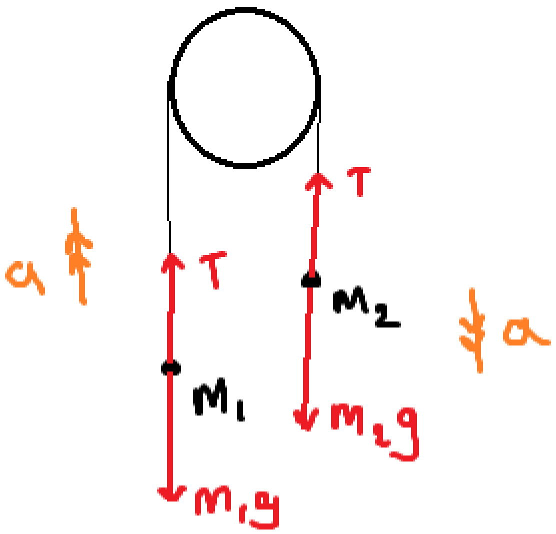


We assume three things with pulleys:

- **The pulley is smooth:** it has no friction, so the tension exerted on m_1 and m_2 is equal.
- **The string is inextensible:** both masses accelerate at the same rate when released
- **The string / rope is light:** it has no mass and does not influence the weight of the system.

When the system is released, if m_2 has a higher mass than m_1, m_2 will move downwards, and m_1 will move upwards at the same acceleration due to the string being inextensible.

Suppose $m_2 > m_1$. m_2 will accelerate downwards, and m_1 will accelerate upwards with an equal magnitude:



Although both bodies are accelerated downwards by gravity, since there is a tension force opposing the gravity, the acceleration of the system will be less than $9.81ms^{-2}$.

/

Since m_1 is accelerating upwards, the tension in the string must be greater than the weight of m_1.

The resultant force on m_1 is $\mathbf{T - m_1 g}$. A body with mass m_1 and acceleration a has a resultant force acting on it of $\mathbf{m_1 a}$.

Thus, $\mathbf{T - m_1 g = m_1 a}$

Since m_2 is accelerating downwards, its weight must be greater than the tension in the string.

Thus, $\mathbf{m_2 g - T = m_2 a}$

For m_1, we can say: $\mathbf{T = m_1 g + m_1 a}$
For m_2, we can say: $\mathbf{T = m_2 g - m_2 a}$

Since the tension is the same for both masses, we can equate the equations:

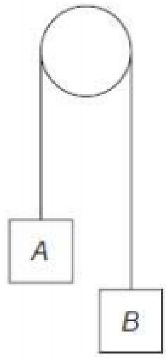
$\mathbf{m_1 g + m_1 a = m_2 g - m_2 a}$

This concept is useful in pulley calculations.

In pulley calculations, we have to combine SUVAT and the dynamics of the system.

Example 1:

2. The diagram shows two objects, *A* and *B*, of mass 2 kg and 5 kg respectively, connected by a light inextensible string passing over a smooth fixed pulley. Initially, the objects are held at rest with the string taut. The system is then released.



(a) Find the magnitude of the acceleration of *A* and the tension in the string. [7]

a)

Because B has a higher mass than A, the system will accelerate downwards when it is released.

For A, we can say:

T - 2g = 2 * a => T = 2a + 2g

For B,

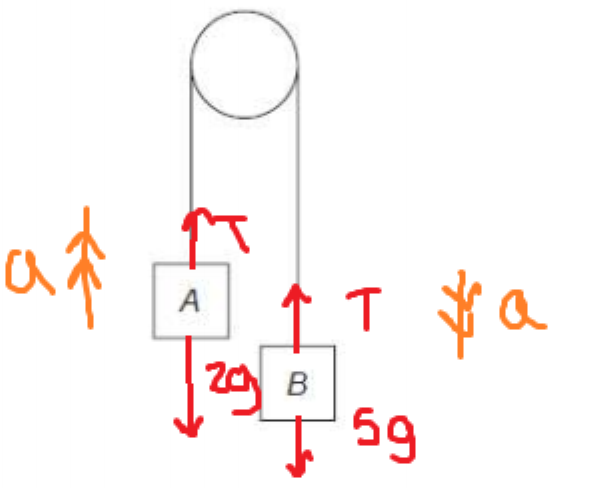
5g - T = 5 * a => T = 5g - 5a

Since the tensions are equal:

2a + 2g = 5g - 5a

7a = 3g

a = 3g / 7 = 4.2ms^{-2}.



Thus, since **T = 2a + 2g, T = 2(4.2) + 2g = 28N.**

-

(b) Before the object A reaches the pulley and 2 seconds after the system is released, the string breaks.

(i) Find the speed of A when the string breaks.
(ii) Given that A does not reach the pulley in the subsequent motion and that A is 18·9 m above the ground when the string breaks, determine the time taken for A to reach the ground. [6]

i)

A has an initial velocity of $0ms^{-1}$ because it is at rest before the system is released.

We have found that A accelerates upwards at $4.2ms^{-2}$.

Thus, we can find its velocity after 2s.

u = 0, v = v, t = 2, a = 4.2

v = 4.2(2) = $8.4ms^{-1}$.

-

ii)

When the string breaks, the tension will go away. This means that the only force acting on A will be gravity, so it will be accelerated downwards at $9.81ms^{-2}$ (ignoring air resistance).

A has a velocity of $8.4ms^{-1}$ in the upwards direction when the string breaks, so it will continue to move upwards until it is decelerated by gravity to a negative velocity. When A is moving at $0ms^{-1}$, it will reach its apex, before descending back to the ground.

When the string breaks, A is 18.9m above the ground, so from the point where the string breaks to the ground, A has a displacement of -18.9m (if upwards is positive). It is being decelerated at $9.81ms^{-2}$ due to gravity, thus:

s = -18.9, u = 8.4, a = -g

We need to find the time taken to reach the ground:

$s = ut + 1/2at^2$

$-18.9 = 8.4t + 1/2(-g)t^2$

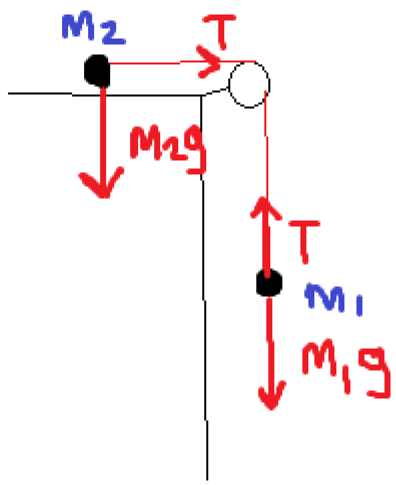
$gt^2/2 - 8.4t - 18.9 = 0$

Using the quadratic formula, **t = 3s**.

This problem is just like a particle being projected from a height 18.9m above the ground with a velocity of $8.4ms^{-1}$.

-

Another type of pulley system is where one body is held over a smooth table, and the other body is left freely hanging:



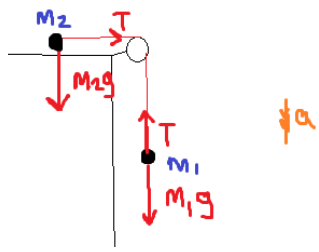
For the particle on the table (m_2), the only force that influences its motion over the table is the tension force, since gravity acts in the vertical plane. There is no friction since we defined the table as *smooth*.

However, since m_1 moves in the vertical plane, it has a downwards force, m_1g, exerted on it. This downwards force pulls m_1 downwards, and m_2 along the table towards the pulley.

If the system accelerates downwards at ams^{-2}, for m_1, we can say:

$$m_1g - T = m_1a$$

For m_2, since the only force influencing its motion in the horizontal plane is tension, we can say that the resultant force on m_2 in the horizontal plane is equal to the tension:

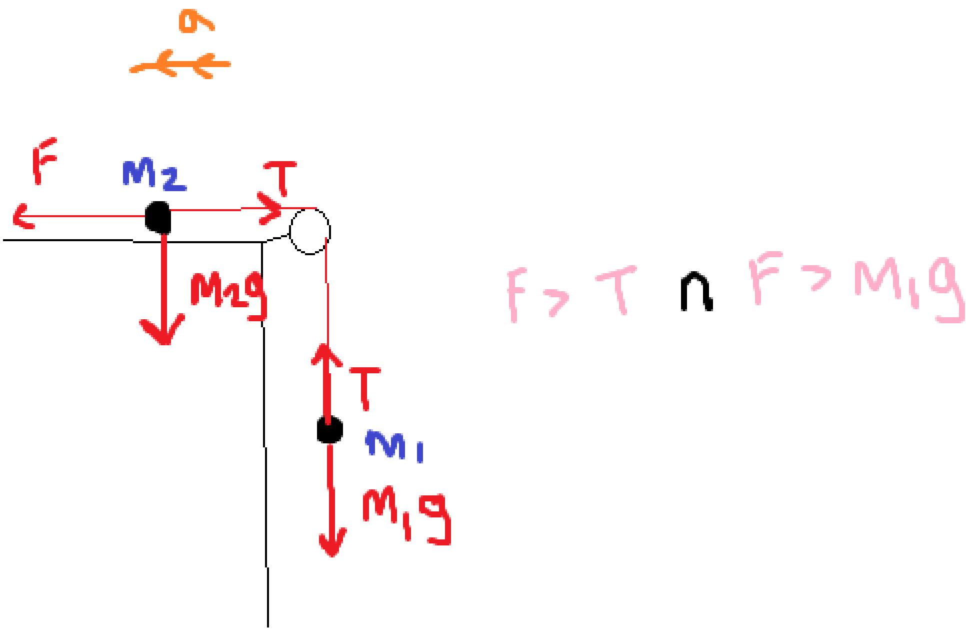


$$T = m_2a$$

Again, since T is equal in both cases, we can say that:

$$m_1g - m_2a = m_1a$$

The only way in-which m_1 could accelerate upwards rather than downwards is if m_2 has a force exerted on it to the left that is greater than the tension in the string **and** the weight of m_1:



Example:

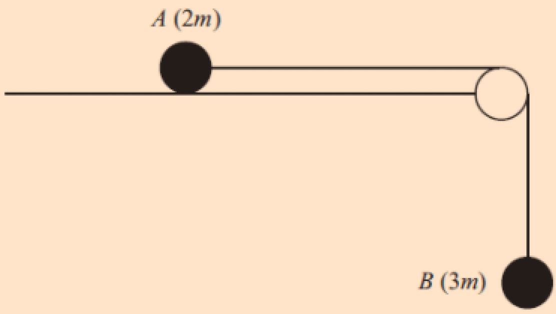


Figure 2

Two particles A and B have masses $2m$ and $3m$ respectively. The particles are attached to the ends of a light inextensible string. Particle A is held at rest on a smooth horizontal table. The string passes over a small smooth pulley which is fixed at the edge of the table. Particle B hangs at rest vertically below the pulley with the string taut, as shown in Figure 2. Particle A is released from rest. Assuming that A has not reached the pulley, find

(a) the acceleration of B,
(5)

(b) the tension in the string,
(1)

(c) the magnitude and direction of the force exerted on the pulley by the string.
(4)

In this question, since we have no actual mass values, our results will be in terms of **m**.

a)

The system will accelerate downwards, so the tension in the string must be lower than the weight of B.

Thus, for B:

3mg - T = 3a

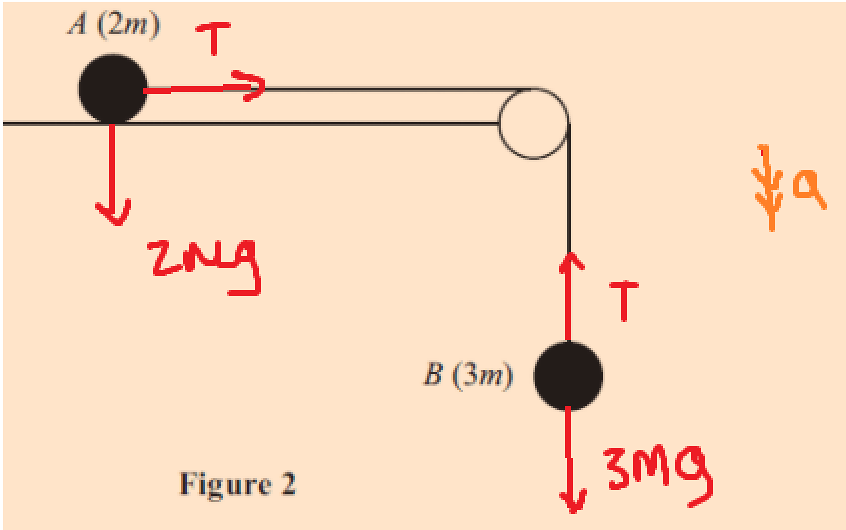
For A:

T = 2a (tension is the only force).

Thus, substitute **T=2a** into **3mg-T=3a** to give:

3mg - 2a = 3a

3mg = 5a => **a= 3mg / 5**



b)

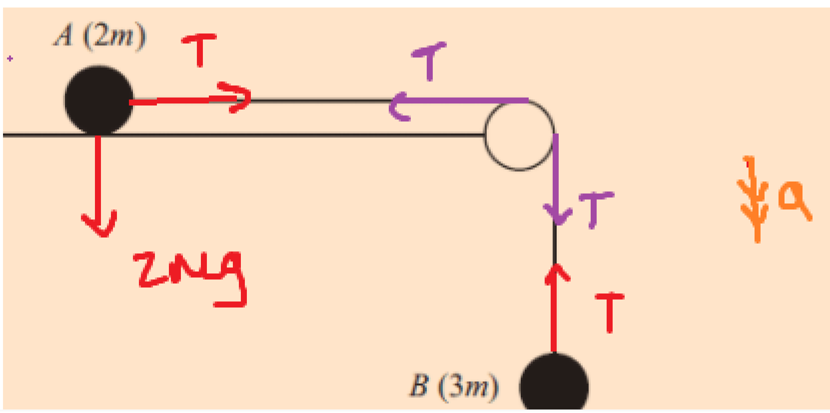
We know **T = 2a**, and **a = 3mg / 5**, so **T = 2(3mg / 5) = 6mg / 5**

c)

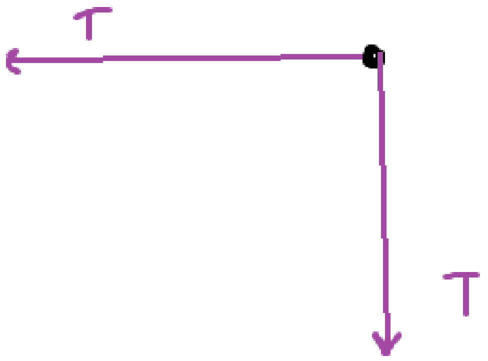
[Additional question]

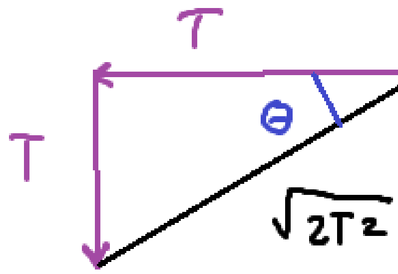
The pulley is causing tension forces to be exerted on A and B.

According to Newton's third law, A and B respond by exerting an equal and opposite tension force on the pulley:



The pulley therefore has a force exerted on it in both the horizontal and vertical planes. To find the resultant of these forces, we can consider a vector triangle:





We can say that the resultant of T is $\sqrt{(T^2+T^2)} = \sqrt{2T^2}$

We previously calculated that **T = 6mg/5**

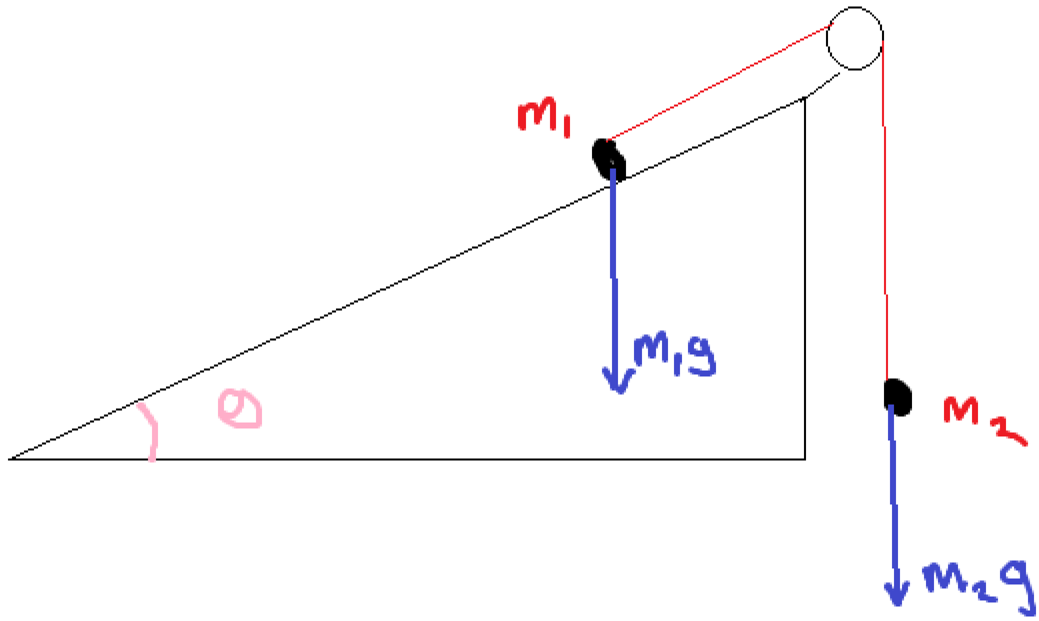
Thus, $\sum T = \sqrt{2(6mg/5)^2} = \sqrt{(2(36m^2g^2 / 25)} = \sqrt{72m^2g^2 / 25}$

Taking the square root of $72m^2g^2 / 25$ gives $\sum T = 6\sqrt{2}mg / 5 = 16.6m$.

The bearing of this force must be **180 + 45 = 225º**.

--

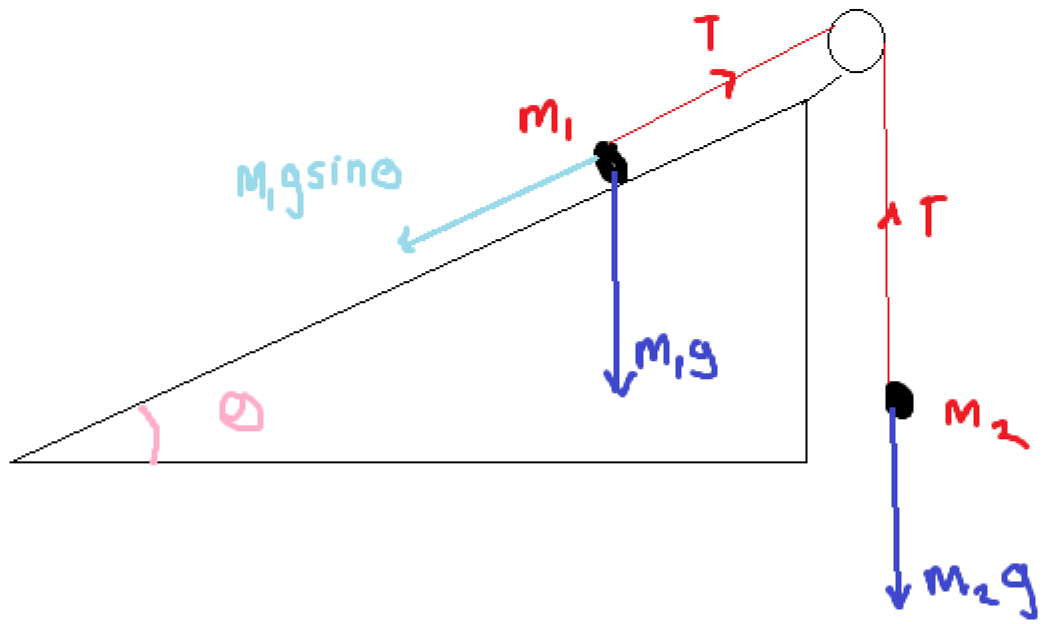
Another type of pulley problem is where the smooth table is at an incline:



When the smooth table was a horizontal plane, the weight of the particle on the table did not influence its motion in the horizontal plane, since the horizontal and vertical planes are considered separately.

However, when a particle is on an inclined plane, the horizontal component of its weight acts down the plane.

In this example, the horizontal component of m_1's weight is **$m_1 g \sin\Theta$**:



From this, we can see that the horizontal component of the weight of m_1 is in the same plane as the tension force (i.e. they are *colinear*).

Now, if the horizontal component of the weight of m_1 is greater than the weight of m_2, the system will be pulled down the inclined plane (like the driving force of a car pulling the freely hanging particle upwards).

Suppose that $\mathbf{m_1 g\sin\Theta > m_2 g}$.

The tension in the string must be lower than the horizontal component of the weight of m_1, thus:

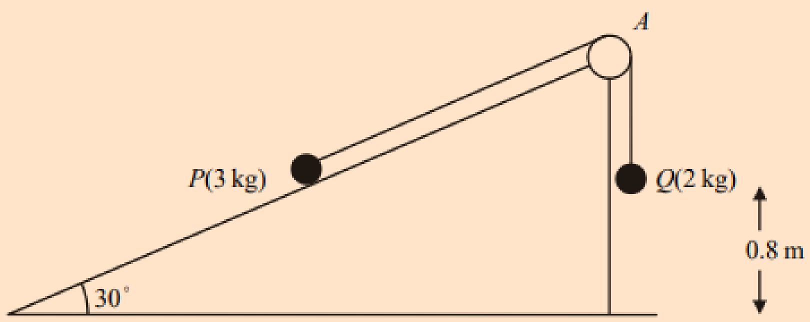
$\mathbf{m_1 g\sin\Theta - T = m_1 a}$

and the tension in the string must be greater than the weight of m_2 since m_2 is accelerating upwards, thus:

$\mathbf{T - m_2 g = m_2 a}$

Example:

7.



The diagram above shows two particles P and Q, of mass 3 kg and 2 kg respectively, connected by a light inextensible string. Initially P is held at rest on a fixed smooth plane inclined at 30° to the horizontal. The string passes over a small smooth light pulley A fixed at the top of the plane. The part of the string from P to A is parallel to a line of greatest slope of the plane. The particle Q hangs freely below A. The system is released from rest with the string taut.

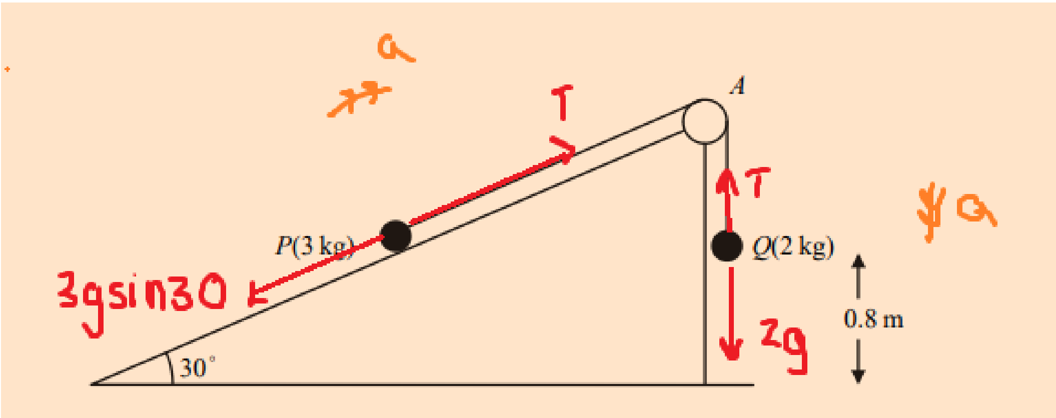
a) **Show that the acceleration of the particle's is 0.98ms^{-2}.**

First let us consider whether the horizontal component of the weight of P is greater or less than the weight of Q:

Weight of Q = 2g = 19.6N.

Horizontal weight of P = 3gsin30 = 14.7N.

The weight of Q is greater, so Q must accelerate downwards, pulling P up the slope.



For Q: **2g - T = 2a => T = 2g - 2a**
For P: **T - 3gsin30 = 3a**

Thus, subbing **T = 2g - 2a** into the equation for P:

2g - 2a - 3gsin30 = 3a

5a = 2g - 3gsin30

a = (2g-3gsin30) / 5 = 0.98ms^{-2}.

-

When the system is released, Q accelerates 0.8m to the ground, and hits the ground before P reaches the pulley.

b) **Find the velocity of Q as it hits the ground**.

The initial velocity of Q before the release of the system is 0ms^{-1}.

Thus, u = 0, a = 0.98, s = 0.8.

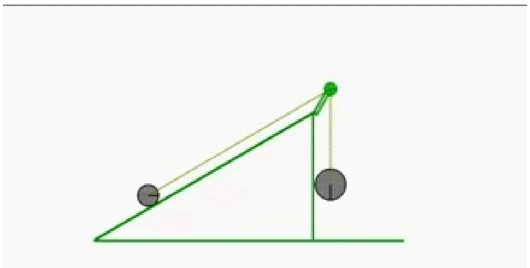
$v^2 = u^2 + 2as$

v = √(u^2+2as) = √(2(0.98)(0.8)) = 1.25ms^{-1}.

-

When Q hits the ground, the string becomes slack.

c) **Find the time taken for the string to become taut again**.



When the string becomes slack, the tension in the string must go away.

This means that P (particle on the slope) no longer has the tension force pulling it up the slope. It now only has the horizontal component of its weight acting down the slope, so it rolls down the slope.

If Q hits the ground with a velocity of $1.25ms^{-1}$, when Q hits the ground, P must also be moving at $1.25ms^{-1}$. The acceleration will not instantly cause P to move down the slope, since it first has to negate the $1.25ms^{-1}$ velocity up the slope.

This means that P will continue to roll up the slope for a short amount of time. When its velocity is $0ms^{-1}$, it will then roll back down the slope. At the position where the string became slack, it must become taut again when P is rolling down the slope, as shown in the above animation.

This means that the displacement between the string becoming slack and then the string becoming taut again must be 0m.

-

Now let us find the acceleration due to gravity when the string becomes taut.

The only force acting on P is its horizontal weight component (since the slope is smooth), so we can say:

3gsin30 = 3 * a

Thus, a = 3gsin30 / 3 = gsin30 = 1/2g = $4.905ms^{-2}$.

This means that P will decelerate at $4.905ms^{-2}$ when the string becomes taut.

Thus,

s = 0, u = 1.25, a = -4.905

This allows us to find the time taken to roll up the slope, then back to the original position,from an initial velocity of $1.25ms^{-1}$ while being decelerated down the slope at $4.905ms^{-2}$.

$s = ut + 1/2at^2 \Rightarrow 0 = 1.25t + 1/2(-g/2)(t^2)$

$1.25t - gt^2/4 = 0$

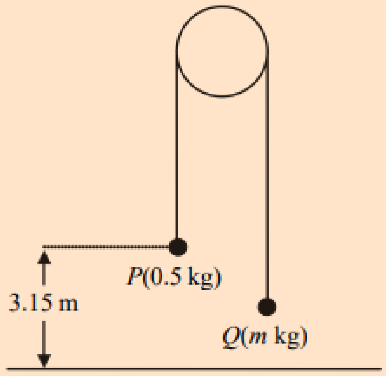
$t(1.25 - gt/4) = 0$

$1.25 - gt/4 = 0 \Rightarrow t = 4(1.25) / g =$ **0.51s**.

-

Another example of 'time taken for string to become taut again':

6.



Two particles P and Q have mass 0.5 kg and m kg respectively, where $m < 0.5$. The particles are connected by a light inextensible string which passes over a smooth, fixed pulley. Initially P is 3.15 m above horizontal ground. The particles are released from rest with the string taut and the hanging parts of the string vertical, as shown in the diagram above. After P has been descending for 1.5 s, it strikes the ground. Particle P reaches the ground before Q has reached the pulley.

(a) Show that the acceleration of P as it descends is 2.8 m s^{-2}.

(3)

(b) Find the tension in the string as P descends.

(3)

(c) Show that $m = \frac{5}{18}$.

(4)

(d) State how you have used the information that the string is inextensible.

(1)

When P strikes the ground, P does not rebound and the string becomes slack. Particle Q then moves freely under gravity, without reaching the pulley, until the string becomes taut again.

(e) Find the time between the instant when P strikes the ground and the instant when the string becomes taut again.

(6)

(Total 17 marks)

a)

P starts from a velocity of $0ms^{-1}$, and descends 3.15m in 1.5s.

Thus, u = 0, s = 3.15, t = 1.5

$s = ut + 1/2at^2$

$3.15 = 1/2\ (a)(1.5)^2$

$a = 3.15 / (\ 1/2 * 1.5^2) =$ **$2.8ms^{-2}$.**

b)

Since P is descending, its weight is greater than the tension.

Thus,

$0.5g - T = 0.5 * 2.8$

$T = 0.5g - (0.5 * 2.8) = 3.505N = 3.5N$ (2SF)

c)

m must accelerate upwards, so the tension in the string is greater than its weight.

Thus, **$T - mg = ma$**

We know $T = 3.5N$ and $a = 2.8ms^{-2}$.

Thus, **$3.5 - mg = 2.8m$**

$m(2.8 + g) = 3.5$

$m = 3.5 / (2.8 + g) = 0.278kg = 0.28kg$ (2SF)

This is very close to *5/18*.

-

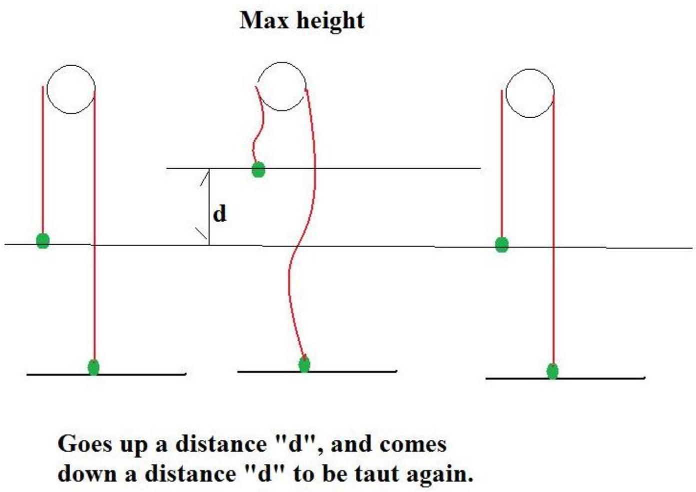
d)

The tension exerted on particle P and Q has been assumed to be equal.

e)

When P hits the ground, the string becomes slack and the tension in the string goes away.

This means that the only force that is acting on Q is gravity, so it begins to accelerate downwards at $9.81ms^{-2}$.



However, Q will have a small velocity before the string becomes slack, so it will continue to travel upwards after the string becomes slack for a small period of time. It will then reach an apex, before being accelerated downwards by gravity.

When Q returns to its position the instant before the string became slack, the string will become taut again.

-

We first need to find the velocity of Q before the string becomes slack.

If P moves 3.15m downwards before hitting the ground, Q must move 3.15m upwards. It starts with a velocity of $0ms^{-1}$, and accelerates upwards at $2.8ms^{-2}$.

Thus, u = 0, s = 3.15, a = 2.8, finding v...

$v = \sqrt{(u^2 + 2as)} = \sqrt{(2(2.8)(3.15))} = 4.2ms^{-1}$.

-

When the string becomes slack, Q moves upwards at $4.2ms^{-1}$ while being accelerated downwards by gravity at $9.81ms^{-2}$. It then reaches an apex, before being accelerated back down to its original position before becoming slack - at which point the string becomes taut.

Thus, s = 0, u = 4.2, a = -g

$s = ut + 1/2at^2$

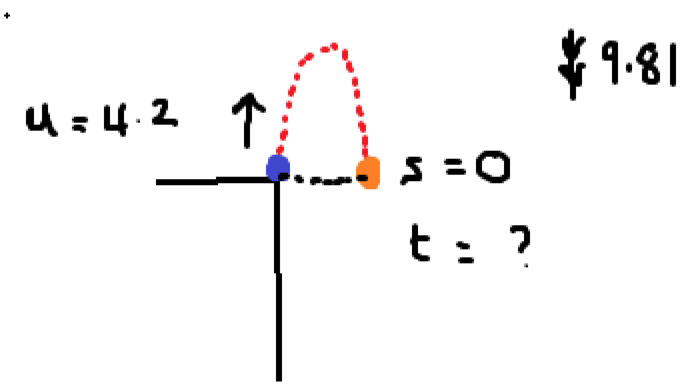
$0 = 4.2t + 1/2(-g)(t^2)$

$t(4.2 - gt/2) = 0$

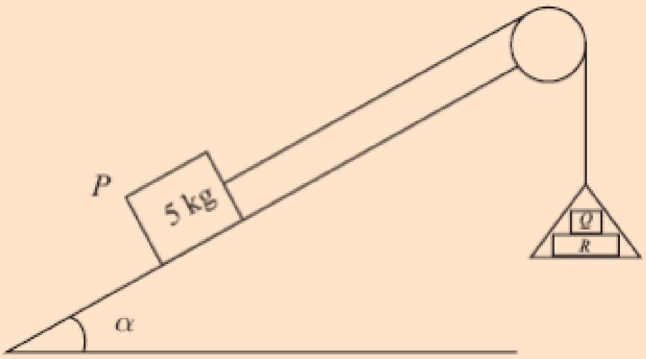
$4.2 - gt/2 = 0$

t = 2(4.2) / 9.81 = **0.86s**.

This is similar to projecting a particle from a certain height above the ground at a speed of $4.2ms^{-1}$, and finding the time it takes to return to the position from which it was projected:



Pulley on a slope example 2:



One end of a light inextensible string is attached to a block *P* of mass 5 kg. The block *P* is held at rest on a smooth fixed plane which is inclined to the horizontal at an angle α, where $\sin\alpha = \frac{3}{5}$. The string lies along a line of greatest slope of the plane and passes over a smooth light pulley which is fixed at the top of the plane. The other end of the string is attached to a light scale pan which carries two blocks *Q* and *R*, with block *Q* on top of block *R*, as shown in Figure 3. The mass of block *Q* is 5 kg and the mass of block *R* is 10 kg. The scale pan hangs at rest and the system is released from rest. By modelling the blocks as particles, ignoring air resistance and assuming the motion is uninterrupted, find

(a) (i) the acceleration of the scale pan,

(ii) the tension in the string,

(8)

(b) the magnitude of the force exerted on block *Q* by block *R*,

(3)

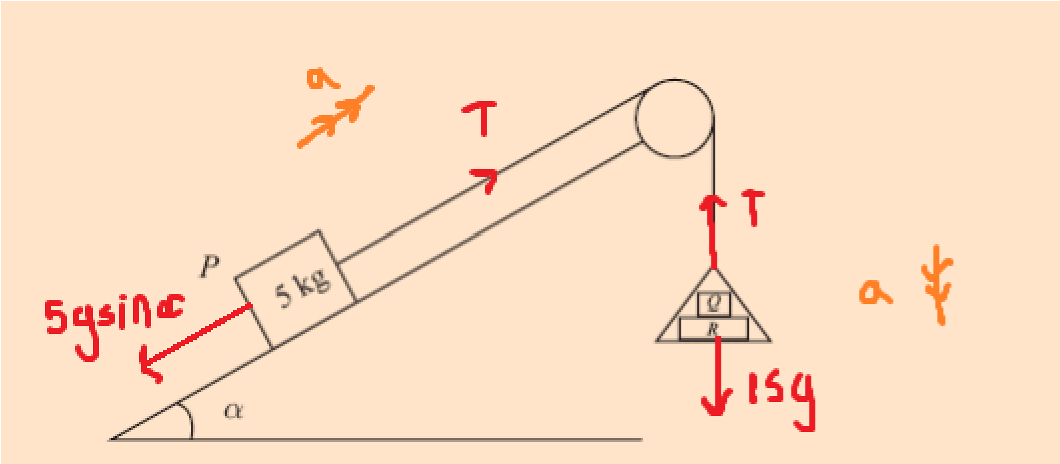
(c) the magnitude of the force exerted on the pulley by the string.

(5)

a) I.

The overall mass of the scale pan is much greater than the horizontal component of the weight of P, so P must accelerate up the slope, whilst Q and R accelerate downwards.

Consider the forces acting on each particle:



The pan is 'light', so it does not contribute to the overall mass of the system.

For the pan:

15g - T = 15a = T = 15g - 15a

For the block **P**:

T - 5gsinα = 5a

Sinα = 3/5

Thus, T - 3g = 5a

Subbing **T = 15g - 15a** into the equation for **P:**

15g - 15a - 3g = 5a

12g = 20a

a = 12g / 20 = **5.9ms^{-2}**

-

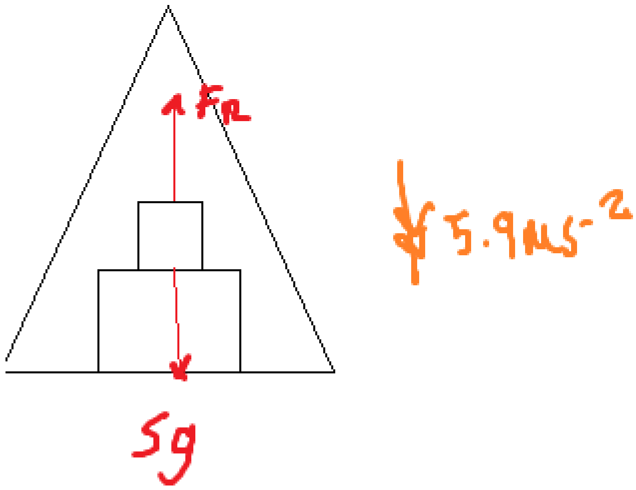
ii.

T = 15g - 15a = 15(9.81) - 15(5.9) = 58.9N

b)

Q is sitting on R, so R exerts a normal reaction force on Q.

Q therefore has two forces acting on it - its weight, and the normal reaction force (let this be F_R):



Since Q is accelerating downwards, its weight is greater than the reaction force.

Thus,

$5g - F_R = 5(5.9)$

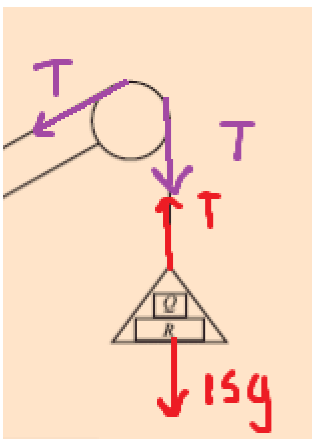
$F_R = 5g - 5(5.9) =$ 19.6N.

c)

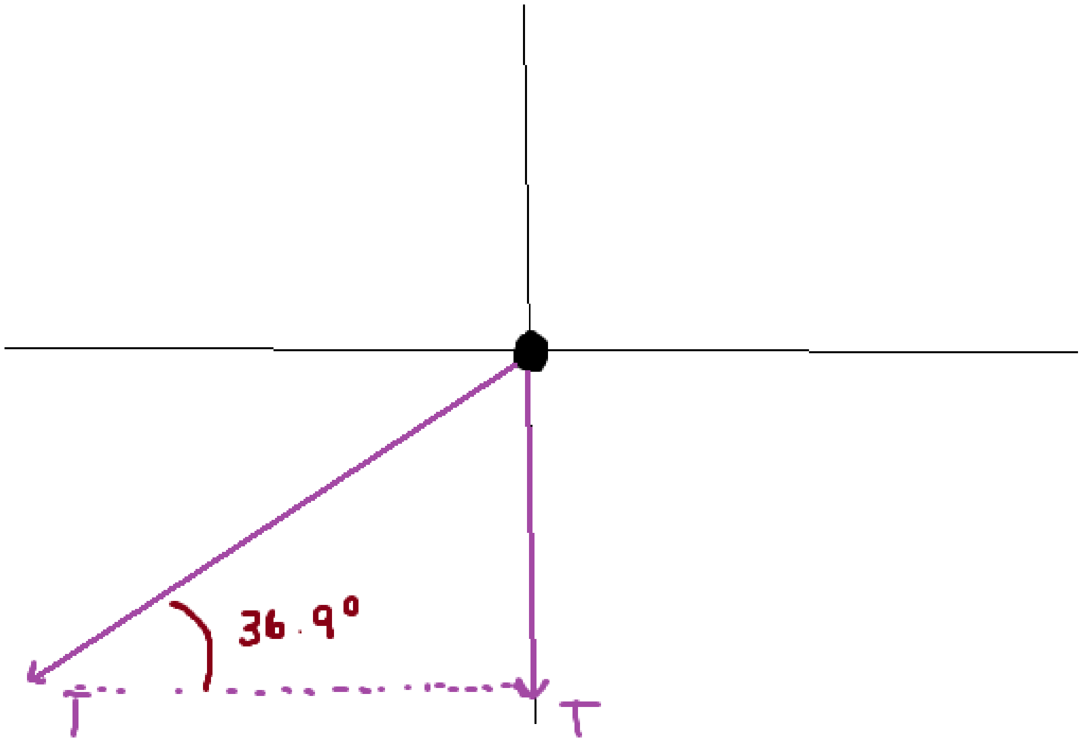
Like in one of the previous examples, the pulley creates the tension forces exerted on the system.

The pulley exerts a tension force at an angle of $\sin^{-1}(3/5) = 36.9^o$ to the normal on block P, so the block responds by exerting an equal and opposite tension force on the pulley.

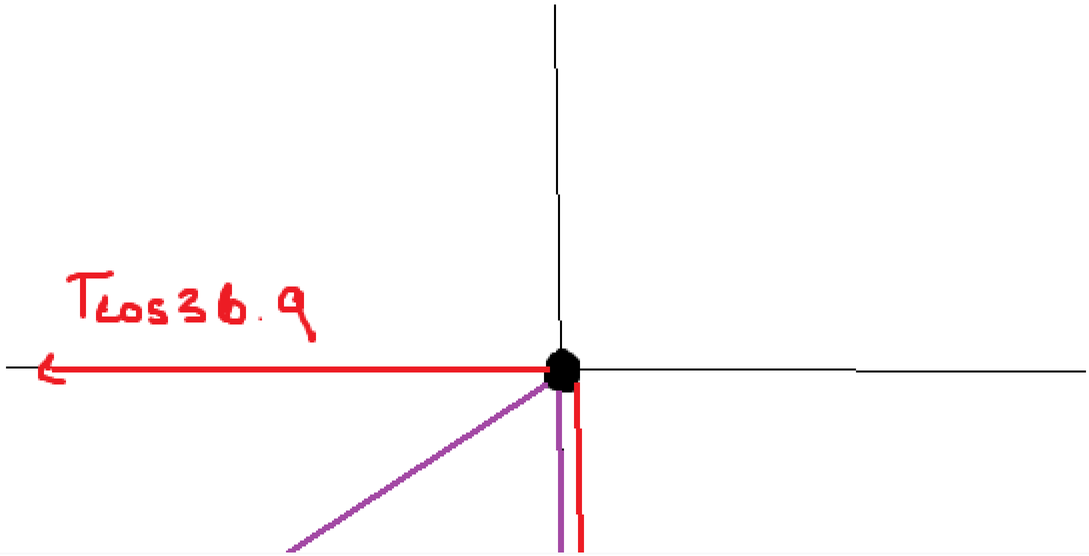
The pulley also exerts an upwards tension force on the pan, so the pan responds by exerting an equal and opposite tension force on the pulley in the downwards direction:



Consider this as a free-body diagram:



We cannot find the resultant tension currently because the forces are not colinear, so we instead need to split the tension acting at an angle into horizontal and vertical components:



The pulley therefore has two downwards forces, **T** and **Tsin36.9** acting on it, and a horizontal force, **Tcos36.9**.

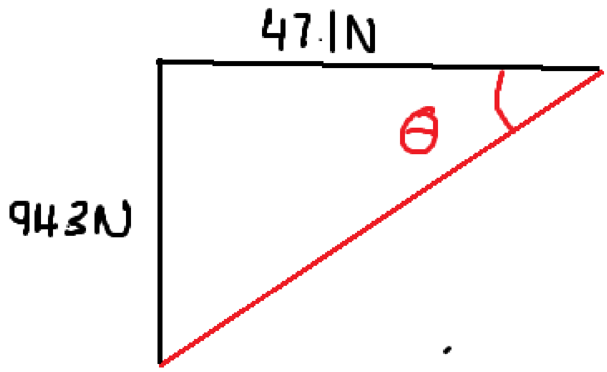
Since T = 58.9:

Total horizontal force = 58.9cos36.9 = 47.1N
Total vertical force = 58.9 + 58.9sin36.9 = 94.3N

This gives a resultant force of **√(47.1² + 94.3²) = 105N**

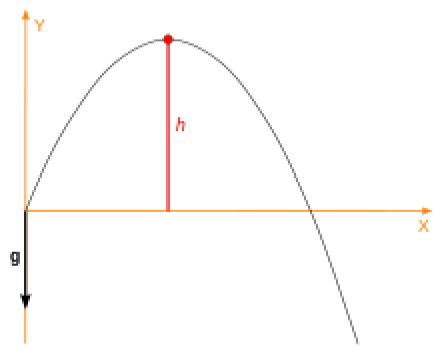
The direction of the force is **$\tan^{-1}$ (94.3 / 47.1) = 63.4°.**

Thus, **bearing = 270 - 63.4 = 206.6°**.



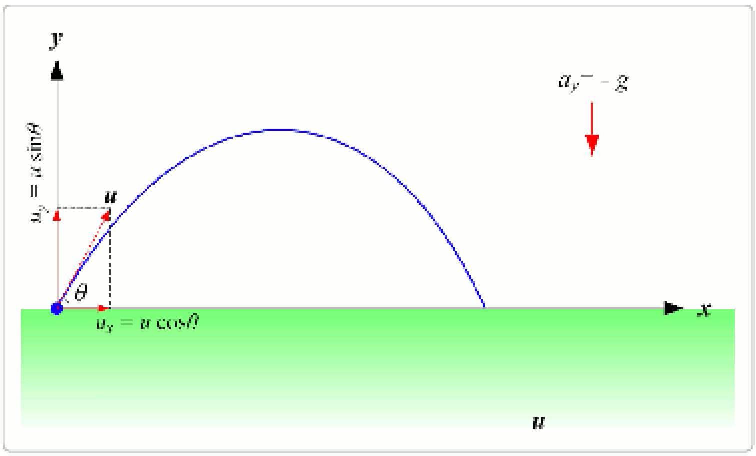
Projectile motion

If a particle is projected at an angle to the horizontal, it behaves like a **projectile**.



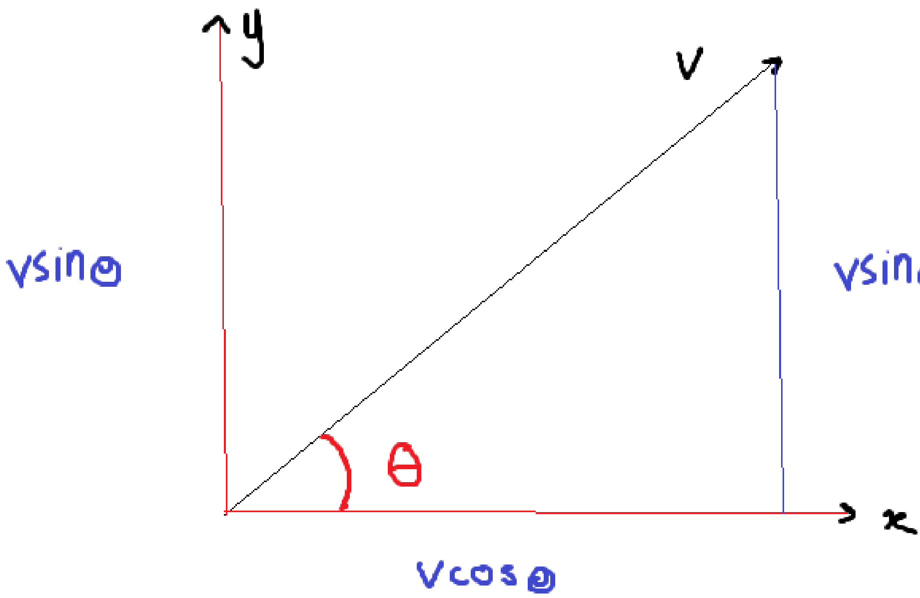
A projectile moves in a **parabolic path**.

When calculating the motion of a projectile, we consider the horizontal and vertical planes independently.



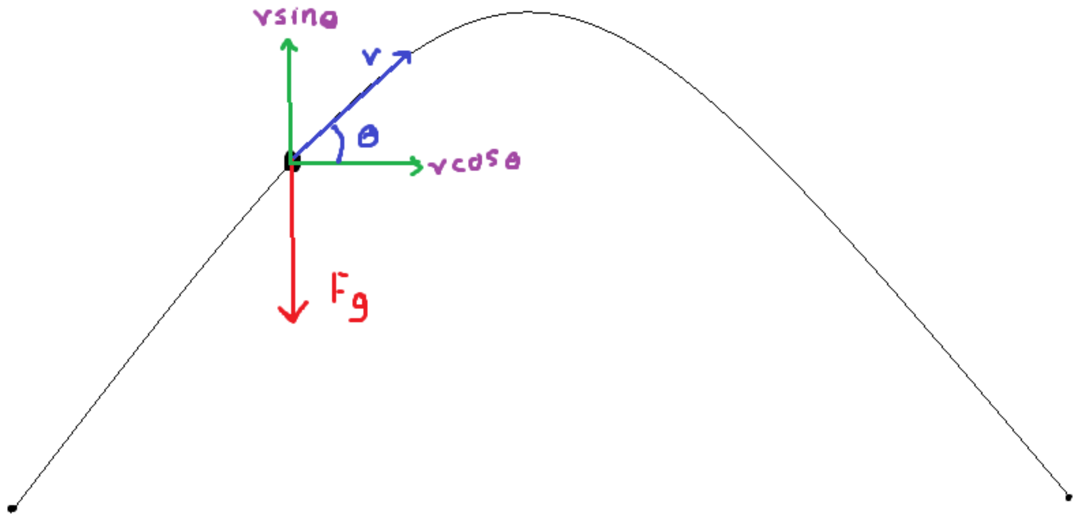
This means that the horizontal and vertical components do not affect each other.

If a particle is projected at an angle of **Θ** with a velocity of **v**ms^{-1}, the horizontal component of its velocity is **vcosΘ**ms^{-1}, and the vertical component of its velocity is **vsinΘ**ms^{-1}:



When we consider projectile motion, we assume that the only force acting on the projectile is gravity; forces such as air resistance or cross winds are not considered.

Gravity only acts in the vertical plane - it pulls a particle downwards towards the centre of the gravitational field. This means that gravity *only affects the vertical velocity component* - it causes a downwards acceleration in the vertical plane; it has no effect on the horizontal motion.



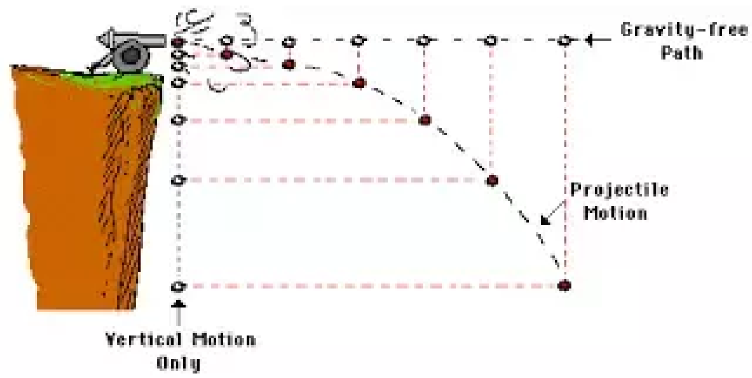
Because we are assuming that the only force acting on a projectile is gravity, and gravity acts in the vertical direction, this means that there is no force on the projectile in the horizontal plane.

As a result, the particle does not accelerate in the horizontal plane - *the horizontal velocity remains constant.*

-

Let us consider a projectile motion situation.

Suppose a cannon is on a cliff with a significant height. Consider three situations:



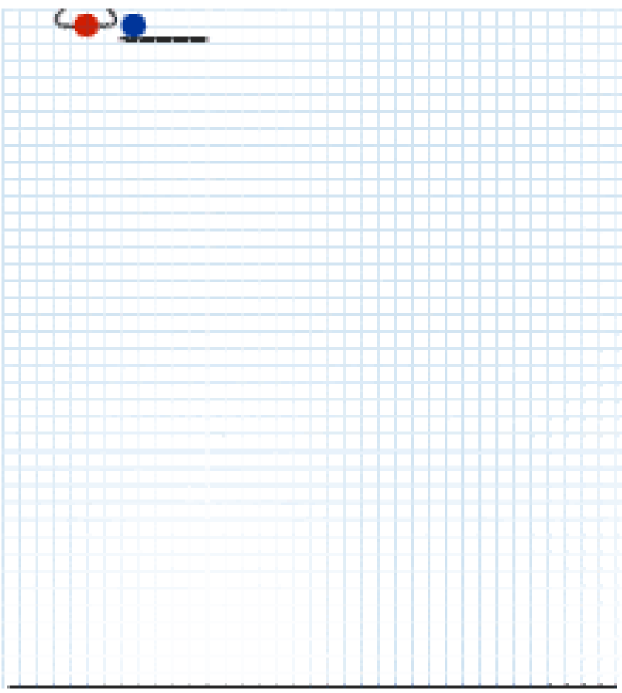
Dropping the cannonball: If a cannonball is dropped vertically downwards, the ball will be subject to gravitational acceleration, so its vertical velocity will increase by $9.81ms^{-1}$ every second.

Firing the cannon ball (gravity included):

- The ball will now have a vertical velocity component and a horizontal velocity component.
- Because the vertical and horizontal components are considered as independent of each other, the ball will still take the same time to hit the ground as it did when it was dropped vertically downwards - it is still subject to the downwards acceleration of $9.81ms^{-2}$.
- However, this time, because there is now a horizontal velocity component, the ball will also travel a horizontal distance - but the horizontal distance travelled is unaffected by the vertical component.

Firing the cannon ball in the same direction (gravity excluded):

- If gravity did not exist and the cannonball is fired in the horizontal direction, it will continue to move in the horizontal plane indefinitely - because we are assuming that there is no acceleration in the horizontal plane to slow the ball down.
- Because gravity does exist on Earth, projectiles take a parabolic path.



The animation to the left shows how the time taken for a projectile to reach the ground is the same when it is dropped vertically downwards or if it is projected horizontally.

This is because the horizontal and vertical velocity components are independent of each other. In both cases, the particle's are projected with a vertical velocity of $0ms^{-1}$, and the vertical acceleration is exactly the same in both cases, so the projectile's will take the same time to reach the ground in both instances.

For example, if they were projected from a height of 15m, considering the vertical velocity:

u = 0, s = 15, a = 9.81

Time taken to reach the ground = √(2s/a) = √(2(15) / 9.81) = 1.75s.

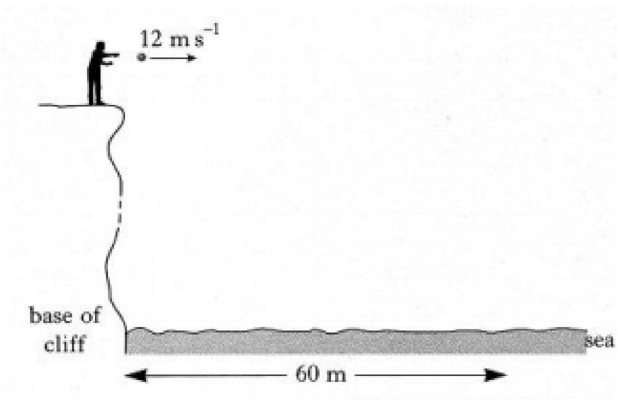
Projectile motion calculations

-

Example 1:

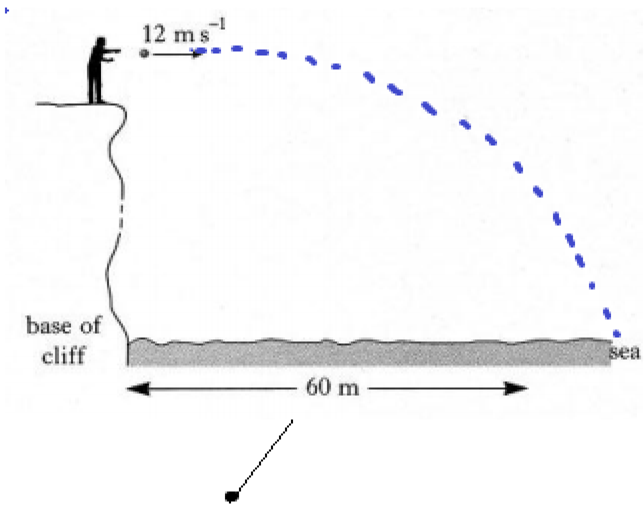
A stone is **thrown horizontally** with a **speed of 12ms^{-1}** over the edge of a vertical cliff.

It hits the sea at a **horizontal distance of 60m** out from the base of the cliff.



Calculate the height from which the stone was projected above the level of the sea.

The stone will take a parabolic path to the sea:



Step 1 - Consider motion in the horizontal plane

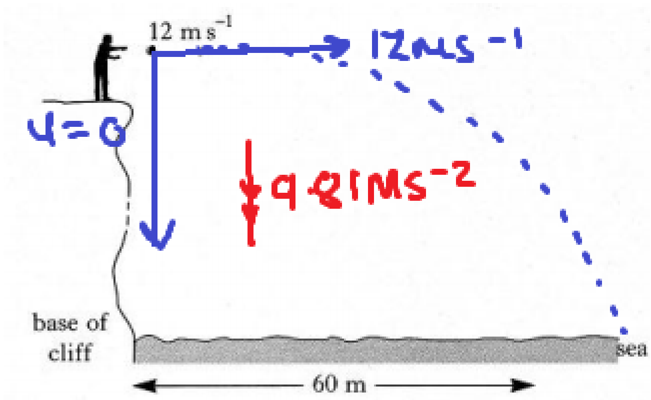
As mentioned previously, in the horizontal plane, a projectile is not subject to acceleration. This means that the horizontal velocity remains constant for the entire path to the sea.

If a particle travels a horizontal distance of 60m at $12ms^{-1}$, the time it takes to complete this path is **60 / 12 = 5s**.

Step 2 - Consider motion in the vertical plane

The projectile starts with a vertical velocity of $0ms^{-1}$, as its initial projection is in the horizontal plane - not the vertical plane.

However, in the vertical plane, the particle is subject to a gravitational acceleration of $9.81ms^{-2}$, so its vertical velocity component will increase by $9.81ms^{-1}$ every second.



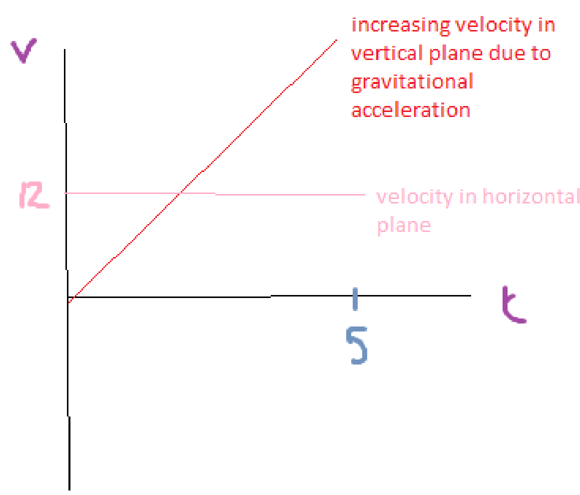
We therefore know three variables about the particle's motion in the vertical plane:

- The particle has an initial velocity of $0ms^{-1}$ in the vertical plane
- It is accelerated at $9.81ms^{-2}$ in the vertical plane
- It takes 5s to hit the sea

u = 0, a = 9.81, t = 5

We can therefore calculate the distance travelled in the downwards direction:

$s = ut + 1/2at^2 = 1/2(9.81)(5^2) =$ 122m => this is the height of the cliff

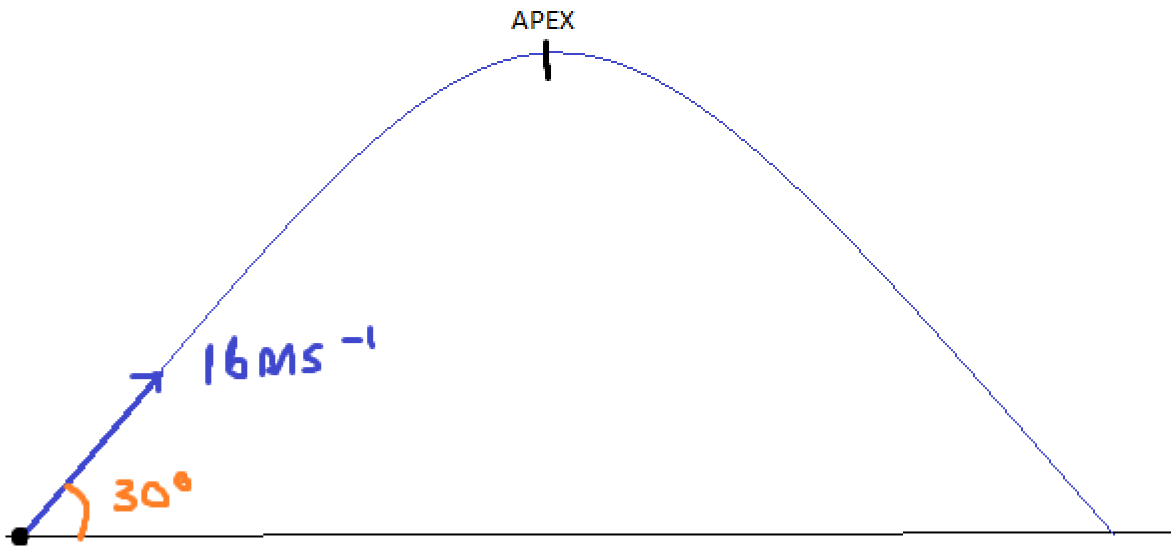


Example 2:

A particle is projected at an angle of 30° from the horizontal at a velocity of $16ms^{-1}$.

a) **Calculate the total horizontal distance travelled by the particle.**

The particle takes a parabolic path:



The particle has a vertical velocity component and a horizontal velocity component.

The vertical velocity component decreases at a rate of 9.81ms^{-1} per second due to gravitational acceleration. This means that the particle's vertical velocity component will eventually be reduced to 0ms^{-1}, at which point the **apex** will be reached.

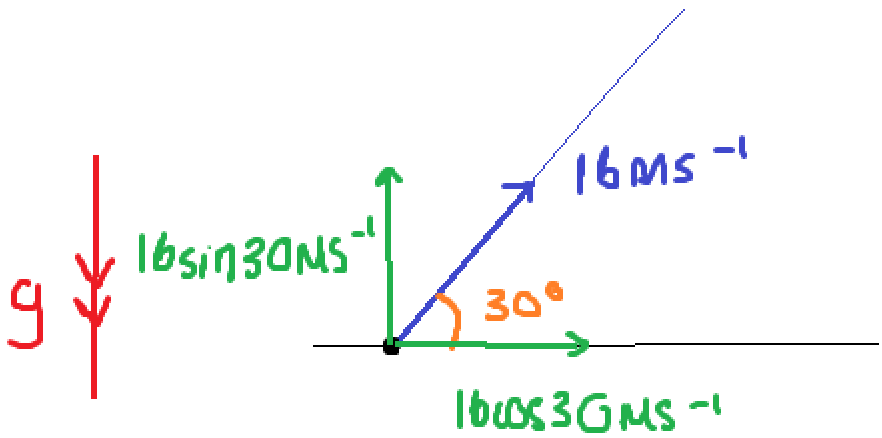
Beyond the apex, the particle accelerates back towards the ground. Because the gravitational acceleration is constant, and there are no other forces exerted on the projectile, the time taken to reach the apex must equal the time taken to return from the apex to the ground.

Because there is no acceleration in the horizontal plane, the horizontal velocity component remains constant throughout the path.

-

We can start by finding the values of the horizontal and vertical velocity components:

- **Vertical : 16sin30**
- **Horizontal : 16cos30**



In the motion, the horizontal component remains constant. This means that the horizontal distance travelled is simply equal to the horizontal velocity multiplied by the time of flight:

Horizontal distance = horizontal velocity * time of flight

We know the horizontal velocity - **16cos30** - but we do not know the time of flight. We can find the time of flight using the vertical component.

Step 1 - *Find the time taken for the entire journey using the vertical component*

If we find the time taken for the particle to reach its apex, and double this time, we find the total time of flight (time taken to reach apex = time taken to return from apex to ground).

At the apex, the vertical velocity of the particle has been reduced to **0ms**$^{-1}$ by gravitational acceleration. It has an initial vertical velocity of **16sin30ms**$^{-1}$, so we can find the time taken for this to happen:

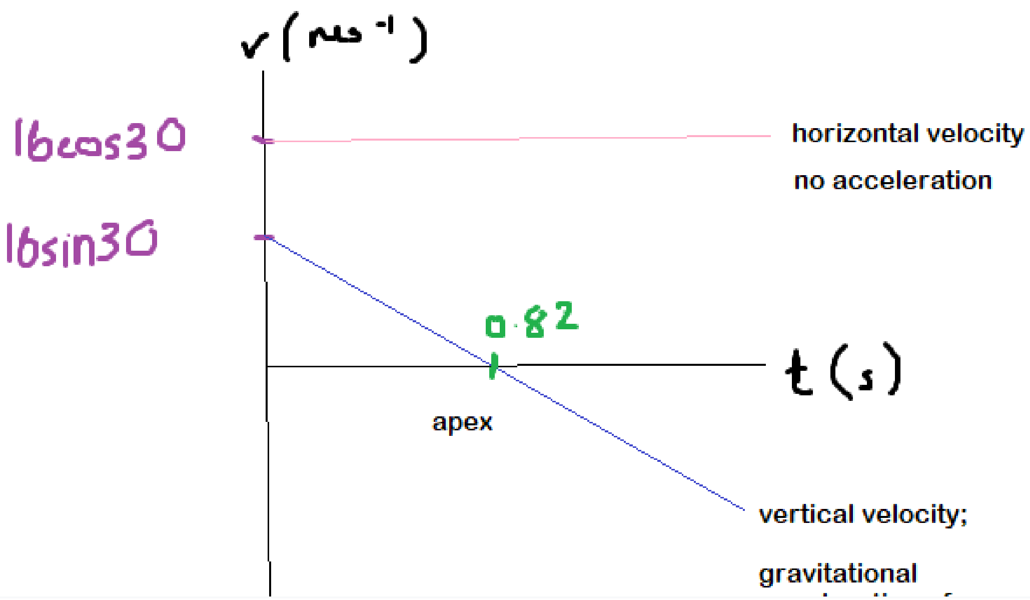
u = 16sin30, a = -9.81, v = 0

t = (v-u) / a = (0-16sin30) / -9.81 = **0.82s**.

The total time of flight is therefore**: 2(0.82) = 1.64s**.

Step 2 - *Find horizontal distance travelled*

The particle is travelling in the horizontal plane at **16sin30ms**$^{-1}$ for **1.64s**, so its horizontal distance travelled is: **1.64(16sin30) = 13.1m**.

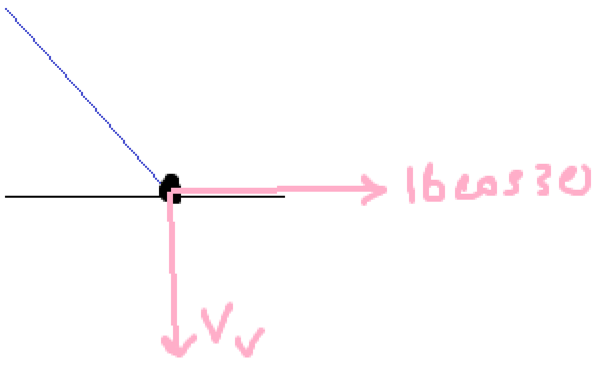


b) **Calculate the resultant velocity of the particle as it hits the ground and the bearing at which this occurs**

When the particle hits the ground, it will have two velocity components:

- A horizontal velocity component
- A vertical velocity component

We need to find the resultant of these components.

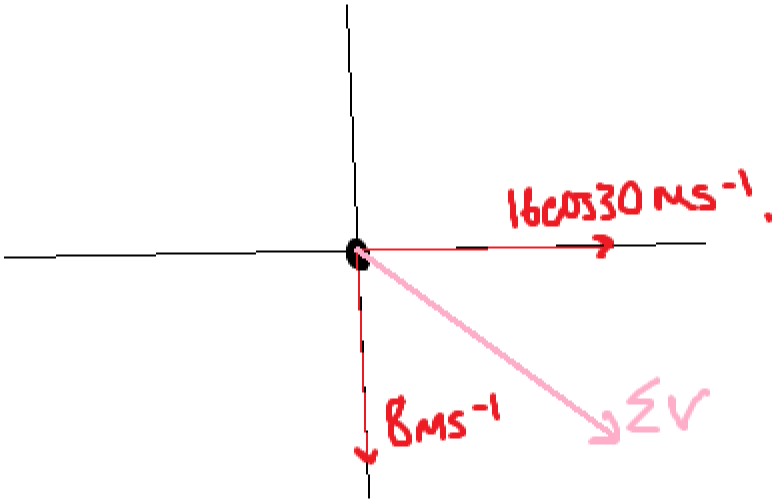


The magnitude of the vertical velocity component when the ball was projected is equal to the magnitude of the vertical velocity component upon the particle hitting the ground.

To prove this, from the apex:

u = 0, a = 9.81, t = 0.82

v = 9.81(0.82) = 8ms^{-1} = 16sin30ms^{-1}.



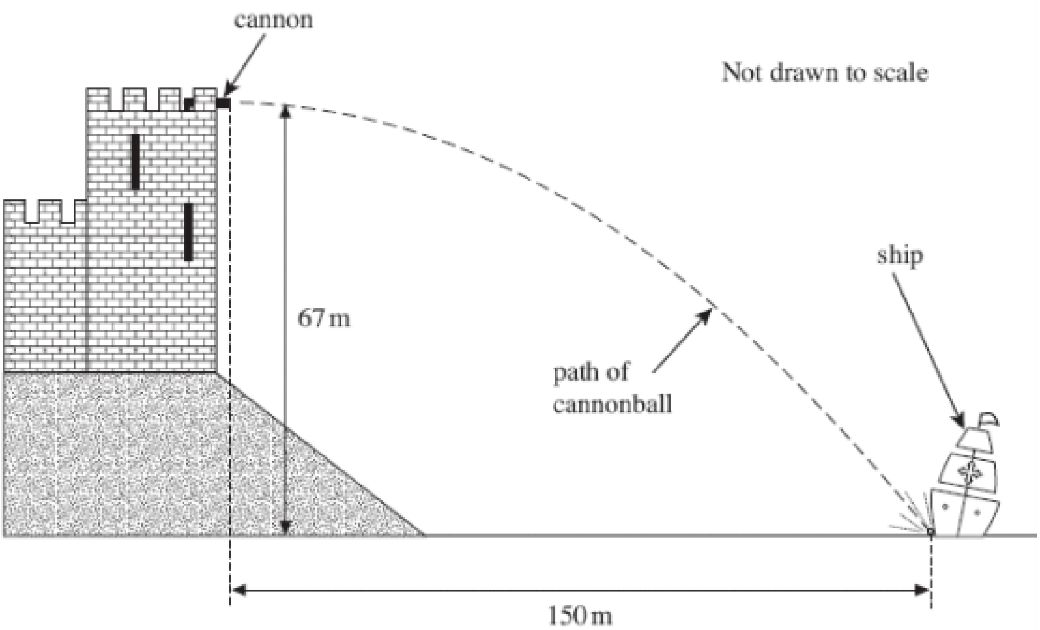
The resultant velocity is therefore $\sqrt{(8^2 + (16\cos30)^2)}$ = **16ms⁻¹** (i.e. the initial velocity).

The direction must now be 30° below the horizontal, which gives a bearing of **120°.**

-

Example 3:

In a castle, overlooking a river, a cannon was once employed to fire at enemy ships. One ship was hit by a cannonball at a horizontal distance of 150 m from the cannon as shown in the figure below. The height of the cannon above the river was 67 m and the cannonball was fired horizontally.



Find the resultant velocity of the cannon ball when it hits the ship, and the direction of this velocity.

We need to find the vertical and horizontal velocity components of the cannonball before it hits the ship.

Since the cannon ball is projected horizontally, its initial vertical velocity must be $0ms^{-1}$.

The cannonball moves 67m vertically during its motion, so we can find the time taken for the cannon ball to descend this distance, and its vertical velocity when it hits the ship:

$u = 0$, $a = 9.81$, $s = 67$

$s = 1/2at^2$ => $\mathbf{t = \sqrt{(2s/a)} = \sqrt{(2(67) / 9.81)} = 3.7s}$.

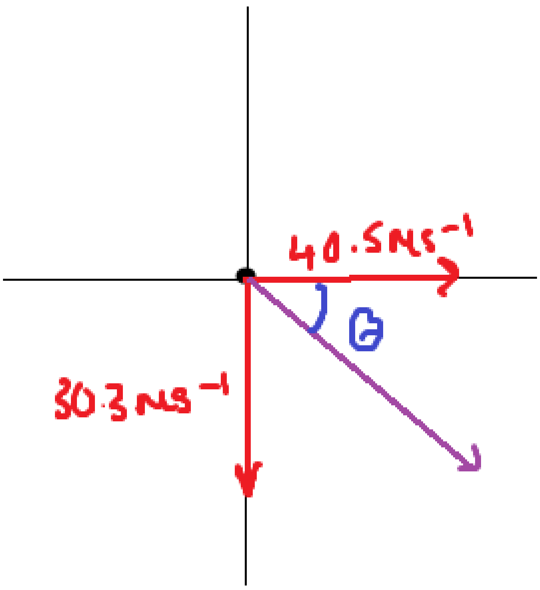
$\mathbf{v = u + at = 9.81(3.7) = 36.3ms^{-1}}$

-

The velocity of the cannonball in the horizontal plane remains constant throughout the motion.

This means that the cannonball travels 150m over 3.7s at a constant velocity, so $\mathbf{v = 150 / 3.7 = 40.5ms^{-1}}$.

The cannonball thus hits the ship with a vertical velocity of $36.3ms^{-1}$ and a horizontal velocity of $40.5ms^{-1}$. This information enables us to find the resultant velocity of the cannonball:



$\Sigma v = \sqrt{(40.5^2 + 30.3^2)} = \mathbf{50.6ms^{-1}}$.

Direction = $\mathbf{tan^{-1} (30.3 / 40.5) = 36.8°}$ below the horizontal (**bearing of 126.8°**).

--

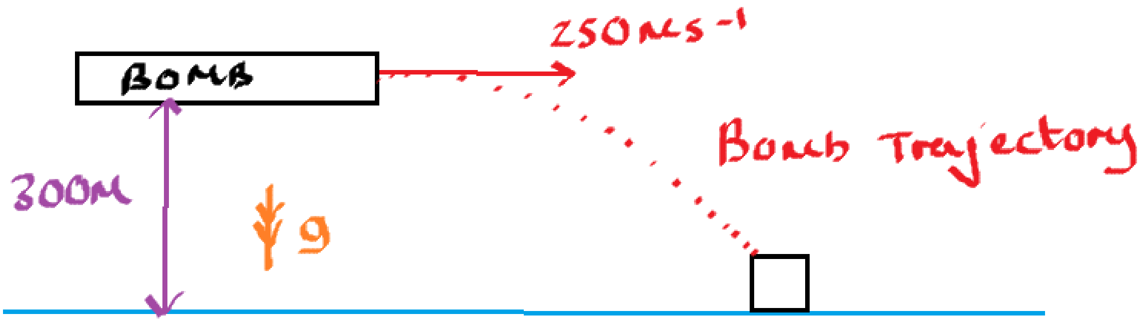
Example 4:

A bomber is travelling in the horizontal plane at an altitude of 300m with a constant velocity of $250ms^{-1}$. A ship is positioned at an altitude of 0m at sea level.

Determine the horizontal distance from which the bomber should release a bomb to hit the ship.

When the bomber released the bomb, the bomb will be released at the horizontal velocity of the bomber ($250ms^{-1}$), and the bomb will have an initial vertical velocity of $0ms^{-1}$.

This means that the bomber must release the bomb behind the ship:



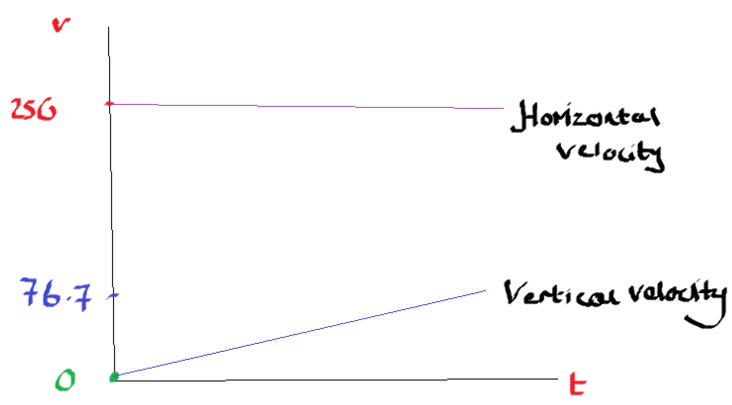
We can first consider the vertical plane, determining how long it takes gravity to accelerate the bomb to the ground.

u = 0 , a = 9.81, s = 300

$s = 1/2at^2 \Rightarrow t = \sqrt{(2s/a)} = \sqrt{(2(300) / 9.81)} = 7.82s$.

If the bomb takes 7.82s to hit the ground, and it is travelling at a constant horizontal velocity of $250ms^{-1}$, the distance from which the bomb must be released is given by:

d = 250 * 7.82 = **1960m**.



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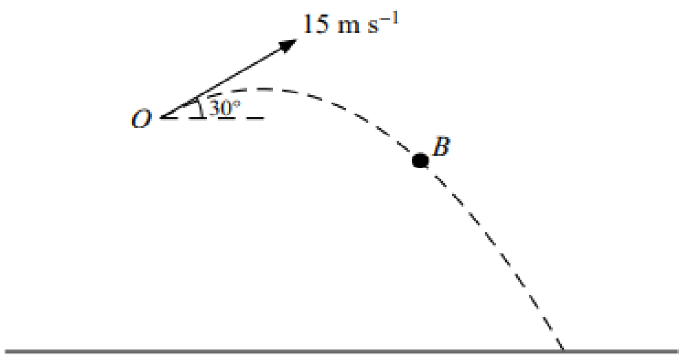
In actuality, the bomb must be released from a distance further back. This is because air resistance opposes gravity, which increases the actual time taken to reach the ground, and hence the horizontal distance required.

You would also have to consider whether the wind is blowing in the direction of the bomb, or opposing it.

Example 5



A small ball B is projected from a point O above horizontal ground, with initial speed $15\,\mathrm{m\,s^{-1}}$ at an angle of projection of 30° above the horizontal (see diagram). The ball strikes the ground 3 s after projection.

(i) Calculate the speed and direction of motion of the ball immediately before it strikes the ground. [5]

(ii) Find the height of O above the ground. [2]

i)

The vertical velocity of the ball is given by **15sin30**.

The ball is projected at **15sin30** in the vertical plane, and is accelerated towards the ground at **9.81ms^{-2}**. It takes **3s** to reach the ground.

Thus, **u = 15sin30, a = -g, t = 3**

We need to find the speed at which the ball hits the ground, which requires us to find its vertical velocity component:

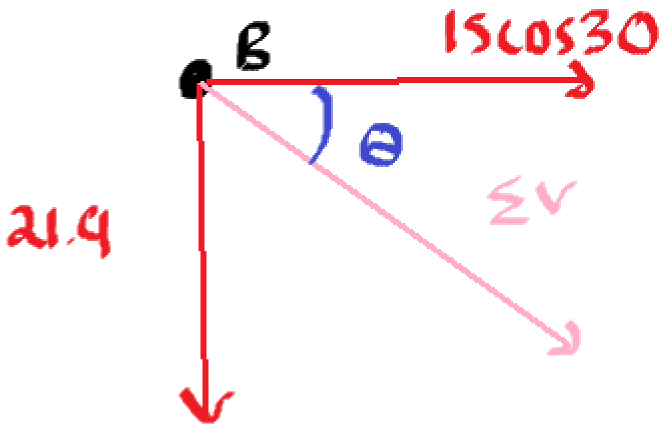
v = u + at = 15sin30 - 3g = -21.9ms^{-1}.

So the ball hits the ground with a vertical velocity component of 21.9ms^{-1}.

To find the resultant velocity (the speed), we also need to know the horizontal velocity.

The horizontal velocity of the ball is **15cos30**. This remains constant throughout the journey since there is no acceleration in the horizontal plane.

Thus,



Σv = $\sqrt{}$**((15cos30)^2 + 21.9^2) = 25.5ms^{-1}.**

Direction = **tan^{-1} (21.9 / 15cos30) = 59.3°.**

The ball hits the ground with a speed of **25.5ms^{-1}** at an angle of **59.3°** below the horizontal.

ii) Considering the vertical plane:

u = 15sin30, a = -9.81, t = 3

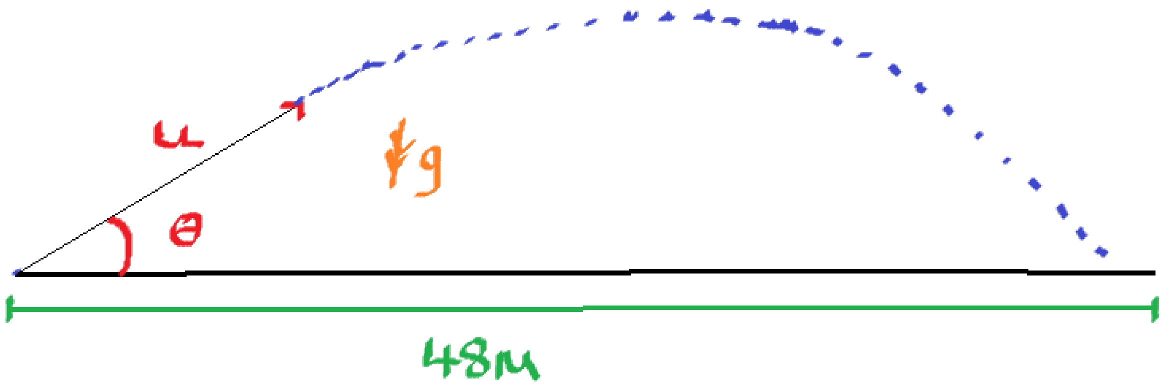
s = ut + 1/2at^2 = -21.6m.

Thus O is **21.6m** above the ground.

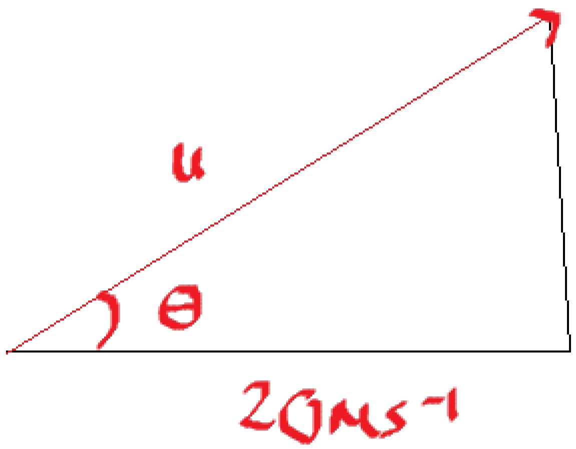
-

Example 6

A golf ball B is projected from a point O on horizontal ground. B hits the ground for the first time at a point 48 m away from O at time 2.4 s after projection. Calculate the angle of projection. [3]



If the particle travels a horizontal distance of 48m over 2.4s, its horizontal velocity is **48 / 2.4 = 20ms^{-1}**.



This means that the vertical velocity component of the particle is **20tanΘ**.

If we consider the displacement of the particle up to its apex in the vertical plane:

u_V = **20tanΘ**
v = 0
t = 1.2
a = -g

Using v = u+at

0 = 20tanΘ + (-g * 1.2)

1.2g = 20tanΘ

Θ = tan^{-1} (1.2g / 20) = 30.5°.

-

Example 7

A rock is launched from a catapult at an angle of 45° at a speed of $22ms^{-1}$. The catapult has a height of 5m. The rock is aimed at a 20m high castle wall that is 40m away.

Determine if the rock hits the wall.

We can first determine the time taken for the rock to travel 40m.

The horizontal component of the rock's velocity is **22cos45**.

Thus, **t = 40 / 22cos45 = 2.57s.**

We now need to determine the vertical displacement of the rock over 2.57s, and determine if this will lead to the rock hitting the wall.

u = 22sin45
a = -g
t = 2.57
s = s

$s = ut + 1/2at^2$ = (22sin45(2.57)) + 1/2(-g)(2.57^2) = 7.6m.

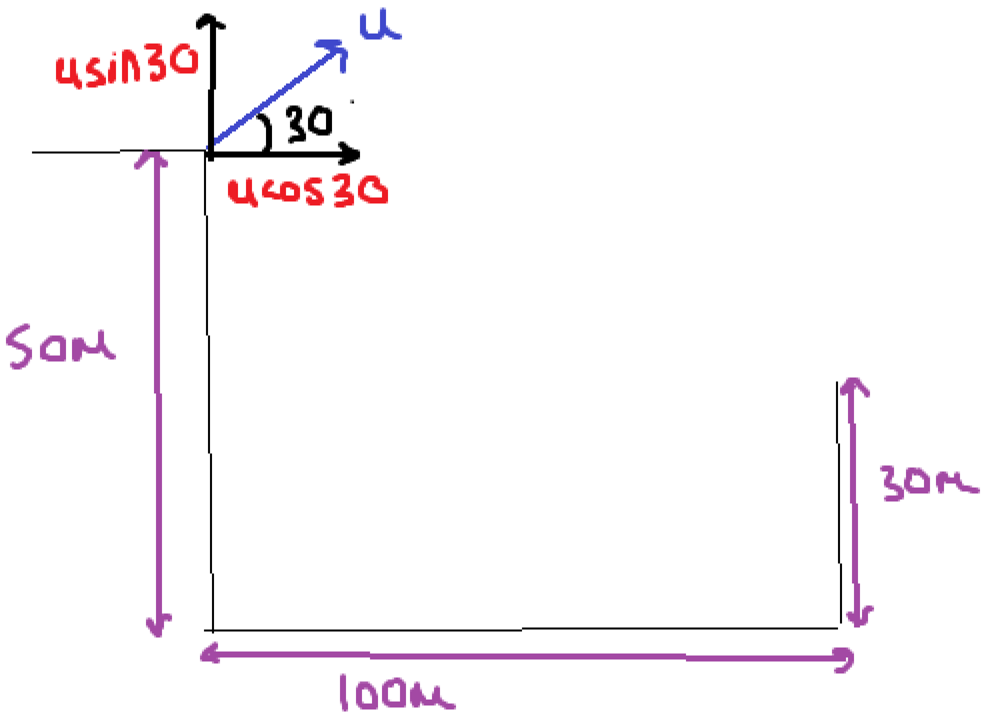
So the rock is displaced by **7.6m** from its original **5m** height at the time when it reaches the horizontal distance of the wall.

This means that its height above the ground at this point is **12.6m**, so it must hit the wall.

Example 8

A particle is projected from a height of 50m above the ground with a speed of ums^{-1} at an angle of 30° above the horizontal. A target is located at a height of 30m above the ground, at a horizontal distance of 100m from the point of projection.

Calculate the value of u required for the particle to hit the target.



Between the origin and the target, there is a displacement of **-20m** if upwards is taken as positive.

We therefore need to find the value of **u** required for the particle to have a vertical displacement of -20m over a time t.

-

We can first consider the horizontal component of motion.

The particle has a horizontal velocity of **ucos30** ms-1, and this remains constant throughout the journey since there is no acceleration in the horizontal plane.

We can therefore say that the time taken to travel from the origin to the target is:

t = 100 / ucos30

-

We can now consider the vertical component. The initial vertical component of the particle's velocity is **usin30** ms^{-1}. The displacement of the particle is **-20m** from the origin to the target, and the time taken to reach the target is **100 / ucos30**. The particle accelerates at **-g**ms^{-2}.

Thus,

s = -20, u = usin30, a = -g, t = 100 / ucos30 ***(this is why finding t in terms of the horizontal distance was necessary - as it means we now only have one unknown in the equation)***

Using $\mathbf{s = ut + 1/2at^2}$, we can solve for **u**:

$$s = ut + \frac{1}{2}at^2$$

$$-20 = u\sin 30\left(\frac{100}{u\cos 30}\right) + \frac{1}{2}(-g)\left(\frac{100}{u\cos 30}\right)^2$$

$$-20 = \frac{100\sin 30}{\cos 30} - \frac{g}{2}\left(\frac{10000}{u^2\cos^2 30}\right)$$

$$-77.7 = -\frac{10000g}{2u^2\cos^2 30}$$

$$155.5\,u^2\cos^2 30 = 10000g$$

$$u = \sqrt{\frac{10000g}{155.5\cos^2 30}} = 29ms^{-1}$$

So the particle must be projected at $29ms^{-1}$ to achieve a displacement of -20m such that it hits the target.

-

Example 9

3

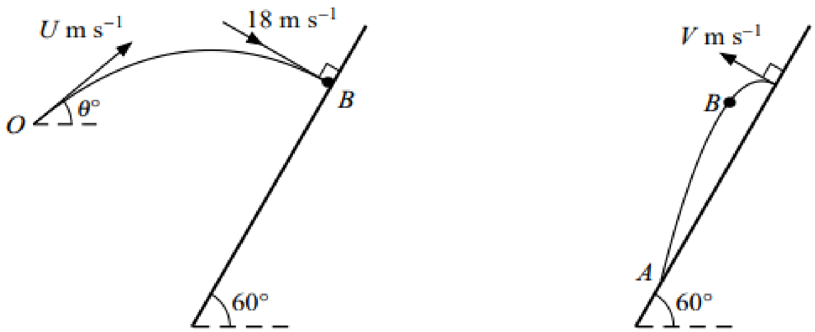


Fig. 1 **Fig. 2**

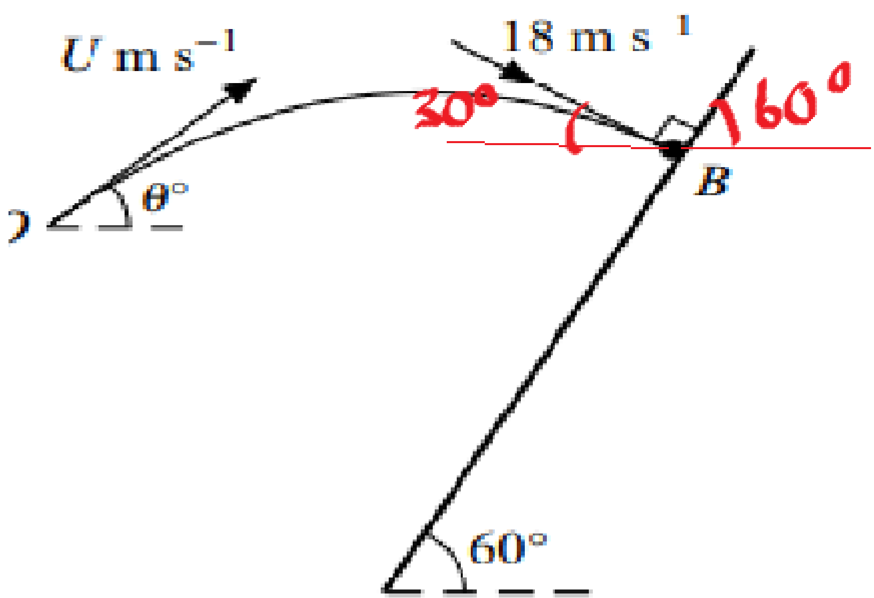
A small ball B is projected with speed $U\,\mathrm{m\,s^{-1}}$ at an angle of $\theta°$ above the horizontal from a point O. At time 2 s after the instant of projection, B strikes a smooth wall which slopes at 60° to the horizontal. The speed of B is $18\,\mathrm{m\,s^{-1}}$ and its direction of motion is perpendicular to the wall at the instant of impact (see Fig. 1). B bounces off the wall with speed $V\,\mathrm{m\,s^{-1}}$ in a direction perpendicular to the wall. At time 0.8 s after B bounces off the wall, B strikes the wall again at a lower point A (see Fig. 2).

(i) Find U and θ. [5]

(ii) By considering the motion of B after it bounces off the wall, calculate V. [4]

i)

Let us first determine the angle at which the ball hits the wall:



The ball hits the wall at an angle of 30° above the horizontal.

This means that the horizontal velocity component of the ball when it hits the wall is **18cos30**.

The horizontal velocity from the initial motion is **UcosΘ**.

This remains constant throughout the motion, so **UcosΘ = 18cos30 => U = 18cos30 / cosΘ**

This is one equation that we can use at a later stage.

-

We also know from the above diagram that the final vertical velocity is **18sin30**.

The initial vertical velocity is **UsinΘ**, and the ball is accelerating at **-9.81**ms^{-2} in the vertical plane for 2s before it hits the wall.

Thus, **u = UsinΘ, v = -18sin30 = -9**, **t = 2**, **a = -g**.

v = u + at

-9 = UsinΘ - 2g

UsinΘ = -9 + 2g => U = (-9+2g) / sinΘ

We now have two equations that express **U**, **U = (9+2g) / sinΘ** and **U = 18cos30 / cosΘ**, so we can equate the equations to find the value of **Θ**, and hence the value of **U**:

$$\frac{-9+2g}{\sin\theta} = \frac{18\cos 30}{\cos\theta}$$

$$\sin\theta(18\cos 30) = \cos\theta(-9+2g)$$

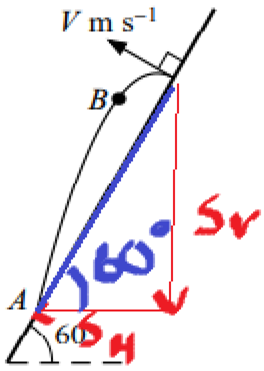
$$\tan\theta = \frac{-9+2g}{18\cos 30}$$

$$\theta = \tan^{-1}\left(\frac{-9+2g}{18\cos 30}\right) = 34.3^{\circ}$$

Thus, **U = 18cos30 / cos34.3 = 18.9ms^{-1}.**

ii)

When the ball bounces off the wall, it travels a certain horizontal distance (let this be **S_H**) and a certain vertical distance (**S_V**), all at an angle of **60°**:

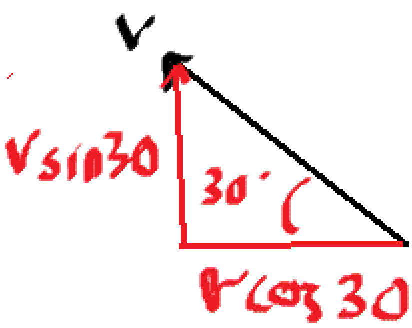


We can say that **tan60 = S_v / S_H.**

We need to determine a value for **S_V** and **S_H** in terms of **V**.

-

Firstly, **V** can be split into horizontal and vertical components. These must be at an angle of 30° since the ball bounces off in the same direction that it was incident in, givign a horizontal component of **Vcos30**, and a vertical component of **Vsin30**:



Vertical distance:

If we take downwards as the positive direction:

s = $\mathbf{S_V}$, u = **-Vsin30** (particle is initially projected upwards, so -ve), a = **g**, t = **0.8**

$\mathbf{S_V = -Vsin30(0.8) + 1/2(-g)(0.8^2) = -0.4V + 0.32g}$

Horizontal distance:

The horizontal velocity is constant since no force acts in the horizontal plane.

Thus, $\mathbf{S_H = Vcos30 * 0.8}$

-

tan60 = $\mathbf{S_v / S_H}$

We can now solve for V:

$$\tan 60 = \frac{S_V}{S_H}$$

$$\tan 60 = \frac{-0.4V + 0.32g}{V\cos 30 \times 0.8}$$

$$\tan 60(V\cos 30 \times 0.8) = -0.4V + 0.32g$$

$$1.2V = -0.4V + 0.32g$$

$$V = \frac{0.32g}{1.6} = 2.0\ ms^{-1}$$

-

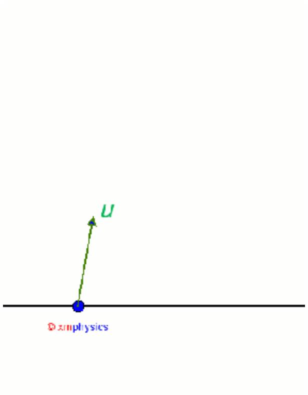
Example 10

A particle P is projected with speed V m s^{-1} at an angle of 60° above the horizontal from a point O on horizontal ground. P is moving at an angle of 45° above the horizontal at the instant 1.5 s after projection.

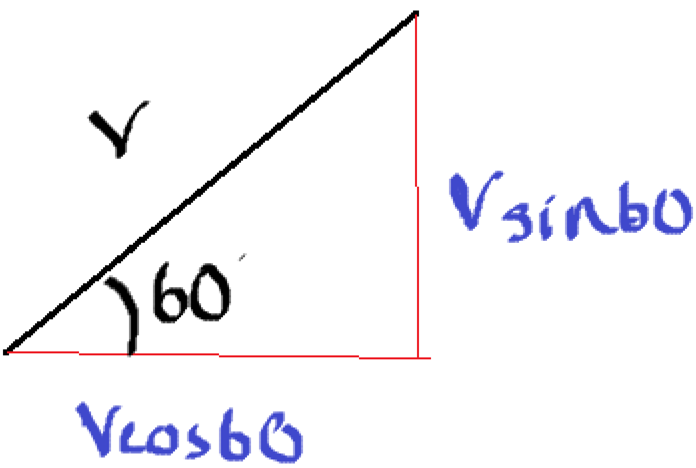
(i) Find V. [3]

(ii) Hence calculate the horizontal and vertical displacements of P from O at the instant 1.5 s after projection. [2]

As the particle progresses closer to the apex, the angle made with the horizontal decreases, which is why the angle has reduced to 45° 1.5s after motion.



We can first consider the initial horizontal and vertical components of the particle's velocity;



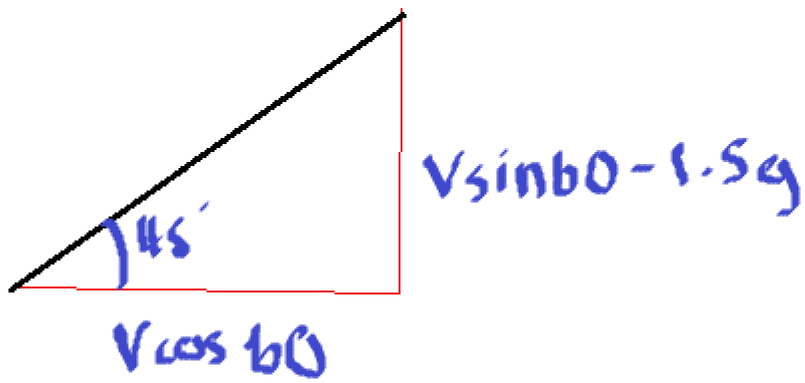
Now let us determine an expression for the vertical velocity of the particle after 1.5s:

u = Vsin60, v = v, a = -g, t = 1.5

v = u + at => v = Vsin60 - 1.5g.

The horizontal component of the velocity, **Vcos60**, remains constant since there is no horizontal acceleration.

Thus, at time t=1.5, the particle has the following velocity components, at an angle of 45° to each other:



Using trigonometry, **tan45 = (Vsin60-1.5g) / Vcos60**.

Now we have an expression that allows us to find **V**:

$$V\cos 60(\tan 45) = V\sin 60 - 1.5g$$

$$V\cos 60 = V\sin 60 - 1.5g$$

$$1.5g = V\sin 60 - V\cos 60$$

$$1.5g = V(\sin 60 - \cos 60)$$

$$V = \frac{1.5g}{\sin 60 - \cos 60} = 40.2ms^{-1}$$

ii)

If V=40.2ms^{-1}, the horizontal component of velocity is **40.2cos60 = 20.1ms^{-1}.**

Over a time t of 1.5s, the horizontal displacement is therefore **20.1 * 1.5 = 30.15m.**

-

The initial vertical velocity component must be **40.2sin60 = 34.8ms^{-1}.**

Thus, u = 34.8, a = -g, t = 1.5, s = s

s = ut + 1/2at^2 = 34.8(1.5) + 1/2(-g)(1.5^2) = **41.2m**.

-

1.3 - Dynamics

Momentum

Momentum introduction

- Momentum is a way of quantifying the motion of a body.

The momentum of a body is found by multiplying its **mass** (m) by its **velocity** (v). Momentum is given the symbol **p**.

Momentum = mass * velocity

p = mv

Units = kg * ms-1 = kgms-1

Suppose we have a 10kg body moving at $10ms^{-1}$ and a 100kg body moving at $10ms^{-1}$.

Clearly, the 100kg body will require a greater force over the same time to bring it to rest; in other words, it has a higher momentum.

- The momentum of the 10kg body is: **10 * 10 = $100kgms^{-1}$**
- The momentum of the 100kg body is: **10 * 100 = $1000kgms^{-1}$**

Momentum can therefore be thought of as how difficult an object is to bring to rest.

A body with a higher momentum requires either:

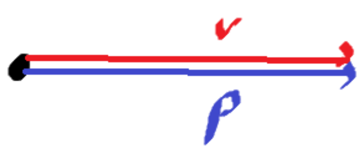
- The same force applied over a longer time to bring it to rest in comparison to a body with lower momentum
- A greater force over the same time to bring it to rest in comparison to a body with higher momentum

We previously said that according to *Newton's First Law*, a body with a higher mass has a higher resistance to a change in motion - i.e. a higher *inertia*. This is the same with momentum; a body with a higher momentum requires a stronger force, or a force applied over a longer time, to change its motion.

-

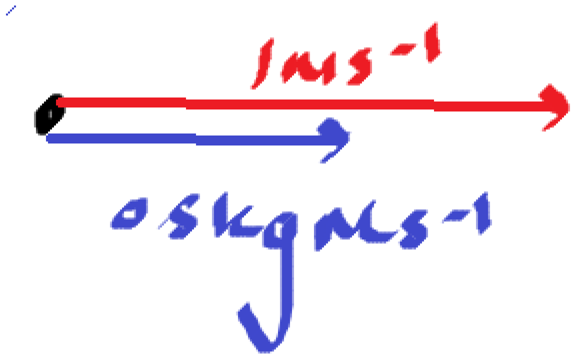
Momentum is a vector quantity - since velocity occurs in a certain direction, and momentum is the product of the mass and velocity of a body.

The momentum of a body is in the same direction as its velocity:



However, the magnitude of a body's momentum and its velocity do not always have to be equal.

For example, if a 0.5kg body is moving at $1ms^{-1}$, its velocity is $1ms^{-1}$, and its momentum is 1*0.5 = $0.5kgms^{-1}$; so its momentum is half its velocity:



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The relation between force and momentum

If a body has a change in velocity, this requires an acceleration. To produce an acceleration, a force must be exerted.

According to *Newton's second law*, **F = ma**.

Acceleration is the rate of change of velocity - **a = Δv / t**.

We can therefore say that: **F = (mΔv) / t**.

-

If a body of constant mass has a change in velocity, it must have a change in momentum.

For example, suppose a 10kg body changes its velocity from $10ms^{-1}$ to $5ms^{-1}$.

Its initial momentum is **10(10) = $100kgms^{-1}$**.

Its final momentum is **10(5) = $50kgms^{-1}$**.

This means that its *change in momentum* is **$-50kgms^{-1}$**. As a single calculation, we can say:

Δp = 10(5) - 10(10) = 10(5-10).

Thus, if a particle of mass **m** has an initial velocity of **u** and a final velocity of **v**, we can say the change in momentum, **Δp** is given by: **Δp = m(v-u)**.

-

Returning to **F = (mΔv) / t**, the change in velocity is the final velocity, **v**, subtracted by the initial velocity, **u** -> **Δv = v - u**

Thus, **F = m(v-u) / t**

We previously showed that **Δp = m(v-u)**, so we can say that **F = Δp / t** => force is the change in momentum over the time (the **rate of change of momentum**)

This is another way of defining Newton's second law.

-

Now let us consider how **F = Δp / t** applies in context.

Firstly, a body will only accelerate when a net force is being applied to it; if the net force goes away completely, it will stop accelerating.

Suppose a 6kg body accelerates from $10ms^{-1}$ to $20ms^{-1}$ over 5s.

The acceleration of the body is **(20-10) / 5 =** $\mathbf{2ms^{-2}}$. The force required for a **6kg** body to accelerate at $\mathbf{2ms^{-2}}$ is **6 * 2 = 12N**.

We can also interpret this in terms of a change in momentum:

- The initial momentum of the body is **6 * 10 =** $\mathbf{60kgms^{-1}}$.
- When the body has accelerated to $20ms^{-1}$, its momentum is now **6 * 20 =** $\mathbf{120kgms^{-1}}$.

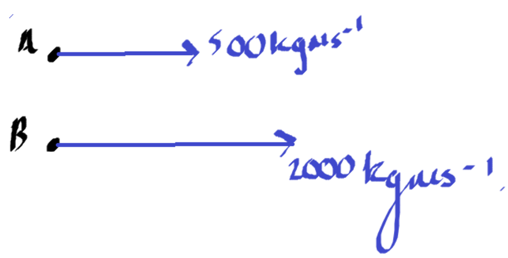
- Its change in momentum, **Δp**, is therefore **120 - 60 =** $\mathbf{60kgms^{-1}}$.
- If a change in momentum of $60kgms^{-1}$ occurs over 5s, the net force applied according to **F = Δp / t** must be **60 / 5 = 12N**.

Suppose that the **12N** net force goes away after the 5s acceleration period. The body will no longer continue to accelerate because it no longer has a force being applied to it. It will move off at a constant velocity of $\mathbf{20ms^{-1}}$.

--

As mentioned previously, a body with a greater momentum requires a greater force to stop over the same period of time than a body with a lower momentum.

For example, suppose we have two bodies:



- Body **A** has a mass of 50kg and a velocity of $10ms^{-1}$.

- Body **B** has a mass of 200kg and a velocity of $10ms^{-1}$.

Obviously, body **B** will require a greater force to bring it to rest in the same time as body **A**.

First, let us consider the force required to bring the bodies to rest in **10 seconds** in terms of **F=ma**.

To bring a body moving at $10ms^{-1}$ to rest in 10 seconds, the body must decelerate at **$1ms^{-2}$**.

Thus,

- The force required to bring body **A** to rest is**:** m * a = 50 * 1 = **50N**.

- The force required to bring body **B** to rest is**:** m * a = 200 * 1 = **200N**.

We can also consider this in terms of the change of momentum of the bodies.

- For body **A**: **Δp = (0 * 50) - (50 * 10) = -500$kgms^{-1}$**. Thus, F = 500 / 10 = **50N**.
- For body **B**: **Δp = (0 * 200) - (10 * 200) = -2000$kgms^{-1}$**. Thus, F = 2000 / 10 = **200N**.

This shows that a more massive body requires a greater force to bring it to rest from the same velocity as a less massive body in the same time.

Suppose a force of 100N is applied to body **A** and body **B**. It would take more time to decelerate body B to rest because it has more momentum:

F = Δp / t = **t = Δp / F**

For body **A**: **t = 500 / 100 = 5s**
For body **B**: **t = 2000 / 100 = 20s**.

Because body **B** is four times more massive than body **A**, it takes four times longer to bring it to rest with the same force from the same initial velocity.

-

As mentioned, a body only accelerates when a force is being applied directly to it; when the force goes away, the acceleration created by that force instantly stops.

Example: Suppose a golf club strikes a 0.05kg golf ball, causing it to accelerate from $0ms^{-1}$ to $50ms^{-1}$, and the golf ball and club are in contact for 3ms. **What is the force exerted on the golf ball?**

The golf ball is only accelerating for the period of time in which the golf ball and club are in contact with each other.

The initial momentum of the golf ball is $0kgms^{-1}$ since it is stationary. Its final momentum is $0.05(50) = 2.5kgms^{-1}$. The golf ball has a change in momentum of **$2.5kgms^{-1}$**.

The time over which this change in momentum occurs is $3ms = 3*10^{-3}s$.

Thus, according to **$F = \Delta p / t$**, the force exerted on the golf ball must be **$2.5 / (3*10^{-3}) = 833N$**.

Once the golf ball and club are no longer in contact, the force goes away and the acceleration stops.

-

Newton's second law in terms of momentum is used widely in safety.

Since **$F = \Delta p / t$**, if a body is in contact with another body for more time, a lower force is exerted.

For example, suppose an egg is dropped onto a) a hard floor, and b) a sponge.

When an egg is dropped onto a hard floor, the floor and the egg are in-contact for a very short period of time, so a large force is exerted.

When an egg is dropped on a sponge, on the other hand, the egg and sponge are in contact for a longer period of time, so a lower force is exerted overall. This makes the egg less likely to crack.

/

Applying this concept to cars, many cars use airbags. When an airbag is deployed, the airbag significantly increases the time of collision, so a lower force is generated for the same change in momentum.

Similarly, many cars have crumple zones. Crumple zones are a hollow space that increase the time of collision, therefore leading to a lower force.

-

So, we can conclude that:

- If the same change in momentum (so the same change in velocity for a body of constant mass) occurs over less time, a greater force is exerted.
- A greater force is required to bring a more massive body to rest from the same velocity as a less massive body because there must be a greater reduction in momentum.

Conservation of momentum and collisions

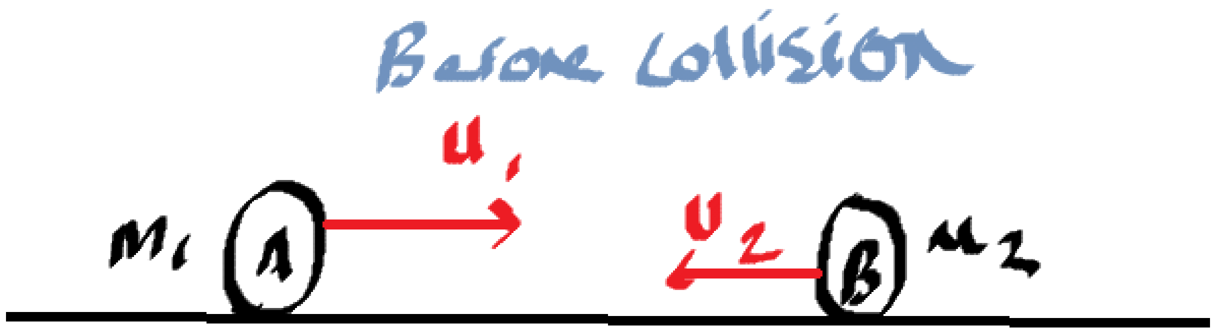
The law of **conservation of momentum** states that the **total momentum** both before and after a collision must be the same **provided that there are no external forces**.

-

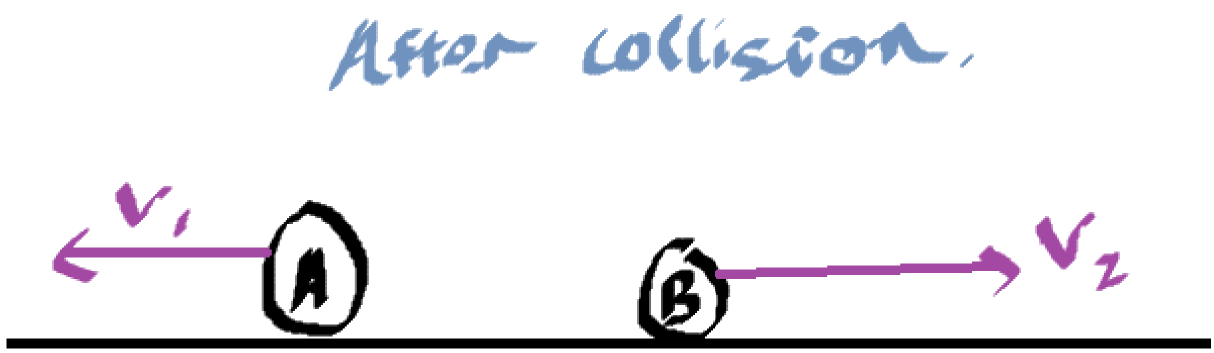
Suppose we have two bodies, **A** and **B**, that collide.

Let **A** have a mass of m_1 and a velocity before the collision of u_1.

Let **B** have a mass of m_2 and a velocity before the collision of u_2.



When **A** and **B** collide, A will move off to the left, and B will move off to the right. Suppose A moves off with a velocity of v_1 and B with a velocity of v_2.



According to the law of the conservation of momentum, the total momentum before the collision must equal the total momentum after the collision.

The total momentum before the collision is:

$\mathbf{p_{before} = p_{A\ before} + p_{B\ before} = m_1u_1 + m_2u_2}$

The total momentum after the collision is:

$\mathbf{p_{after} = p_{A\ after} + p_{B\ after} = m_1v_1 + m_2v_2}$

We can say $\mathbf{p_{before} = p_{after}}$, so: $\mathbf{m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2}$

-

We can also consider this in terms of the change in momentum of A (let this be $\mathbf{\Delta p_A}$) and the change in momentum of B (let this be $\mathbf{\Delta p_B}$).

The change in momentum of A is the final momentum of A subtracted by the initial momentum of A: $\mathbf{\Delta p_A = m_1v_1 - m_1u_1}$

The change in momentum of B is the final momentum of B subtracted by the initial momentum of B: $\mathbf{\Delta p_B = m_2v_2 - m_2u_2}$.

From $\mathbf{m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2}$, we can rearrange this to say:

$\mathbf{m_1v_1 - m_1u_1 = - m_2v_2 + m_2u_2}$

$\mathbf{m_1v_1 - m_1u_1 = - (m_2v_2 - m_2u_2)}$

Thus, $\mathbf{\Delta p_A = - \Delta p_B}$

Proof of conservation of momentum

We are now going to prove the idea of conservation of momentum by deriving the formula, $\mathbf{\Delta p_A = - \Delta p_B}$.

Suppose two particles, **A** and **B**, collide with each other. A will exert a force on B (let this be $\mathbf{F_{AB}}$). According to N3L, B will exert an equal and opposite force on **A** (let this be $\mathbf{F_{BA}}$).

Because the forces are equal and opposite, and force is a vector, we can say: $\mathbf{F_{AB} = -F_{BA}}$.

As mentioned previously (*in the **relation between force and momentum***), when a body has a force exerted on it, it causes an acceleration, which causes a change in velocity, and therefore a change in momentum.

Since $\mathbf{F = \Delta p / t}$, $\mathbf{\Delta p = Ft}$.

The change in momentum of **A** during the collision (**Δp_A**) must be the force exerted on A, **F_{BA}**, over the time of collision:

$\Delta p_A = F_{BA} * t$, so **$F_{BA} = \Delta p_A / t$**

The change in momentum of **B** during the collision (**Δp_B**) must be the force exerted on B, **F_{AB}**, over the time of collision:

$\Delta p_B = F_{AB} * t$, so **$F_{AB} = \Delta p_B / t$**

Since **$F_{AB} = -F_{BA}$, $\Delta p_B / t = - \Delta p_A / t$**.

The time of collision must be the same for both bodies, so we can cancel **t** to say:

$\Delta p_B = - \Delta p_A$ or **$\Delta p_A = - \Delta p_B$**

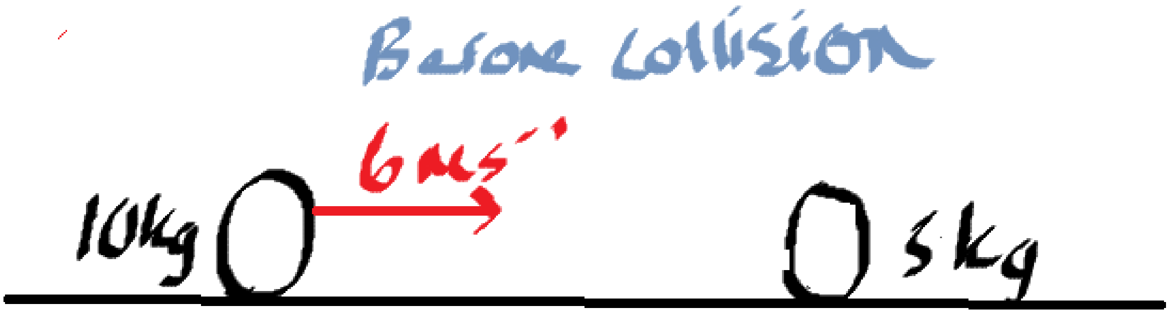
This is what we derived previously when discussing conservation of momentum, so we have therefore proved the concept using Newton's Third and Second Laws.

-

Now let us consider examples of conservation of momentum calculations.

Example 1

A 10kg ball is rolling across a smooth table at $6ms^{-1}$ when it hits a stationary 5kg ball. This causes the 5kg ball to move in the original direction as the 10kg ball with a velocity of $8ms^{-1}$. The 10kg ball moves off at vms^{-1}. a) **Find v.**



It is first important that we define a positive and negative direction, because momentum is a vector. In this example, we will let the RHS be positive; so R(->).

The total momentum before the collision must equal the total momentum after the collision.

The total momentum before is: **10(6) + 5(0) = 60**. (the 5kg ball has no momentum since it is stationary).

The total momentum after is: **10(v) + 5(8) = 10v + 40**

Thus, **60 = 10v + 40 => v = (60-40) / 10 = 2ms^{-1}.**

This velocity is positive, so the 10kg ball must have its velocity reduced 6ms^{-1} to 2ms^{-1}, and it remains moving in its original direction.

-

The balls are in-contact for 10ms. *b)* ***Find the force exerted by the 10kg ball on the 5kg ball.***

The 5kg ball accelerates from $0ms^{-1}$ to $8ms^{-1}$ in 10ms.

This is an acceleration of **8 / ($10*10^{-3}$) = $800ms^{-2}$**. The force required for a 5kg ball to accelerate at $800ms^{-2}$ is **5 * 800 = 4000N**.

We can also find F by considering $F = \Delta p / t$.

The 5kg ball has a change in momentum of **8(5) = $40kgms^{-1}$**.

The force required to produce a change in momentum of $40kgms^{-1}$ over 10ms is **40 / ($10*10^{-3}$) = 4000N**.

So the 5kg ball has a 4000N force exerted on it by the 10kg ball.

/

The 10kg ball must also have a 4000N force exerted on it according to N3L. We can prove this by considering the change in momentum of the 10kg ball:

The 10kg ball decelerates from **$6ms^{-1}$** to **$2ms^{-1}$** in **10ms**; so the force exerted on it is:

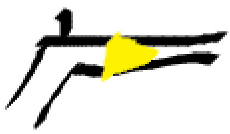
$F = 10(2-6) / (10*10^{-3}) = -4000N$.

This is equal and opposite to the force exerted on the 5kg ball.

Example 2

A rifle of mass 12kg fires a 0.005kg bullet at $450ms^{-1}$. **Calculate the recoil velocity of the rifle**.

Initially, the rifle and bullet are stationary, so they have a total momentum of $0kgms^{-1}$.



This means that the total momentum after the firing must add to $0kgms^{-1}$.

As a result of this, if the bullet moves off with a positive velocity, the gun must recoil back with a negative velocity, because this allows the total momentums of the gun and bullet to cancel out to $0kgms^{-1}$. This is why a gun recoils.

If we let the velocity of the rifle after the firing be **vms^{-1}**, we can say:

$p_{bullet} + p_{rifle} = 0$

$(0.005 * 450) + (12v) = 0$

$12v = -2.25$

$v =$ **$-0.19ms^{-1}$.**

Example 3

A train wagon of mass 1200kg collides with a stationary train wagon of mass 800kg. After the collision, the two wagons become connected, and move off with a velocity of $12ms^{-1}$.
Find the initial velocity of the 1200kg train wagon.

If we let the initial velocity of the 1200kg train wagon be **u**ms^{-1}, the total momentum before the collision is: **1200u**.

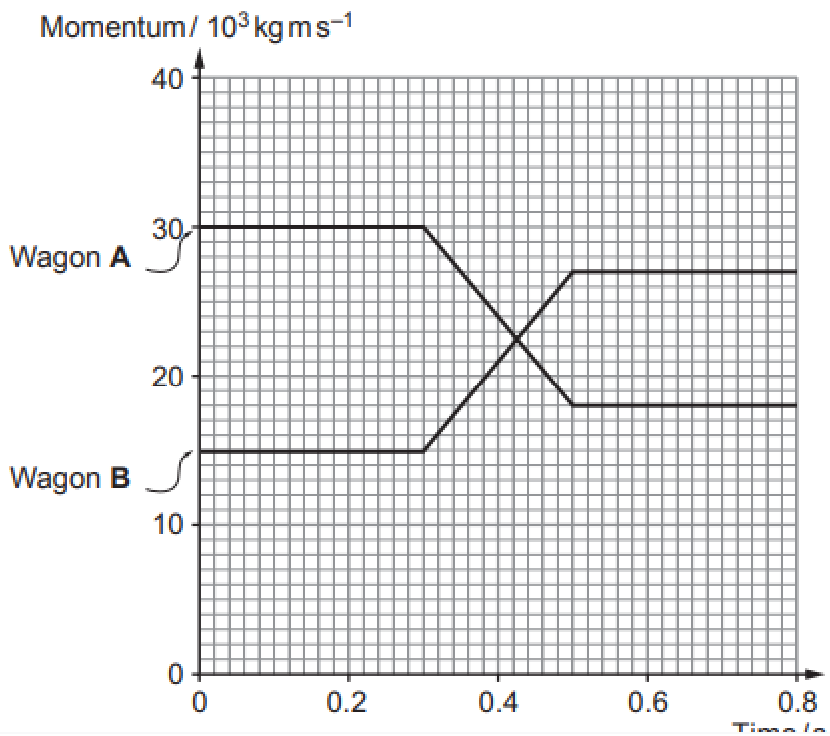
The total momentum after the collision can be expressed as $\mathbf{m_1v_1 + m_2v_2}$. Since the wagons become connected after the collision, they must have equal velocities, thus final momentum = $\mathbf{v(m_1+m_2)}$. Using the figures provided, $\mathbf{p_{after} = 12(800+1200) = 24000kgms^{-1}}$.

Thus, since $\mathbf{p_{before} = p_{after}}$, **1200u = 24000 => u = $20ms^{-1}$.**

-

Example 4

The graph shows how the momentum of two colliding railway wagons (**A** and **B**) varies with time. The collision takes place between 0.30 s and 0.50 s as shown. The wagons remain joined together after impact.



The principle of conservation of momentum states that momentum is conserved in a collision *provided there are no external forces*.

The total momentum of the wagons before the collision is: **($30*10^3$) + ($15*10^3$) = 45000kgms^{-1}.**

The total momentum after the collision is: (**$18*10^3$) + ($27*10^3$) = 45000kgms^{-1}.**

No momentum is lost in the collision, so there cannot be any non-negligible external forces.

(ii) Calculate the final velocity of the two wagons given that the total mass of wagon **A** and wagon **B** is 25000 kg. [3]

The final momentum is **45000kgms^{-1}**.

This must be the sum of the final momentums of both wagons. If both of the wagons move off with the same velocity, we can say:

4500 = v(25000)

v = 45000 / 25000 = 1.8ms^{-1}.

Determine from the graph the resultant force on wagon **A** during the collision. [3]

Wagon A has an initial momentum of 30000kgms^{-1} and a final momentum of 18000kgms^{-1}, so this equates to a change in motion of **-12000kgms^{-1}**.

This change in momentum occurs over 0.20s, so **F = -12000 / 0.20 = -60000N**.

The force experienced by wagon **B** during the collision is equal and opposite to the force experienced by wagon **A**. State which law of motion this is an example of **and** explain how the graph confirms this law. [2]

This is an example of Newton's third law of motion.

The graph confirms this because B has a gain in momentum of **12000kgms^{-1}**, while A has a loss in momentum of **12000kgms^{-1}**. The wagons have an equal (but opposite) change in momentum over the same period of time, so the same force must be exerted.

Example 5

(b) (i) A head-on inelastic collision occurs between a neutron and a lithium atom, ${}^{6}_{3}$Li.

The nucleus of the atom absorbs the neutron, to form the heavier isotope ${}^{7}_{3}$Li. Using the data in the diagram, calculate the *velocity* of the ${}^{7}_{3}$Li atom, adding an arrow to the (right hand) diagram, to show its direction of motion. [4]

Taking right as positive:

The total momentum before the collision is: **3150(1.67*10^{-27}) + -225(9.98*10^{-27}) = 3.02*10^{-24}kgms^{-1}**.

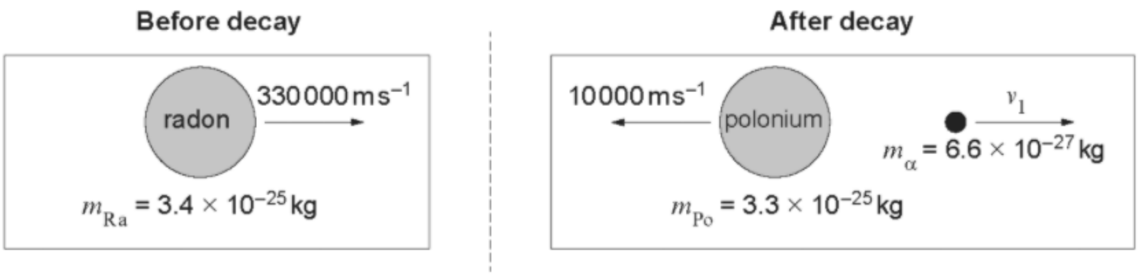
The total momentum after the collision is **v(11.6*10^{-27})** where **v** is the velocity of the lithium atom.

Thus, **3.02*10^{-24} = v(11.6*10^{-27})**

v = (3.02*10^{-24}) / (11.6*10^{-27}) = 260ms^{-1}

Example 6

A radon nucleus travelling at 330 000 ms^{-1} decays to produce a polonium nucleus and an alpha particle as shown.



(a) Use the principle of conservation of momentum to calculate the velocity (v_1) of the alpha particle. [3]

p_{before} = p_{after}

Taking right as positive, R(->):

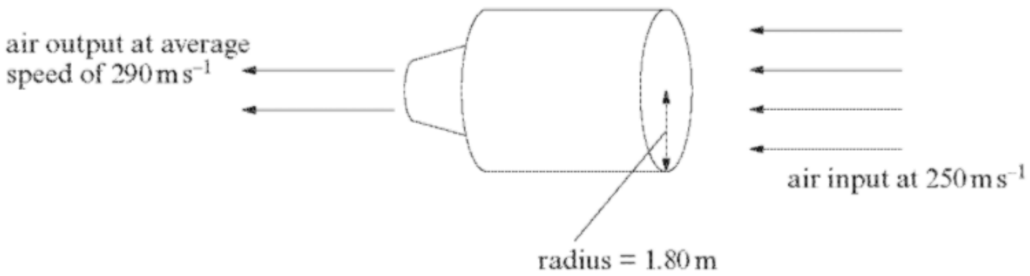
330000(3.4*10^{-25}) = -10000(3.3*10^{-25}) + v_1(6.6*10^{-27})

1.12*10^{-19} = -3.3*10^{-21} + 6.6*10^{-27}v_1

v_1 = (1.12*10^{-19} + 3.3*10^{-21}) / (6.6*10^{-27}) = **1.75*10^{7}ms^{-1}**

Example 7

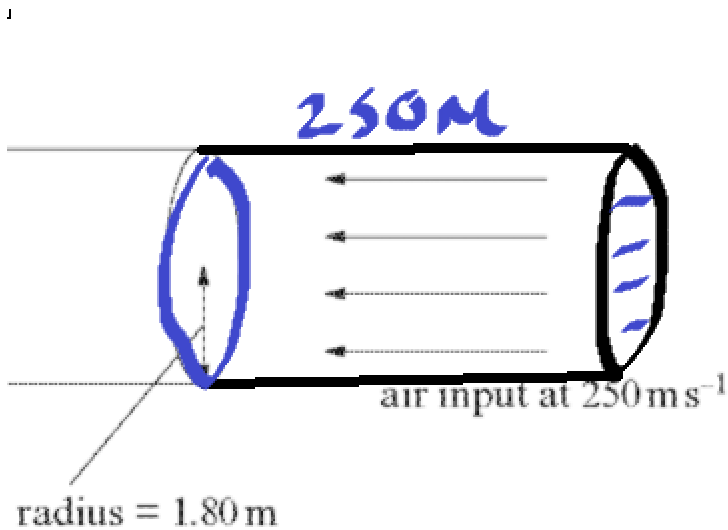
A Rolls Royce jet engine operates by collecting air into the jet engine at a speed of $250\,m\,s^{-1}$ and ejecting it with an average speed of $290\,m\,s^{-1}$.



(a) The radius of the jet engine is 1.80 m as shown and the density of air entering it is $0.4\,kg\,m^{-3}$. Show that the mass of air entering the jet engine per second is approximately 1 000 kg. [3]

Density = mass / volume.

The volume of air entering the engine each second is a cylinder of radius 1.80m and length 250m:



The volume of this cylinder is: $\pi(1.80^2) * 250 = 810\pi\ m^3$.

So if $810\pi\ m^3$ of air of density $0.4 kgm^{-3}$ enters the engine each second, the mass of air entering is: $810\pi * 0.4 =$ **1020kg** (approximately 1000kg).

(b) Calculate the forward thrust produced by this jet engine. [2]

Each second, 1020kg of air is accelerated from $250ms^{-1}$ to $290ms^{-1}$ by the engine.

This acceleration of the air is therefore **(290-250) / 1 = 40ms^{-2}**.

The force required to cause 1020kg of air to accelerate at 40ms^{-2} is **1020 * 40 = 40800N.**

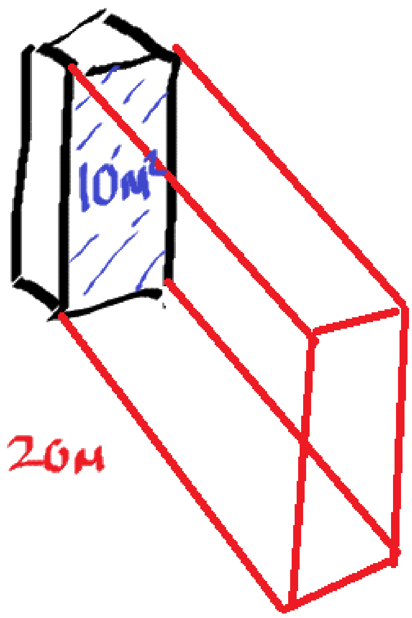
The air will exert an equal and opposite force of 40800N on the airplane, allowing it to be pushed forwards (i.e. thrust).

-

Example 8

> A wind blows steadily against a tree. The area of the tree perpendicular to the direction of the wind is 10 m^2 and the velocity of the wind is 20 m s^{-1}.

Find the magnitude of the force exerted on the tree by the wind assuming that all of the air is stopped by the tree (density of air at this altitude = 1.23kgm^{-3})



Each second, 20m of air hits a 10m^2 section of the tree.

Thus, the volume of air hitting the tree per second is **10(20) = 200m^3.**

This has a mass of **200 * 1.23 = 246kg.**

If the air is completely stopped by the tree, this means that 246kg of air is having its velocity reduced from 20ms^{-1} to 0ms^{-1} per second.

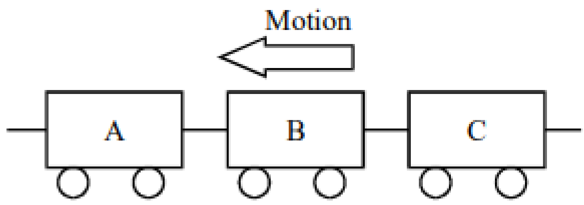
The air is thus being decelerated at: **20 - 0 / 1 = 20ms^{-2}.**

This requires a force of magnitude = **20(246) = 4920N**.

-

Example 9

The diagram shows three trucks which are part of a train. The mass of each truck is 84 000 kg.



The train accelerates uniformly in the direction shown from rest to 16 m s^{-1} in a time of 4.0 minutes. Calculate the resultant force on each truck.

Each truck is accelerating at a rate of: **(16-0) / (4(60)) = 1/15 ms^{-2}.**

To accelerate a truck of mass 84000kg at 1/15ms^{-2}, a force of **84000(1/15) = 5600N** is required.

The force exerted by truck B on truck C is 11 200 N. Draw a free-body force diagram for truck B, showing the magnitudes of all the forces. Neglect any frictional forces on the trucks.

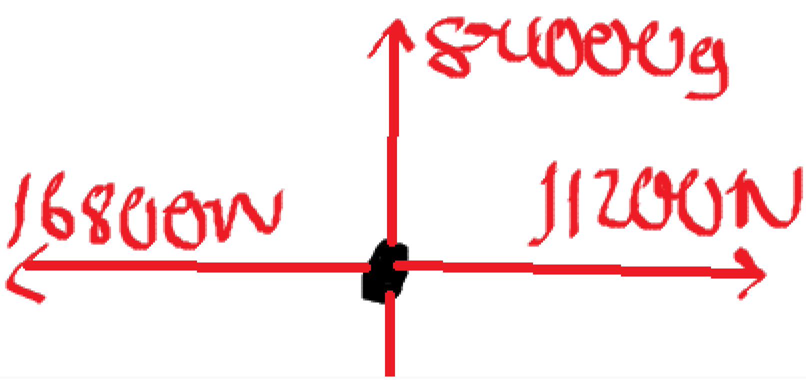
If truck B exerts an 11200 N force on truck C (tension), truck C exerts an equal and opposite force on truck B (opposite to direction of motion).

The resultant force on truck B is 5600N in the direction of travel. Therefore, there must be a force acting in the direction of motion on B that is greater than the tension force exerted by C; let this be **F**.

Thus, **F - 11200 = 5600 => F = 16800N.**

The truck therefore has 4 forces acting on it:

- A 16800N force in the direction of motion
- A 11200N force against the direction of motion
- A weight of 84000g
- A reaction force of 84000g (equal to weight since there is no motion in the vertical plane)



Example 10

A solar-powered ion propulsion engine creates and accelerates xenon ions. The ions are ejected at a constant rate from the rear of a spacecraft, as shown in Fig. 2.1.
The ions have a fixed mean speed of $3.2 \times 10^4\,m\,s^{-1}$ relative to the spacecraft.
The initial mass of the spacecraft is 5.2×10^3 kg.

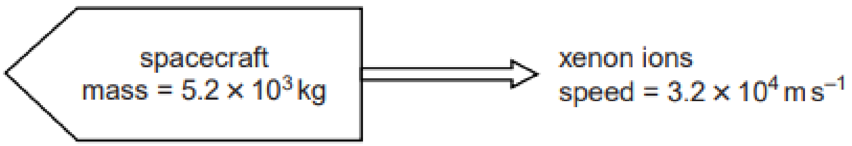


Fig. 2.1

[Mass of xenon ion = $2.18*10^{-25}$kg]

The engine is designed to eject 9.5×10^{18} xenon ions per second.
Determine the initial acceleration of the spacecraft.

If the initial momentum is $0kgms^{-1}$, the xenon ions are being ejected in one direction, so for momentum to be conserved, the spacecraft must accelerate in the other direction - which allows it to be propelled forwards.

The total mass of xenon ions ejected from the spacecraft per second is **$(9.5*10^{18})(2.18*10^{-25})$ = $2.07*10^{-6}$kg**.

Suppose the spacecraft has a velocity of $0ms^{-1}$ before it has ejected any xenon ions. This means that the total momentum of the system is $0kgms^{-1}$.

This must be equal to the total momentum after **$2.07*10^{-6}$kg** of xenon is ejected from the jet at $3.2*10^4ms^{-1}$.

We can therefore say:

$(5.2*10^3 * v) + (2.07*10^{-6} * 3.2*10^4) = 0$ *where **v** is the velocity of the spacecraft after ejection.*

Solving for **v**:

$v = -(2.07*10^{-6} * 3.2*10^4) / (5.2*10^3) = -1.3*10^{-5}ms^{-1}$.

This means that the spacecraft speeds up from $0ms^{-1}$ to $1.3*10^{-5}ms^{-1}$ in 1s, so its acceleration is **$1.3*10^{-5}ms^{-2}$.**

Example 11

A tennis ball is hit by a racket as shown in Fig. 1.2.

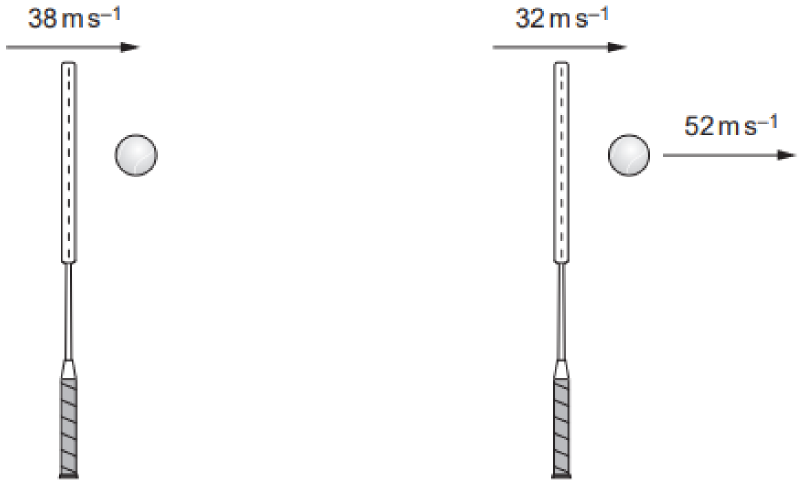


Fig. 1.2a **Fig. 1.2b**

Fig. 1.2

The mass of a tennis ball is 0.058 kg. During a serve the racket head and the ball are in contact for 4.2 ms. Just before contact, the racket head is travelling towards the ball at 38 m s^{-1} and the ball is stationary. Fig.1.2a shows the situation just before contact. Immediately after contact, the racket head is travelling in the same direction at 32 m s^{-1} and the ball is travelling away from the racket at 52 m s^{-1}. This is shown in Fig. 1.2b.

(i) Calculate the mean force provided by the racket on the ball.

The tennis ball accelerates from 0ms^{-1} to 52ms^{-1} in 4.2ms, so the acceleration of the tennis ball is **52 / (4.2*10^{-3}) = 12400ms^{-2}.**

The force required for a 0.058kg body to accelerate at 12400ms^{-2} is **12400(0.058) = 720N.**

(ii) Estimate the mass of the racket.

Using conservation of momentum,

$p_{before} = p_{after}$

Let the mass of the racket be **m**.

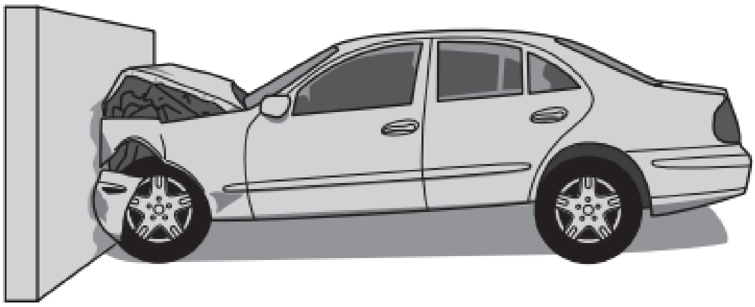
38m = 32m + 52(0.058)

6m = 3.016

m = 0.50kg.

--

The crumple zone of a car is a hollow structure at the front of the car designed to collapse during a collision. In a laboratory road-test, a car of mass 850 kg was driven into a concrete wall. A video recording of the impact showed that the car, initially travelling at 7.5 m s^{-1}, was brought to rest in 0.28 s when it hit the wall.



(i) Calculate the average force exerted by the wall on the car.

The car decelerates from 7.5ms^{-1} to 0ms^{-1} in 0.28s, so its deceleration is **7.5 / 0.28 = 26.8ms^{-2}**.

For an 850kg car to decelerate at **26.8ms^{-2}**, a force of **850(26.8) = 23000N** is required.

In a different test, another car of mass 850 kg is travelling at a speed of 7.5 m s^{-1}. It makes a head-on collision with a stationary car of mass 1200 kg. Immediately after the impact, both cars move off together with a common speed *v*. Calculate this speed.

850(7.5) = v(850+1200)

v = (850 * 7.5) / (850 + 1200) = **3.1ms^{-1}.**

-

Example 12

Fig. 1.2 shows a helicopter viewed from above.

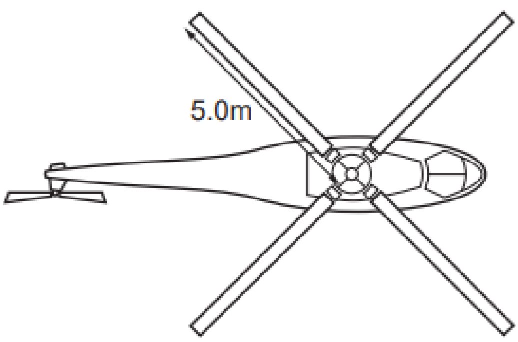


Fig. 1.2

The blades of the helicopter rotate in a circle of radius 5.0 m. When the helicopter is hovering, the blades propel air vertically downwards with a constant speed of 12 m s^{-1}. Assume that the descending air occupies a uniform cylinder of radius 5.0 m.

The density of air is 1.3 kg m^{-3}.

(i) Show that the mass of air propelled downwards in a time of 5.0 seconds is about 6000 kg.

Each second, the propellers propel 12m of air downwards in a cylinder shape.

The volume of this cylinder is **$\Pi(5^2)$ * 12 = 940m^3.**

Thus, the mass of air being propelled downwards per second is **940 * 1.3 = 1220kg.**

Thus, over 5 seconds, the mass of air propelled downwards is **5(1220) = 6100kg** - which is approximately **6000kg**.

(ii) Calculate the force provided by the rotating helicopter blades to propel this air downwards

The change in momentum of the **6100kg** column of air (assuming it is initially stationary) is **6100(12) = 73200kgms^{-1}.**

This change in momentum occurs of 5s, thus the force required is **73200 / 5 = 14500N.**

(iii) Calculate the mass of the hovering helicopter

The air must exert an equal and opposite force of 14500N in the upwards direction on the aircraft according to N3L.

When the aircraft is hovering, it is in equilibrium, so the weight of the aircraft is equal to the upwards thrust force.

Thus, weight = **14500N**.

Weight = mg => m = weight / g = 14500 / 9.81 = 1480kg.

-

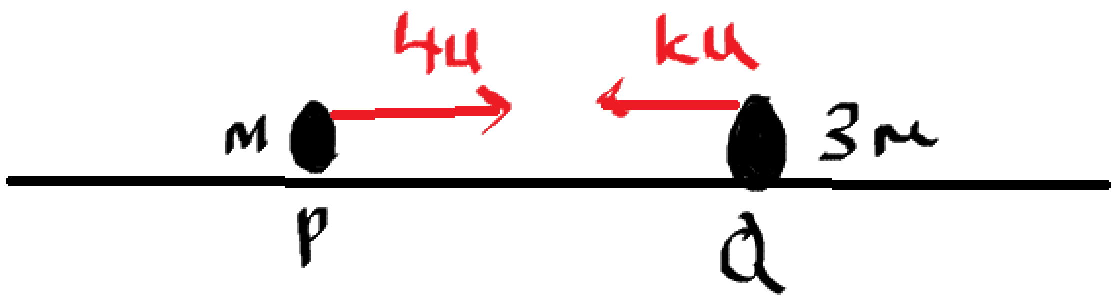
Example 13

Particle P has mass m kg and particle Q has mass $3m$ kg. The particles are moving in opposite directions along a smooth horizontal plane when they collide directly. Immediately before the collision P has speed $4u$ m s^{-1} and Q has speed ku m s^{-1}, where k is a constant. As a result of the collision the direction of motion of each particle is reversed and the speed of each particle is halved.

(a) Find the value of k.

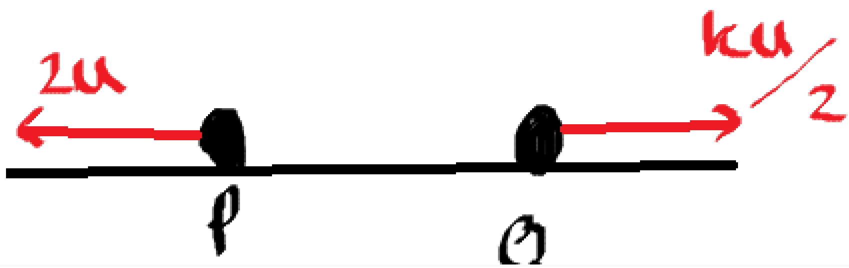
(4)

Consider the particles before the collision:



If we take the right as positive, the total momentum of the particles is: **4mu - 3mku.**

After the collision, the speed of P halves to **2u** and the speed of Q halves to **ku / 2**. They are now travelling in opposite directions:



The total momentum of the particle's is now: **-2mu + 3m(ku/2) = -2mu + 3/2mku**

The total momentum before and after is equal, so:

4mu - 3mku = -2mu + 3/2mku

6mu = 9/2mku

6 = 9/2k

k = 4/3.

/

Example 14

A 70g bullet moving at 400ms^{-1} to the east collides with a 1.2kg block and becomes embedded in the block. a) **Find the velocity of the bullet-block system after collision**.

The initial momentum of the bullet is **(0.070 * 400) = 28kgms^{-1}.**

This must equal the momentum of the bullet and the block (which now move at a single velocity) after the collision. If the velocity is **v**, **p = (1.2 + 0.070) * v**

Thus, **1.27 v = 28 => v = 22ms^{-1}.**

-

It took 10ms for the bullet to be reduced from its velocity of 400ms^{-1} to the velocity after impact. b) **Find the force exerted by the block on the bullet.**

The bullet decelerates from 400ms^{-1} to 22ms^{-1} in 10ms, so **a = (400-22) / (10*10^{-3}) = 37800ms^{-2}.**

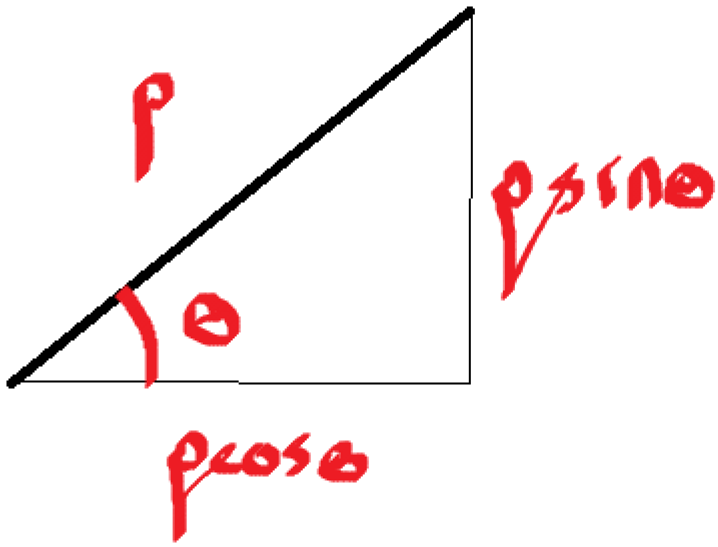
Thus, **F = (0.070 * 37800) = 2650N.**

Momentum in two dimensions

Previously, we considered problems where bodies were moving in a single plane - i.e. one dimension.

Momentum is a vector, and it can therefore also occur at an angle. Suppose we have a momentum **p** at an angle of **Θ** above the horizontal.

This can be split into horizontal and vertical components: **pcosΘ** and **psinΘ**:



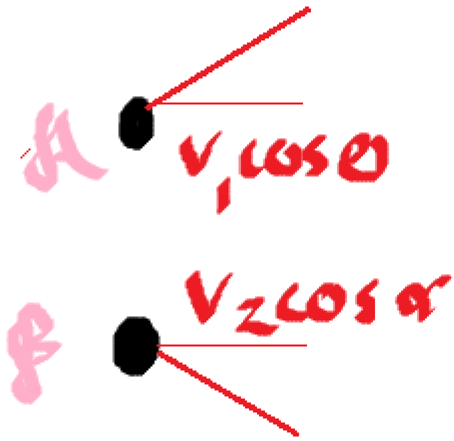
This is useful because if a collision occurs at an angle, or produces a vector at an angle, we can say that **momentum is conserved in the horizontal plane and in the vertical plane**. The horizontal and vertical planes are independent of each other, so the conservation of momentum in one plane does not affect the other plane.

Example:

Suppose a ball **A** of mass $\mathbf{m_1}$ is moving in the horizontal plane at $\mathbf{u_1}$ ms^{-1}, when it hits a ball **B** of mass $\mathbf{m_2}$ that is initially stationary. Suppose that after the collision, the ball **A** moves off at an angle of **Θ** above the horizontal with a velocity of $\mathbf{v_1}$ and the ball **B** moves off at an angle of **α** below the horizontal with velocity $\mathbf{v_2}$.

As mentioned, in the collision, momentum is conserved in the horizontal plane, and momentum is conserved separately in the vertical plane.

Horizontal plane:



Before the collision, the ball **A** is moving in the horizontal plane and has a horizontal momentum of m_1u_1. The ball **B** is stationary, so it has no momentum.

After the collision, the balls move off at an angle to the horizontal. The horizontal velocity component of **A** is $v_1\cos\Theta$ and the horizontal velocity component of **B** is $v_2\cos\alpha$. This means that the horizontal momentum of **A** after the collision is $m_1v_1\cos\Theta$ and the horizontal momentum of **B** is $m_2v_2\cos\alpha$. The total momentum in the horizontal plane after the collision is therefore $m_1v_1\cos\Theta + m_2v_2\cos\alpha$.

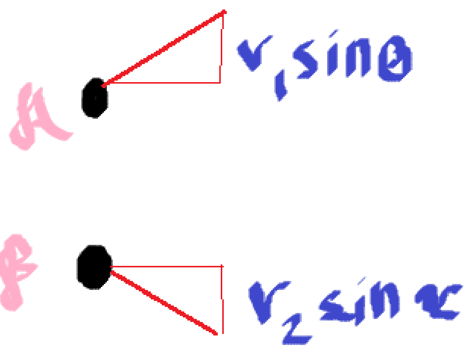
Momentum is conserved in the horizontal plane, so we can say:
$m_1u_1 = m_1v_1\cos\Theta + m_2v_2\cos\alpha$.

Since the balls are moving in the same direction in the horizontal plane throughout the motion, all of our momentum values are positive.

Vertical plane:

Initially, the balls are not moving in the vertical plane, and therefore they have a vertical velocity of $0ms^{-1}$. This means that there is no initial momentum in the vertical plane - all of the momentum is in the vertical plane.

After the collision, the balls have a vertical velocity of $\mathbf{v_1 sin\Theta}$ and $\mathbf{v_2 sin\alpha}$:



A and **B** are moving in opposite directions in the vertical plane; **A** is moving up vertically, and **B** is moving down. If we take upwards as positive, this means that **A** has a vertical momentum of $\mathbf{m_1 v_1 sin\Theta}$ and **B** has a vertical momentum of $\mathbf{-m_2 v_2 sin\alpha}$ since it is moving in the negative direction.

Thus, if we consider that momentum is conserved in the vertical plane, we can say:

$\mathbf{m_1 v_1 sin\Theta - m_2 v_2 sin\alpha = 0}$

We can also find the magnitude of each ball's momentum after the collision.

For example, if **A** has a horizontal momentum of $\mathbf{m_1 v_1 cos\Theta}$ and a vertical momentum of $\mathbf{m_1 v_1 sin\Theta}$, its total momentum is $\sqrt{(m_1 v_1 cos\Theta)^2 + (m_1 v_1 sin\Theta)^2}$

-

Suppose we project a 0.5kg ball at 25.0ms^{-1} at an angle of 32^{o} above the horizontal, and we want to find its overall momentum after 1.5s.

In this case, we would consider ideas of projectile motion.

Since there is no acceleration in the vertical plane, the ball's velocity remains at **25cos32** for the entire journey.

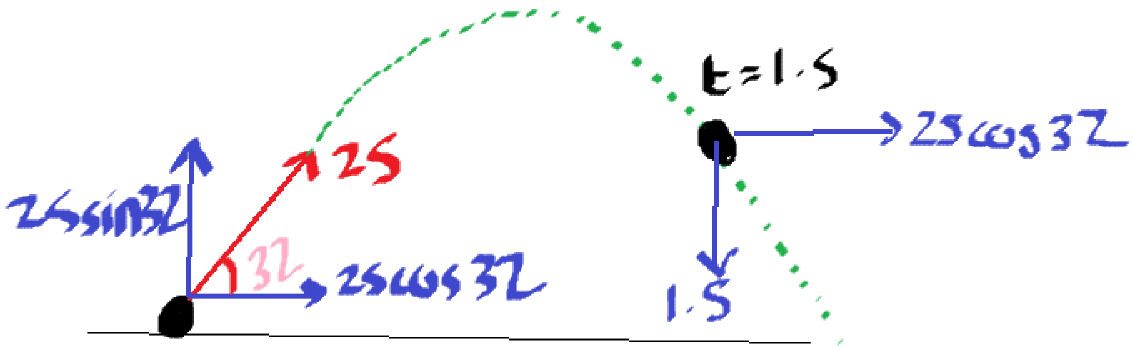
Considering the vertical plane, where the vertical velocity decreases over time due to gravitational acceleration:

u = 25sin32
a = -g

After 1.5 seconds, t = 1.5.

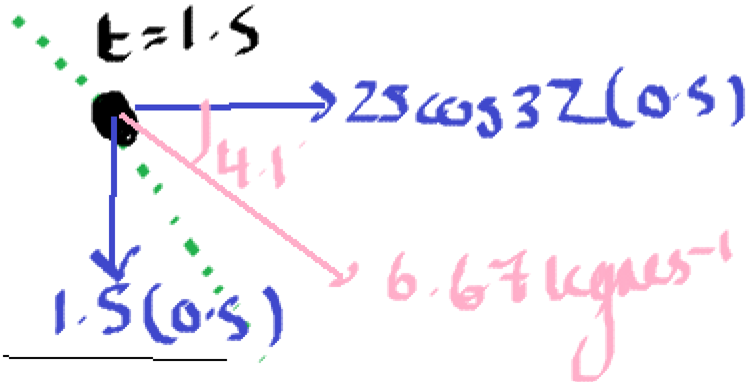
Thus, **v = 25sin32 + (-9.81 * 1.5) = -1.5ms^{-1}**.

The velocity is negative so the ball has now passed its apex and is being accelerated back towards the ground.



The momentum of the ball in the horizontal plane at **t = 1.5** is **25sin32(0.5) = 6.63kgms^{-1}.**
The momentum of the ball in the vertical plane at **t = 1.5** is **1.5(0.5)** = **0.75kgms^{-1}**

This leads to a total momentum of √**(6.63^2 + 0.75^2)** = **6.67kgms^{-1}**, at an angle of **tan^{-1} (1.5 / 25cos32) = 4.1°** below the horizontal.



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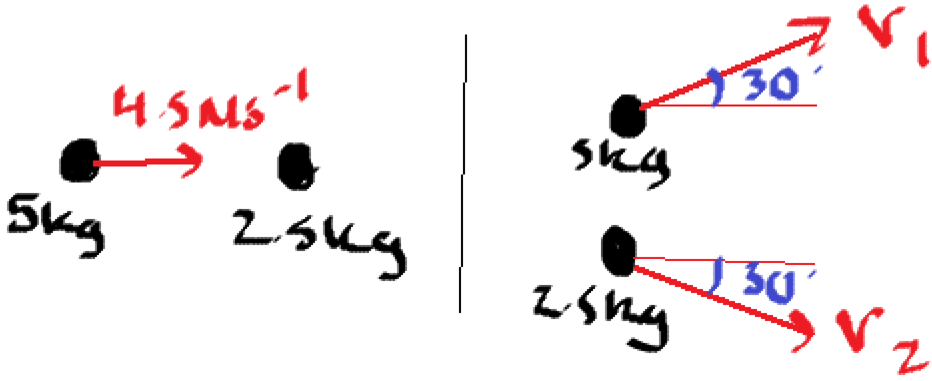
Now let us consider examples of conservation of momentum in two dimensions:

Example 1

A 5kg body slides along a smooth horizontal surface with a velocity of 4.5ms^{-1}. It collides with a stationary 2.5kg body. After the collision, one body moves off at 30° above the horizontal, and the other moves off 30° below the horizontal.

Find the speed of the bodies after collision.

Suppose the speed of the 5kg body after collision is $\mathbf{V_1}$ and the speed of the 2.5kg body after collision is $\mathbf{V_2}$:



Before the collision, the total horizontal momentum is **5(4.5) = 22.5kgms^{-1}.**

After the collision, the total horizontal momentum is: **V_1cos30(5) + V_2cos30(2.5) = (5√3/2)V_1 + (5√3/4)V_2 = 5√3/4 (2V_1 + V_2)**

Thus, since momentum is conserved in the horizontal plane: **22.5 = 5√3/4 (2V_1 + V_2)**, which simplifies to **6√3 = 2V_1 + V_2**

Before the collision, the total vertical momentum is **0kgms^{-1}** since the motion is in the horizontal plane.

After the collision, the total vertical momentum is **5V_1sin30 - 2.5V_2sin30 = 5/2V_1 - 5/4V_2**

Thus, we can say **5/2V_1 - 5/4V_2 = 0**, so **5/2V_1 = 5/4V_2**, so **2V_1 = V_2**.

We can now substitute **V_2 = 2V_1** into the equation **6√3 = 2V_1 + V_2**, and solve for **V_1**.

6√3 = 2V_1 + 2V_1 => 6√3 = 4V_1 => V_1 = 2.6ms^{-1}.

Thus, **V_2 = 2(2.6) = 5.2ms^{-1}.**

These are the speeds of the two bodies after collision.

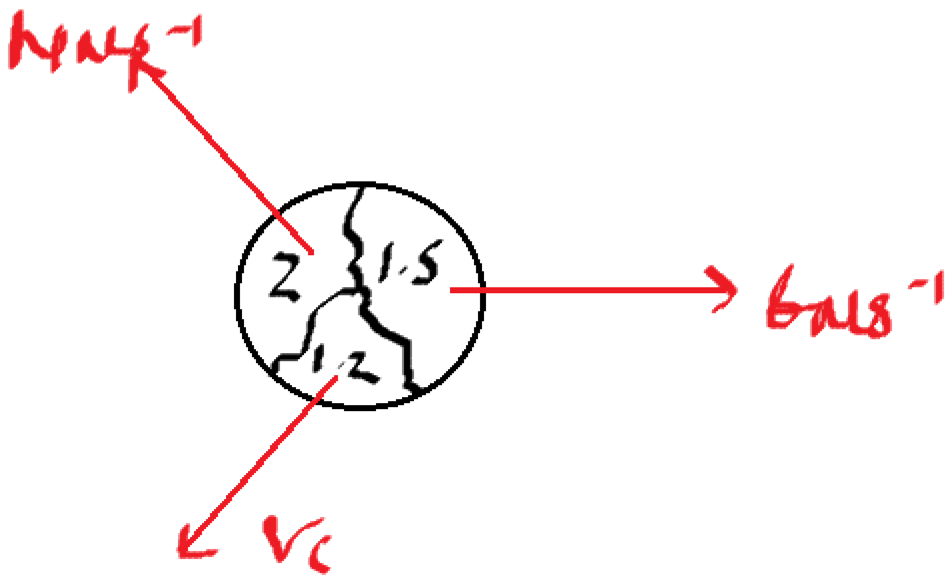
--

Example 2

A 4.7kg cannonball is fired vertically upwards in the air. The cannon ball reaches its apex, and explodes, causing three fragments to be ejected:

- A 2kg fragment is ejected at a bearing of 330° with velocity 4ms^{-1}
- A 1.5kg fragment is ejected at a bearing of 090° with velocity 6ms^{-1}.

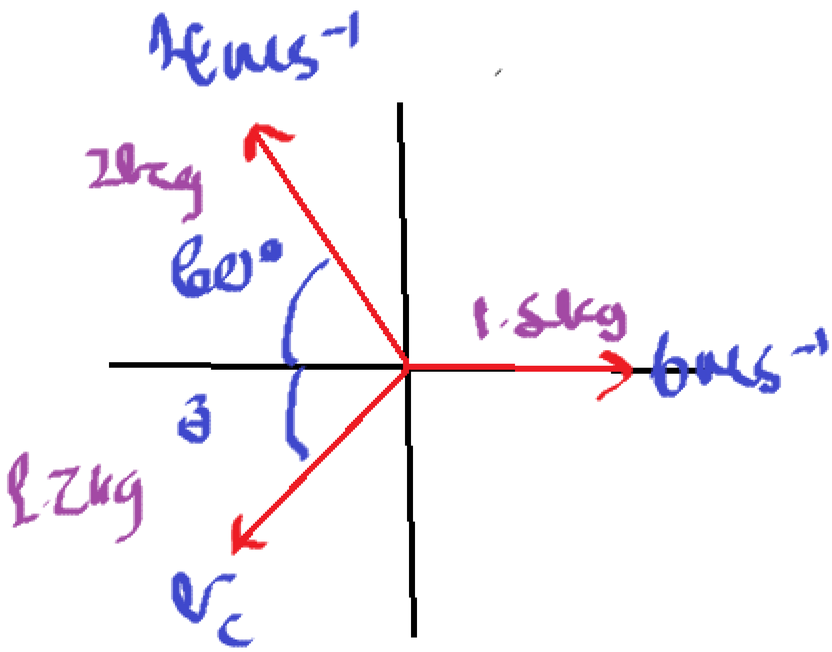
- A 1.2kg fragment is ejected at an angle of **Θ** below the west direction with a velocity of $\mathbf{V_c}$ ms^1.



Find V_c and Θ.

In this problem, the total momentum before the explosion must equal the total momentum after the explosion. The fragments are ejected at angles to the horizontal, so we therefore need to consider the horizontal and vertical components of momentum.

First let us consider a free body diagram for the explosion:



Momentum must be conserved in both the horizontal and vertical planes. Before the explosion, the cannonball is at its apex, so it has a momentum of 0kgms^{-1} in both the horizontal and vertical planes.

Horizontal plane:

If we take the right as positive, considering conservation of momentum:

$$1.5(6) - 2(4\cos 60) - 1.2(V_c \cos\theta) = 0$$

$$9 - 4 = 1.2 V_c \cos\theta$$

$$\boxed{\frac{5}{1.2\cos\theta} = V_c}$$

We now have one expression for V_c.

Vertical plane:

The 1.5kg fragment has no vertical velocity since it is moving solely in the horizontal plane. We therefore only need to consider the 1.2kg and 2kg fragments.

Taking upwards as positive, considering conservation of momentum:

$$2(4\sin 60) - 1.2(V_c \sin\theta) = 0$$

$$4\sqrt{3} = 1.2 V_c \sin\theta$$

$$\boxed{V_c = \frac{4\sqrt{3}}{1.2\sin\theta}}$$

We can now equate our expressions for V_c and find the value of **Θ:**

$$\frac{4\sqrt{3}}{1.2\sin\theta} = \frac{5}{1.2\cos\theta}$$

$$\frac{24\sqrt{3}}{5}\cos\theta = 6\sin\theta$$

$$\frac{\left(\frac{24\sqrt{3}}{5}\right)}{6} = \tan\theta \Rightarrow \theta = 54.18^{\circ}$$

Thus, $\mathbf{V_c}$ **= 4√3 / 1/2sin54.18 = 7.12ms^{-1}**

So, the fragment is ejected at 7.12ms^{-1} at a bearing of **215.9°.**

--

Impulse

Newton's second Law states that force is the **rate of change of momentum**, F = Δp / Δt.

We can rearrange this to show that the change in momentum is the force multiplied by the time over which the force is applied:

$$\Delta p = F\Delta t$$

For example, if a 500N force is applied over 20s, the change in momentum of a body must be **500(20) = 10000kgms^{-1}.**

The force multiplied by the time interval over which the force is applied is known as the **impulse** (symbol **J**).

J = impulse = FΔt (units: Ns or kgms^{-1})

So we can say that a change in momentum is an **impulse**.

A change in momentum is the final momentum minus the initial momentum, so **J = mv - mu = m(v-u).**

Thus, **J = FΔt = m(v-u)**

-

Take a 50kg body moving at 10ms^{-1} and a 500kg body moving at 10ms^{-1}. Suppose a 500N force is applied on both bodies to bring them to rest.

The 500kg body has a momentum 10x greater than the 50kg body. This means that there must be a much greater *impulse* (change in momentum) exerted to bring it to rest. This requires the force to be applied over a longer period of time.

-

If we know the force applied to a body and the time over which it occurs, we can find the velocity of the body.

Example: suppose a **0.5kg** body has a **10N** force applied over **3s** which causes it to accelerate from **2ms^{-1}** to **vms^{-1}**. **What is v?**

The impulse of the body is: **J = 10(3) = 30Ns.**

Impulse is the change in momentum, **J = m(v-u)**, so **30 = 0.5(v-2) => 60 = v - 2 => v = 62ms^{-1}.**

Example: a **100N** force is applied to a ball over **20ms**, causing it to accelerate from 10ms^{-1} to 120ms^{-1}. **Find the mass of the ball.**

J = 100(20*10^{-3}) = 2Ns.

2 = m(120-10) => m = 2 / 110 = **0.02kg**.

Example: A 10kg body has an impulse of **60Ns** over **25ms**. **What force is required to produce this impulse?**

60 = F * ($25*10^{-3}$) => **F = 2400N.**

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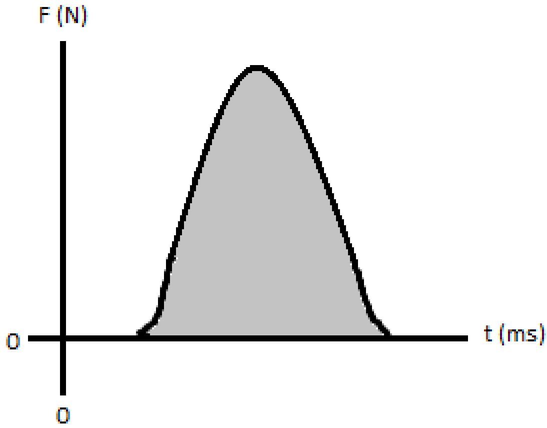
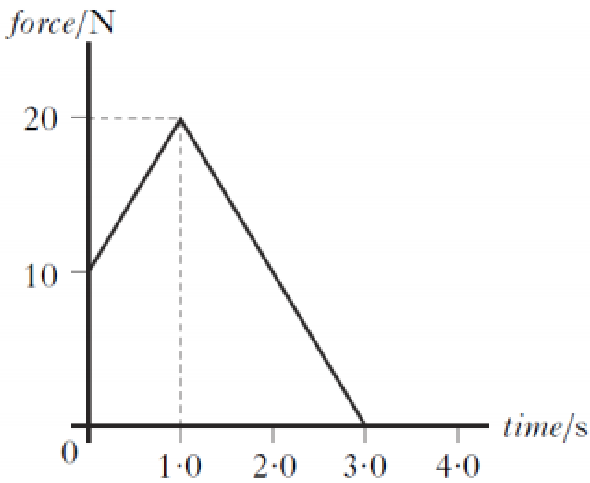
In most collisions, the force exerted over time is not a constant, but instead changes over time; for example:



Since **impulse = FΔt**, **the impulse during a collision is the area under a force-time graph**. This is where the key usefulness of impulse derives from.

The graph below shows the **force acting** on a mass of **5.0kg**.



In the graph on the left, the area under the graph is **1/2(10+20) + (1/2*20*2) = 35Ns**.

If the body starts from rest, then has this impulse, we can calculate its final velocity after the impulse.

Impulse = change in momentum = m(v-u).

35 = 5(v-0) => v = $7ms^{-1}$.

So the variable force shown in the graph on the left causes the body to accelerate from **$0ms^{-1}$** to **$7ms^{-1}$** in 3s. The body only accelerates when the force is being applied.

Previously, we had just considered situations where the force acting on the body is constant, and therefore the acceleration is constant. A variable force, on the other hand, causes a variable acceleration.

Describing the graph above, the force initially increases at a constant rate, so the acceleration increases at a constant rate since **F = ma**. Then, after 1s, the force begins to decrease at a constant rate, so the acceleration decreases at a constant rate. At 3s, there is no force applied to the body, so a=0.

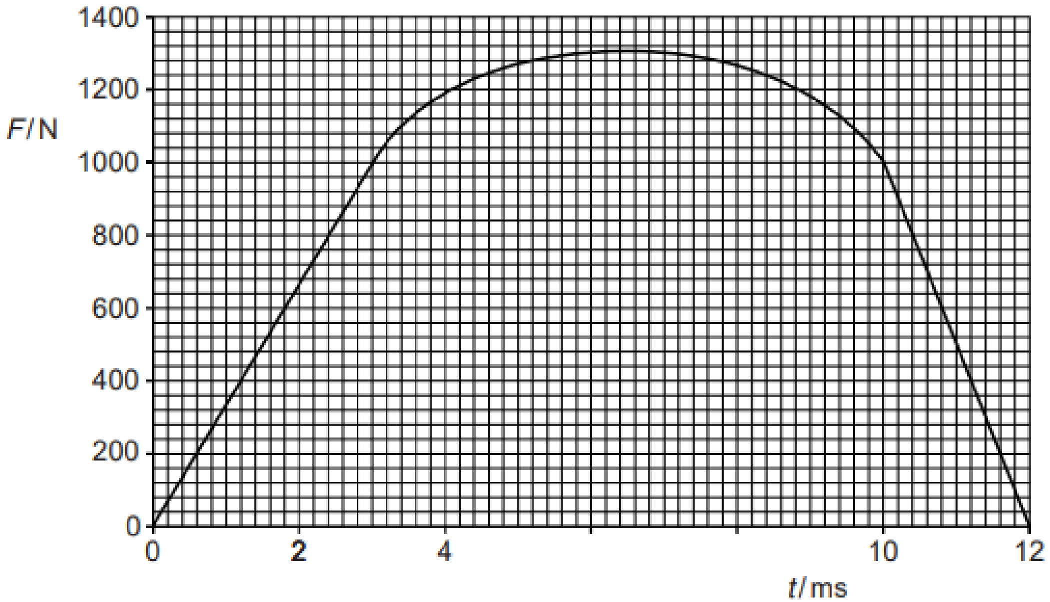
-

Note: If a **force-time graph** is non-polygonal, to determine the impulse, it would be necessary to either:

- Draw polygons to estimate the area
- Integrate the equation of the curve

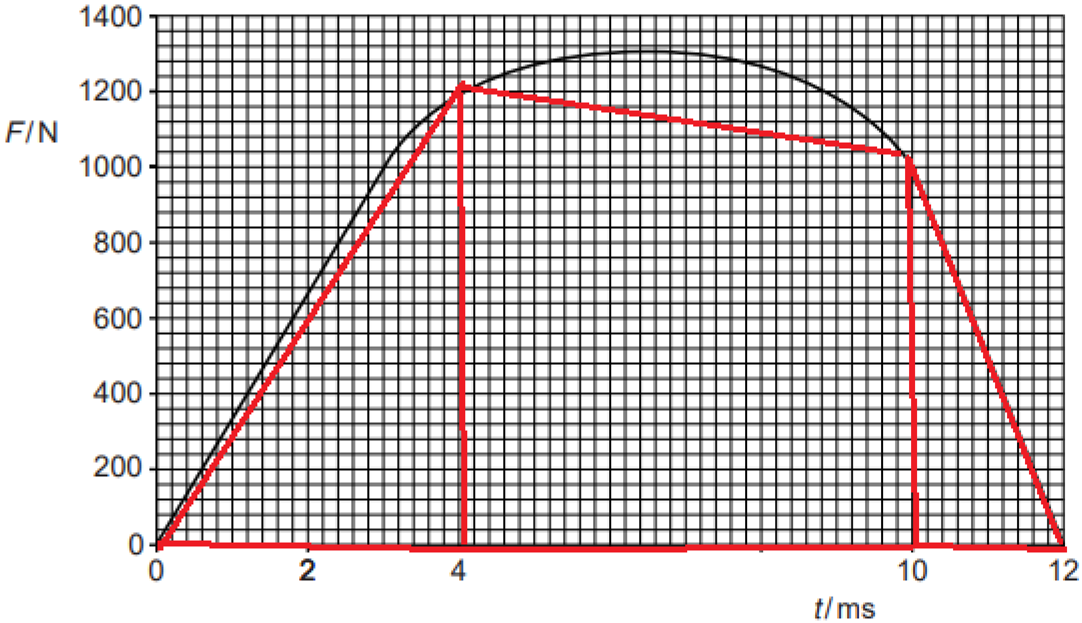
Example

A small rocket is used to detach a satellite of mass 180 kg from the spacecraft. Fig. 2.2 shows the variation of the force *F* created by the rocket on the satellite with time *t*.



determine the change in the velocity of the satellite as a result of the force *F* applied for the period of 12 ms.

Drawing polygons within the graph allows us to estimate the impulse:



Impulse = (1/2 * 4 ** 1240) + (1/2 (1240 + 1040) * 6) + (1/2 (2 * 1040)) = 10360Nms = 10.36Ns

Impulse = change in momentum. The impulse is exerted on a **180kg** satellite, so if a 180 kg body has a change in momentum of **10.36Ns**, we can say:

10.36 = 180(v-0) => v = **0.058ms^{-1}.**

-

Example

A snooker ball is at rest on a smooth horizontal table. It is hit by a snooker cue. Fig. 1.2 shows a simplified graph of force *F* acting on the ball against time *t*.

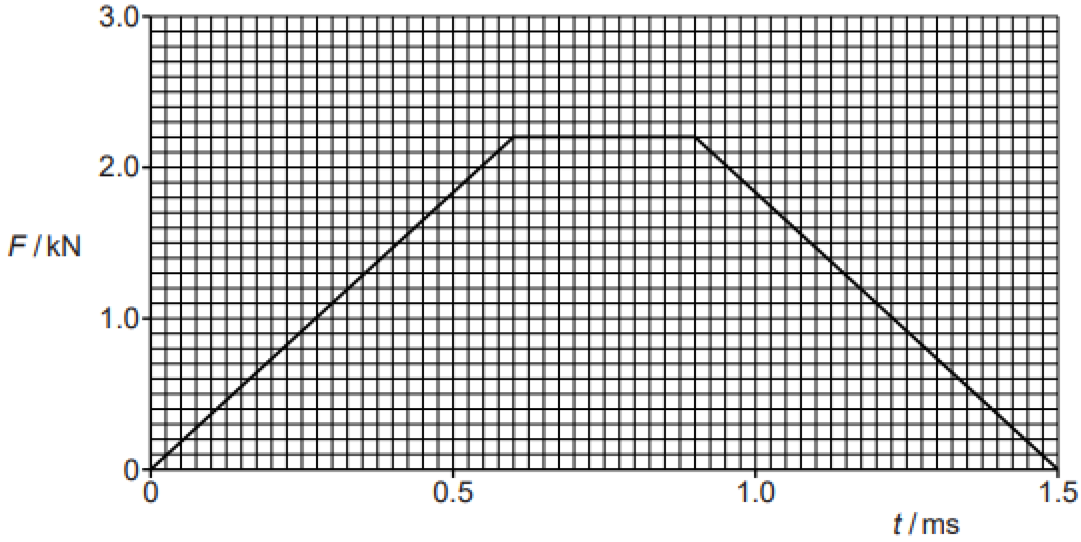


Fig. 1.2

(i) Describe how the velocity of the ball varies between *t* = 0.6 ms and *t* = 0.9 ms.

Between 0.6 and 0.9ms, the ball has a constant force exerted on it. This leads to a constant acceleration, and therefore a constant increase in velocity.

The mass of the snooker ball is 140 g. Calculate the final speed of the snooker ball as it leaves the cue.

The impulse exerted on the ball is **1/2 * (1.5 + 0.3) * 2200 = 1980Nms = 1.98Ns**

1.98 = m(v-u).

u = 0 since the ball is initially at rest.

So **1.98 = 0.140(v)** => **v = 14.1ms^{-1}.**

Example

A constant force of 80N acts on a 10kg block for 5s. This causes the block to accelerate from rest over a smooth horizontal plane. After 5s, the force is withdrawn, and the block now has no forces acting on it in the horizontal plane. a) **Find the distance travelled by the block in 10s.**

Firstly, if a 80N force acts over 5s, this will create an impulse, or change in momentum, of: **80(5) = 400Ns.**

Impulse is the change in momentum, so **400 = 10(v) => v = 40ms^{-1}.**

So after the 5s period, the velocity of the block is **40ms^{-1}**.

We can therefore find the distance travelled in the first 5s using SUVAT:

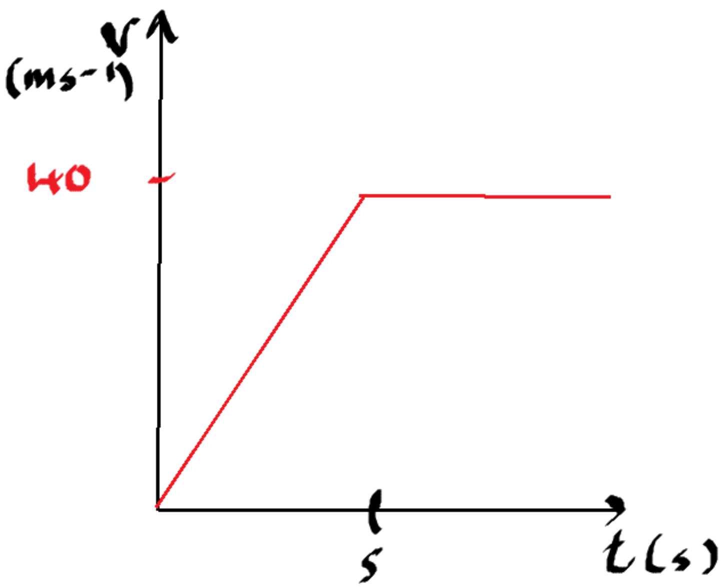
u = 0, v = 40, t = 5.

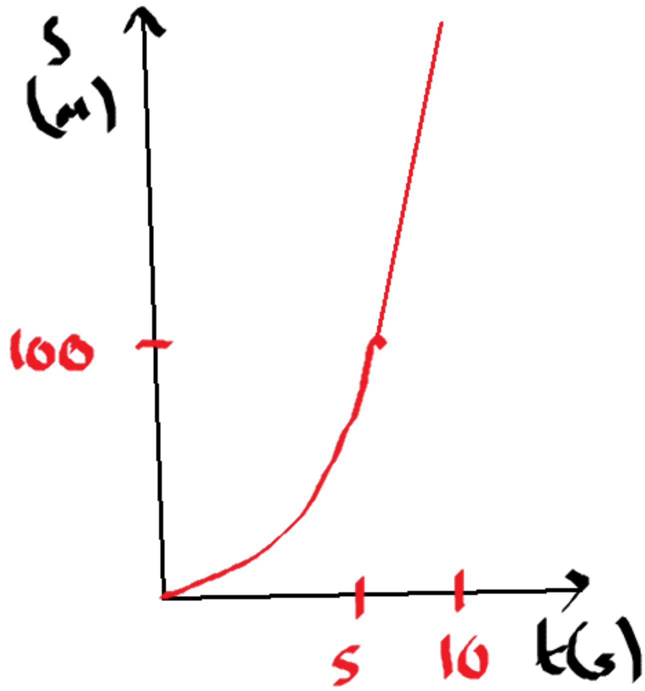
s = 1/2 (40) * 5 = 100m.

After the 5s acceleration period, the force on the block is withdrawn, so the block no longer continues to accelerate. Because the block has no forces that can decelerate or accelerate it, it will remain at a constant horizontal velocity of **40ms^{-1}** after **t=5**.

Thus, the distance travelled between **t=5** and **t=10** is **5(40) = 200m.**

The total distance travelled up to t = 10 is thus **300m**.



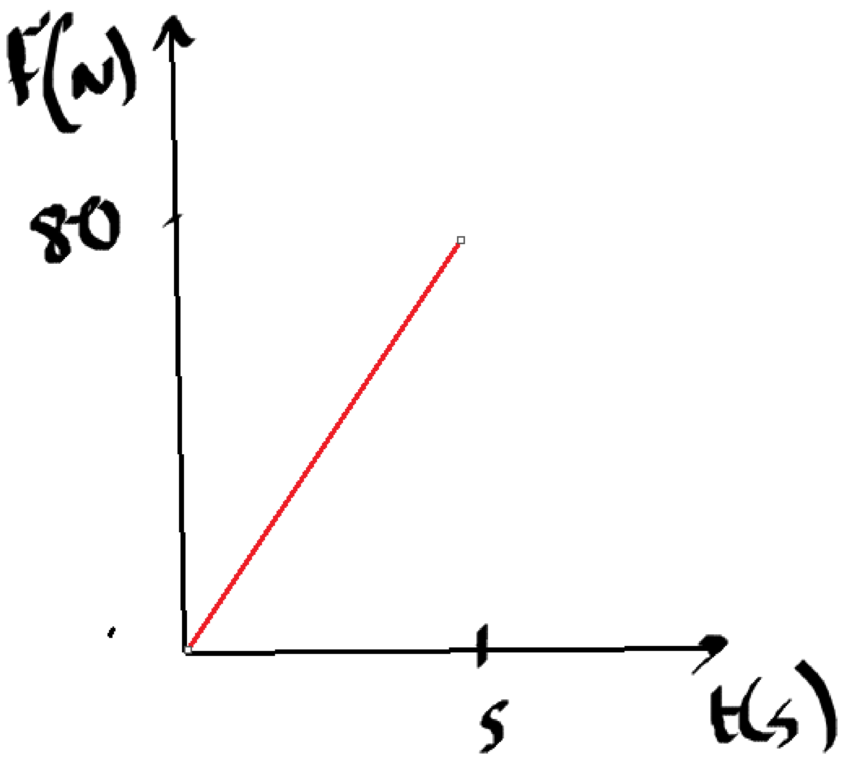


-

Now suppose that the force exerted on the block over the 5s period increases from 0N to 80N at a constant rate. b) **If the block starts from a velocity of 6ms^{-1} before the force is applied, find its velocity after 5 seconds.**

Now, rather than a constant 80N force, the force is increasing to 80N over time, so the impulse will be half of the impulse when a constant 80N force is applied (see graph below). The acceleration of the block will increase as the force increases.

Plotting force against time gives the following graph:



The impulse is the area under the graph, which is **1/2 * 80 * 5 = 200Ns**. This is half the impulse when a constant 80N force is applied. This is because the average force during this time period is **40N**, so the impulse is **40(5) = 200Ns**.

The impulse is equal to the change in momentum, so **200 = 10(v-6) => 200 = 10v - 60 => 260 = 10v, v = 26ms^{-1}.**

-

In a collision between two bodies, we say that one body exerts an **impulse** on the other body. In other words, the force exerted causes a change in momentum.

In a collision, the colliding bodies exert equal and opposite forces on each other. Since **Δp = FΔt**, and the force, **F**, on both colliding bodies is equal, and they collide over the same time, the change in momentum, or impulse, exerted on both bodies must also be equal and opposite. We proved this earlier by showing that $\Delta p_A = -\Delta p_B$.

So when two bodies collide, they have an equal and opposite change in momentum; so if one body has an increase in momentum of 200kgms^{-1}, the other must have a decrease in momentum of 200kgms^{-1} relative to the other body.

Example

Particle P has mass 3 kg and particle Q has mass m kg. The particles are moving in opposite directions along a smooth horizontal plane when they collide directly. Immediately before the collision, the speed of P is 4 m s^{-1} and the speed of Q is 3 m s^{-1}. In the collision the direction of motion of P is unchanged and the direction of motion of Q is reversed. Immediately after the collision, the speed of P is 1 m s^{-1} and the speed of Q is 1.5 m s^{-1}.

(a) Find the magnitude of the impulse exerted on P in the collision.

(3)

Impulse is the change in momentum, so we simply need to find the change in momentum of P, which is **final momentum - initial momentum**.

P has an initial momentum of **3(4) = 12kgms^{-1}.**

P has a final momentum of **3(1) = 3kgms^{-1}.**

Its momentum has therefore changed by **-9kgms^{-1}** - which is a **magnitude** of **9kgms^{-1}.**

So $|\mathbf{J}|$ **= 9kgms^{-1}.**

-

(b) Find the value of m.

(3)

According to Newton's third law, the impulse exerted on P by Q must be equal and opposite to the impulse exerted on Q by P.

If we take the direction of P as positive, the direction that Q is initially travelling in is negative.

The initial momentum of Q is therefore **m(-3) = -3m.**

The final momentum of q is **m(1.5) = 1.5m** (velocity is now positive since it has changed direction).

Thus, the impulse exerted on Q is **1.5m - - 3m.**

This is equal to **9kgms^{-1} =>** so **1.5m + 3m = 9 => 4.5m = 9 => m = 2kg.**

Two particles A and B, of mass $5m$ kg and $2m$ kg respectively, are moving in opposite directions along the same straight horizontal line. The particles collide directly. Immediately before the collision, the speeds of A and B are 3 m s^{-1} and 4 m s^{-1} respectively. The direction of motion of A is unchanged by the collision. Immediately after the collision, the speed of A is 0.8 m s^{-1}.

(a) Find the speed of B immediately after the collision.

(3)

To find the speed of B (let this be **v**), we can use conservation of momentum.

Taking the initial direction of A as positive:

5m(3) + 2m(-4) = 5m(0.8) + 2m(v)

15m - 8m = 4m + 2mv

3m = 2mv

v = 3m / 2m = 1.5ms^{-1}.

In the collision, the magnitude of the impulse exerted on A by B is 3.3 N s.

(b) Find the value of m.

(3)

A has a change in momentum, or impulse, of 3.3kgms^{-1}.

The impulse exerted on A must be in the opposite direction to its direction of travel since it is causing A to slow down.

If we take the velocity of A as positive, the impulse is therefore negative.

-

The initial momentum of A is **5m(3) = 15m**
The final momentum of A is **5m(0.8) = 4m**

We can therefore say that **4m - 15m = -3.3**

-11m = -3.3 => m = 0.3kg.

-

We can also find the exact same value of **m** by considering the impulse of B. If B exerts a 3.3Ns impulse on A, A must exert an equal and opposite impulse of 3.3Ns on B.

The velocity of B changes from $4ms^{-1}$ to $1.5ms^{-1}$ in the opposite direction to the original direction. The impulse of 3.3Ns must act in the opposite direction to the original direction.

Taking the initial direction as positive: **-3.3 = 2m(-1.5) - 2m(4)**
-3.3 = -3m - 8m

-11m = -3.3=> m = 0.3kg

-

1.4 - Energy Concepts

Energy is the capacity for doing **work**.

Energy can occur in many forms - but all of these forms are related by work. Examples of the forms in which energy can occur are:

- *Kinetic energy*
- *Gravitational Potential Energy*
- *Elastic Potential Energy*
- *Electrical energy*

A key concept that we will discuss later is that **energy can never be created or destroyed** - it can only be converted from one form into another form.

If a closed system has **5000J** of energy (where J, joule, is the unit of energy), that closed system will always have **5000J** of energy as long as it is closed. This energy does not always have to be in the same form; for example, the **5000J** of energy may exist as **3000J** of kinetic energy and **2000J** of gravitational potential energy; but the overall energy that is stored in the system can never increase or decrease from **5000J**. Only an *external force* can convert energy from the closed system into a different form.

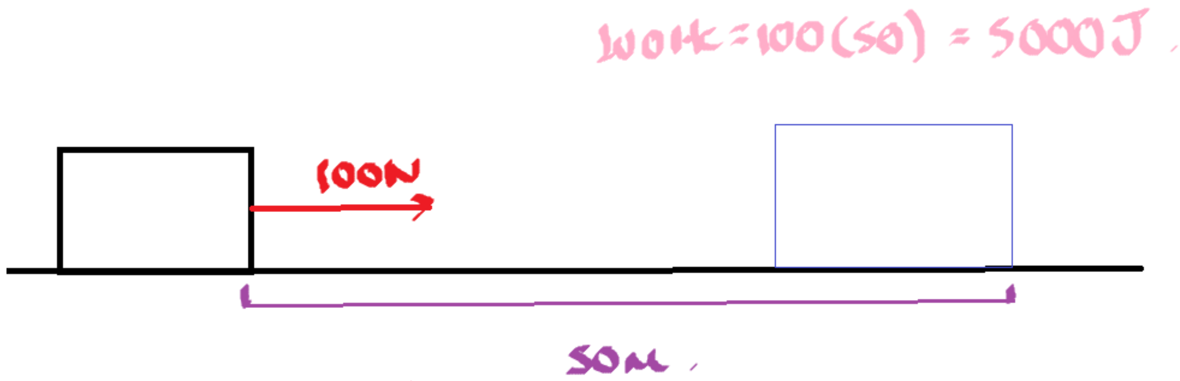
When energy is transferred into another form, *work* is done. Work is defined as the product of the force (*in the direction of travel*) over the distance for which the force is applied.

Work = Force * distance
W = Fd

The unit of work, or energy, is the **joule**, J. This is a *derived SI unit*. The base SI units of work are: **F * d = m * a * d = kg * ms^{-2} * m =** **kgm^2s^{-2}**. So **kgm^2s^{-2} = J**.

Suppose you have a force of 100N applied to a body, and it causes it to accelerate over a distance of 50m.

According to **W = Fd**, the *work done* to this body is **100(50) = 5000J**.



This means that an energy of 5000J has been transferred to the block to cause it to move over this distance.

The definition of one joule, the unit of energy, is: '**the work done when a body is moved over 1m by a force of 1N**'.

If a force of **2N** is applied over the same distance of **1m**, double the amount of work must have been done.

Whenever any sort of energy transfer occurs, for example, kinetic energy converting to potential energy, work must be done. **Work causes an energy transfer.**

-

Suppose that a force of **1000N** transfers **50000J** of energy to a body. The distance over which the body must have moved is **d = W / F = 50000 / 1000 = 50m**.

-

Suppose the kinetic energy of a body reduces from 2000J to 800J over a distance of 50m. The work done to the body to cause this energy conversion must have been **1200J**, so the force applied to reduce the kinetic energy of the body is **F = W / d = 1200 / 50 = 24N**.

-

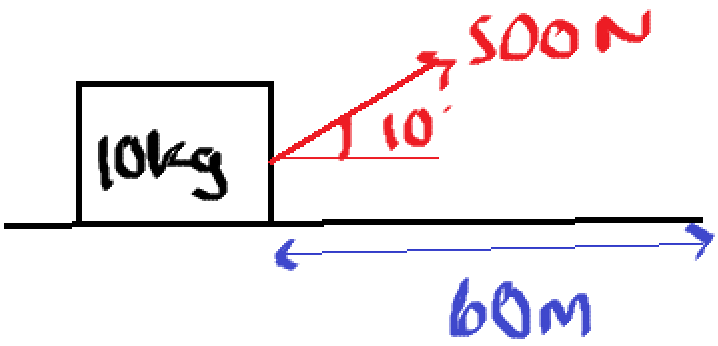
Work always causes an energy transfer; if 1000J of work is done on a body, that 1000J must have been transferred from one form (the form doing work) into one or more other forms.

-

We previously said that the work done on a body is the force **in the direction of travel** multiplied by the distance over which this force is applied.

This means that we must always multiply the force component in the direction of travel by the distance.

Example:



In the diagram on the left, a **500N** force is applied at **10°** above the horizontal, causing acceleration to the right.

The **500N** force is not acting in the direction of travel, so we therefore need to resolve the force into its horizontal component, and multiply this value by the distance travelled.

Horizontal component of force **= 500cos10** => this is in the direction of travel.

The horizontal component of the force acts over a **60m** distance, so **work = 500cos10 * 60 = 29600J**.

-

So if a force **F** acts at an angle **Θ** to the horizontal, causing acceleration over a distance of **d**, we can say that **work done = FcosΘd**, as multiplying **F** by **cosΘ** tells us the force component in the direction of travel.

This is a more refined definition of work.

If the force is in the direction of travel **Θ = 0°**. **Cos0** = **1**, so the **cosΘ** in the equation cancels to give **work done = Fd**, as we have seen previously.

If a force acts perpendicular to the direction of motion (e.g. the normal force when moving horizontally along a plane), the work done is 0J since **cos90 = 0**.

-

Now let us define two key types of energy:

- Kinetic Energy
- Gravitational potential energy

Kinetic energy

If a body is moving, it is said to have **kinetic energy**. The kinetic energy, E_K, of a body is given by:

$$E_K = 1/2mv^2.$$

Where m = mass of body, v = velocity, E_K = kinetic energy

For example, a body of mass **50kg** moving at velocity **$10ms^{-1}$** has kinetic energy **$1/2 * 50 * 10^2 = 2500J$**.

Work causes an energy transfer. When a certain amount of work is done on a body, it causes the energy to be transferred into another form.

Suppose a force **F** accelerates a block over a smooth plane for a distance of **d**.

This force is going to cause the object to gain velocity, and therefore it gains kinetic energy.

If all of the work done causes a transfer to kinetic energy (i.e. if there is no friction), we can say that **work done = kinetic energy**, or **$Fd = 1/2mv^2$.**

For example, suppose a constant force of **50N** is applied to a **10kg** body that is initially at rest, causing it to accelerate over a distance of **25m**. *What is its velocity after **25m**, ignoring the effects of friction?*

The work done to the body is **50 * 25 = 1250J.**

Assuming that there is no friction to deplete the kinetic energy, all of the work done must cause a transfer to kinetic energy. Thus, the kinetic energy of the body is **1250J**.

We can therefore say that **$1250 = 1/2 * m * v^2 \Rightarrow 1250 = 1/2 * 10 * v^2 \Rightarrow$** **$v = \sqrt{(2(1250) / 10)} = 15.8ms^{-1}$.**

-

This problem can also be considered in terms of kinematics. If a force of **50N** acts on a **10kg** body, it causes an acceleration of **50 / 10 = 5ms^{-2}.**

Thus, over a distance of **25m**, using SUVAT, we can find the final velocity:

u = 0, s = 25, v = v, a = 5

$v^2 = u^2 + 2as$ => $v = \sqrt{(u^2+2as)}$ = $\sqrt{(2(25)(5))}$ = 15.8ms^{-1}.

Ultimately, energy is a model that combines kinematics and dynamics that can be applied to situations that kinematics and dynamics are less effective at modelling.

*Proof that **work done = kinetic energy***

Indeed, kinematics (motion), dynamics (forces) and energy are closely related to each other, and we can prove that:

a) The work done on a body is transferred to kinetic energy, **$Fd = 1/2mv^2$**
b) Kinetic energy is **$1/2mv^2$**

by considering kinematics and dynamics equations.

Suppose a constant force of **F** acts on a body of mass **m**, causing it to accelerate at **a** ms^{-2} from an initial velocity of **0ms^{-1}** to a velocity of **vms^{-1}**, over a distance of **d**.

Newton's second law states **F = ma**, so **a = F / m**.

Using SUVAT, **u = 0, v = v, a = a, s = d**

Thus, **$v^2 = u^2 + 2as$**, u = 0, so **$v^2 = 2ad$**, so **$a = v^2 / 2d$**.

Equating our expressions for acceleration, **$v^2 / 2d = F / m$**

So, **$mv^2 = 2Fd$**.

Thus, **$Fd = mv^2 / 2 = 1/2mv^2$.**

This proves that when a force acts over a distance, causing a body to accelerate, the work done to the body must equal the kinetic energy at the end of the distance travelled.

Suppose a force of **800N** acts over **50m** to cause a body to accelerate from rest to **20ms^{-1}.**
What is the mass of the body?

$800 * 50 = 1/2 * 20^2 * m$

$m = 2(800*50) / 20^2 = 200$kg.

-

However, what if a body does not start from rest?

Suppose a 500kg car has an initial velocity of $10ms^{-1}$, and a driving force acts over a distance of 50m, causing it to accelerate to $40ms^{-1}$. **What is the driving force required to produce this acceleration?**

The initial kinetic energy of the body is **$1/2 * 10^2 * 500 = 25000J$**
The final kinetic energy of the body is **$1/2 * 40^2 * 500 = 400000J$**

The car therefore gains a kinetic energy of **400000 - 25000 = 375000J**. The work done on the car is therefore equal to **375000J**, since work done is simply an energy transfer.

If a body gains **375000J** of energy over a distance of **50m**, we can say:

W = fd

So **375000 = F * 50 => F = 7500N**.

-

So if a body of mass **m** has an initial velocity of **ums^{-1}** and a final velocity of **vms^{-1}**, and it accelerates, its change in kinetic energy is given by:

$1/2mv^2 - 1/2mu^2$.

A transfer of energy is work, so this gain in kinetic energy must equal work, which is **Fd.**

Thus, **$Fd = 1/2mv^2 - 1/2mu^2$**.

This is known as the **work-energy relationship**.

-

An example of where this may apply is in the deceleration of a car. Suppose a 200kg car brakes, reducing its velocity from $30ms^{-1}$ to $2ms^{-1}$ over a distance of 50m. **What braking force is required to achieve this?**

The loss in kinetic energy of the car is **$(1/2 * 200 * 30^2) - (1/2 * 200 * 2^2) = 89600J$**. This energy is not destroyed; it is transferred to another form - which is likely heat and sound energy.

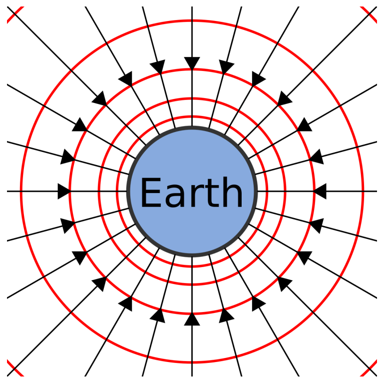
An energy transfer is work, so **89600J** of work has been done on the car.

Thus, **89600 = F * 50 => F = 1790N.**

-

Gravitational potential energy

Another key type of energy is **gravitational potential energy**.



When a body in a gravitational field is at a certain height, it has *gravitational potential energy*.

The term 'potential' in terms of gravitational potential energy refers to how a body at a certain height has a potential to fall to a lower height. For example, a ball on a shelf has a *potential* to return to the ground. When the ball falls to the ground, its gravitational potential energy will decrease and be converted into another form. Any type of energy transfer is **work**, so a body with a gravitational potential energy has a capacity to do work.

-

To find gravitational potential energy, which we will express as $\mathbf{E_g}$, we multiply the mass of a body by its height by the gravitational field strength.

$$E_g = mgh$$

Where E_g = gravitational potential energy, m = mass, g = gravitational field strength, h = height

[The gravitational field strength on Earth is given by **$g = 9.81ms^{-2}$**]

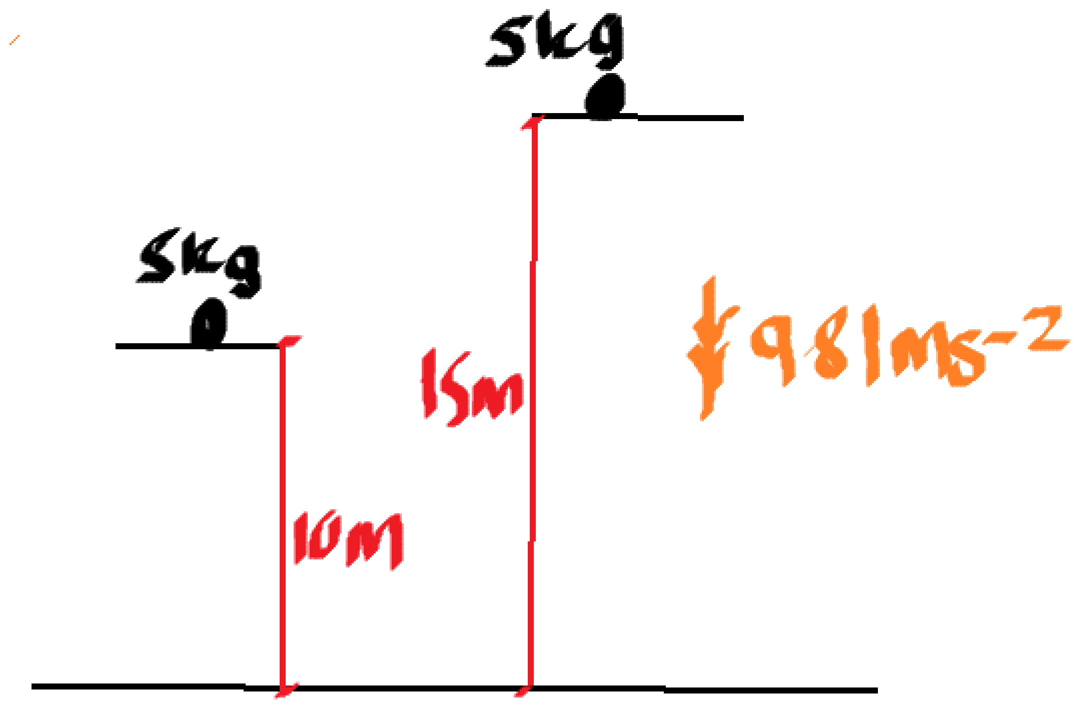
*We can prove this equation is homogeneous. The base SI units for energy are **kgm^2s^{-2}**.*

$m * g * h = kg * ms^{-2} * m = kgm^2s^{-2}$.

So, a body with a mass of **50kg** at a height of **10m** on Earth has a gravitational potential energy of **50 * 9.81 * 10 = 4905J.**

-

This equation shows us that if a body of the same mass has a greater height, it has a greater gravitational potential energy.



For example, a 5kg ball at a height of 10m has a lower gravitational potential energy than a 5kg ball at a 15m height; with the difference in gravitational potential energy being **(5 * 15 * 9.81) - (5 * 10 * 9.81) = 245J**.

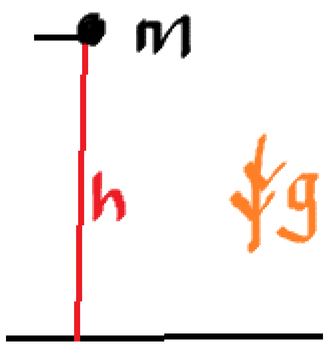
-

Proof that ***E_g = mgh = work***

The equation **E_g = mgh** is the same as the work equation.

Suppose a body of mass **m** falls over a height **h**.

The force acting on the body when it is falling is its weight, **mg**. It is falling a distance of **h**.



Since **work = force * distance**, and **force = mg**, and **distance = h**, we can say:

Work = mgh.

So the work done, or *energy transfer*, on a body falling in a gravitational field is given by **mgh.** Similarly, the work that must be done to lift a body to a higher height must be **mgh**.

So, since we previously proved that **work done = kinetic energy** when a body moves a distance of **d** from an initial velocity of **u** to a velocity of **v**, when a body falls in a gravitational field, since **work done = mgh,** we can say that **mgh = kinetic energy** => in other words, the gravitational potential energy of a falling body is converted into kinetic energy. Similarly, when a body rises, its kinetic energy is converted into gravitational potential energy.

For example, suppose a body with a gravitational potential energy of 500J falls to the ground. Assuming no air resistance, all of this energy is converted into kinetic energy.

-

An important point is that a body always wants to fall from a high *gravitational potential* to a low *gravitational potential*. For example, a river always flows downstream from a higher altitude.

This is ultimately because the acceleration is towards the centre of a gravitational field, so all bodies on Earth will naturally fall towards the centre of Earth's gravitational field.

For a body to move from a low gravitational potential to a high gravitational potential, it requires an energy input, since it is going against the pull of gravity.

For example, take a body of mass 10kg:

- When the body is 2m above the ground, **E_g = 10 * 2 * 9.81 = 196J**
- When the body is 12m above the ground, **E_g = 10 * 12 * 9.81 = 1177J**

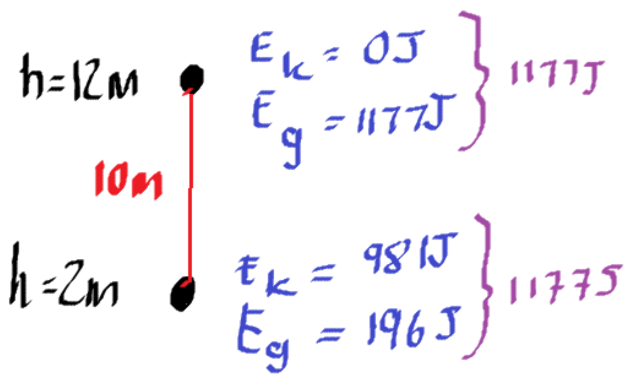
Thus, to move the body from a height of 2m to a height of 12m, it must gain an energy of **1177 - 196 = 981J.**

This means that **981J** of energy must be inputted, or **981J** of work must be done, to lift the body from 2m to 12m.

When the body falls from a height of 12m above the ground to a height of 2m above the ground, rather than gaining **981J** of gravitational potential energy, it will now lose **981J** of gravitational potential energy.

Since energy can never be created or destroyed, this **981J** of energy must be converted into another form; and because when a body falls, it has velocity, the gravitational potential energy must be converted into kinetic energy.

So suppose that the 10kg body drops from a height of **12m** to a height of **2m**. If we assume that there is no air resistance, 100% of the gravitational potential energy must be converted into kinetic energy.



Therefore, at the height of **2m** above the ground, the ball will have a **kinetic energy** of **981J**, as well as the gravitational potential energy of **196J**.

From its kinetic energy, we can calculate its velocity.

$981 = 1/2(10)v^2 \Rightarrow v = \sqrt{(2(981) / 10)} = 14ms^{-1}$.

We can also prove this with kinematics equations.

If the body falls from an initial velocity of $0ms^{-1}$ over a distance of 10m, with a gravitational acceleration of $9.81ms^{-2}$:

$u = 0, v = v, s = 10, a = 9.81$

$V^2 = u^2 + 2as \Rightarrow v = \sqrt{(0^2 + 2(9.81) * 10)} = 14ms^{-1}$.

This again shows the interconnectedness of energy, dynamics and kinematics.

-

If we dropped the ball from a height of 22m rather than 12m, because the ball initially has a higher gravitational potential energy, it will have a higher kinetic energy upon reaching a height of 2m.

Its change in E_g between the heights of 22 and 2m is **$(22*9.81*10)-(2*9.81*10) = 1962J$.**

This change in gravitational potential must be converted into kinetic energy; thus:

$1962 = 1/2(10)v^2 \Rightarrow v = \sqrt{(0^2 + 2(9.81)(20))} = 19.8ms^{-1}$.

-

Conservation of energy

We have already indirectly used the concept of **conservation of energy**.

Conservation of energy states that the energy in a *closed system* (a system with no external forces) must always remain constant; energy can never be created or destroyed - only transferred from one form into another.

Take a closed system at one time, $\mathbf{t_1}$ and another time $\mathbf{t_2}$. Suppose the total energy of the system at $\mathbf{t_1}$ is $\mathbf{E_1}$ and the total energy of the system at $\mathbf{t_2}$ is $\mathbf{E_2}$.

According to conservation of energy, the energy in a closed system cannot be increased or decreased, so:

$E_1 = E_2$.

An example of where this can be applied is a falling body.

When a body falls from a height, its potential energy decreases (since E_g = mgh and h is decreasing), and its kinetic energy increases (since it is accelerated by gravity, so its velocity increases).

If we assume that there is no air resistance, as a body falls, all of its gravitational potential energy is converted into kinetic energy, since **mgh** is simply a form of work (as shown in the ***gravitational potential energy section***), and work is equal to **$1/2mv^2$** (as shown in the ***kinetic energy*** section), so **$mgh = 1/2mv^2$**.

So when a body falls a distance of **h** under a gravitational force **mg**, since work = force * distance, and work = kinetic energy, there must be a conversion of potential energy into kinetic energy.

At a given height, the total ***mechanical*** energy of a body is a sum of its gravitational potential energy and its kinetic energy: **$E_{total} = E_g + E_k$.**

For example, a 12kg body at a height of 8m, with a velocity of $9ms^{-1}$ has a potential energy of **12(9.81)(8) = 942J**, and a kinetic energy of **$1/2(12)(9^2) = 486J$**, so it has a total energy of **1428J**.

According to conservation of energy, this value of **E_{total}** must remain constant in a closed system; so if the potential energy decreases (i.e. the body falls), its kinetic energy must increase to allow energy to be conserved.

If the body above falls to a height of 4m, it now has a gravitational potential energy of **4g(12) = 471J**. Its total mechanical energy must have remained constant (at **1428J**), and the mechanical energy is a sum of the potential and kinetic energy. Thus, **$1428 = E_k + 471 \Rightarrow E_k = 957J$**.

-

Example

A 50kg body is initially at a height of 30m, falling at velocity $2ms^{-1}$. ***What will its velocity be*** *(without using kinematics equations)* ***at a height of 5m?*** ***[ignore the effects of air resistance]***

At a height of 30m, the body has a gravitational potential energy of **30(50)g = 1500g J**. It also has a kinetic energy of **$1/2 * 50 * 2^2 = 100J$**.

Its total energy is therefore **1500g + 100 = 14820J**.

When the body falls to a height of 5m, its gravitational potential energy reduces. Since energy cannot be destroyed, only converted into another form, this lost gravitational potential energy must have been converted to kinetic energy.

At a height of 5m, the body has a gravitational potential energy of **50 * g * 5 = 250g J**.

Since the energy of a closed system must remain constant, the total energy of the body at a height of 30m must equal the total energy of the body at the height of 5m.

At the height of 5m, the total energy of the body is its gravitational potential energy, **250g J**, plus its kinetic energy, E_K.

Thus, $250g + E_k = 14820 \Rightarrow E_k = 12400J$.

Using $E_k = 1/2mv^2$, $12400 = 1/2 * 50 * v^2 \Rightarrow v = 22.3ms^{-1}$.

Another way to view this is that the body moves a distance of **25m** under a weight force, **50(9.81) = 490.5N**. Since **work = force * distance**, the energy transfer into kinetic energy is equal to **490.5(25) = 12260J** (this is just like the *work-energy relationship* shown previously). The body initially has a kinetic energy of $1/2(50)(2^2) = 100J$.

Thus, its total kinetic energy at a height of **5m** is **12260(100)** = **12360J = 12400J** *(3sf)*.

/

We can also prove that the velocity of the body at a height of 5m is $22.3ms^{-1}$ with kinematics:

s = 25, a = 9.81, u = 2

$v = \sqrt{(2^2 + 2(9.81)(25))} = 22.2ms^{-1}$ *(not exactly* $22.3ms^{-1}$ *due to rounding)*.

To generalise this, suppose we have a body that falls down to a lower height (ignoring the effects of air resistance).

Let its gravitational potential energy at a height h_1 be E_{g1}, and let its gravitational potential energy at a height h_2 be E_{g2} (where $E_{g1} > E_{g2}$). Let its kinetic energy at h_1 be E_{k1}, and let its kinetic energy at h_2 be E_{k2} (where $E_{k2} > E_{k1}$).

As the body falls, its gravitational potential energy is converted into kinetic energy.

When it is at **h_1**, its total energy is **E_{g1} + E_{k1}**.

When it has descended to **h_2**, its total energy is **E_{g2} + E_{k2}**.

Energy is always conserved, so the total energy at the high point and the low point must be equal, *assuming no air resistance*.

Thus, **$E_{g1} + E_{k1} = E_{g2} + E_{k2}$**.

If the body starts from rest, its initial kinetic energy is 0, so we can say: **$E_{g1} = E_{g2} + E_{k2}$**.

If the body does have an initial kinetic energy, and reaches a height of 0m above the ground, its final gravitational potential energy is 0, so we can say: **$E_{g1} + E_{k1} = E_{k2}$**.

If the body starts from rest and hits the ground, we can say **$E_{g1} = E_{k2}$** => in other words, there is a complete conversion of potential energy to kinetic energy.

Example

A 10kg body is falling downwards. At the instant when the body is 200m above the ground, it has a velocity of $30ms^{-1}$. The body continues to fall down. At a certain instant, it has a kinetic energy of 9000J. **Calculate the height of the body at this point** [ignore the effects of air resistance].

The total energy of the body at the height of 200m is **(200 * 9.81 * 10) + (1/2 * 10 * 30^2) = 24120J**.

This total energy remains the same throughout the freefall, so **$E_k + E_g = 24120$.**

Thus, when **E_k = 9000J, E_g = 24120 - 9000 = 15120J**.

Since E_g = mgh, **15120 = 10 * 9.81 * h**, so **h = 154m**.

So the body has fallen a height of **46m**.

-

Another way to view this problem is that at a height of 200m, the body has a kinetic energy of **1/2(10)(30^2) = 4500J**.

As the body falls, its potential energy is converted directly into kinetic energy.

The body's kinetic energy is **9000J** at our desired height, so it must have gained **4500J** of kinetic energy.

If a body gains **4500J** of kinetic energy over a distance of **d** with a weight force acting on it of **10(9.81) = 98.1N**, we can say:

Kinetic energy = work
4500 = 98.1 * d
d = 45.9m.

So, the body must have fallen a distance of **45.9m** from its original height of **200m** to gain **4500J** of kinetic energy, so its height when it has **9000J** of kinetic energy is **200-45.9 = 154.1m.**

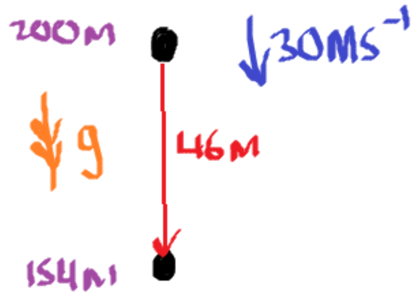
/

Find the time after the instant when the body is 200m above the ground when the kinetic energy of the body is 9000J.

We just found that the body is at a height of **154m** when its kinetic energy is **9000J**. It has therefore fallen **46m** from its original position.

Thus, we can say:

u = 30, s = 46, t = t, a = 9.81



$46 = 30t + 1/2(9.81)t^2$

$4.905t^2 + 30t - 46 = 0$

Using the quadratic formula, **t = 1.27s**.

--

If a body instead moves from a lower height to a higher height, its gravitational potential energy will increase.

Suppose a body is thrown upwards. As the height of the body increases, its gravitational potential energy increases, while its kinetic energy decreases due to gravitational deceleration.

The kinetic energy of the body is therefore being converted into gravitational potential energy. Again, if we ignore air resistance, the total energy of the body must remain the same; so if the kinetic energy is decreasing, the gravitational potential energy must be increasing.

This is similar to a decelerating car. If a car's kinetic energy reduces due to a braking force, we can say that the **loss in kinetic energy = work done**; and gravitational potential energy is just a form of work.

-

Example

A 10kg ball is thrown upwards from the ground at $50ms^{-1}$. When its velocity has reduced to $10ms^{-1}$ in the upwards direction, **what is its height?**

The kinetic energy of the ball upon release is **$1/2 * 10 * 50^2 = 12500J$.**

When its velocity is $10ms^{-1}$, its kinetic energy is **$1/2 * 10 * 10^2 = 500J$**.

Its kinetic energy has therefore been reduced by **12000J**. This energy cannot be completely destroyed - it must have been converted into another form. Since we are ignoring air resistance, the **12000J** of kinetic energy must have been converted completely into potential energy.

Thus, **$12000 = mgh = 10 * 9.81 * h$.**

$h = 12000 / (10 * 9.81) = 122m$.

This can be proven with kinematics equations.

u = 50
v = 10
a = -9.81

$v^2 = u^2 + 2as \Rightarrow s = (v^2 - u^2) / 2a = (10^2 - 50^2) / 2(-9.81) =$ **122m**.

-

Example

A body of mass 14kg is thrown vertically upwards from the ground with a velocity of **vms^{-1}**. At the instant when its velocity has decreased to $15ms^{-1}$ in the upwards direction, its height is 30m. **Find v.**

Since we are ignoring air resistance, the total energy of the body must remain constant.

At the instant when its velocity is $15ms^{-1}$, the body has a total energy of: **$(1/2 * 15^2 * 14) + (14 * 9.81 * 30) = 5700J$**.

When the body is initially thrown upwards, it has no gravitational potential energy - only kinetic energy.

Thus, its initial kinetic energy is **5700J**.

So, **$5700 = 1/2 * 14 * v^2 \Rightarrow v = 28.5ms^{-1}$**.

-

Another way to approach this is to consider the bodies gain in gravitational potential energy.

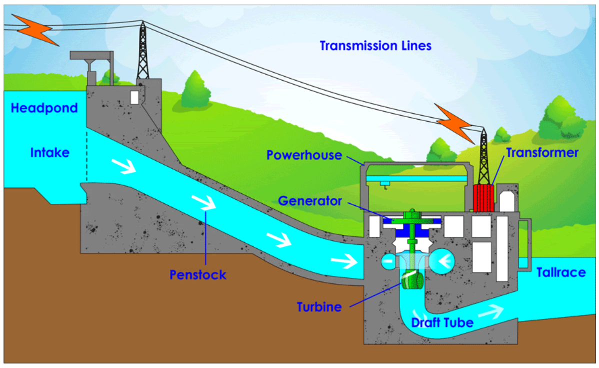
At a height of 30m, it has a gravitational potential energy of **30 * 9.81 * 14 = 4120J**.

It has therefore gained **4120J** of gravitational potential energy, and lost **4120J** of kinetic energy.

Its kinetic energy at a height of **30m** is **1/2 * 15^2 * 14 = 1575J**, so its initial kinetic energy must have been **4120 + 1575 = 5695J**, and from this we can find **v**.

The idea that gravitational potential energy is converted into another form when the height decreases is utilised in hydroelectric power stations.

In a hydroelectric power station, water flows from a high height to a lower height. As the water flows downwards, its gravitational potential energy is converted into kinetic energy. This kinetic energy is used to spin a turbine, which can be used to generate power.



The effects of friction

When a body is subject to friction such as air resistance, the velocity of the body decreases. This is because friction is a force that opposes the direction of travel, so it creates an acceleration in the opposite direction to the direction of travel, reducing the velocity of the body.

-

We can prove that a frictional force causes a depletion in kinetic energy.

Suppose we have a body of mass **m** with a frictional force **F** acting against it. The acceleration of the body due to the frictional force is **F/m**. If the body has an initial velocity of **u** and a final velocity of **v** (v<u since the frictional force causes deceleration), and travels a distance of **d**, we can say:

u = u, v = v, a = -F/m, s = d

$v^2 = u^2 + 2(-F/m)(d)$

$v^2 - u^2 = -2Fd/m$

$2Fd/m = u^2 - v^2$

$2Fd = mu^2 - mv^2$

$Fd = 1/2mu^2 - 1/2mv^2$.

So, the work done by the frictional force is equal to the loss in kinetic energy.

-

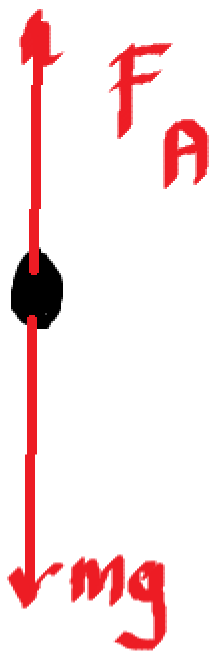
This concept applies to a falling body on Earth. Earth has an atmosphere, which causes falling bodies to be subject to air resistance. Since air resistance reduces the velocity of a body, it must reduce its kinetic energy. Since energy cannot be created or destroyed, only converted into another form, this lost kinetic energy must be converted into another form - which is waste energy such as heat and sound energy.

Air resistance is causing the kinetic energy of the body to reduce; it is causing an energy transfer, and therefore it is causing work to be done on the body, which depletes its kinetic energy.

Therefore, when a body falls to the ground in Earth's gravitational field, we cannot assume that all of its gravitational potential energy is converted into kinetic energy, because some of the kinetic energy will be depleted due to air resistance.

So some of the gravitational potential energy is converted into kinetic energy, and some into waste energy due to air resistance; but the total gravitational potential energy must equal the total energy upon hitting the ground due to conservation of energy.

Thus, **gravitational potential energy = kinetic energy + energy loss due to air resistance**, or $E_g = E_k + W_F$ (where W_F is the work done by friction).



*A falling body on Earth, or any other celestial body with an atmosphere, has two main forces acting on it - weight (**mg**) in the downwards direction, and air resistance (**F_A**) in the upwards direction.*

A force does work on a body. The weight force is causing the body to accelerate, so it causes it to gain kinetic energy ($Fd=1/2mv^2$). The work done by the air resistance, on the other hand, decreases the kinetic energy.

We can therefore not assume that all of a body's gravitational potential energy is converted into kinetic energy; as some of the kinetic energy is lost due to air resistance.

-

Example: suppose a 10kg body is dropped from a height of 50m. As the body falls to the ground, the work done by air resistance is **550J**. **What is the velocity of the ball upon hitting the ground**?

The gravitational potential energy of the body at a height of **50m** is **50(10)(9.81) = 4905J.**

If there was no air resistance, we would assume that all of this gravitational potential energy is converted into kinetic energy upon hitting the ground.

However, not all of the gravitational potential energy is converted due to the work done by air resistance.

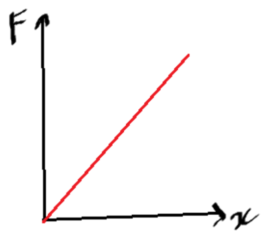
We can say that **4905 = 550 +** $\mathbf{E_k}$ => so $\mathbf{E_k}$ **= 4355J**.

Thus, **4355 = 1/2 * 10 *** $\mathbf{v^2}$ => v = **29.5**$\mathbf{ms^{-1}}$**.**

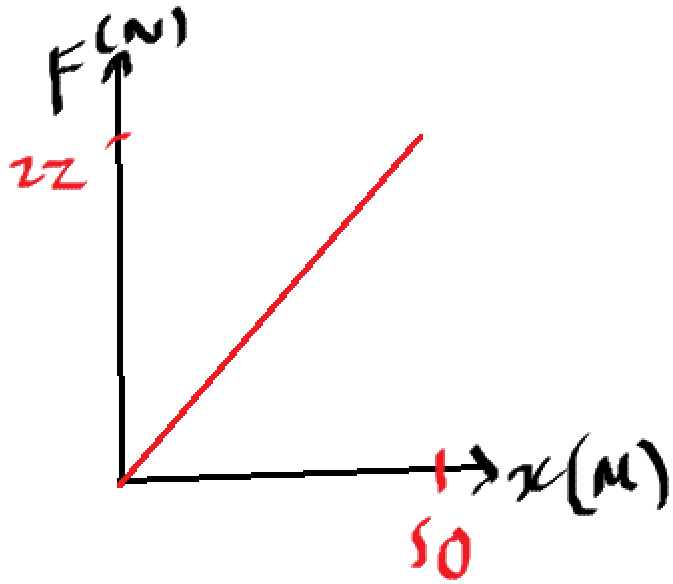
-

We are usually given the work done by air resistance, because the force exerted by air resistance is not constant; it increases over time as the body's downwards velocity increases. It is often difficult to work with variable forces.

However, a graph could also be used to show the air resistance over the distance to the ground:



Since work = force * distance, the work done by air resistance is equal to the area under the graph.



For the previous example, if we assumed that air resistance increases at a constant rate, we would get the following graph for air resistance over time:

The total work done by air resistance is **1/2 * 22 * 50 = 550J** (as stated in the question).

Air resistance exerted on rising bodies

Air resistance also occurs when a body is rising in Earth's gravitational field. When a body rises in a gravitational field, the weight force decelerates the body, reducing its kinetic energy. This work done must lead to an energy conversion into gravitational potential energy. However, when air resistance exists, not all of the kinetic energy of the body is converted into potential energy.

When a body is moving upwards, it contacts air particles, which creates a resistive force in the opposite direction to the direction of motion.

This resistive force causes the kinetic energy of the body to reduce over time, so not all of the body's kinetic energy is converted into potential energy - some is lost as waste heat / sound energy.

We can therefore say $\mathbf{E_k = E_g + W_f}$.

Example

A 0.5kg ball is thrown upwards at $20ms^{-1}$, and rises to a height of **h**m, and the work done by air resistance during this motion is **20J**. **Find h**.

If there was no air resistance, we would expect all of the ball's kinetic energy to be converted into potential energy at the apex.

However, some of the kinetic energy of the ball is being converted into sound/thermal energy due to air resistance.

$\mathbf{E_k = 1/2 * 20^2 * 0.5 = 100J}$.

If **20J** of this kinetic energy is lost due to air resistance, **80J** must be converted into potential energy.

Thus, **80 = 0.5 * 9.81 * h** => **h = 16.3m.**

Without this air resistance, **h** would be **100 / (0.5 * 9.81) = 20.4m.**

Example

A 10kg block is thrown upwards at $18ms^{-1}$. It reaches a height of 15m. **Find the work done by air resistance on the block.**

The initial kinetic energy of the block is **1/2 * 10 * 18^2 = 1620J.**

If there was no air resistance, this would be converted to **1620J** of potential energy, and thus the height reached would be **1620 / (9.81 * 10) = 16.5m**.

However, the block only reached a height of **15m**, so air resistance must have depleted the velocity of the ball.

At a height of **15m**, the block has a potential energy of **15 * 9.81 * 10 = 1472J**.

So **1620J** of kinetic energy is converted to **1472J** of potential energy. Energy must be conserved, so the rest of kinetic energy must have been converted into heat / sound energy due to air resistance.

The kinetic energy loss due to air resistance is therefore **1620 - 1472 = 148J**.

-

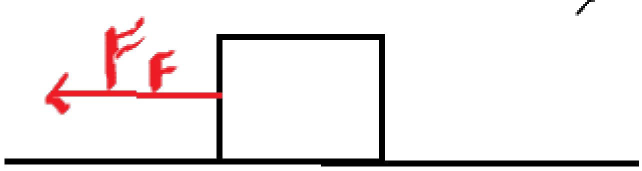
This is why considering these problems in terms of energy is useful. The above problems could not be answered with kinematics concepts.

Frictional forces with the ground

A loss of kinetic energy due to friction can also occur when a body is moving along a rough surface.

We previously assumed that if a force acts on a body over a certain distance, all of the work done by that force must be converted into kinetic energy.

However, in reality, some kinetic energy is lost due to friction.



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When a driving force is exerted on a body on a horizontal plane, it can cause the body to accelerate across the horizontal plane. This force does a certain amount of work.

If the plane is rough, there will also be a frictional force acting against the body. This frictional force will do a certain amount of work to deplete the kinetic energy of the body (as proven here).

This means that not all of the work done by the driving force is transferred to kinetic energy; as some of the kinetic energy is depleted due to the work done by friction.

Following conservation of energy, we can therefore say:

Work done = Gain in kinetic energy + Work done by friction
$W_T = E_K + W_F$.

Example

A 570kg car initially at rest has a constant driving force of 1500N applied over a distance of 120m. The constant frictional force acting against the car is 300N. **Find the velocity of the car after travelling 120m from rest.**

The work done on the car by the driving force is **120 * 1500 = $1.8*10^5$J**.

If there was no friction, we would assume that all of this energy is converted as kinetic energy. However, there is also a **300N** force acting against the car to deplete its kinetic energy.

If a **300N** force acts over a **120m** distance, the work done by that force is **300 * 120 = $3.6*10^4$J**.

Thus, the kinetic energy of the car after the **120m** distance is **$(1.8 * 10^5) - (3.6 * 10^4) = 1.44*10^5$J**.

Thus, **$1.44 * 10^5 = 1/2 * 570 * v^2$** => **$v = 22.5ms^{-1}$**.

When a car is decelerating with no driving or braking force acting on it, the frictional force must be doing work to slow down the car.

Example

A 50kg car with its engine off has an initial velocity of $40ms^{-1}$, but this is reduced to $20ms^{-1}$ over a distance of 300m due to a constant frictional force. **What is the value of the frictional force?**

The kinetic energy of the car reduces by **$(1/2 * 50 * 40^2) - (1/2 * 50 * 20^2) = 30000$J**.

The only force acting on the car to do this work is friction. Thus, the frictional force is doing 30000J of work over a distance of 300m.

The force required to do **30000J** of work over a distance of **300m** is **F = 30000 / 300 = 100N**.

-

Another way to view this is with kinematics:

u = 40, v = 20, s = 300

$a = (20^2 - 40^2) / 2(300) = -2ms^{-2}$.

This deceleration is caused by the frictional force.

When a **$2ms^{-2}$** acceleration acts on a **50kg** body, the force exerted must be **50(2) = 100N.**

Kinetic energy decreases over time

Suppose a ball is left to bounce. We see that the maximum height the ball reaches decreases with each successive bounce.

This is because when the ball contacts the ground, it transfers some of its kinetic energy to the ground. Additionally, as the ball moves through the air, air resistance depletes its kinetic energy.

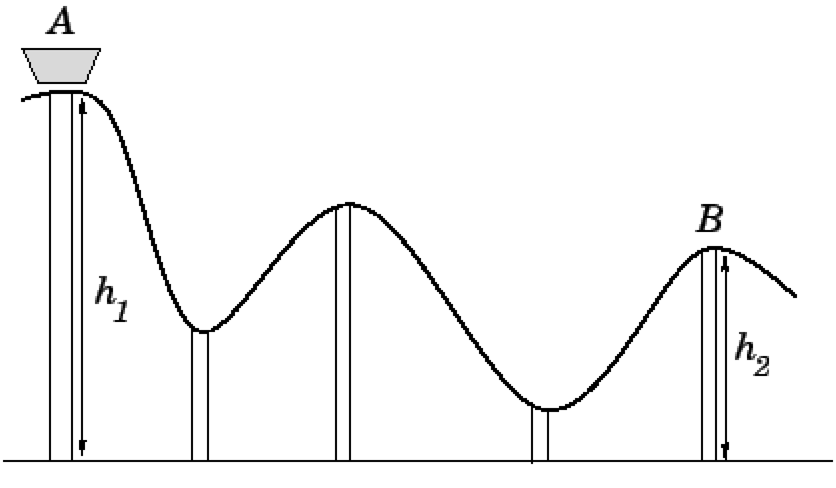
Consequently, during the next bounce, since kinetic energy is converted into potential energy as the ball rises, and the kinetic energy is lower, the potential energy must be lower - leading to a lower maximum height.

If there was no air resistance, and the surface was completely smooth, the ball would continue to bounce to the same height after each bounce indefinitely, as none of its energy would be lost to the surroundings.

-

The same principle applies to rollercoasters.

Suppose a rollercoaster cart is lifted to a height **h_1** by a motor, and then released. If the cart is



left to move freely without any further mechanical support for the rest of the path, this means that the height of the next rollercoaster peak cannot be as high as the first peak; and the third peak cannot be higher than the second peak.

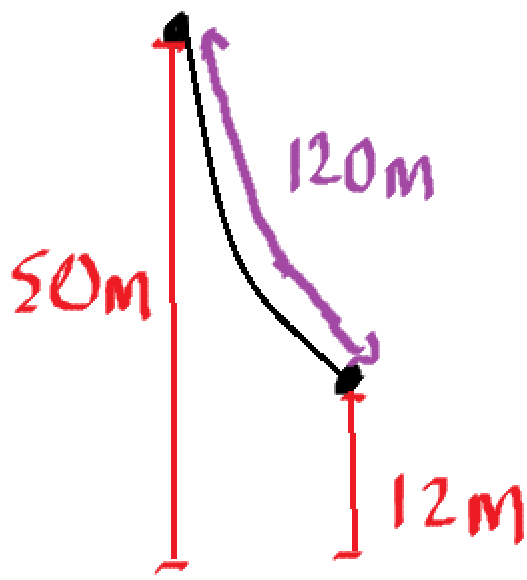
When the cart is at the top of the first peak, it has a certain gravitational potential energy. When it falls down the peak, its potential energy reduces due to a decreasing height, and energy cannot be completely destroyed, so this potential energy must be converted into kinetic energy.

As the cart moves along the track, there is friction between the cart and track, which transfers the kinetic energy of the cart into sound / thermal energy. When the cart begins climbing the next peak, its kinetic energy is reconverted into potential energy as its height increases and its speed decreases due to gravitational acceleration. However, because the kinetic energy has been reduced due to friction, the maximum potential energy that the cart can have is lower, and consequently it cannot reach the same height as the first peak.

For the cart to be able to reach a second peak of height equal to the first peak without mechanical support, it would have to lose no energy due to friction. The second peak could not possibly be higher than the first peak without an energy input, as this would require energy to be generated without being converted from a single form - which is impossible according to the law of the conservation of energy.

Example

A motor lifts a rollercoaster cart of mass 550kg to a height of 50m. The rollercoaster is released, and falls to a height of 12m as shown in the diagram. The distance the cart travels is 120m, and there is a constant frictional force of 300N acting against the cart. a) **Find the velocity of the cart at the height of 12m.**



When the rollercoaster falls from h=50m to h=12m, its gravitational potential energy is converted into kinetic energy. However, there is not a complete conversion, because the frictional force does work against the cart, depleting some of its kinetic energy. Since energy must be conserved, the total loss in gravitational potential energy must equal the total gain in kinetic energy plus the energy loss due to friction.

If a force of **300N** acts over **120m**, the total work done by that force must be **300 * 120 = 36000J**. This is the total energy loss due to friction.

The roller coaster has an initial potential energy of **550(50)(9.81) = $2.7*10^5$J**.

Its potential energy at a height of 12m is:
550(12)(9.81) = 6.47*10^4J.

The loss in potential energy from the height of 50m to the height of 12m is therefore **(2.7*10^5) - (6.47*10^4) = 2.05*10^5J**.

If there was no friction, this must have been converted completely into kinetic energy, but **36000J** is lost due to friction, so the kinetic energy at the height of 12m is **(2.05*10^5) - 36000 = 1.69*10^5J**.

Thus, **1.69*10^5 = 1/2 * 550 * v^2** => **v = 24.8ms^{-1}.**

b) Find the maximum height above the ground that the cart can reach from its height of 12m, neglecting friction.

If friction is neglected, we can assume that all of the **1.69*10^5J** of kinetic energy at the height of 12m is converted into potential energy.

Thus, **1.69*10^5 = 550 * g * h**
h = 31.3m.

-

An important point is that the conservation of energy is independent of the path taken.

Consider two situations:

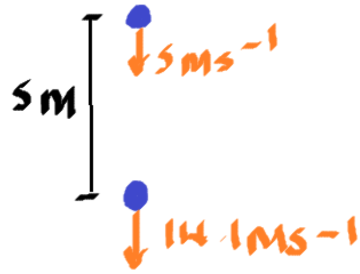
- A ball of mass 5kg travelling at 10ms^{-1} downwards falls downwards by a distance of 5m.
- A ball of mass 5kg travelling at 10ms^{-1} moves from the top of slope **A** to the top of slope **B**, where slope **B** is 5m lower in altitude than slope **A**.

What is the final velocity of the ball in both situations, assuming no friction or air resistance in both situations?

In the first situation, the change in potential energy must be equal to the gain in kinetic energy.

The change in potential energy is **5 * 9.81 * 5 = 245J**.

The final kinetic energy is therefore **245 + (1/2 * 10^2 * 5) = 495J**, giving a velocity of **495 = 1/2 * 5 * v^2** => **v = 14.1ms^{-1}**.



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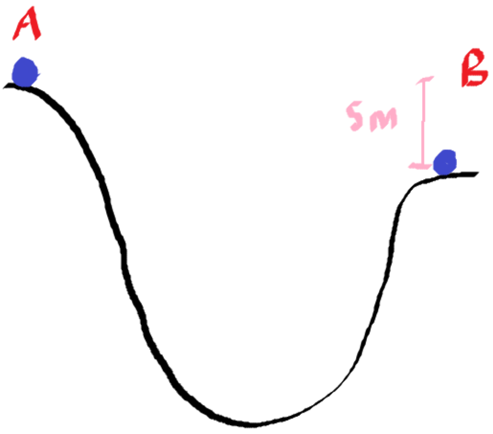
In the second situation, the ball again has a change in potential energy of **245J** from slope **A** to slope **B**. This is converted into kinetic energy, giving a final kinetic energy of **495J** and thus the same velocity.

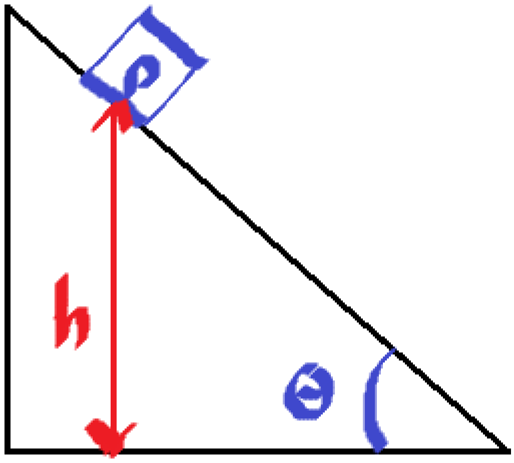


So, in both situations, even though the paths taken are different, there is still the same change in potential energy and therefore the same gain in kinetic energy. Energy is conserved no matter what path is taken. This is why energy is often a more useful model than kinematics or dynamics.

Work and energy on a slope

Let us apply the principles of conservation of energy to movement down a slope.

Take a particle **P** on top of a slope that is angled at Θ to the horizontal. The particle is at a height of **h** on the slope.



Because the particle is at a certain height, it has a gravitational potential energy.

When the particle is released, it will move down the slope due to the acceleration created by its horizontal weight component.

Its vertical height now decreases, so its potential energy decreases, while its kinetic energy increases.

If the slope is rough, some of its kinetic energy will be converted into heat and sound energy due to friction.

At the bottom of the slope, all of the potential energy of the particle is lost, and it has been converted into kinetic energy, as well as some heat and sound energy if friction exists.

Thus, we can say that **potential energy at top of slope = kinetic energy at bottom of slope + energy loss due to friction**, or **$E_g = E_k + W_F$**.

Once the particle reaches the bottom of the slope, it will continue to move off with some kinetic energy, **E_k**.

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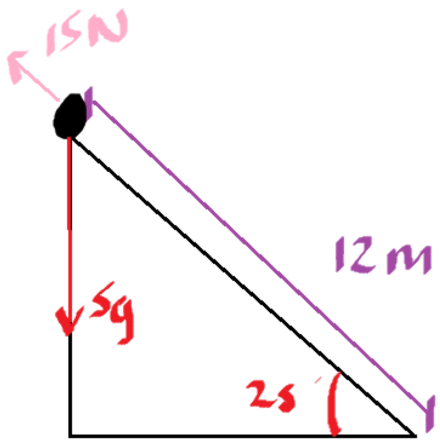
Example

A particle of mass 5kg is positioned at the top of a 12m long rough slope angled at 25° to the horizontal. The particle is released from rest, and accelerates to the bottom of the slope. The frictional force exerted on the particle is 15N.

Find the velocity of the particle at the bottom of the slope.



First let us solve the problem using dynamics.

The weight component of the particle acting down the slope is **5gsin25**.

The resultant force acting on the particle is therefore **5gsin25 - 15 = 5.73N**

This acts on a particle of mass **5kg**, so its acceleration must be **5.73 / 5 = 1.15ms^{-2}.**

The particle has an initial velocity of **0ms^{-1}** and a final velocity of **vms^{-1}**, and it moves a distance of **12m**.

Thus, s = 12, u = 0, a = 1.15, v = v

V = √(2(12)(1.15)) = **5.25ms^{-1}.**

Now let us consider energy principles to determine this value:

Using trigonometry, the vertical height of the particle on the top of the slope is **12sin25**m.

Its potential energy is therefore **12sin25 * 9.81 * 5 = 249J.**

If the slope was smooth (no friction), all of this potential energy would be converted into kinetic energy. However, the slope is actually rough, so some of the kinetic energy is lost due to friction.

The particle travels a distance of **12m** down the slope with a constant frictional force of **15N** acting against it, so the work done by friction is **12(15) = 180J**.

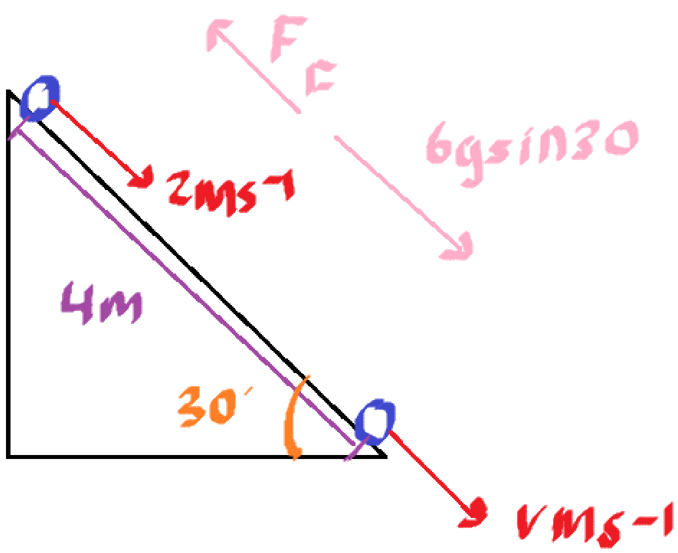
This means that the kinetic energy at the bottom of the slope is **249 - 180 = 69J**.

Thus, **69 = 1/2 * 5 * v^2** => **v = 5.3ms^{-1}**.

Example

*A particle P of mass 6kg is projected down a rough slope at a velocity of 2ms^{-1}. It hits the bottom of the slope with a velocity of **v**ms^{-1}. The slope is angled at 30º to the horizontal, and the particle moves a distance of 4m down the slope. The slope exerts a constant frictional force of 13N on the particle.*

a) Determine the value of v.



The vertical height of the particle is **4sin30 = 2m.**

Thus, at the top of the slope, its potential energy is **6(2)(9.81) = 118J.**

The particle moves a distance of **4m** down the slope with a frictional force of **13N** acting against it, so the kinetic energy lost due to friction is **4(13) = 52J**.

This means that the particle gains **118 - 52 = 66J** of kinetic energy when moving down the slope.

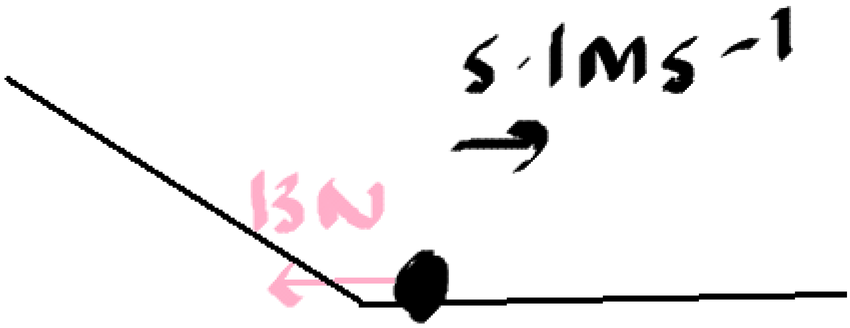
However, it also starts off with a kinetic energy of **1/2 * 6 * 2^2 = 12J**, so its total kinetic energy at the bottom of the slope is **66 + 12 = 78J.**

78 = 1/2 * 6 * v^2 => **v = 5.1ms^{-1}.**

When the particle hits the bottom of the slope, it starts moving freely along a rough horizontal plane with the same frictional force of 13N with an initial velocity of vms^{-1}. ***Find the time taken for the particle to be brought to rest****.*

The only force acting on the particle is now the frictional force, because there is no weight component in the horizontal plane.

This force acts against the particle, causing it to decelerate:



The deceleration of the particle is therefore **13 / 6 ms^{-2}.**

The velocity of the particle at the bottom of the slope is 5.1ms^{-1}, and we want to find the time taken for this deceleration to reduce the velocity to 0ms^{-1}.

u = 5.1, v = 0, a = -13/6, t = t

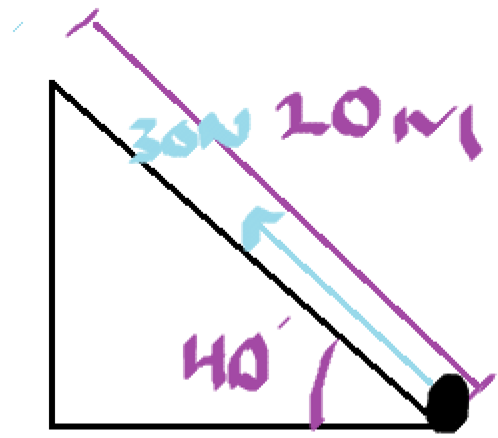
t = (0 - 5.1) / (-13/6) = **2.35s.**

Now let us consider what happens when a body is pushed up a slope.

If a body is pushed up a slope by a certain force **F**, it will gain both kinetic energy and potential energy. The total energy transfer, or **work done**, on the particle must equal the sum of the kinetic and potential energy if we ignore friction:

$$\mathbf{W = E_K + E_g}$$

For example, suppose a 4kg body is pushed 20m up a smooth slope inclined at 40° to the horizontal with a force of 30N, starting from rest.



If a force of **30N** acts over a distance of **20m**, the total work done is **20(30) = 600J**.

This work done must be converted into both kinetic energy and potential energy.

So, **600 =** $\mathbf{E_K + E_g}$. The slope is smooth, so no kinetic energy is lost due to friction.

The height at the top of the slope is **20sin40 = 12.9m.**

Thus, $\mathbf{E_g}$ **= 12.9 * 9.81 * 4 = 506.2J**.

This means that $\mathbf{E_K}$ **= 600 - 506.2 = 93.8J.**

Therefore, we can find the velocity of the body: **93.8 = 1/2 * 4 *** $\mathbf{v^2}$ => **v = 6.9**$\mathbf{ms^{-1}}$**.**

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Another way to view this is that the weight force of **4gsin40** acting down the slope does a work of **20(4gsin40) = 506.2J**. This work is done against the direction of the pulling force, so it depletes the work done by the pulling force by **506.2J**. Thus, the gain in kinetic energy is **600 - 506.2 = 93.8J**.

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We can find the same velocity using kinematics and dynamics:

The force acting on the particle is **30N**. The particle also has a weight component of **4gsin40 = 25.2N** acting down the slope. Thus, the net force is **30 - 25.2 = 4.8N.**

This acts on a **4kg** body, so **a = 4.8 / 4 = 1.2ms^{-2}.**

u = 0, a = 1.2, v = v, s = 20

v = √(2 * 20 * 1.2) = **6.9ms^{-1}.**

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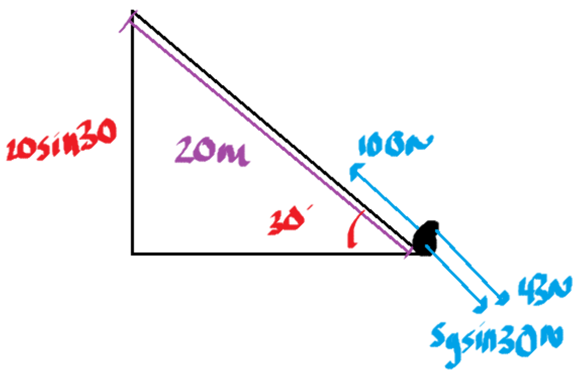
Now suppose that a body is pushed up a *rough* slope. The slope now has friction. This means that not all of the energy transfer (work) is to potential energy and kinetic energy - some of the kinetic energy is lost as sound / heat energy.

So we can say that **work done = E_k + E_g + Energy loss due to friction**

Example

A 5kg block is pushed from rest up a rough slope inclined at 30° to the horizontal by a force of 100N. The block is pushed a distance of 20m up the slope. The frictional force acting against the block is 43N.

Find the kinetic energy, and hence the velocity, of the block at the top of the slope.



If a force of 100N acts over a distance of 20m, the work done by that force is **100(20) = 2000J**.

At the top of the slope, the potential energy of the block is **5(9.81)(20sin30) = 490.5J**.

The energy loss due to friction is equal to **43(20) = 860J**.

So, **2000J** of work is done, which is converted into some kinetic energy, some potential energy, and some heat/sound energy due to friction.

So, **2000 = E_K + E_g + E_F**

2000 = E_K + 490.5 + 860 => **E_K = 649.5J.**

649.5 = 1/2 * 5 * v^2 => **v = 16.1ms^{-1}.**

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We can also prove this with kinematics and dynamics.

The particle has three forces acting on it in the plane of the slope:

- A 100N force up the slope
- A 43N frictional force opposing the direction of motion
- The horizontal component of weight acting down the slope, **5gsin30**.

The block is accelerating up the slope, so we can say the net force is **100 - 43 - 5gsin30 = 32.5N** up the slope.

This leads to an acceleration of **32.5 / 5 = 6.5ms^{-2}.**

Thus, u = 0, s = 20, a = 6.5

v = √**(2 * 6.5 * 20) = 16.1ms^{-1}.**

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If the body had an initial kinetic energy before moving up the slope, we would have to add this to the final kinetic energy to find the total kinetic energy at the top of the slope.

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Energy conservation on slope examples

When considering energy conservation on a slope, or in other examples of energy conservation, we have to consider the total energy transfer involved, and how much energy must be inputted to cause this energy transfer.

Example 1

Take a situation where the amount of potential energy of a particle at the top of a slope is **800J**. The particle is released and moves to the bottom of the slope. The kinetic energy of the particle at the bottom is **400J**, and the work done by friction is **900J** from the top to the bottom of the slope. **What work must have been done by a driving force to allow these energy transfers?**

The potential energy of the particle is converted into kinetic energy and heat / sound energy due to friction. However, the total energy transfer to the bottom of the slope is **1300J**, yet there is only **800J** of potential energy available. This means that an additional **500J** of work must be done to cause this energy transfer.

Now let us consider this in terms of the total energy input and the total energy output:

The total energy input is a sum of the potential energy (**E_g**) and the work done by the driving force (**W_D**). The total energy transfer is the kinetic energy (**E_K**) and the work done by friction (**W_F**).

We can therefore say: **$E_g + W_D = E_K + W_F$**

Thus, **$800 + W_D = 400 + 900$** => **$W_D = 500$J**.

Example 2

Suppose a particle at the top of a slope has **200J** of potential energy. The particle is released to the bottom of the slope, and a driving force does **1200J** of work. If the kinetic energy of the particle at the bottom of the slope is **800J**, **what is the work done by friction?**

The total energy that is transferred is a sum of the potential energy and the work done by the driving force, **1400J**. **800J** of this energy is transferred to kinetic energy, which means that **600J** of work must have been done by the frictional force.

$E_g + W_D = E_K + W_F$

$200 + 1200 = 800 + W_F$ => **$W_F = 600$J**.

Example 3

A particle is pushed up a slope by a tension force **F**. The total work done by friction in pushing the particle to a height **h** is **450J**, and the gain in potential energy is **620J**. If the total work done by the tension force is **1300J**, **what is the kinetic energy of the particle at the top of the slope?**

The total energy that is inputted is the work done by the tension force. This is transferred to potential energy, work done by friction, and kinetic energy.

Thus, **$W_T = E_K + E_g + W_F$** where **W_T** is the work done by the tension force.

$1300 = E_K + 450 + 620$ => **$E_K = 230$J**.

Example 4

A particle at the bottom of a slope has **350J** of kinetic energy. It climbs up the slope under a driving force **F**. When the velocity of the particle has reduced to $0ms^{-1}$, it has gained a potential energy of **900J**, and the total work done by friction was **1400J**. **What work must have been done by the driving force?**

350J of kinetic energy has the potential to transfer into **350J** of potential energy and heat / sound energy. However, the total energy conversion is **900 + 1400 = 2300J**. This means that an additional **2300 - 350 = 1950J** of work must have been done by the driving force.

$E_K + W_D = W_F + E_g$ => $350 + W_D = 1400 + 900$ => $W_D = 1950J$.

Example 5

A particle at the bottom of a slope has **800J** of kinetic energy. The particle climbs up the slope, with an additional driving force, **F**, and at the point where the particle has **200J** of kinetic energy, it has gained **1300J** of potential energy and **600J** of work has been done by friction. **What work must have been done by the driving force?**

The particle's kinetic energy has reduced by **600J**. With no driving force, this has the potential to convert to **600J** of potential energy and heat / sound energy. However, the total energy transfer is **1900J**, so an additional **1300J** of work must have been done by the driving force.

Thus, $E_K + W_D = E_g + W_F$ => $600 + W_D = 1300 + 600$ => $W_D = 1300J$.

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Now, we can consider these problems with actual calculations.

Example 1

A 10kg particle is at the top of a 12m long slope inclined at 12° to the horizontal. The particle falls down the slope from rest, with a driving force ***F*** *supporting its motion. At the bottom of the slope, the particle has a velocity of* $9ms^{-1}$*. The frictional force acting on the particle is 14N.* **By considering conservation of energy, determine the value of the driving force F**.

The potential energy of the particle at the top of the slope is **12sin12 * g * 10 = 245J**.

The kinetic energy at the bottom of the slope is $1/2 * 9^2 * 10 = 405J$, and the total work done by the frictional force is **14(12) = 168J**.

The total energy input is $245 + W_D$ where W_D is the work done by the driving force, and the total energy output is **405 + 168 = 573J**.

This means that **573 - 245 = 328J** of work must have been done by a driving force.

W = Fd => **F = 328 / 12 = 27.3N**.

Example 2

A particle of mass 8kg is at the bottom of a rough slope inclined at 32° to the horizontal. The particle climbs the slope with an initial velocity of $4ms^{-1}$ under a driving force of 100N. At the point where the particle has travelled 9m up the slope, it has a velocity of $10ms^{-1}$. **Find the value of the constant frictional force exerted on the particle.**

The total work done by the driving force is **100(9) = 900J**.

The gain in kinetic energy of the particle is **(1/2 * 8 * 100) - (1/2 * 8 * 16) = 336J**, and the total gain in potential energy is **9sin32 * 8 * 9.81 = 374J**.

The total energy input is **900J**, and the total energy transfer is **336 + 374 + the energy loss due to friction.** Thus, the energy loss due to friction is **900 - 336 - 374 = 190J**.

190 = F * 9 => **F = 21.1N**.

Example 3

A particle of mass 12kg is moving down a slope inclined at 31° to the horizontal under a constant driving force of **400N**. At point A, the particle has a velocity of $42ms^{-1}$, and at point B, the particle has a velocity of **vms^{-1}**. The distance AB is 40m. The frictional force exerted on the particle is a constant **30N**. **Determine the value of v by considering energy changes.**

If a driving force of **900N** acts over a distance of **40m**, the work done is **400(40) = 16000J**.

The change in potential energy between point **A** and point **B** is **12 * 40sin31 * 9.81 = 2430J**.

This means that the total energy that can be transferred to other forms between A and B is **16000 + 2430 = 18430J**.

The work done by the frictional force between **A** and **B** is **30(40) = 1200J**. This means that the transfer to kinetic energy is **18430 - 1200 = 17230J**.

The total kinetic energy at point **B** is therefore **$(1/2 * 42^2 * 12) + 17320 = 27904J$**.

Thus, **$27904 = 1/2 * 12 * v^2$** => **$v = 68.2ms^{-1}$**.

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Example 4

A particle of mass 4kg is moving freely up a slope inclined at 12° to the horizontal. At point A on the slope, the particle has a velocity of $9ms^{-1}$, and at point B, it has a velocity of $5ms^{-1}$. The frictional force acting against the particle is a constant 20N. **Determine the distance AB by considering energy changes**.

The particle is not under any driving force, so we do not have to consider the work done by the driving force. We will let the distance AB be **x**.

The change in kinetic energy of the particle between A and B is (**1/2 * 81 * 4) - (1/2 * 25 * 4) = 112J**.

This is converted into potential energy and heat / sound energy due to friction.

$\mathbf{E_K = E_g + W_F}$.

The change in height of the particle between A and B is **xsin12**. The gain in potential energy is therefore **xsin12 * g * 4**. The energy loss due to friction is **20x**.

Thus, **112 = 4gxsin12 + 20x** => **x(4gsin12 + 20) = 112** => **x = 3.98m**.

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We can also prove this with dynamics.

The total force acting against the particle is **20 + 4gsin12 = 28.16N**.

Thus, **a = 28.16 / 4 = 7.04**$\mathbf{ms^{-2}}$.

u = 9, v = 5, a = -7.04

$\mathbf{5^2 = 9^2 + 2(-7.04)(x)}$ => **x = 3.98m**.

Example 5

A particle of mass 13kg is moving down a slope inclined at 28° to the horizontal, with a braking force of **FN** acting against it, as well as a frictional force of **15N**. The particle has a velocity of $5ms^{-1}$ at point A and a velocity of $8ms^{-1}$ at point B. The distance between A and B is 30m. **Find the value of F**.

The particle loses potential energy, and this is converted into kinetic energy and work done by friction. There is also a braking force that does work on the particle.

Hence, $\mathbf{E_g = E_K + W_F + W_B}$ where $\mathbf{W_B}$ is the work done by the braking force.

The change in potential energy between A and B is **13 * g * 30sin28 = 1796J**.

The gain in kinetic energy is $\mathbf{(1/2 * 8^2 * 13) - (1/2 * 5^2 * 13) = 253.5J}$.

The work done by friction is **15(30) = 450J**.

So **1796J** of potential energy is converted into **253.5 + 450 = 703.5J** of other energy forms. The remaining **1796 - 703.5 = 1092.5J** must have been depleted due to the braking force.

Thus, **1092.5 = F * 30** => **F = 36.4N**.

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The above example can be proven with dynamics.

1 - find acceleration of particle

u=5, v=8, s=30

$8^2 = 5^2 + 2(30)(a)$ => **a = 0.65ms^{-2}.**

2 - consider net force

The particle has a force of **13gsin28** acting down the slope, and this is opposed by a frictional force and the braking force.

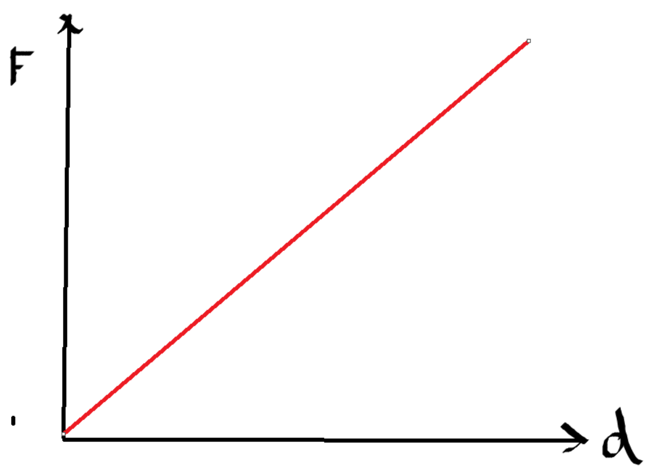
Thus, **13gsin28 - 15 - F = 13 * 0.65** => **F = 36.4N**

Force-distance graph

We have previously considered work when the force applied to a body is constant.

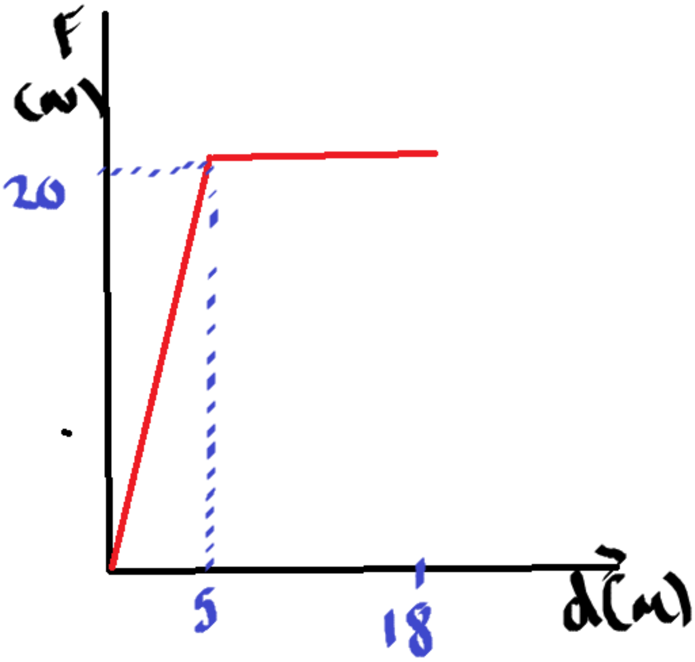
However, the force applied to a body can also vary over the distance for which it is applied.

This can be shown in a **force-distance graph:**



Since **work = force * distance**, this is the same as the area under the graph.

Work done = area under force-distance graph



Consider the graph to the left.

Suppose a 10kg body initially at rest has this variable force applied to it across a smooth horizontal plane.

The total area under the graph tells us the work done to the body:

Work done = (1/2 * 20 * 5) + (13 * 20) = 310J.

If **310J** of work is transferred to a body, and the body is only gaining kinetic energy, the body must gain **310J** of kinetic energy.

Thus, **310 = 1/2 * 10 * v^2**
v = 7.87ms^{-1}.

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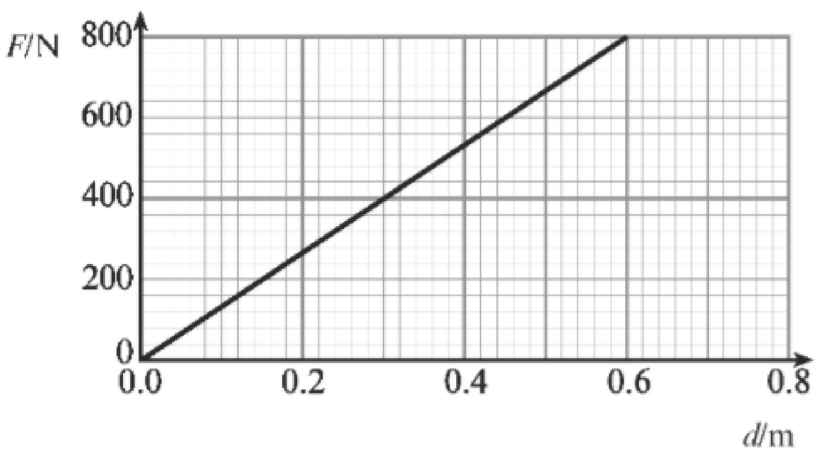


The graph on the left shows how the distance a bow string is pulled back varies with the force **F** that is pulling the bow back.

Suppose that the distance the bowstring is pulled back is **0.6m**. The force exerted must be 800N.

The work done to achieve this is the area under the graph**, 1/2 * 800 * 0.6 = 240J**.

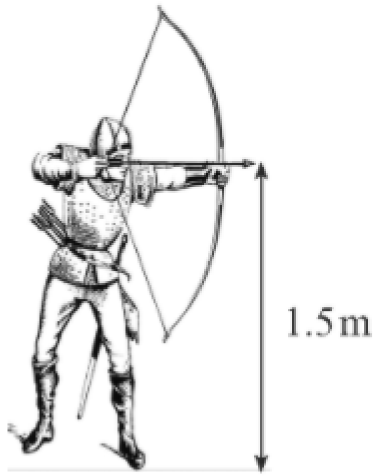
This means that **240J** of work has been done on the bowstring, so it is storing 240J of energy.

Suppose an arrow of mass **$20*10^{-3}$kg** is being drawn back in the bow by 0.6m. It is then released in the horizontal direction at a height of 1.5m above the ground.

When the bow is released, the **240J** of energy in the string is converted into kinetic energy in the arrow.

$240 = 1/2(50*10^{-3})v^2$

v = 98ms^{-1} => horizontal velocity of arrow upon release



We can then use projectile motion concepts to determine how far the arrow travels horizontally.

Its initial vertical velocity is **0ms^{-1}**, and it descends a distance of **1.5m**.

u = 0, s = 1.5, a = 9.81, t = t

$1.5 = 1/2(9.81)t^2$

t = 0.55s.

The bow travels with a constant horizontal velocity of **98ms^{-1}** over a time of **0.55s**, so distance travelled = 98(0.55) = **53.9m**.

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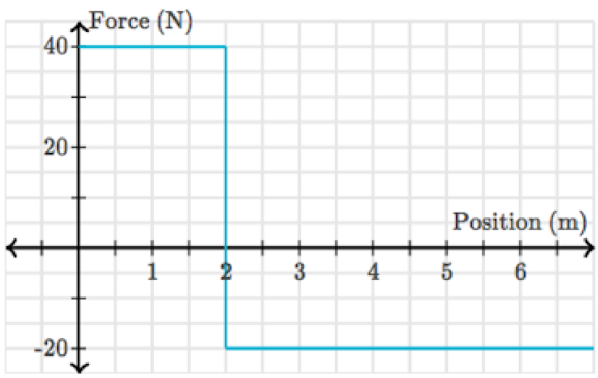
If a body is moving along the horizontal plane under a force **F** in the direction of travel, the work done to the body is transferred to kinetic energy.

$$Fd = 1/2mv^2.$$

Now suppose that the net force **F** changes such that it opposes the direction of motion. This causes the velocity of the body to reduce, so its kinetic energy reduces. In such a situation, ‘negative work’ is being done to the body, since the body now transfers its kinetic energy to the surroundings rather than gaining kinetic energy.

*We previously said that the work done is the force component in the direction of travel multiplied by the distance, or **work done = FdcosΘ**.*

*If a force is acting opposite to the direction of motion, **Θ=180**. **Cos 180 = -1**, so the work done becomes negative.*



For example, take the following graph:

The force applied is initially positive, and it is therefore causing the body to gain kinetic energy.

At a position of 2m, the force becomes negative, so the force is now acting against the direction of motion, reducing the kinetic energy of the body.

Suppose the force-position graph above is applied to a particle of mass 15kg that is initially moving at $2ms^{-1}$ along a smooth horizontal plane.

Up to a position of **2m**, the work done is **40(2) = 80J**. When work is done to a body moving along a smooth horizontal plane, all of the work done must be transferred as kinetic energy.

This work done will cause the body to accelerate, and gain **80J** of kinetic energy.

Its initial kinetic energy is **$1/2 * 2^2 * 15 = 30J$**, so it now has **110J** of kinetic energy.

At a position of 2m, the force drops to -20N, so it is now in the opposite direction to the direction of travel, opposing the motion.

The area under the graph between **2** and **6m** is **-20(4) = -80J**.

This means that the body loses **80J** of kinetic energy due to this opposing force, so its overall kinetic energy is now **110-80 = 30J**. This **80J** of energy is lost to the surroundings as heat / sound energy.

Thus, its velocity is: **$30 = 1/2(15)v^2$** => **$v = 2ms^{-1}$.**

The particle gained **80J** of kinetic energy, then lost **80J** of kinetic energy, so its velocity is now back to **$2ms^{-1}$.**

Power

Work does not define the time over which a force is applied; only the distance over which the force is applied.

Power is a way of measuring the work done over the time taken to do that work.

Power = work done / time

Si units: ***Power = (F * d) / t = kgms-2 * m / s = kgm2s-3***

The units for power are **watts (W)**, or **Js^{-1}**. One watt is defined as doing 1 joule of work over one second.

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Suppose a 50N force acts on a 10kg block initially at rest, causing it to move a distance of 20m along a smooth horizontal plane.

The work done on the body is **50(20) = 1000J.**

The time taken for the journey can be found with SUVAT:

$a = F / m = 50 / 10 = 5ms^{-2}$.

u = 0, a = 5, s = 20

$20 = 1/2(5)t^2$

t = **2.83s**.

So, **1000J** of work is being done over **2.83s**, meaning the power input into the body is **1000/2.83 = 353Js^{-1} = 353W.**

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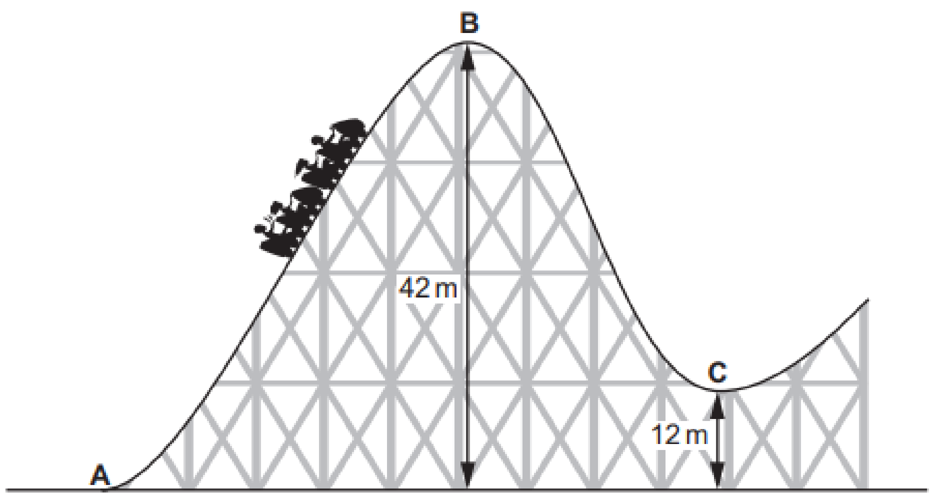
Consider the following question:

The diagram shows part of a rollercoaster ride at a theme park.



(a) A motor with a power output of 65 kW and a chain mechanism pulls the carriages of mass 2 600 kg from **A** to **B** in a time of 32 s.

(i) Show that the work done by the motor in 32 seconds is approximately 2 MJ. [1]

The motor allows the carriages to be lifted to a higher gravitational potential energy; it is doing work on the carriages.

The motor has a power output of **65kW**, so it does **65000J** of work **per second**.

This means that in a time of **32s**, it will do **65000(32) = 2.08*10^6J** of work = **2.08MJ**.

-

In many mechanical systems, such as in the motor example shown above, not all of the energy inputted is outputted as useful energy.

When the rollercoaster carriage climbs to a height of 52m, it has a gravitational potential energy of **2600(9.81)(42) = 1.07*10^6J = 1.07MJ.**

So **2.08MJ** of energy was inputted by the motor to bring the carriage to a height of **42m**, but only **1.07MJ** was transferred as gravitational potential energy. Since energy must be conserved, the remaining **2.08 - 1.07 = 1.01MJ** of energy must have been lost to the surroundings as waste heat / sound energy. This energy would have been lost in the motor itself, and due to the friction of the carriage.

This leads to the idea of **efficiency.**

Efficiency measures what percentage of the total energy / power input is outputted as useful energy / power.

Efficiency = (useful (energy or power) output / total (energy or power) input) * 100

In the example above, the useful energy output is **1.07MJ** and the total energy input is **2.08MJ**, so the efficiency of the motor is **100(1.07 / 2.08) = 51.4%.**

We could also view this in terms of power. The rollercoaster carriage gains **1.07MJ** of energy over **32 seconds**, so the useful power transferred to the carriage is **(1.07*10^6) / 32 = 33400W = 33.4kW.**

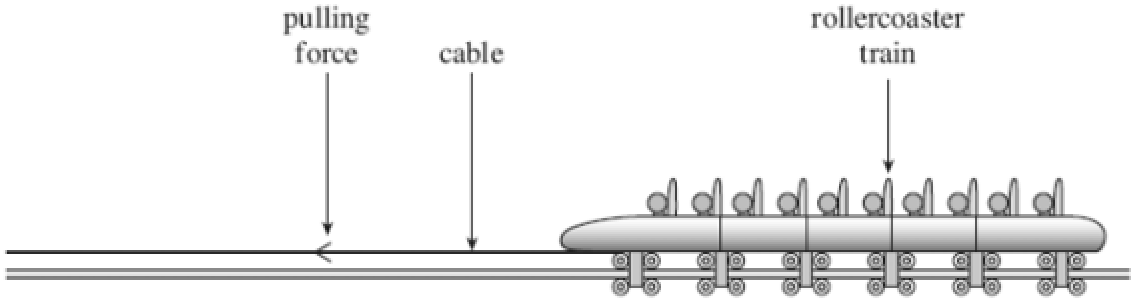
The power of the motor is **65kW**, so the efficiency is **100(33.4 / 65) = 51.4%.**

-

Let us consider some examples of efficiency and power:

Example

The figure below shows a rollercoaster train that is being accelerated when it is pulled horizontally by a cable.



(a) The train accelerates from rest to a speed of $58ms^{-1}$ in 3.5 s. The mass of the fully loaded train is 5800 kg.

[Assume no friction]

a) Calculate the tension force in the cable.

The tension force in the cable accelerates the rollercoaster train.

The acceleration of the train is **58 / 3.5 = $16.6ms^{-2}$.**

This acts on a **5800kg body**, thus **T = 16.6(5800) = 96300N.**

This assumes that there is no frictional force.

b) Hence, calculate the work done on the train during the acceleration period by the tension force.

We need to determine the distance over which the tension force acts, which is the distance during the acceleration period.

u = 0, a = 16.6, v = 58, t = 3.5

s = 1/2 * 58 * 3.5 = 101.5m.

Since **W = Fd, W = 96300(101.5) = $9.8*10^6$J**.

The kinetic energy of the train after the acceleration period is **1/2 * 5800 * 58^2 = $9.8*10^6$J**. This shows that none of the energy used to accelerate the rollercoaster was lost due to friction.

c) Given that the efficiency of the rollercoaster acceleration system is 20%, calculate the total power input to accelerate the rollercoaster.

The kinetic energy of the rollercoaster train after acceleration is **$9.8*10^6$J** => this is useful energy.

The fact that the efficiency is only **20%** tells us that 80% of the energy inputted into the acceleration system must have been wasted, and only **$9.8*10^6$J** was outputted as useful energy.

If we say the total energy input is **E_I**,

We can say that **$(9.8*10^6 / E_i) * 100 = 20$**

$E_i = (9.8*10^6) / 0.2 = 4.9*10^7$J.

This is the total energy that must be inputted to convert **$9.8*10^6$J** of kinetic energy for a **20%** efficient system.

The question asks us to find the total *power* input.

The acceleration occurs over a time of **3.5s**, so **$4.9*10^7$J** of energy is inputted over **3.5s**, giving a power input of **$4.9*10^7 / 3.5 = 1.4*10^7$W = 14MW.**

Example

It has been predicted that in the future large offshore wind turbines may have a power output ten times that of the largest ones currently in use. These turbines could have a blade length of 100 m or more. A turbine such as this is shown in the diagram below.

(a) At a wind speed of 11 m s^{-1} the volume of air passing through the blades each second is 3.5×10^5 m^3.

[Density of air = 1.2kgm^{-3}]

Determine the kinetic energy of the air entering the wind turbine per second.

The mass of air entering the turbine per second is **$3.5*10^5$ * 1.2 = 420000kg.**

Thus, the kinetic energy of the air is **1/2 * 420000 * 11^2 = $2.54*10^7$J**

This is entering the wind turbine per second, so the power input is **$2.54*10^7$W** or **25.4MW**.

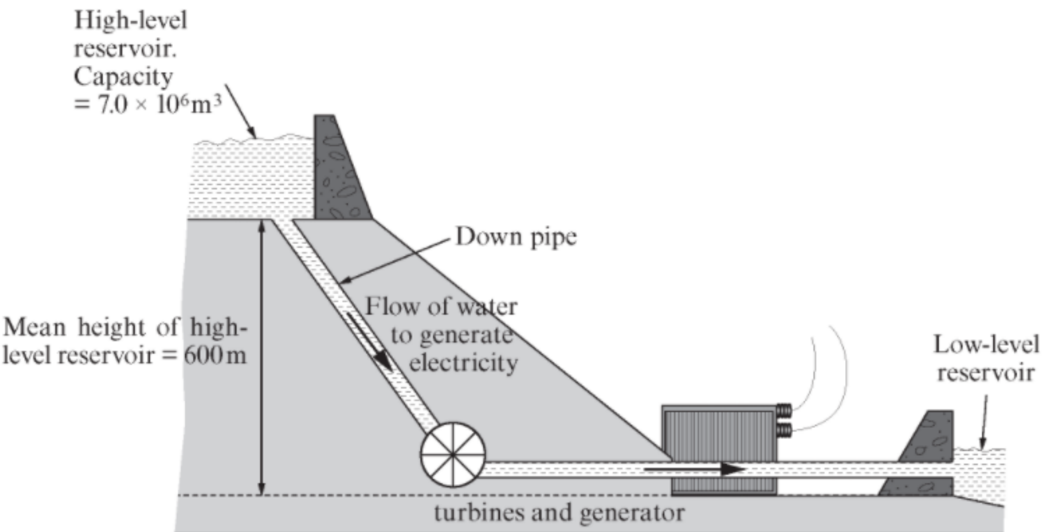
It has been predicted that the turbine would produce an electrical power output of 10 MW in these wind conditions. Calculate the percentage efficiency of the turbine in converting this kinetic energy into electrical energy.

There is **25.4MW** of energy inputted into the wind turbine per second, but only **10MW** is converted into electrical energy.

Thus, **efficiency = 100(10 / 25.4) = 39.4%.**

Example

The hydroelectric power station at Dinorwig in North Wales is the largest of its kind in Europe. A simplified diagram showing the main features of the plant is shown.



(a) Use the information in the diagram to show that the gravitational potential energy stored in the high-level reservoir is approximately 4×10^{13} J.
[Density of water = 1000 kg m^{-3}]. [2]

The volume of water in the reservoir is **$7*10^6$m^3**.

Thus, the mass of water is **$7*10^6$*1000 = $7*10^9$kg.**

This means that the gravitational potential energy of the water is **$7*10^9*9.81*600 = 4.12*10^{13}$J**.

(b) The power plant has six 300 MW generators. Calculate the longest time for which the stored energy could provide power at maximum output given that the generation process is 90% efficient [i.e. 10% of the gravitational potential energy stored in the high level reservoir is wasted]. [3]

Six 300MW generators would require **300(6) = 1800MW** of energy input. This is **1800MJ** per second.

The energy stored in the reservoir in MJ is **$(4.12*10^{13}) / 10^6 = 4.12*10^7$MJ**.

90% of this energy is provided to the generators, so this is an energy of **$0.9(4.12*10^7)= 3.71*10^7$MJ**.

So the generators require **1800MJ** per second, and there are **$3.71*10^7$MJ** available in total, so the total operating time is **$3.71*10^7$ / 1800 = 20600s = 5.7hours.**

(c) (i) Calculate the mean rate of flow of water (in kg s^{-1}) through the turbines of the power station when it is operating at full power. [1]

The total mass of water is **$7*10^9$kg.**

It takes **20600s** for this mass to flow through the turbines, so each second, the mass through the turbines is **$(7*10^9) / 20600 = 3.4*10^5$kg**.

Mean rate of water flow = **$3.4*10^5 kgs^{-1}$.**

(ii) After passing through the turbines the water enters the lower lake at a speed of 20 m s^{-1}. Use your answer to (c)(i) to calculate the kinetic energy per second [power] of this water. [1]

If **$3.4*10^5$kg** of water enters the lower lake each second, this has a kinetic energy of:

$1/2 * (3.4*10^5) * 20^2 = 6.8*10^7$J.

(iv) Hence show that your answer to (c)(ii) represents between 30% and 40% of the wasted power. [1]

As the water flows down from the reservoir, its potential energy is converted into kinetic energy. The flow of water spins turbines, which converts the kinetic energy into electrical energy.

If the turbines were 100% efficient, all of this kinetic energy would be converted into electrical energy. However, the water has a velocity $20ms^{-1}$ when it enters the lower reservoir, so this means that it still has some kinetic energy - meaning that not all of the kinetic energy was converted into electrical energy.

-

We must start by finding the total energy wasted per second.

The energy stored in the reservoir is **$4.12*10^7$MJ**, but only **$3.71*10^7$MJ** is converted, so **$4.1*10^6$MJ** of energy is lost.

This energy is converted over **20600s**, so each second, **$4.1*10^6$ / 20600 = 199MJ** is wasted.

The kinetic energy of the water is **68MJ**, so the percentage of waste energy that the kinetic energy of the water makes up is **100(68 / 199) = 34.2%**.

(v) Where else would energy be wasted during the generating process? [1]

The remaining **65.8%** of energy could be wasted due to the friction between the water and the pipe depleting some of its kinetic energy. It could also be lost due to friction with the turbine.

Power = force * velocity

Another way of expressing power is **power = force * velocity**.

This is because **power = work done / time = (F * d) / t**.

Distance / time is equal to **velocity**. Thus, **Power = F * v**.

--

An example of where this can be applied is in a car.

Suppose a car has a driving force of 800N and is moving with a velocity of $30ms^{-1}$. The power at this instant is **800 * 30 = 2400W** => so the driving force is inputting **2400J** each second.

Another example is a cyclist with a driving force of 30N, moving with a velocity of $10ms^{-1}$. The power exerted by the cyclist is **30 * 10 = 300W**.

It is important that the power exerted by a force is always the force only, and not the net force.

For example, if a car has a driving force of 400N and a frictional force of 50N and is moving at $20ms^{-1}$, and we want to find the power exerted by the engine to create that driving force, we are only concerned with the 400N driving force, not the frictional force.

So, **P = 400 * 20 = 8000W**.

Power output of a vehicle

Often, we are given the **power** of a vehicle. This can be used to determine its velocity or the driving force.

The power output of a vehicle determines how fast it can travel. The higher the power output, the more work that can be done per second, and therefore the higher the maximum velocity.

Example

Suppose a motorcycle has a power output of **24kW** and is moving with a velocity of **$40ms^{-1}$**. ***What is the driving force of the car?***

Using **P = Fv,** we can say **F = P / v**, so the driving force is **$(24 * 10^3) / 40 = 600N$**.

Example

Suppose a car is travelling at a constant speed across a horizontal plane with a friction of **1600N** acting against it. The maximum power output of the car is **40kW**. ***What is its velocity?***

If the car is moving at a constant velocity, it is not accelerating, so the net force on the car is 0.

Thus, the driving force is equal to the resistive force, so the driving force is **1600N**.

Using **P = Fv**, **$v = P / F = (40*10^3) / 1600 = 25ms^{-1}$.**

Considering maximum velocity

If a car is moving at its *maximum velocity*, it cannot be accelerating. This means that the net force on the car is 0, so the resistive force equals the driving force:

Example

Suppose a car of mass **1000kg** has a maximum velocity of **35ms^{-1}** on a level road against a resistance of **400N**. ***What is the maximum power output of the car's engine?***

Because the car is at its maximum velocity, the resistive force equals the forward driving force. Thus, the driving force is **400N**.

P = Fv, so **P = 400 * 35 = 14000W = 14kW**.

-

Once finding the maximum power output of a vehicle, we can apply it to find the velocity / acceleration of the vehicle in other situations.

If a vehicle is moving is operating at its maximum power output, but its velocity is not at its maximum, the vehicle must be accelerating to that maximum.

Following on from the last question, suppose the vehicle is moving with a velocity of **30ms^{-1}** at its maximum power output with a resistive force of **300N**. ***What is the acceleration of the vehicle at this instant?***

Since the vehicle is at its maximum power output, but it is not at its maximum velocity, it must be accelerating to its maximum velocity.

Using the maximum power output of **14kW**, the driving force when **v = 30ms^{-1}** must be **14000 / 30 = 467N**.

Thus, the net force acting on the car is **467 - 300 = 167N**.

This leads to an acceleration of **167 / 1000 = 0.167ms^{-2}**.

--

The maximum velocity of a vehicle can change depending on:

1- The frictional force of the surface the vehicle is travelling on

As shown previously, when a vehicle is moving at maximum velocity, the driving force is equal to the resistive forces.

Thus, if the frictional force increases, the driving force must also increase. Since **P = Fv**, a higher driving force with the same power output will lead to a lower maximum velocity.

2 - The incline of the slope on which the vehicle is travelling

If a vehicle is travelling on an incline, it not only has a frictional force acting against it, but the weight component acting down the incline. This means that the total resistive force is greater, and since the resistive forces are equal to the driving force at maximum velocity, the driving force must also be higher.

Thus, since **P = Fv**, since **F** is higher, if the same maximum power output, **P**, is maintained, **v** must be lower.

-

Suppose the same **1000kg** car with a maximum power output of **14kW** is moving up a slope inclined at **8.2°** to the horizontal, with the same frictional force of **400N**. ***What is the maximum velocity of the car on this slope?***

The maximum velocity of the car will now be lower because the car has both the frictional force of **400N** and the component of weight of the car acting down the slope, **1000gsin8.2 N**.

The total force acting against the car's motion is therefore **400 + 1000gsin8.2 = 1800N**.

Because the car is moving at its maximum velocity, the forward driving force is equal to the resistive forces.

Thus, driving force = **1800N**.

Using **P = Fv** => **v = P / F = 14000 / 1800 = 7.8ms^{-1}** => this is much lower than the maximum velocity of **35ms^{-1}** when the car is travelling on level ground.

-

If we increased the incline of the slope, the weight component would increase. This leads to an increase in the driving force at maximum velocity, so the maximum velocity is lower.

-

Let us consider another problem:

> The engine of a lorry of total mass 2 tonnes is working at 50 kW. The lorry is travelling at a constant speed of 20 ms^{-1}, along a level road. Find the total resistance to the motion.

If a vehicle is travelling at **20ms^{-1}** and its power output is **50kW**, the driving force is **50000 / 20 = 2500N**.

Since the vehicle is travelling at a constant velocity, the driving force equals the resistive force, so the resistive force is also **2500N**.

If the power is increased to 60 kW, find the acceleration of the lorry at the instant it is moving with speed 20 ms^{-1} assuming that the resistances to motion remain constant.

The power of the lorry is now higher, so it has a potential to reach a higher maximum velocity.

When the lorry is moving at **$20ms^{-1}$** with a power output of **60kW**, the driving force is **60000 / 20 = 3000N**.

The resistive force is still **2500N**, so the net force on the lorry is now **500N**.

Thus, the lorry will accelerate at **500 / 2000 = $0.25ms^{-2}$**.

-

A car of mass 1000 kg moves on a horizontal road against a resistance of 600 N with the engine working at a rate of 8 kW. Find the acceleration of the car at the instant it is moving with a speed of 10 ms^{-1}.

The driving force is **8000 / 10 = 800N**.

Thus, **800 - 600 = 1000 * a** => **a = $0.2ms^{-2}$**.

-

A car of mass 1500 kg travels up a slope of inclination α to the horizontal, where $\sin \alpha = \frac{1}{49}$, against constant frictional resistances of 3600 N. Find the maximum speed of the car given that the engine works at a rate of 80 kW.

If the car is moving at its maximum velocity, the total forces down the slope are equal to the driving force.

Thus, **driving force = 3600 + 1500g(1/49) = 3800N**.

Using **P = Fv** => **v = $80*10^3$ / 3800 = $21.1ms^{-1}$.**

After reaching the top of the slope the power is switched off and the car descends a slope of inclination β to the horizontal against the same constant frictional resistances at constant speed. Calculate $\sin \beta$.

The car now has a weight component of **1500gsin**β acting down the slope and a frictional force of **3600N** acting up the slope (since the car is now moving down the slope).

There is no longer a driving force since the engine has been switched off.

Since the car is moving at constant velocity, the weight and frictional forces must be equal.

Thus, **3600 = 1500gsin**β

Sinβ **= 3600 / 1500g**

β **= 14.2°**.

-

The resistive forces opposing the motion of a train of total mass 50 tonnes are 5000 N.

(a) Find the power necessary to keep the train moving along a straight level track at a constant speed of 10 ms^{-1}.

(b) If this power is suddenly increased by 10 kW when the train is moving along the level track at 10 ms^{-1}, find the initial acceleration of the train.

(c) When the train climbs a hill, of inclination α to the horizontal, at a constant speed of 8 ms^{-1}, the engine of the train is working at a rate of 180 kW. Find the value of sin α.

For part c), assume the frictional force is still 5000N.

a)

The resistive forces equal the driving force at constant velocity.

Thus, **P = 10 * 5000 = 50000W = 50kW**.

b)

The train now has a power output of **60kW**.

The driving force at **10**ms^{-1} is therefore **60000 / 10 = 6000N**.

Thus, **6000 - 5000 = (50*10**^{3}**) * a**

a = 0.02ms^{-2}.

c)

The driving force of the train at this speed is **180*10**^{3} **/ 8 = 22500N**.

The train is moving at constant velocity, so the driving force equals the forces acting down the slope.

The forces acting down the slope are the frictional force and the horizontal component of weight.

Thus, **22500 = 5000 + (50*10^3*g*sinα)**

Sinα = (22500 - 5000) / (50*10^3*g) = 0.0357

[Thus, **α = 2.05°**.]

-

Summary of work, energy and power

- Energy can occur in many forms: kinetic, potential, electric etc.
- Energy is the capacity to *do work*.
- **Work** is a measure of the force applied to a body in the direction of travel multiplied by the distance for which the force is applied, **W = FdcosΘ** - the **cosΘ** allows us to obtain the force component in the direction of travel.
- Conservation of energy states that energy cannot be created or destroyed, only transferred into another form.
- When energy is transferred from one form to another, work is being done. For example, when the chemical energy in the engine of a car is converted into motion - i.e. kinetic energy, work is being done.
- When a body is accelerated by a force over a certain distance, it gains kinetic energy. When a body is decelerated by a force over a certain distance, it loses kinetic energy.
- If this occurs across a smooth horizontal plane, we can say **FdcosΘ = 1/2mv^2 - 1/2mu^2**; or **work = change in kinetic energy.**
- However, if there is also friction to consider, the total work done is not completely converted into kinetic energy, as some of the energy is lost as heat / sound energy due to friction.
- These principles also apply in the vertical plane. When a body falls a distance **h**, the force acting on it is weight, **mg**. Thus, since **work = force * distance**, **work = mgh**. So, if a body falls in a gravitational field, we can say **work = mgh = 1/2mv^2 - 1/2mu^2**.
- All bodies have a gravitational potential energy given by **mgh**.
- A body with a higher mass at the same height as a body with a lower mass has a greater gravitational potential.
- When a body falls, this gravitational potential energy is converted into kinetic energy. When a body rises, its kinetic energy is converted into gravitational potential energy.
- When forces act against a body rather than in the direction of motion, they cause work to be done such that the kinetic energy is depleted (the work energy principle, **FdcosΘ = 1/2mv^2 - 1/2mu^2** applies exactly the same, but just in reverse since the particle is losing velocity due to the opposing force). One example of such an

opposing force is friction; if a frictional force of **F** acts over a distance of **d**, it causes the kinetic energy of a body to be depleted.

- When a body falls a height **h**, not all of its gravitational potential energy is converted into kinetic energy if there is an atmosphere (like on Earth). This is because air resistance, a form of friction, does work in the upwards direction, which reduces the kinetic energy of the falling body.
- We can therefore say that when a body falls a certain height, **h**, the work done, **mgh**, is equal to the kinetic energy + the energy loss due to air resistance. **mgh = 1/2mv^2 + W$_f$** where **W$_f$** is the work done by friction.
- When a body rises, not all of its kinetic energy is converted into potential energy, since air resistance is a force that again does work against the body, depleting its kinetic energy. Thus, for a rising body, **1/2mv^2 = mgh + W$_f$.**
- The above equations simply show the transfer, or gain / loss, of kinetic energy or potential energy when rising or falling. If a body already has some kinetic / potential energy, we need to add / subtract this from its existing gravitational or kinetic energy it has depending on the context.
- When a body is pushed up a slope by a force **F**, it gains both kinetic and potential energy. The total work done to the body must be the sum of the kinetic and potential energy it gains. **W = E$_g$ + E$_k$**. However, if there is also friction to consider, some of the kinetic energy can be lost as heat / sound energy, so not all of the work done is converted to gravitational and potential energy. Thus, **W = E$_g$ + E$_k$ + W$_f$** where **W$_f$** is the work done by friction.
- If a body moves down a slope with a driving force **F**, the work done by the driving force + the loss in potential energy is converted into kinetic energy and work done by friction; **W + E$_g$ = E$_k$ + W$_F$**.
- The power is the work done over the time to do that work.
- Electrical systems such as motors, or turbines, create a force and therefore do work. However, they are not always 100% efficient at doing work. The efficiency is given by **E = 100u / t**, where **u** is the useful energy output, **t** is the total energy input.
- Ultimately, energy is a way of conceptualising certain problems that are not easily described with dynamics and kinematics.

Energy concepts questions

1

A 500kg car decelerates from a velocity of 30ms^{-1} to a velocity of 5ms^{-1} over a distance of 52m.

a) **Determine the braking force required to achieve this, assuming that there is no friction.**

Using the work-energy relationship,

(1/2 * 30^2 * 500) - (1/2 * 5^2 * 500) = F$_b$ * 52, where F$_b$ is the braking force.

F_b = **4210N**.

-

If the car decelerates from 30ms^{-1} to 5ms^{-1} over a distance of 52m, but now with a frictional force of 300N acting against it, **b) determine the braking force required.**

The car now has two forces opposing its motion, a frictional force and a braking force. Both forces reduce the kinetic energy of the car.

The reduction in kinetic energy of the car is therefore caused by both the frictional force and the braking force.

The change in kinetic energy is given by: (1/2 * 30^2 * 500) - (1/2 * 5^2 * 500) = **2.19*10^{5}J**.

If a frictional force of 300N acts over a distance of 52m, the work done by that force is **300(52) = 15600J**.

So **15600J** of the **2.19*10^{5}J** of kinetic energy is lost due to friction. The rest, **(2.19*10^5) - 15600 = 2.03*10^{5}J** must be lost due to the braking force.

Thus, **2.03 * 10^5 = F_b * 52** => **F_b = 3911N**.

Another way to view this is that the *total force* required to bring the car from 30ms^{-1} to 5ms^{-1} over a distance of 52m is **4210N**, as calculated in the first part.

The total force is shared between the braking force and the frictional force in this part of the question, so **300 + F_b = 4210** => **F_b = 3910N**.

-

2

A ball of mass 5kg falls from a height of **h**m to a height of 5m. At the height of **h**m the body has a velocity of 0ms^{-1}, and at the height of 5m, the body has a velocity of 32ms^{-1}. **Find the value of h**, **ignoring the effects of air resistance**.

The sum of the potential and kinetic energy of the ball must remain constant throughout the fall.

The total energy of the ball at a height of 5m is: **(5 * 9.81 * 5) + (0.5 * 5 * 32^2) = 2805J**.

When the ball is at a height of **hm**, the only energy it has is potential energy, thus its potential energy must be **2805J**.

2805 = 5 * 9.81 * h

h = 57.2m.

-

3

*A ball of mass 4.8kg falls from a height of 50m to a height of 5m. At a height of 50m, it has a velocity of 10ms^{-1}, and at a height of 5m, it has a velocity of **v**ms^{-1}. The work done by air resistance when descending from 50m to 5m is 1200J. **Find v.***

The total energy of the ball at a height of 50m must equal the total energy of the ball at a height of 5m plus the energy loss due to air resistance, according to conservation of energy.

The total energy at a height of 50m is**: (50 * 9.81 * 4.8) + (0.5 * 10^2 * 4.8) = 2595J**.

The energy loss due to air resistance during the fall is **1200J**, so the total energy of the ball at a height of **5m** is **2595 - 1200 = 1395J**.

The total energy of the ball is the sum of its kinetic energy and its potential energy.

Thus, **(4.8 * 5 * 9.81) + (1/2 * 4.8 * v^2) = 1395**

v = 22ms^{-1}.

-

4

*A ball of mass **mkg** is dropped from rest from a height of 21m to the ground. The work done by air resistance during the fall is 800J, and the velocity of the ball when it hits the ground is 15ms^{-1}.* **Find m**.

The potential energy of the ball is converted into kinetic energy, but 800J of this kinetic energy is lost due to friction.

The potential energy of the ball is given by **m * 9.81 * 21**.

Its kinetic energy at the ground is **0.5 * m * 29^2.**

Thus, **(0.5 * m * 15^2) + 800 = m * 9.81 * 21**

800 = 206m - 113m

800 = 93m

m = 8.6kg.

-

5

A ball of mass 2.4kg is thrown vertically upwards at ***vms⁻¹***. *When the ball is 15m above its original position, it has a velocity of 13ms⁻¹. The work done by air resistance up to the height of 15m is 200J.* **Find v.**

The kinetic energy of the ball is converted into potential energy as it rises, but there is not a complete conversion due to air resistance.

The gain in potential energy up to a height of 15m is **(15 * 9.81 * 2.4) = 353J**.

If there was no air resistance, this would mean that **353J** of kinetic energy was converted into potential energy. However, **200J** of kinetic energy was lost due to friction, so the kinetic energy change between a height of 0m and a height of 15m is **353 + 200 = 553J**.

Its kinetic energy at a height of 15m is **$1/2 * 13^2 * 2.4$ = 203J**, so the ball must have had an initial kinetic energy of **203 + 553 = 756J**.

Thus, **$756 = 1/2 * 2.4 * v^2$** => **$v = 25.1ms^{-1}$.**

6

A ball of mass ***m kg*** *is thrown vertically upwards at $20ms^{-1}$. At its apex, it has a height of 16.6m. Given the work done by air resistance is 200J,* **find m**.

At the apex, the ball has no velocity and therefore no kinetic energy. The ball's initial kinetic energy has therefore been converted into potential energy plus the energy loss due to friction.

$E_{k\ initial} = E_g + W_A$ where **W_A** is the work done by air resistance.

$0.5 * m * 20^2 = (m * 9.81 * 16.6) + 200$

200m = 163m + 200

37m = 200

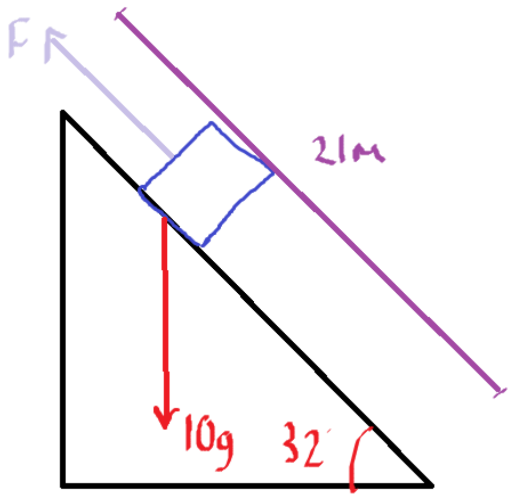
m = 5.4kg.

-

7

A block of mass 10kg is at the top of a slope of length 21m inclined at 32° to the horizontal. The block is released from rest, and accelerates down the slope. At the bottom of the slope, it has a velocity of $13ms^{-1}$. The frictional force acting against the block is **F N**.

a) **Find the value of F, the frictional force.**



The gravitational potential energy of the block at the top of the slope is **21sin32 * 9.81 * 10 = 1092J**.

The kinetic energy of the block at the bottom of the slope is **1/2 * 13^2 * 10 = 845J**.

This means that the kinetic energy loss due to friction is **1092 - 845 = 247J**.

The work done by friction is over 21m.

Thus, **F * 21 = 247**

F = 11.8N.

b) **Find the velocity of the block when it is 12m down the slope.**

Method 1: energy changes

The change in potential energy of the block when moving down the slope by **12m** is **(12sin32 * 9.81 * 10) = 624J**.

This loss in potential energy must have been converted to some heat / sound energy due to friction, and the rest must have been transferred as kinetic energy.

If the frictional force of **11.8N** acts over a distance of **12m**, the work done by that force is **12 * 11.8 = 142J**.

So **142J** of the **624J** of potential energy is converted into heat / sound energy, meaning **624 - 142 = 482J** must have converted into kinetic energy.

Thus, **482 = 1/2 * 10 * v^2** => **v = 9.8ms^{-1}**.

Method 2: dynamics

The block has two forces acting on it in the plane of the slope:

- A weight component of **10gsin32** down the slope
- A frictional force of **11.8N** up the slope.

The net force is **10gsin32 - 11.8 = 40.2N**.

Thus, **40.2 = 10 * a** => **a = 4.02ms^{-2}.**

Using SUVAT,

s = 12, u = 0, a = 4.02, v = v

v = √(2 * 12 * 4.02) = 9.8ms^{-1}.

-

*Upon reaching the bottom of the slope, the block slides along a horizontal plane, with the same frictional force found in **a)** acting against it.* **c) Find the time taken for the block to come to rest.**

The only force now acting on the block is the frictional force in the opposite direction to the direction of motion.

Thus, the deceleration of the block is **11.8 / 10 = 1.108ms^{-2}.**

The initial velocity of the block upon hitting the bottom of the slope is 13ms^{-1}.

Thus, **u = 13, v = 0, a = -1.108**

t = -13 / -1.108 = **11.7s**.

-

8

*A 4kg block is pulled up a slope inclined at 12° to the horizontal with a tension force of **TN**. The block starts from rest, and is pulled up a distance of 15m. The work done by friction on the block is 30J. Given that the velocity of the block at the end of the 15m distance is 13ms^{-1}, find **T**.*

The tension force does a certain amount of work over the distance of 15m. We can therefore say that **W_T = T * 15**, where **W_T** is the work done by the tension force.

We need to find **W_T** so that we can find **T**.

The work done by the tension force is converted into kinetic energy, potential energy and the work done by friction.

Thus, **$W_T = E_g + E_K + W_F$**.

W_T = (15sin12 * 9.81 * 4) + (0.5 * 13^2 * 4) + 30

W_T = 490J.

Thus, **T = 490 / 15 = 33N** (2sf)

-

9

A block of mass 8kg is pulled up a rough slope inclined at 30º to the horizontal by a rope. The tension in the rope is 80N. The block is pulled a distance of 20m up the slope, and has a kinetic energy of 750J at the point 20m up the slope. **Find the frictional force, F, acting against the block.**

The tension force does a work of **80 * 20 = 1600J**.

This work done is converted into kinetic and potential energy, but some of the energy is lost due to friction.

Hence, **1600 = E_K + E_g + W_f.**

The work done by friction, **W_f** is equal to the frictional force multiplied by the distance over which it occurs: **W_f = F * 20**.

1600 = 750 + (20sin30 * 9.81 * 8) + (F * 20).

F = 3.3N.

-

*The rope suddenly snaps when the block is 20m up the slope and the block continues to move up the slope, before sliding down the slope with the same frictional force found in part **a)** acting against it.* **b) Determine the time it takes for the block to reach the bottom of the slope**.

The block has an initial velocity up the slope when the rope snaps, so it will not instantly start sliding down the slope when the rope snaps.

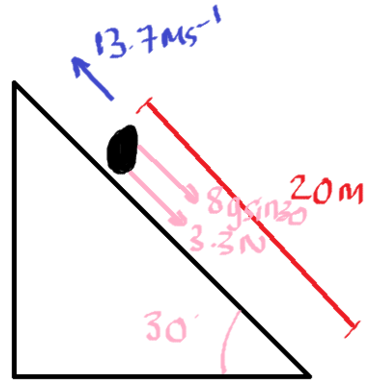
The velocity of the block before the rope snaps can be found using its kinetic energy.

750 = 0.5 * 8 * v^2 => **v = 13.7ms^{-1}.**

The block has two forces acting against it, the component of weight down the slope, and the frictional force. This gives a total force of **3.3 + 8gsin30 = 42.5N**.

Thus, its deceleration is **a = 42.5 / 8 = 5.3ms^{-2}.**

Thus, taking the movement up the slope as positive, we can say: **u = 13.7, a = -5.3, s = -20**.



$-20 = 13.7t + 1/2(-5.3)t^2$

$-20 = 13.7t - 2.65t^2$.

$2.65t^2 - 13.7t - 20 = 0$.

t = 6.36s.

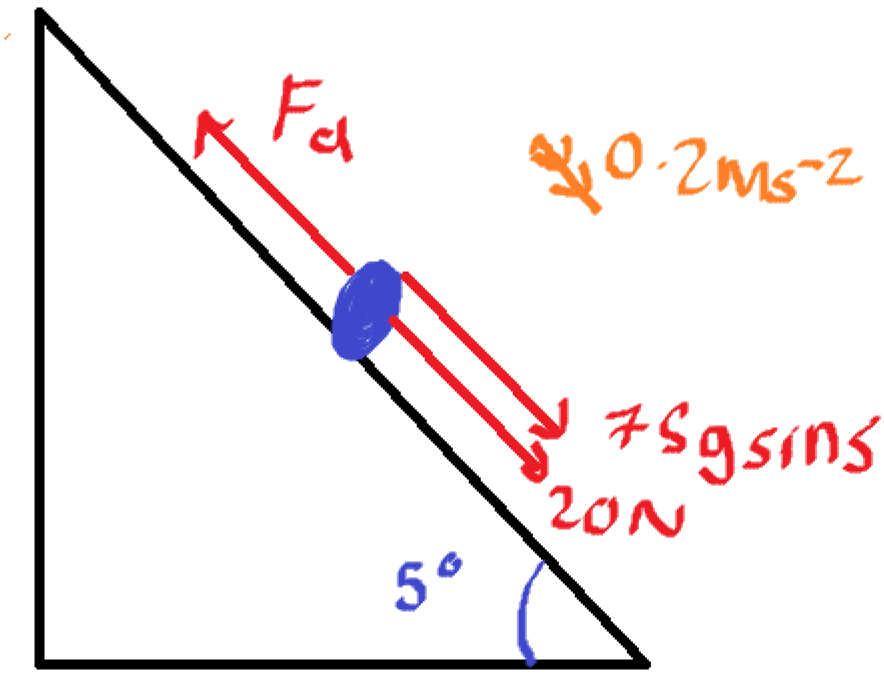
10

A cyclist and her cycle have a combined mass of 75 kg. The cyclist is cycling up a straight road inclined at 5° to the horizontal. The resistance to the motion of the cyclist from non-gravitational forces is modelled as a constant force of magnitude 20 N. At the instant when the cyclist has a speed of 12 m s^{-1}, she is decelerating at 0.2 m s^{-2}.

(a) Find the rate at which the cyclist is working at this instant.

(5)

The cyclist must be exerting a force up the slope (let this be F_d) that is resisting the weight force acting down the slope and the frictional force. However, because the cyclist is decelerating the sum of the weight and frictional force must be greater than this driving force.



Using Newton's second law, **net force = ma**, thus: **(75gsin5 + 20) - F_d = 75 * 0.2**.

Solving for **F_d**: **F_d = 69.1N**.

So the cyclist is cycling up the slope with a force of **69.1N**.

The velocity of the cyclist is **12ms^{-1}** at the given instant. This means that each second, the cyclist is moving a distance of **12m**.

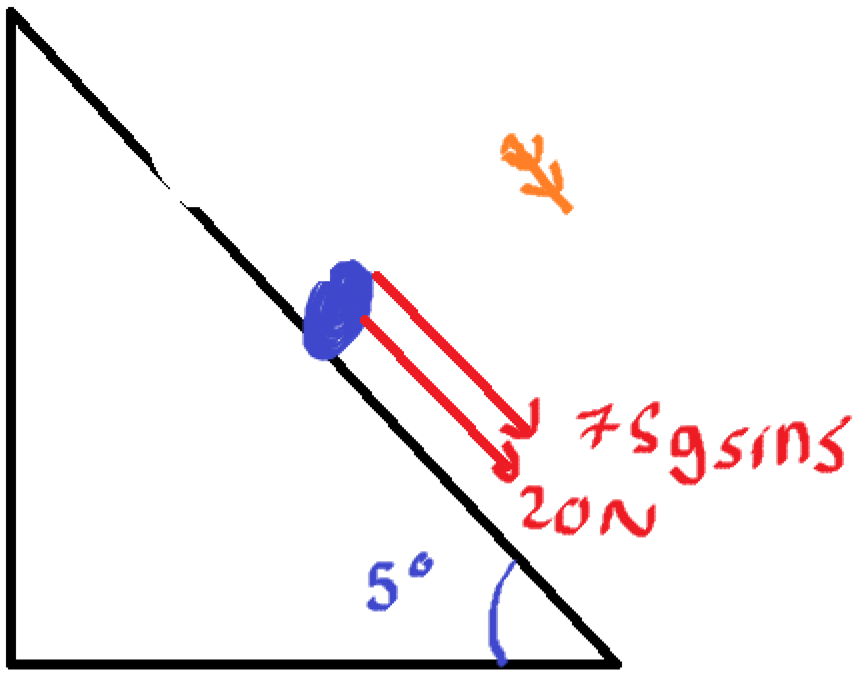
Thus, the work done per second, which is the power, is equal to **69.1 * 12 = 830Js^{-1}.**

When the cyclist passes the point A her speed is 8 m s^{-1}. At A she stops working but does not apply the brakes. She comes to rest at the point B.
The resistance to motion from non-gravitational forces is again modelled as a constant force of magnitude 20 N.

(b) Use the work-energy principle to find the distance AB.

(5)

The cyclist now has two forces acting on them, the weight force down the slope, and the frictional force down the slope.



The total force acting against the cyclist is **75gsin5 + 20 = 84.1N**.

The cyclist has an initial velocity up the slope of $8ms^{-1}$, so they will continue to move up the slope until their velocity is reduced to $0ms^{-1}$.

Using the work-energy principle, the change in kinetic energy must be equal to the work done to cause that change in kinetic energy. The work is done by the **84.1N** force.

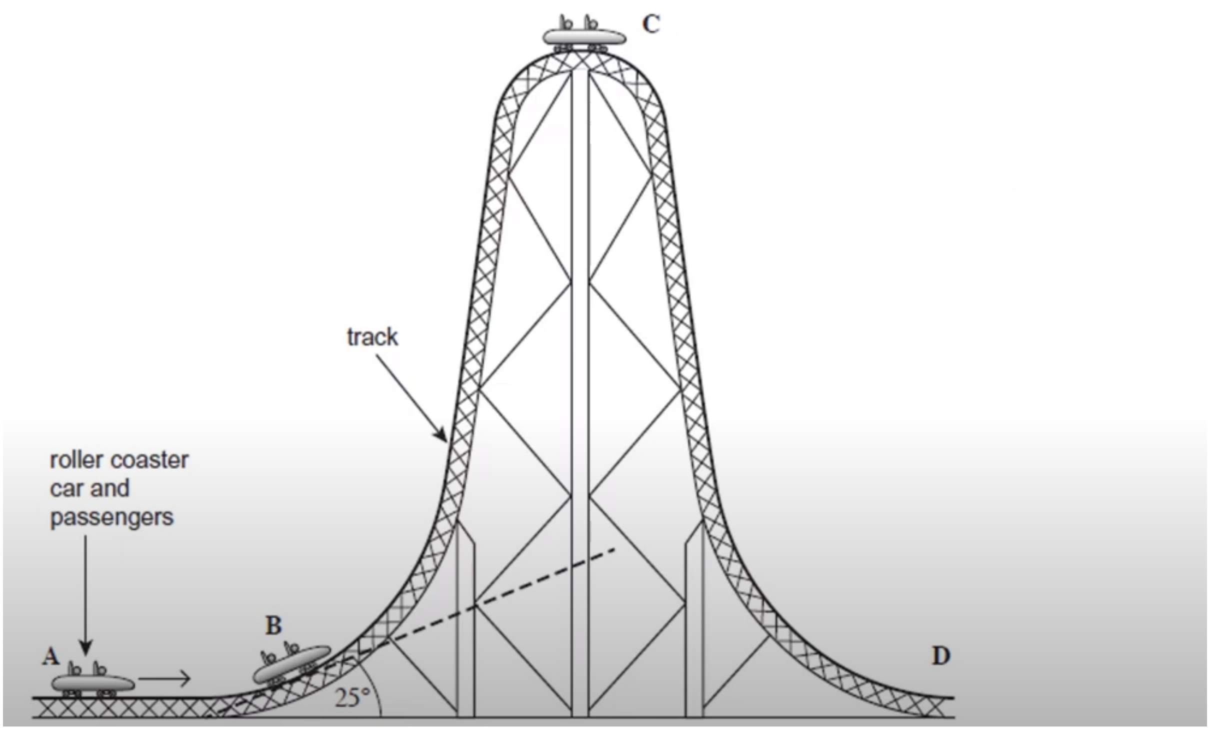
Thus, **$(1/2 * 75 * 8^2) - (1/2 * 75 * 0^2) = 84.1 * d$**

d = 28.5m.

-

11

Q2. The following figure shows a roller coaster car which is accelerated from rest to a speed of 56 m s^{-1} on a horizontal track, **A**, before ascending the steep part of the track. The roller coaster car then becomes stationary at **C**, the highest point of the track. The total mass of the car and passengers is 8300 kg.



a) **Find the maximum height of point C.**

The maximum height is a situation where all of the kinetic energy of the roller coaster is converted into potential energy.

The kinetic energy at the bottom of the car at the bottom of the track is **1/2 * 8300 * 56^2 = 1.301*10^7J**.

Assuming a complete conversion into potential energy, **1.301*10^7 = 8300 * 9.81 * h**

h = 160m (3sf).

-

The point C is actually 140m above the ground. **b) Find the work done by friction on the car.**

The gravitational potential energy at a height of 140m is **140 * 9.81 * 8300 = 1.14*10^7J**. Thus, the loss in energy due to friction is **1.301*10^7 - 1.14*10^7 = 1.61*10^6J** (2SF).

-

The car descends from point C to D, and 87% of its energy at C is converted into kinetic energy. **c) Find the velocity of the car at D.**

The potential energy of the car at the top of the slope is **1.14*10^7J**, as calculated in **b)**.

87% of this is converted into kinetic energy, thus **E_k at D = 0.87*1.14*10^7 = 9.92*10^6J**.

Thus, **9.92*10^6 = 1/2 * 8300 * v^2** => **v = 48.9ms^{-1}**.

-

12

Q3. The driver of a car travelling with a velocity, *v*, along a level road, applies the brakes. The brakes lock and the car skids to a stop. The skidding distance, *d*, is given by

$$d = kv$$

where *k* is a constant.

(a) For a car travelling at a speed of 30 ms^{-1}, *d* is 45 m. Calculate the value of *d* when the car is travelling at 15 ms^{-1}.

We can use the data given to find **k**: **45 = k * 30** => **k = 1.5**.

Thus, **d = 1.5 * 15 = 22.5m**

-

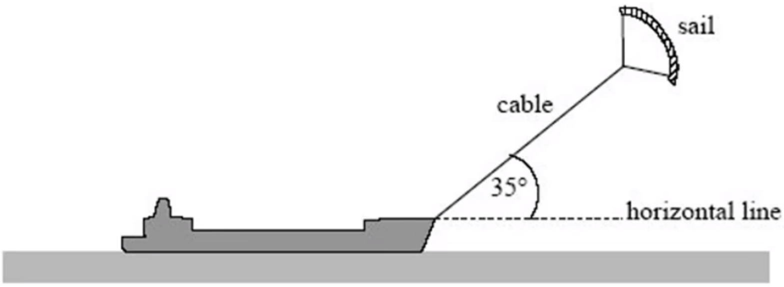
(b) The mass of the car and its passengers is 700 kg. Calculate the average skidding force that would bring the car to a stop from an initial speed of 30 ms^{-1} in a skidding distance of 45 m.

Using the work-energy relationship,

45 * F = (1/2 * 700 * 30^2) => **F = 7000N**.

13

(b) The diagram below shows a ship fitted with a sail attached to a cable. The force of the wind on the sail assists the driving force of the ship's propellers.



The cable exerts a steady force of 2.8 kN on the ship at an angle of 35° above a horizontal line.

(ii) The ship is moving at a constant velocity of 8.3 m s^{-1} and the horizontal component of the force of the cable on the ship acts in the direction in which the ship is moving.
Calculate the power provided by the wind to this ship, stating an appropriate unit.

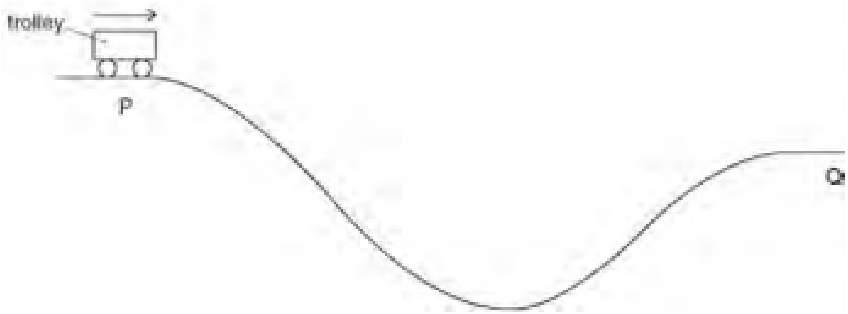
The work done must always be in the direction of travel. Thus, if the force exerted is 2800N at 35° to the horizontal, the horizontal component of this force is **2800cos35**

Power is equal to **Fv**, thus **P = 2800cos35 * 8.3 = 19000W = 19kW** (2sf)

-

14

3. A trolley runs from P to Q along a track. At Q its potential energy is 50 kJ less than at P.



At P, the kinetic energy of the trolley is 5 kJ. Between P and Q the work the trolley does against friction is 10 kJ.
What is the kinetic energy of the trolley at Q?

The trolley loses 50kJ of potential energy from P to Q. This must have been converted into kinetic energy, but 10kJ of this kinetic energy was lost due to friction, so 40kJ of the potential energy is converted into kinetic energy.

The initial kinetic energy of the trolley is 5kJ, so the total kinetic energy at Q is **5 + 40 = 45kJ**.

-

15

8. A twig from a tree drops from a 200m high cliff on to a beach below. During its fall, 40% of the twig's energy is converted into thermal energy.

Find the twig's velocity upon hitting the beach.

Let the twig have a mass of **m**. At 200m, it has a potential energy of **200 * m * 9.81**.

Its kinetic energy at the bottom of the cliff is **$1/2 * v^2 * m$**.

40% of the potential energy is converted into thermal energy, so 60% must be converted into kinetic energy.

Thus, **$0.6 (200 * m * 9.81) = 1/2 * v^2 * m$**.

Because **m** is common on both sides, we can cancel it out.

Thus, **$0.6(200 * 9.81) = 1/2 * v^2$.**

$v = 48.5ms^{-1} = 49ms^{-1}$ (2sf)

16

10. A motorist travelling at $10ms^{-1}$ can bring his car to rest in a distance of 10 m.
If he had been travelling at $30ms^{-1}$, in what distance could he bring the car to rest using the same braking force?
A 17 m **B** 30 m **C** 52 m **D** 90m

We can use the work-energy principle to find the maximum braking force of the car. We do not know the mass of the car, so we shall leave this as **m**.

$F * 10 = (1/2 * 10^2 * m)$

F = 50m / 10 = 5m.

The same force applies for decelerating from $30ms^{-1}$. Thus,

$d * 5m = (1/2 * 30^2 * m)$

5md = 450m

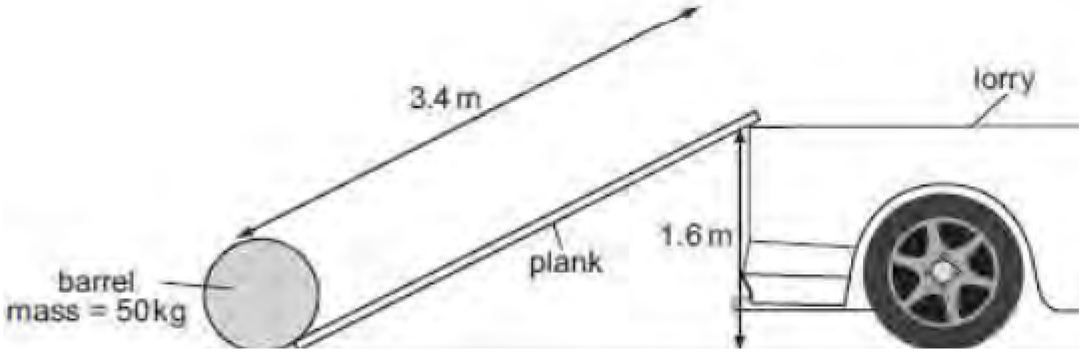
5d = 450

d = 90m.

-

17

A barrel of mass 50 kg is loaded onto the back of a lorry 1.6 m high by pushing it up a smooth plank 3.4 m long.



Find the minimum work done to push the barrel this distance.

First, let us find the angle of the slope:

Angle of slope = $\sin^{-1}$ (1.6 / 3.4) = 28.1°.

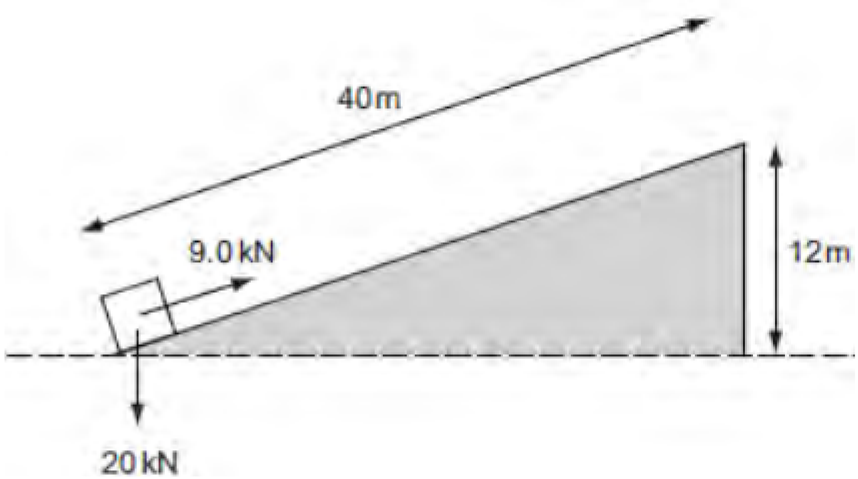
The weight of the barrel acting down the slope is therefore **50gsin28.1 = 231N**.

For the barrel to be pushed up the slope, the minimum force required must be equal to **231N**. This allows the barrel to move up the slope at a constant velocity. If the force exerted is any less than this, the barrel will roll down the slope.

Thus, a force of **231N** is being applied over a distance of **3.4m**, so **work = 231 * 3.4 = 790J** (2sf).

18

A constant force of 9.0 kN, parallel to an inclined plane, moves a body of weight 20 kN through a distance of 40 m along the plane at constant speed. The body gains 12 m in height, as shown.



Find the work done by friction on the body.

The angle of the slope is **$\sin^{-1}$ (12 / 40) = 17.5°**.

Thus, the horizontal weight component is **20sin17.5 = 6.01N**.

The body also has a frictional force acting down the slope.

Since the body is moving at constant velocity, the forwards and backwards forces must be equal.

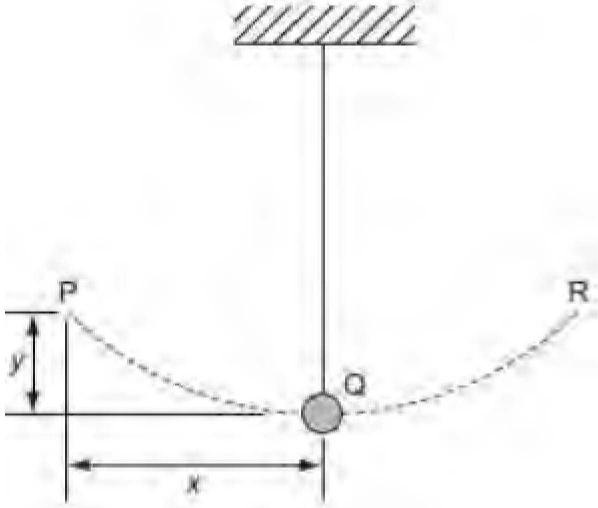
Thus, **F_F + 6.01 = 9** => **F_F = 2.99kN** where F_F is the frictional force.

This acts over a distance of **40m**, so the work done by the force is **2.99 * 40 = 120kJ** (2SF).

-

19

A pendulum bob oscillates between P and R.



Assuming the gravitational potential energy lost in moving from P to Q is converted into kinetic energy, what is the speed of the bob at Q?

A $\sqrt{2gx}$ **B** $2gx$ **C** $\sqrt{2gy}$ **D** $2gy$

The potential energy of the bob at P is **m * g * y**, where **y** is the height of the bob.

This is converted completely into kinetic energy, so **mgy = 1/2 * m * v^2.**

We can cancel **m** to give:

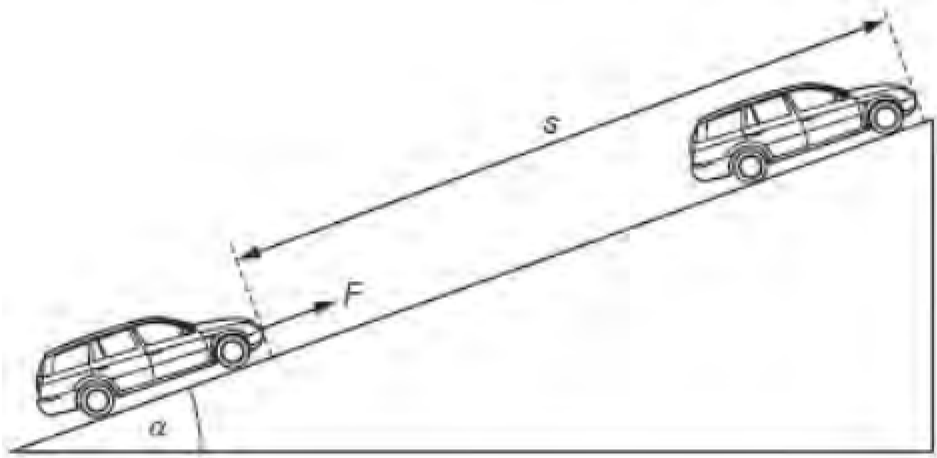
gy = 1/2v^2.

2gy = v^2

v = √(2gy).

-

45. A constant force F, acting on a car of mass m, moves the car up the slope through a distance s at constant velocity v. The angle of the slope to the horizontal is α.



Which expression gives the efficiency of the process?

A $\frac{mgs\sin\alpha}{Fv}$ **B** $\frac{mv}{Fs}$ **C** $\frac{mv^2}{2Fs}$ **D** $\frac{mg\sin\alpha}{F}$

The frictional force acting against the car causes some of its kinetic energy to be depleted. We will let this frictional force be $\mathbf{F_F}$.

Since the car is moving at constant velocity, the driving force must equal the forces opposing the motion, the frictional force and the component of weight down the slope.

Thus, $\mathbf{F = F_F + mg\sin\alpha}$

Therefore, $\mathbf{F_F = F - mg\sin\alpha}$.

The work done by the frictional force is therefore **s(F-mgsinα)**.

If a question refers to *efficiency* rather than *percentage efficiency*, we just leave it as a fraction rather than multiplying by 100.

The driving force does a total work of **Fs**. If the total work done is **Fs** and **s(F-mgsinα)** of energy is wasted, the energy wasted as a fraction is **s(F-mgsinα) / Fs.**

Cancelling **s** gives **(F-mgsinα) / F**.

If this fraction of the energy is wasted, the fraction of useful energy (i.e. the efficiency) must be **1 - ((F-mgsinα) / F)**. This simplifies as follows:

$$1 - \frac{F - mg\sin\alpha}{F}$$

$$= \frac{F - (F - mg\sin\alpha)}{F}$$

$$= \frac{mg\sin\alpha}{F}$$

The answer is therefore **D**.

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21

Fig. 3.1 shows part of a fairground ride with a carriage on rails.

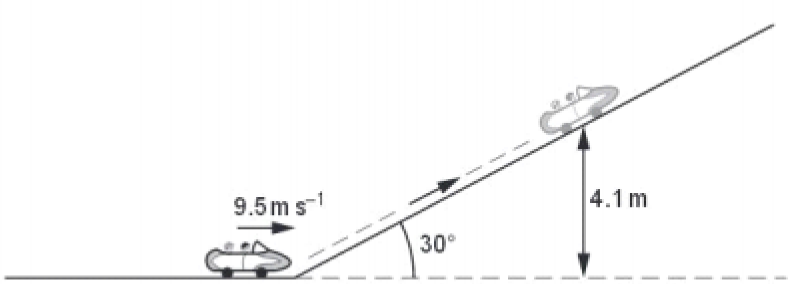


Fig. 3.1

Find the resistive force acting against the carriage.

The carriage and passengers have a total mass of 600 kg. The carriage is travelling at a speed of $9.5\,m\,s^{-1}$ towards a slope inclined at 30° to the horizontal. The carriage comes to rest after travelling up the slope to a vertical height of 4.1 m.

The kinetic energy of the carriage at the bottom of the slope is **1/2 * 9.5^2 * 600 = 27100J**.

The potential energy at the top of the slope is **600 * 9.81 * 4.1 = 23500J**.

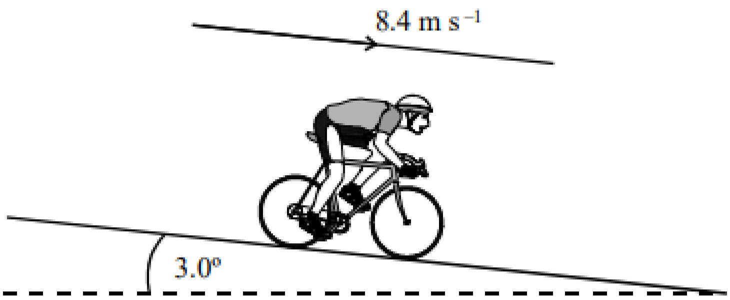
Thus, the energy loss due to friction is **27100 - 23500 = 3600J**.

The distance travelled up the slope is **4.1 / sin30 = 8.2m**.

So, **3600J** of the carriage's energy is depleted over a distance of **8.2m**. This enables us to find the work done on the carriage by friction: **3600 = 8.2 * F** => **F = 439N**.

22

A cyclist is free-wheeling down a long slope which is at 3.0° to the horizontal. He is travelling, without pedalling, at a constant speed of 8.4 m s^{-1}.



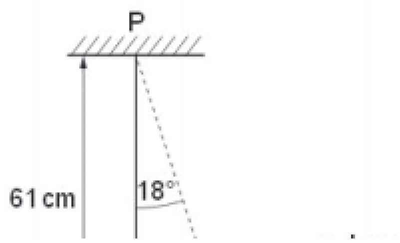
The combined mass of the cyclist and bicycle is 90 kg. Calculate the gravitational potential energy (g.p.e.) lost per second.

The cyclist moves 8m down the slope per second. This corresponds to a decrease in height of **8sin3 = 0.42m** per second.

Thus, the change in g.p.e per second is **0.42 * 9.81 * 90 = 371J**.

23

A small sphere of mass 51 g is suspended by a light inextensible string from a fix point P.
The centre of the sphere is 61 cm vertically below point P, as shown in Fig. 3.1.

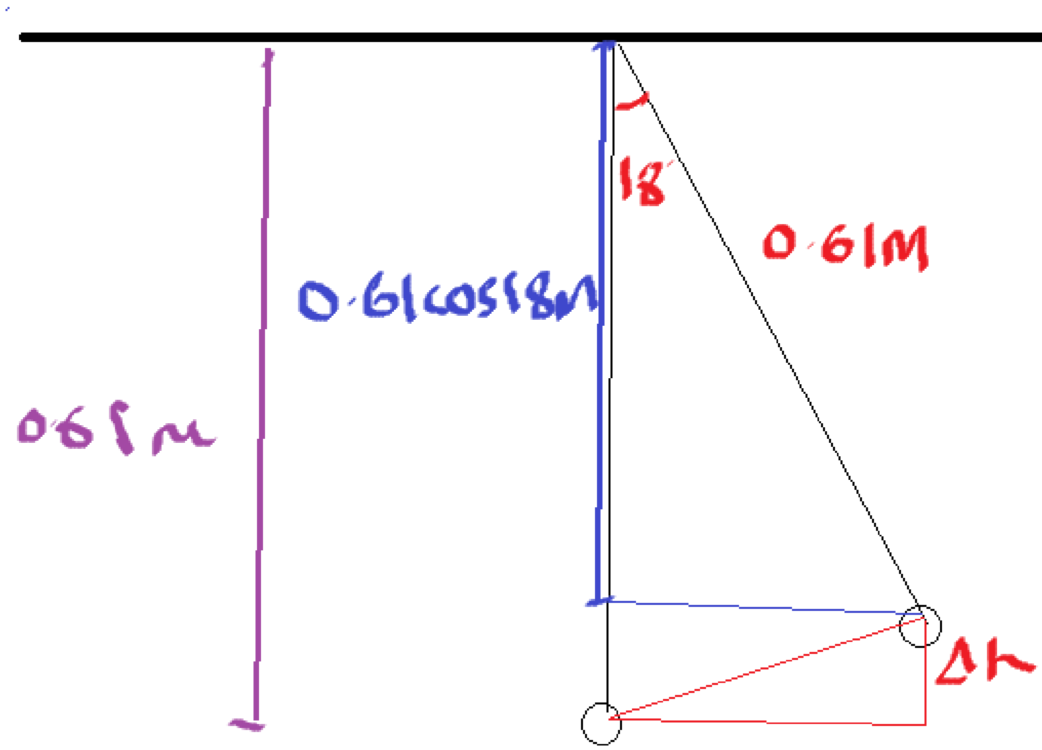


(i) the gain in gravitational potential energy of the sphere,

Fig. 3.1

By considering the geometry of this situation, we can prove that the height increase of the ball is **61 - 61cos18 = 3cm.**

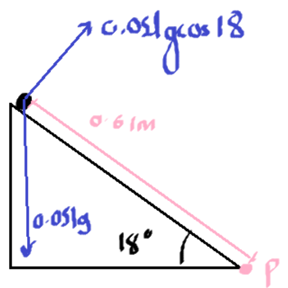
Thus, the gain in **E_g = 0.03 * 9.81 * 0.051 = 0.015J**.



(ii) the moment of the weight of the sphere about point P.

The moment is the product of the force and the **perpendicular distance** to the force.

The weight component perpendicular to the slope is like the vertical component of weight on a slope:



Thus, **moment = 0.61 * 0.051gcos18 = 0.305Nm**.

-

The pendulum is released from the angle of 18°. **Calculate its velocity when it has returned to its original position, neglecting air resistance.**

The potential energy of the ball is converted into kinetic energy.

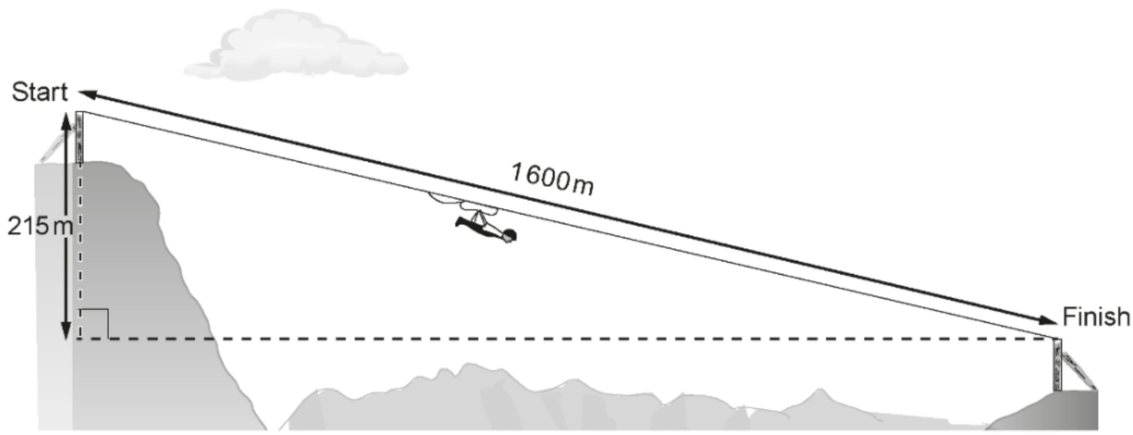
Thus, the kinetic energy at the original position is **0.015J**.

0.015 = 1/2 * 0.051 * v^2 => **v = 0.77**ms^{-1}.

-

24

(*b*) The longest zip-wire ride in the UK is in Snowdonia, North Wales. It is 1600 m long and the vertical drop from start to finish is 215 m as shown. The diagram is not to scale.



(i) A person of mass 70 kg arrives at the finish travelling at 35 m s^{-1}, having started from rest. Use this data and information from the diagram opposite to determine the mean force opposing the motion of the person. [4]

The loss in potential energy of the person is **70 * 9.81 * 215 = 147640J**.

The kinetic energy of the person at the bottom of the zip-wire is **1/2 *** 35^2 *** 70 = 42875J**.

Thus, the energy lost to the surroundings is **147640 - 42875 = 104765J**.

This energy is lost over a distance of **1600m**, so the opposing force is **104765 / 1600 = 66N** (2sf)

(ii) The time taken to travel from start to finish is 46 s. Calculate the mean rate at which energy is transferred to the surroundings during the journey. [2]

The energy lost as heat / sound energy is **104765J**. Thus, the energy lost per second is **104765 / 46 = 2277.5Js^{-1} = 2300W** (2sf).

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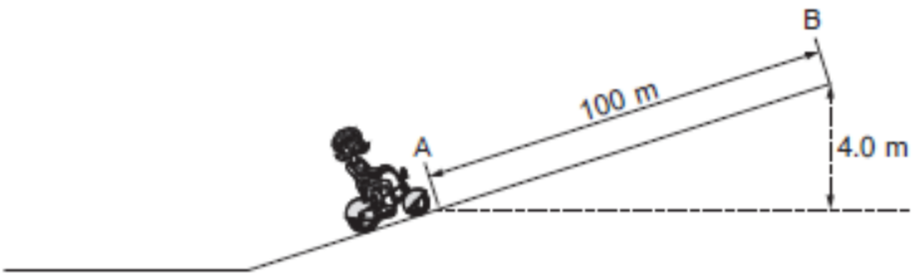
25

Helen is riding an electric bike (a bike that is assisted by an electric motor) up a hill at a speed of 4.5 m s^{-1}. At point A she starts the electric motor and accelerates uniformly reaching a speed of 9.2 m s^{-1} at B. Whilst accelerating she also gains a height of 4.0 m as shown in the diagram below.



(a) Show that the time taken for Helen's journey between A and B is approximately 15 s. [2]

u=4.5, v = 9.2, s = 100

100 = 1/2 (4.5 + 9.2) * t

t = 100 / (1/2(4.5+9.2)) = 14.6s.

(b) Helen and the bike have a combined mass of 95 kg. Determine the gain in total energy between A and B. [3]

The gain in kinetic energy of the bike is **(1/2 * 9.2^2 * 95) - (1/2 * 4.5^2 * 95) = 3060J**.

The gain in potential energy is **95 * 9.81 * 4 = 3730J**.

The total energy gain is therefore **3060 + 3730 = 6790J**.

(c) (i) If the bike's electric motor operates at 36 V and 7.0 A calculate the electrical energy used by the motor between A and B. [2]

Using **P = IV, P = 7 * 36 = 252W**.

Thus, over a time of **14.6s**, **work done = 252 * 14.6 = 3680J**.

(ii) Helen, by pedalling, also provides 5500 J of work between A and B. Determine the efficiency of the electric motor. *Ignore all resistive forces on Helen and the bike.* [2]

The total energy gain is **6790J**. If **5500J** of this energy is provided by Helen, the useful energy provided by the motor is **6790 - 5500 = 1290J**.

The total energy inputted by the motor is **3680J**, but only **1290J** is useful energy.

Thus, **efficiency = 100(1290 / 3680) = 35.1%.**

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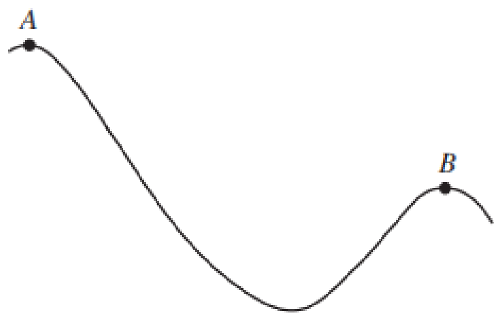
26

The diagram shows two points *A* and *B* on a roller coaster ride in an amusement park.



The heights of *A* and *B* above ground level are 30 m and 22 m respectively. The length of the track between *A* and *B* is 88 m. The resistance to motion of the carriage may be assumed to have a constant magnitude of 132 N. A carriage, of total mass 240 kg, has speed 2 ms^{-1} at *A*. Calculate the speed of the carriage at *B*. [8]

The loss in potential energy between A and B is **240 * 8 * 9.81 = 18835J**.

However, this is not converted completely into kinetic energy.

The work done by friction between A and B is **132(88) = 11620J**.

Thus, the gain in kinetic energy is **18835 - 11620 = 7215J**.

The initial kinetic energy is **1/2 * 2^2 * 240 = 480J**.

The total kinetic energy at B is therefore **480 + 7215 = 7695J**.

Thus, **7695 = 1/2 * 240 * v^2**

v = 8ms^{-1}.

-

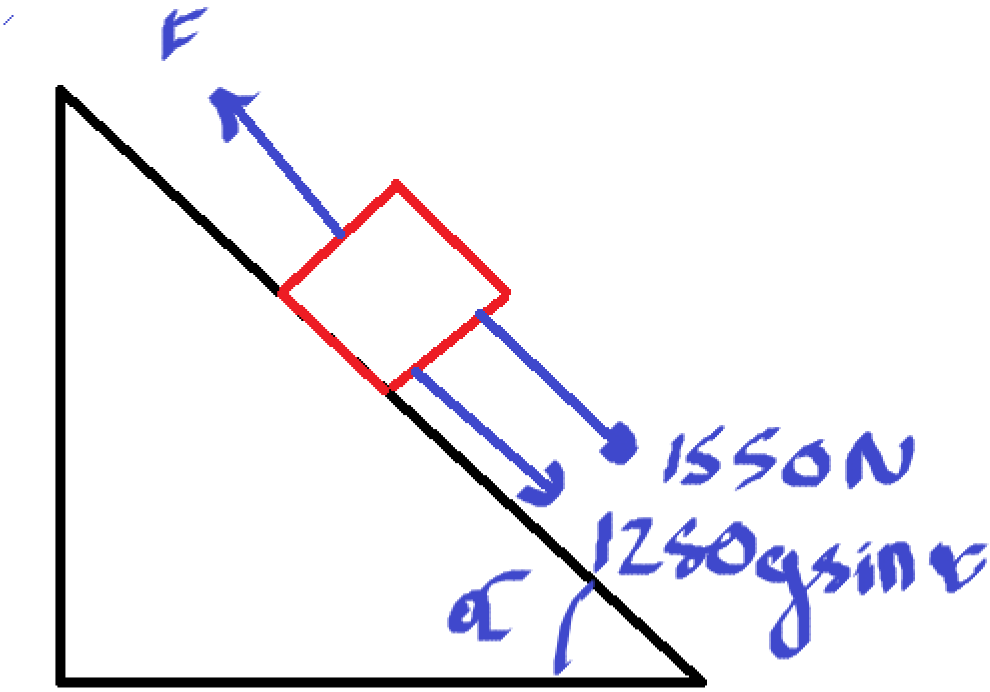
Refer to here for help on the following section.

27

A car, of mass 1250 kg, is travelling up a hill inclined at an angle α to the horizontal at a constant speed of 7·5 ms^{-1}. The car's engine is working at a rate of 30 kW and the resistance to motion of the car is 1550 N. Find the value of α, giving your answer in degrees correct to one decimal place. [6]

The car has three forces acting on it:

- The component of weight acting down the slope, **1250gsinα**
- The frictional force of **1550N** acting down the slope
- A driving force acting up the slope; let this be **F**.



The driving force **F** produces a power output of **30kW**.

Since **P = Fv**, we can say **30000 = F * 7.5** => **F = 4000N**.

Because the car is travelling at constant velocity, the total upwards force must equal the total force down the slope.

Thus, **4000 = 1550 + 1250gsinα**

2450 = 1250gsinα

Sinα = 0.1998

α = 11.5°.

-

28

A car of mass 1200 kg is towing a trailer of mass 800 kg up a slope inclined at an angle α to the horizontal, where $\sin \alpha = \frac{1}{28}$. The resistance to motion acting on the car is 150 N and that acting on the trailer is 100 N. The car's engine is working at 45 kW.

(a) Calculate the acceleration of the car and trailer when the speed is 25 ms^{-1}. [6]

(b) Determine the tension in the rigid tow-bar connecting the car and the trailer when the speed is 25 ms^{-1}. [4]

a)

Using **P = Fv**, the driving force when the velocity is **25**ms^{-1} and the power is **45kW** is **(45*10^3) / 25 = 1800N**.

If we model the car and trailer as a single body with mass **2000kg**, the forces opposing the motion are an overall friction of **250N** and a weight component of **2000g(1/28)**.

Thus, since the car is accelerating, we can say **1800 - 250 - 2000g(1/28) = 2000 * a**

a = 0.425ms^{-2}**.**

<u>*Extra:*</u>

Because the system is accelerating, the car and trailer cannot be at their maximum velocity for this power output. The maximum velocity occurs when the driving force is equal to the resistive forces.

The total resistive force is **250 + 2000g(1/28) = 951N**.

Thus, at maximum speed, the driving force is **951N**.

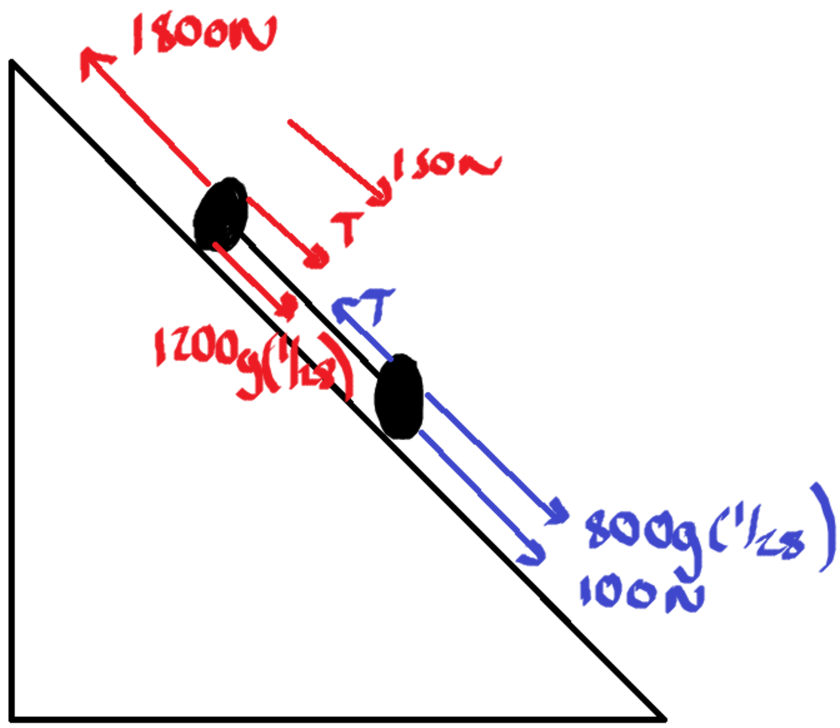
Using **P = Fv**, **v = P / F = 45000 / 951 = 47.3ms**$^{-1}$.

b)

Now we consider the car and the trailer as separate connected particles.

The car has a **1800N** driving force when its velocity is 25ms^{-1}. Opposing its motion is a weight component of **1200g(1/28)**, a frictional force of **150N**, and the tension force, **T**.

The trailer is being pulled up by the tension force **T**, and its motion is opposed by a weight component of **800g(1/28)** and a resistive force of **100N**.



The net force acting on the car is equal to **ma**.

Thus, **1800 - 1200g(1/28) - T - 150 = 1200(0.425)**

T = 1800 - 1200g(1/28) - 150 - 1200(0.425) = 720N.

We can prove this by considering the forces on the trailer:

T - 800g(1/28) - 100 = 800(0.425)

T = 720N.

-

29

A car of mass 900 kg can produce a maximum power of 45 kW. The car experiences a constant resistance to motion of magnitude 1800 N.

(a) Calculate the maximum speed of the car when travelling on a horizontal road. [3]

(b) The car travels up a slope inclined at an angle of 4° to the horizontal. Assuming maximum power is employed, calculate, correct to two decimal places, the acceleration of the car at the instant when its speed is $15\,ms^{-1}$. [5]

(c) The car travels a distance of 800 m. Calculate the work done against resistance. [2]

a)

The maximum speed of the car occurs when the **driving force = resistive force**.

This is because **P = Fv**, so the higher the value of **F**, the lower the value of **v**.

Thus, $\mathbf{45*10^3 = 1800 * v}$ => $\mathbf{v = 25ms^{-1}}$.

b)

If the speed of the car is $\mathbf{15ms^{-1}}$, the driving force can be found using **P = Fv**:

$\mathbf{45*10^3 = F * 15}$ => **F = 3000N**.

The car has a resistive force of **1800N** as well as a weight component of **900gsin4** acting against it.

Thus, using **F=ma**, **3000 - 1800 - 900gsin4 = 900 * a**
$\mathbf{a = 0.65ms^{-2}}$.

c)

If a frictional force of **1800N** is applied over a distance of **800m**, the work done by that force is $\mathbf{1800(800) = 1.44*10^6J}$.

30

An object of mass 8 kg slides in a straight line from point A to point B on a rough horizontal floor. At A, the speed of the object is $7\,ms^{-1}$. It is brought to rest at B by a constant frictional force between the object and the floor. The distance AB is 15 m.

Find the frictional force exerted by the floor.

The change in kinetic energy of the object is **$(1/2 * 7^2 * 8) = 196J$**.

This change in energy occurs over a distance of **15m**, so the frictional force is **196 / 15 = 13.1N**.

31

A vehicle of mass 4000 kg is travelling up a slope inclined at an angle α to the horizontal, where $\sin\alpha = \frac{2}{49}$. The engine of the vehicle is working at a constant rate of 90 kW.

(a) Calculate the resistance to the motion of the vehicle at the instant when its speed is $4{\cdot}8\,ms^{-1}$ and its acceleration is $1{\cdot}2\,ms^{-2}$. [6]

(b) Determine the maximum velocity of the vehicle when the resistance to motion has magnitude 12 800 N. [4]

a)

We can first find the driving force using **$P = Fv$, $90*10^3 = F * 4.8$** => **F = 18750N.**

The car has a driving force of **18750N** acting up the slope, a horizontal component of weight of **4000g(2/49)** acting down the slope, and a frictional force acting down the slope.

If we let the frictional force be **F_f**:

18750 - 4000g(2/49) - F = 4000 * 1.2

F = 12350N.

b)

As shown previously, a vehicle has its maximum velocity when **driving force = resistive forces.**

The total force acting down the slope is the frictional force, **12800N**, and the horizontal component of weight **4000g(2/49)**. This must equal the driving force (let this be **F_D**).

Thus, **F_D = 12800 + 4000g(2/49) = 14400N**.

Using **P = Fv,**

$90*10^3$ = 14400 * v => **v = 6.25ms^{-1}.**

32

A vehicle of mass 6000 kg is moving up a slope inclined at an angle α to the horizontal, where $\sin\alpha = \frac{6}{49}$. The vehicle's engine exerts a constant power of *P* W. The constant resistance to motion of the vehicle is *R* N. At the instant the vehicle is moving with velocity $\frac{16}{5}$ ms^{-1}, its acceleration is 2 ms^{-2}.

The maximum velocity of the vehicle is $\frac{16}{3}$ ms^{-1}.

Determine the value of *P* and the value of *R*. [9]

The maximum velocity of the vehicle on this slope is **16/3ms^{-1}**.

At this maximum velocity, the driving force (let this be **F_1**) is equal to the total resistive forces.

Thus, **F_1 = R + 6000g(6/49) = R + 7207**

We know that **P = Fv**, so **P = (R+7207) * 16/3 = 16/3R + 38437**

This can be used later.

-

When the vehicle is accelerating, the net force acting on it is the driving force (let this be **F_2**) subtracted by the total resistive forces. This is equal to **ma**.

Thus, **F_2 - 6000g(6/49) - R = 6000 * 2**

F_2 - R = 19210.

The power output of the vehicle, **P = 16/3R + 38437**, has remained constant. The driving force, **F_2**, at this velocity is therefore **P / v = (16/3R + 38427) / (16/5) = 5/3R + 12000**.

Substituting **F_2 = 5/3R + 12000** into **F_2 - R = 19210** gives:

(5/3R + 12000) - R = 19210

2/3R = 7210

R = 10800N.

Thus, **P = 16/3(10800) + 38437 = 96000W = 96kW**.

-

A train of mass 2×10^5 kg has a constant speed of 20 ms^{-1} up a hill inclined at $\theta = \sin^{-1}\left(\frac{1}{50}\right)$ to the horizontal when the engine is working at 8×10^5 W. Find the resistance to motion of the trian.

The driving force of the train is **8 * 10^5 / 20 = 40000N**.

The driving force is equal to the total forces down the slope.

Thus, **40000 = (2*10^5*9.81*sin1.15) + F** where **F** is the frictional force.

F = 622N.

-

33

By burning a charge, a cannon fires a cannon ball of mass 12 kg horizontally. As the cannon ball leaves the cannon, its speed is 600 ms^{-1}. The recoiling part of the cannon has a mass of 1600 kg.

(a) Determine the speed of the recoiling part immediately after the cannon ball leaves the cannon. [3]

(b) Find the energy created by the burning of the charge. State any assumption you have made in your solution. [4]

(c) Calculate the constant force needed to bring the recoiling part to rest in 1·2 m. [2]

a)

This question requires the use of the conservation of momentum. The total momentum before the explosion, **0kgms^{-1}**, must equal the total momentum after the explosion.

Hence, **(600 * 12) + (1600v) = 0**

v = -(600*12) / 1600 = -4.5ms^{-1}.

The recoiling part of the cannon therefore moves at **4.5ms^{-1}** in the opposite direction to the direction of the cannon ball.

b)

The energy created by the burning of the charge must be converted into kinetic energy in the cannon and the cannonball.

This assumes that none of the energy in the burning is lost as heat energy.

The kinetic energy of the cannon after the explosion is **1/2 * 4.5^2 * 1600 = 16200J**.

The kinetic energy of the cannonball is **1/2 * 600^2 * 12 = 2.16*10^6J**.

The total kinetic energy transferred is therefore **2.16*10^6 + 16200 = 2.176*10^6J**.

c)

The cannon has an initial kinetic energy of **16200J**.

When it recoils, the ground will exert a frictional force, which will deplete its kinetic energy.

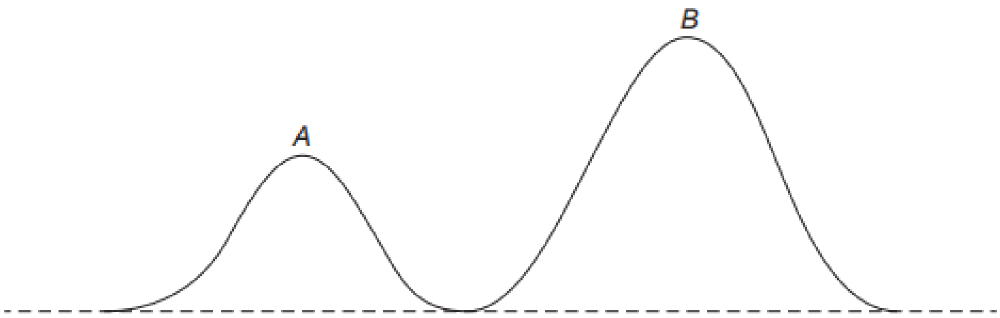
According to the work-energy principle, the change in kinetic energy must equal the work done.

The cannon loses **16200J** of kinetic energy in going to rest, thus **16200 = F * 1.2** => **F = 13500N**.

--

d)

The diagram below shows two points *A* and *B* on a mountain bike track.



The heights of *A* and *B* above ground level are 20 m and 22 m respectively. The length of the track between *A* and *B* is 16 m. The resistance to motion of a biker on the track may be modelled by a constant force of magnitude 50 N. The total mass of the biker and his bike is 70 kg. The speed of the biker at *A* is v ms^{-1}. Find the minimum value of v if the biker is to reach *B* without pedalling. [7]

The biker must gain a potential energy of **(22 * 9.81 * 70) - (20 * 9.81 * 70) = 1373J**.

If there was no friction, this would require the biker to have a kinetic energy of **1373J** at the top of **A**, as all of this kinetic energy would be converted into potential energy.

However, a frictional force of **50N** acts over a distance of **16m**, so the work done by friction is **50(16) = 800J**.

This means that the total kinetic energy of the biker at the top of **A** must be **1373 + 800 = 2173J**.

This allows the biker to gain **1373J** of potential energy, whilst also losing **800J** of kinetic energy due to friction.

The *minimum value* of **v** assumes that the velocity of the biker at the top of **B** is **0ms^{-1}**; if it was any greater than this, the initial kinetic energy would be higher, and hence the initial velocity would be higher.

So, with a kinetic energy of **2173J** at **A**, the biker's velocity can be found:

2173 = 1/2 * 70 * v^2 => **v = 7.88ms^{-1}**.

-

34

A vehicle of mass 3000 kg has an engine that is capable of producing power up to 12 000 W. The vehicle moves up a slope inclined at an angle α to the horizontal, where $\sin\alpha = 0{\cdot}1$. The resistance to motion experienced by the vehicle is constant at 460 N.

(a) Find the maximum acceleration of the vehicle when its velocity is 3 ms^{-1}. [4]

(b) The vehicle now travels at a velocity of v ms^{-1} against an additional braking force of $10v$ N. The other resistance to motion remains constant at 460 N. Determine the maximum value of v. Give your answer correct to 2 decimal places. [5]

a)

Using **P = Fv**, the driving force of the vehicle when **v = 3ms^{-1}**, is **F = 12000 / 3 = 4000N**.

The forces acting against the vehicle are a resistance to motion of **460N** and a weight component of **3000g(1/10)**.

The vehicle is accelerating, hence **4000 - 460 - 3000g(1/10) = 3000 * a**

597 = 3000a => **a = 0.199ms^{-2}**.

-

b)

The maximum velocity of the vehicle occurs when the total forwards force (i.e. the driving force) is equal to the total backwards forces. The backwards forces are the resistive force of **460N**, the weight component of **3000g(1/10)** and the braking force of **10v**.

If the velocity of the vehicle is **v** and its power output is **12000W**, the driving force exerted by the vehicle is **12000 / v**.

This driving force equals the total backwards force, hence:

12000 / v = 460 + 3000g(1/10) + 10v.

Solving for **v:**

12000 / v = 3403 + 10v

12000 = 3403v + 10v^2

10v^2 + 3403v - 12000 = 0.

Using the quadratic formula gives **v = 3.49ms^{-1}**.

-

35

A vehicle of mass 4000 kg is moving up a hill inclined at an angle α to the horizontal, where $\sin\alpha = \frac{1}{20}$. At time $t = 0$ s, the speed of the vehicle is 2 ms^{-1}. At time $t = 8$ s, the vehicle has travelled 30 m up the hill from its initial position and its speed is 5 ms^{-1}. The vehicle's engine is working at a constant rate of 43 000 W. Find the total work done against the resistive forces during this 8 second period. [8]

If the car's engine is working at **43000W**, this means that it inputs **43000J** per second. Thus, over **8s**, the total input is **43000(8) = 344000J**.

This work done is not converted completely into kinetic energy, since some of the energy is converted into potential energy or lost due to the work done by friction.

We can say: $\mathbf{W_E = E_{pot} + E_k + W_f}$ where $\mathbf{W_E}$ is the work done by the engine, and $\mathbf{W_f}$ is the work done by friction.

Using trigonometry, the gain in height of the vehicle is **30(1/20) = 1.5m**. Thus, the gain in potential energy is **1.5 * g * 4000 = 58860J**.

The gain in kinetic energy of the vehicle is **(1/2 * 4000 * 5^2) - (1/2 * 4000 * 2^2) = 42000J**.

Thus, **344000 = 58860 + 42000 + W_F** => **W_F = 243000J** (3sf).

-

36

A particle of mass m approaches a slope with initial speed v_0. It moves up the slope conserving energy.

- (i) What is its initial energy?
- (ii) To what vertical height h_f does it rise when it finally stops?

Assume the slope is frictionless.

i)

The particle initially only has kinetic energy, thus **E = 1/2 * m * v_0^2 = mv_o^2 / 2**

ii)

The kinetic energy of the particle is converted completely into potential energy assuming no friction.

Thus, **mv_o^2 / 2 = mgh_f**

Cancelling **m**:

V_0^2 / 2 = gh_f

h_f = V_0^2 / 2g.

What is the speed of the particle at a height of $3h_f$ / 4 ? .

Let the speed be **v_1**.

The kinetic energy of the particle has been converted into some potential energy, but there is still some kinetic energy remaining.

Thus, **mv_o^2 / 2 = mg($3h_f$ / 4) + (1/2 * m * v_1^2)**

Cancelling **m** gives **v_o^2 / 2 = $3h_f g$ / 4 + v_1^2 / 2**

Multiplying by 2:

$\mathbf{v_0^2 = 3h_f g / 2 + v_1^2}$

Thus, $\mathbf{v_1 = \sqrt{(v_0^2 - 3h_f g / 2)}}$

37

By considering energy, show that the speed v reached when a particle is dropped from rest through a height h is $v^2 = 2gh$. Show that if it initially has a downward speed u, then its final speed is $v^2 = u^2 + 2gh$. [Notice how this energy method yields the result $v^2 - u^2 = 2as$ familiar from kinematics.]

[Ignore the effects of friction]

Part 1:

When a particle is dropped from rest, and there is no air resistance, the change in potential energy is converted into kinetic energy.

The change in potential energy is **mgh**. This is equal to the gain in kinetic energy, $\mathbf{1/2mv^2}$.

Hence, $\mathbf{mgh = 1/2mv^2}$ => $\mathbf{gh = 1/2v^2}$ => $\mathbf{2gh = v^2}$.

Part 2:

When the particle has an initial kinetic energy, its final kinetic energy is a sum of the initial kinetic energy and the kinetic energy gained due to the change in potential energy.

The initial kinetic energy is $\mathbf{1/2mu^2}$.

The change in kinetic energy is equal to the change in potential energy, which is **mgh**.

The final kinetic energy is $\mathbf{1/2mv^2}$.

Thus, $\mathbf{mgh + 1/2mu^2 = 1/2mv^2}$

$\mathbf{gh = 1/2v^2 - 1/2u^2}$

$\mathbf{2gh = v^2 - u^2}$.

So $\mathbf{v^2 = u^2 + 2gh}$.

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38

A car of mass 700 kg is travelling up a hill which is inclined at a constant angle of 5° to the horizontal. At a certain point P on the hill the car's speed is $20\,\mathrm{m\,s^{-1}}$. The point Q is 400 m further up the hill from P, and at Q the car's speed is $15\,\mathrm{m\,s^{-1}}$.

(i) Calculate the work done by the car's engine as the car moves from P to Q, assuming that any resistances to the car's motion may be neglected. [4]

The gain in potential energy of the car is **700g*400sin5 = 2.39*10^{5}J**.

The gain in potential energy is equal to the loss in kinetic energy assuming that there is no work done by the engine.

However, the change in kinetic energy is only **(1/2 * 20^2 * 700) - (1/2 * 15^2 * 700) = 61250J**.

This means that the car must have put in an extra amount of work of **(2.39*10^5) - 61250 = 1.78*10^{5}J**.

-

Assume instead that the resistance to the car's motion between P and Q is a constant force of magnitude 200 N.

(ii) Given that the acceleration of the car at Q is zero, show that the power of the engine as the car passes through Q is 12.0 kW, correct to 3 significant figures. [3]

If the acceleration of the car at **Q** is zero, the driving force must equal the resistive forces.

Thus, **F = 700gsin5 + 200 = 798N**.

The velocity at **Q** is **15ms^{-1}**.

Thus, **P = 798(15) = 11970W = 12.0kW** (3s.f.)

-

Given that the power of the car's engine at P is the same as at Q, calculate the car's retardation at P. [3]

[Retardation is deceleration]

The velocity at **P** is **20ms^{-1}**, so the driving force is **P / v = 12000 / 20 = 600**.

The net force acting on the car is therefore **(200 + 700gsin5) - 600 = 198N**.

Thus, **198 = 700a** => **a = 0.283ms^{-2}**.

-

39

Marco is riding his bicycle at a constant speed of $12\,\mathrm{m\,s^{-1}}$ along a horizontal road, working at a constant rate of 300 W. Marco and his bicycle have a combined mass of 75 kg.

(i) Calculate the wind resistance acting on Marco and his bicycle. [2]

Using **P = Fv**, the driving force of Marco is **300 / 12 = 25N**.

Since Marco is moving at a constant velocity, the wind resistance must also be **25N**.

Nicolas is riding his bicycle at the same speed as Marco and directly behind him. Nicolas experiences 30% less wind resistance than Marco.

(ii) Calculate the power output of Nicolas. [2]

The wind resistance experienced by Nicolas is **25(0.7) = 17.5N**.

Assuming Nicolas moves at a constant velocity, his driving force is **17.5N**.

Thus, **P = 17.5 * 12 = 210W**.

-

The two cyclists arrive at the bottom of a hill which is at an angle of 1° to the horizontal. Marco increases his power output to 500 W.

(iii) Assuming Marco's wind resistance is unchanged, calculate his instantaneous acceleration immediately after starting to climb the hill. [5]

Upon hitting the bottom of the hill, Marco has a velocity of **12ms^{-1}**.

With a power output of **500W**, the driving force must be **500 / 12 = 125/3N**.

The forces acting against Marco are a weight component of **75gsin1** and a resistance of **25N**.

Thus, the net force on Marco is **(125/3) - (75gsin1 + 25) = 3.82N**.

3.82 = 75a => **a = 0.0509ms^{-2}**.

-

Marco reaches the top of the hill at a speed of $13\,\mathrm{m\,s^{-1}}$. He then freewheels down a hill of length 200 m which is at a constant angle of 10° to the horizontal. He experiences a constant wind resistance of 120 N.

(iv) Calculate Marco's speed at the bottom of this hill. [5]

The change in potential energy of Marco from the top to the bottom of the hill is **200*sin10*g*75 = 25552J**.

If there was no friction, this would be converted completely into kinetic energy, but the work done by friction is **120(200) = 24000J**.

Thus, the gain in kinetic energy of Marco is **25552 - 24000 = 1552J**.

The initial kinetic energy of Marco is **1/2 * 13^2 * 75 = 6337.5J**, so his final kinetic energy is **6337.5 + 1552 = 7889.5J**.

7889.5 = 1/2 * 75 * v^2 => **v = 14.5$\mathrm{ms^{-1}}$**.

-

40

A skier of mass 80 kg is pulled up a slope which makes an angle of 20° with the horizontal. The skier is subject to a constant frictional force of magnitude 70 N. The speed of the skier increases from $2\,\mathrm{m\,s^{-1}}$ at the point A to $5\,\mathrm{m\,s^{-1}}$ at the point B, and the distance AB is 25 m.

(i) By modelling the skier as a small object, calculate the work done by the pulling force as the skier moves from A to B. [5]

The work done by the pulling force is converted into kinetic energy, potential energy, and some heat/sound energy due to friction.

The gain in potential energy is **80 * 9.81 * 25sin20 = 6710J**.

The work done by friction is **70(25) = 1750J**.

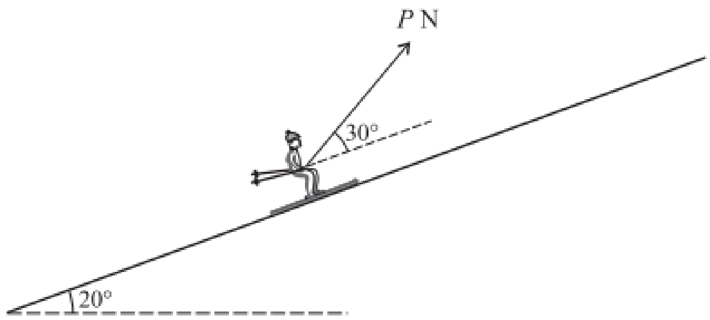
This means that the pulling force must put in a work of **8460J** to counteract these energy losses.

However, the skier also gains velocity, with a change in kinetic energy of **(1/2 * 80 * 25) - (1/2 * 80 * 4) = 840J**.

This means that the total work put in must be **8460 + 840 = 9300J**.

-

(ii)



It is given that the pulling force has constant magnitude P N, and that it acts at a constant angle of 30° above the slope (see diagram). Calculate P. [3]

The acceleration of the skier between **A** and **B** can be found with SUVAT:

u = 2, v = 5, s = 25

$5^2 = 2^2 + 2(a)(25)$ => **$a = 0.42ms^{-2}$**.

The component of the pulling force acting in the direction of travel is **Pcos30**.

The forces acting against the skier are a weight component of **80gsin20** and a frictional force of **70N**.

Thus, the net force on the skier is **Pcos30 - 80gsin20 - 70**. This is equal to **ma**, thus:

Pcos30 - 80gsin20 - 70 = 80 * 0.42

Pcos30 = 372

P = 430N

-

41

A model train has mass 100 kg. When the train is moving with speed v m s^{-1} the resistance to its motion is $3v^2$ N and the power output of the train is $\frac{3000}{v}$ W.

The train moves with constant speed up a straight hill inclined at an angle α to the horizontal, where $\sin\alpha = \frac{1}{98}$.

(iii) Calculate the speed of the train. *(Q5, Jan 2007)* [5]

The train has a driving force of **F** acting up the slope, a weight component of **100g(1/98) = 10N** acting down the slope, and a frictional force of **$3v^2$** acting down the slope.

The driving force is found with **P / v**. So, if **P = 3000 / v**, **F = (3000 / v) / v = $3000/v^2$**.

Since the train is moving at constant velocity, the driving force **$3000/v^2$**, must equal the resistive forces.

Thus, **$3000 / v^2 = 3v^2 + 10$**

$3000 = 3v^4 + 10v^2$

Let **$v^2 = u$**.

$3000 = 3u^2 + 10u$

$3u^2 + 10u - 3000 = 0$

$u = 30ms^{-1}$.

$V^2 = 30$ => **$v = 5.48ms^{-1}$**.

--

42

A rocket of mass 250 kg is moving in a straight line in space. There is no resistance to motion, and the mass of the rocket is assumed to be constant. With its motor working at a constant rate of 450 kW the rocket's speed increases from $100\,m\,s^{-1}$ to $150\,m\,s^{-1}$ in a time t seconds.

(i) Calculate the value of t. [4]

(ii) Calculate the acceleration of the rocket at the instant when its speed is $120\,m\,s^{-1}$. [4]

(Q3, June 2007)

i)

The gain in kinetic energy of the rocket is **$(1/2 * 250 * 150^2) - (1/2 * 250 * 100^2) = 1.56*10^6$J**.

The work done by the motor is converted completely into kinetic energy due to the lack of friction.

The work done by the motor over a time **t** is **$(450*10^3)$t**.

Thus, **$(450*10^3)$t = $1.56*10^6$J** => **t = 3.47s**.

-

ii)

The driving force at **120ms^{-1}** is **$450*10^3$ / 120 = 3750N**.

Thus, **3750 = 250a** => **a = 15ms^{-2}**.

-

43

A cyclist and her bicycle have a combined mass of 70 kg. The cyclist ascends a straight hill *AB* of constant slope, starting from rest at *A* and reaching a speed of 4 m s^{-1} at *B*. The level of *B* is 6 m above the level of *A*.

During the ascent the resistance to motion is constant and has magnitude 60 N. The work done by the cyclist in moving from *A* to *B* is 8000 J.

(iii) Calculate the distance *AB*. [4]

We will let the distance **AB** be **x**.

If the body did not gain any potential energy, and no energy was lost as heat / sound energy due to friction, all of the 8000J of energy would be converted into kinetic energy.

However, there is also work done by the gravitational force and work done by the frictional force that depletes this gain in kinetic energy.

The work done by the frictional force is **60x**.

The work done to gain potential energy is **6 * 70 * 9.81 = 4120J**.

The gain in kinetic energy is **1/2 * 4^2 * 70 = 560J**.

Thus, **8000 - 4120 - 60x = 560**.

60x = 3320J.

x = 55.3m.

-

44

A car of mass 800 kg experiences a resistance of magnitude kv^2 N, where k is a constant and v m s^{-1} is the car's speed. The car's engine is working at a constant rate of P W. At an instant when the car is travelling on a horizontal road with speed 20 m s^{-1} its acceleration is 0.75 m s^{-2}. At an instant when the car is ascending a hill of constant slope 12° to the horizontal with speed 10 m s^{-1} its acceleration is 0.25 m s^{-2}.

(i) Show that $k = 0.900$, correct to 3 decimal places, and find P. **[7]**

The power is increased to 1.5P W.

(ii) Calculate the maximum steady speed of the car on a horizontal road. **[3]**

(Q4, Jan 2009)

First consider the motion of the car on the horizontal straight.

The driving force of the car is equal to **P / v = P / 20**.

The resistive force is **$k(20^2) = 400k$**.

Thus, (**P / 20) - 400k = 800(0.75)**

(P/20) = 600 + 400k

P = 12000 + 8000k.

Now consider the motion of the slope.

The driving force is **P / 10**. The resistive force is **$k(10^2) = 100k$**. There is also a weight component of **800gsin12**.

Thus, **(P/10) - 100k - 800gsin12 = 800(0.25)**

(P/10) = 1832 + 100k

P = 18320 + 1000k.

Equating our equations for **P** gives **12000 + 8000k = 18320 + 1000k**

7000k = 6320 => **k = 0.903**.

Thus, **P = 18320 + 1000(0.903) = 19223W = 19.2kW** (3sf)

-

ii)

The power is now **1.5(19.2) = 28.8kW**.

The forces acting on the car on a horizontal straight are a driving force and a resistive force.

The resistive force is **$0.903v^2$**.

When the car is at maximum speed, the resistive force is equal to the driving force.

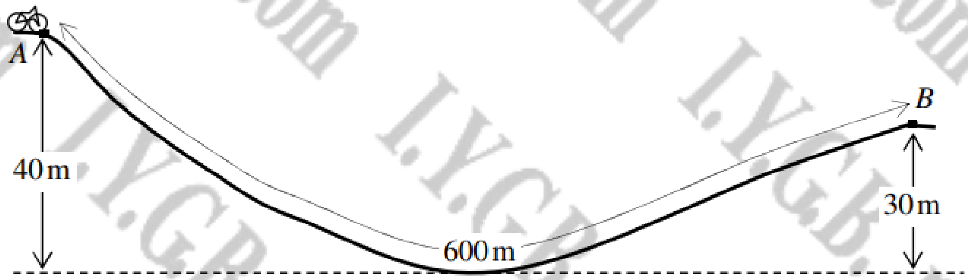
Thus, **$0.903v^2 = F$**.

Using **P = Fv** => **$v = P / F = 28800 / 0.903v^2$**.

$0.903v^3 = 28800$

v = 31.7ms⁻¹.

45



The figure above shows the path of a cyclist on a section of a road from A to B, where the distance AB is 600 m.

The cyclist leaves point A at the top of a hill with a speed of 10 ms^{-1} and descends a vertical distance of 40 m to the bottom of the hill. The cyclist then ascends a vertical distance of 30 m to the top of another hill at point B. The speed of the cyclist at B is 12 ms^{-1}.

The combined mass of the cyclist and his bike is 80 kg.

The cyclist and his bike are modelled as a single particle subject to a constant non gravitational resistance of 25 N, throughout the motion.

Find the work done by the cyclist.

theory:

If the cyclist had no friction acting against them, the total transfer to kinetic energy would be the change in potential energy + the work done by the cyclist. However, the cyclist also has friction acting against them, so the change in potential energy and the work done is not completely converted into kinetic energy; some is lost due to friction.

Suppose the change in potential energy was 100J, but the work done by friction was 150J, and the gain in kinetic energy was 30J. This means that an additional 80J of work must have been put in to achieve this gain in kinetic energy.

Thus, we can say $\mathbf{E_g + W_d = E_k + W_f}$, where $\mathbf{W_d}$ is the work done by the body (cyclist in this case), and $\mathbf{W_f}$ is the work done by friction.

This equation applies to the above situation.

-

The loss in potential energy of the cyclist is **(40 * g * 80) - (30 * g * 80) = 7848J**.

The work done by friction is **25(600) = 15000J**. The gain in kinetic energy is **(1/2 * 80 * 12^2) - (1/2 * 80 * 10^2) = 1760J**.

Thus, **7848 + W_d = 15000 + 1760** => **W_d = 8912J**.

--

46

A gun is fired at a 3 cm thick solid wooden door.

The bullet, of mass 7g, travels through the door and has its speed reduced from 450 ms^{-1} to 175 ms^{-1}.
Assuming uniform resistance, what is the force of the wood on the bullet?

The change in kinetic energy of the bullet is **(1/2 * 0.007 * 450^2) - (1/2 * 0.007 * 175^2) = 601.6J**.

This means that the work done by the resistive force exerted by the door is **601.6J**.

If **601.6J** of work is done over **0.03m**, the force exerted is **601.6 / 0.03 = 20053N**.

47

In a science experiment, a 50g mass slides down a 60^0 incline of length 0.5m.

If the mass is given an initial speed of 2 ms^{-1} down the plane and its final speed is measured as 3 ms^{-1}, what is the magnitude of the frictional force opposing the mass?

The change in potential energy of the mass is **0.05 * 0.5sin60 * 9.81 = 0.212J**.

The gain in kinetic energy is **(1/2 * 9 * 0.050) - (1/2 * 4 * 0.050) = 0.125J**.

This means that the work done by friction is **0.212 - 0.125 = 0.087J**.

Thus, **0.087 = F * 0.5** => **F = 0.174N**.

48

A pump forces up water at a speed of 8ms^{-1} from a well into a reservoir at a rate of 50 kg s^{-1} .

If the water is raised a vertical height of 40 m, what is the work done per second?

If the water gains 8ms^{-1} of velocity, its gain in kinetic energy per second is **1/2 * 50 * 8^2 = 1600J**.

It is also gaining a potential energy of **50 * 9.81 * 40 = 19620J** per second.

This means that the total work that must be done per second by the pump is **19620 + 1600 = 21220Js^{-1}** => this is the power of the pump.

-

49

A car of mass 1500 kg is travelling up a hill on a straight road, with the engine of the car working at the constant rate of 13 kW for 1 minute.

During this minute the car increases its speed from 7 ms^{-1} to 24 ms^{-1} and in addition to the work done against gravity, 80000 J of work is done against resistances to motion parallel to the direction of motion of the car.

Calculate the vertical displacement of the car in this 1 minute interval.

The total work done by the engine in 1minute is **13000(60) = 780000J**.

This work done is converted into kinetic energy, potential energy, and the work done by friction.

The gain in kinetic energy of the car is **(1/2 * 24^2 * 1500) - (1/2 * 7^2 * 1500) = 395250J**.

If we let the vertical displacement be **x**.

780000 = E_K + E_g + W_F

780000 = 395250 + (1500 * g * x) + 80000

304750 = 1500gx => **x = 20.7m**.

-

50

A particle of mass 0.5kg is projected with a speed of 5 ms^{-1} from a point A on a rough plane, inclined at an angle of $20°$ to the horizontal.

The particle slides up the plane and comes to instantaneous rest at a point B on the plane which is 2 m away from A.

The points A and B lie on the line of greatest slope on the plane. The particle is subject to a constant non gravitational resistance of R N, throughout the motion.

a) Determine the value of R, correct to three significant figures.

The kinetic energy of the particle at the bottom of the slope is **1/2 * 25 * 0.5 = 6.25J**.

This is converted into potential energy and the work done by friction.

The gain in potential energy of the particle is **2sin20 * 9.81 * 0.5 = 3.355J**.

Thus, the work done by friction is **6.25 - 3.355 = 2.895J**.

This work done occurs over a distance of **2m**, so the force exerted is **2.895 / 2 = 1.45N**.

The angle of the plane is now increased to $40°$ but the value of R remains unchanged.

The particle is next projected with a speed of 8 ms^{-1} from A and slides up the plane coming to instantaneous rest at a distance d m away from A.

b) Find the value of d, correct to three significant figures.

The kinetic energy of the particle is **1/2 * 64 * 0.5 = 16J** upon projection.

This is converted into potential energy and the work done by friction, **16 = W_F + E_g**.

The gain in potential energy is **dsin40 * g * 0.5**, and the work done by friction over a distance **d** is **1.45d**.

Thus, **16 = 1.45d + 0.5gdsin40**

16 = d(1.45 + 0.5gsin40) => **d = 3.48m**.

-

51

A woman and her bike are modelled a single particle of combined mass 72 kg.

The woman cycles with constant speed of 5 ms^{-1}, **up** a straight road, which lies on the line of greatest slope of a plane inclined at an angle θ to the horizontal, where $\sin\theta = \frac{2}{21}$.

The total non gravitational resistance experienced by the cyclist is assumed to be constant at 25 N.

a) Find the power generated by the woman when cycling up the hill.

The driving force must equal the resistive forces.

Thus, **F = 25 + (72g(2/21)) = 92.27N**.

P = Fv = 92.27 * 5 = 461W.

-

The woman then turns her bike around at some point A on the road. She freewheels down the same road starting with a speed of 5 ms^{-1}. She passes through some point B on that road with a speed v ms^{-1}.

The total non gravitational resistance experienced by the cyclist is assumed to be the same as in part (a).

b) Given that the distance AB is 180 m, find the value of v.

Method 1: kinematics and dynamics

The forces acting on the biker are the weight component acting down the slope, and the frictional force of 25N.

Thus, **72g(2/21) - 25 = 72 * a** => **a = 0.587ms^{-2}**.

Thus, **u = 5, a = 0.587, v = v, s = 180**

v = √(5^2 + 2(0.587)(180)) = 15.4ms^{-1}.

Method 2: energy changes

The change in potential energy between A and B is **72 * g * 180(2/21) = 12108J**.

The work done by friction is **25(180) = 4500J**.

Thus, the conversion of potential energy into kinetic energy is **12108 - 4500 = 7608J**.

This gives a final kinetic energy of **(1/2 * 25 * 72) + 7608 = 8508J**.

Thus, **8508 = 1/2 * 72 * v^2** => **v = 15.4ms^{-1}**.

Elastic and inelastic collisions

In a collision, momentum is always conserved provided that there are no external forces. If a body of mass $\mathbf{m_1}$ and velocity $\mathbf{u_1}$ collides with a body of mass $\mathbf{m_2}$ with velocity $\mathbf{u_2}$, causing the body of mass $\mathbf{m_1}$ to move off with a velocity $\mathbf{v_1}$ and $\mathbf{m_2}$ to move off with a velocity of $\mathbf{v_2}$, we can say:

$$m_1u_1+m_2u_2=m_1v_1+m_2v_2$$

Although momentum is conserved in a collision where no external forces are exerted, kinetic energy does not always have to be conserved.

There are two types of collision:

- **Elastic collision:** a collision where kinetic energy is conserved and momentum is conserved.
- **Inelastic collision:** a collision where kinetic energy is not conserved and momentum is conserved.

Elastic collisions

An **elastic collision** is a collision where both momentum and kinetic energy are conserved.

This means that the total kinetic energy of the colliding bodies before the collision must equal the total kinetic energy of the colliding bodies after the collision - so no energy is lost as heat / sound energy to the surroundings.

Again take the situation where a body of mass m_1 and velocity u_1 collides with a body of mass m_2 with velocity u_2, causing the body of mass m_1 to move off with a velocity v_1 and m_2 to move off with a velocity of v_2.

The total kinetic energy of the colliding bodies before the collision is $1/2m_1u_1^2 + 1/2m_2u_2^2$.

The total kinetic energy of the colliding bodies after the collision is $1/2m_1v_1^2 + 1/2m_2v_2^2$.

Since kinetic energy is conserved in an elastic collision, we can say:

$$1/2m_1u_1^2 + 1/2m_2u_2^2 = 1/2m_1v_1^2 + 1/2m_2v_2^2$$

Referring to this link, we can prove that since kinetic energy is conserved in an elastic collision,

$$u_1 + v_1 = u_2 + v_2$$

For example, if body A has an initial velocity of $4ms^{-1}$, and a velocity of $5ms^{-1}$ in the opposite direction after collision with body B, and body B has an initial velocity of $2ms^{-1}$ in the opposite direction to the initial direction of A, the final velocity of B is:

$4 - 5 = -2 + v_2 \Rightarrow v_2 = 1ms^{-1}$.

In elastic collision problems, we are often told the initial velocities of the colliding bodies, and asked to find the final velocities (so two unknowns) given that the collision is elastic.

Such questions can be answered by considering both conservation of momentum and conservation of kinetic energy.

Example

A 4kg ball moving east at a speed of 5ms^{-1} strikes a 2kg ball at rest. The collision is perfectly elastic. ***Find the velocities of the balls after collision.***

We will let the velocity of the **4kg** ball after collision be $\mathbf{v_1}$ and the velocity of the **2kg** ball after collision be $\mathbf{v_2}$.

Applying conservation of momentum, $\mathbf{4(5) + 0 = 4(v_1) + 2(v_2)}$ => $\mathbf{20 = 4v_1 + 2v_2}$.

Because the collision is elastic, we can say $\mathbf{u_1 + v_1 = u_2 + v_2}$.

Thus, $\mathbf{5 + v_1 = 0 + v_2}$ => $\mathbf{5 + v_1 = v_2}$

We can now substitute $\mathbf{v_2 = 5 + v_1}$ into the equation $\mathbf{20 = 4v_1 + 2v_2}$ and solve for $\mathbf{v_1}$**:**

$\mathbf{20 = 4v_1 + 2(5+v_1)}$

$\mathbf{20 = 4v_1 + 10 + 2v_1}$

$\mathbf{6v_1 = 10}$ => $\mathbf{v_1 = 10 / 6 = 1.67ms^{-1}}$.

Therefore, $\mathbf{v_2 = 5 + 1.67 = 6.67ms^{-1}}$.

Two bumper cars at an amusement park collide elastically as one approaches the other directly from the rear. The car in front (CarA) has a mass of $550kg$ and the car behind it (CarB) has a mass of $450kg$. The car in front was traveling at $3.70m/s$ while the car behind hit him with a velocity of $4.50m/s$. What are their final velocities after the collision?

We will let the velocity of car A after collision be $\mathbf{v_1}$ and the velocity of car B after collision be $\mathbf{v_2}$.

Thus, applying conservation of linear momentum: $\mathbf{550(3.70) + 450(4.50) = 550(v_1) + 450(v_2)}$

$\mathbf{4060 = 550v_1 + 450v_2}$.

Because the collision is elastic, we can say $\mathbf{u_1 + v_1 = u_2 + v_2}$, thus $\mathbf{3.70 + v_1 = 4.50 + v_2}$, which re-arranges so $\mathbf{v_2 = v_1 - 0.80}$.

Substituting $\mathbf{v_2 = v_1 - 0.80}$ into $\mathbf{4060 = 550v_1 + 450v_2}$ gives:

$\mathbf{4060 = 550v_1 + 450(v_1-0.80)}$.

4060 = 550v_1 + 450v_1 - 360.

4420 = 1000v_1

v_1 = 4.42ms^{-1}.

Thus, **v_2 = 4.42 - 0.80 = 3.62ms^{-1}**.

-

Another type of question is to prove that a collision is elastic. In this instance, we have to prove that the kinetic energy before the collision is equal to the kinetic energy after the collision.

Example:

A helium nucleus of mass 4*m* collides head-on with a stationary proton of mass *m*. Use the information in Figure 11.11 to show that this is an elastic collision.

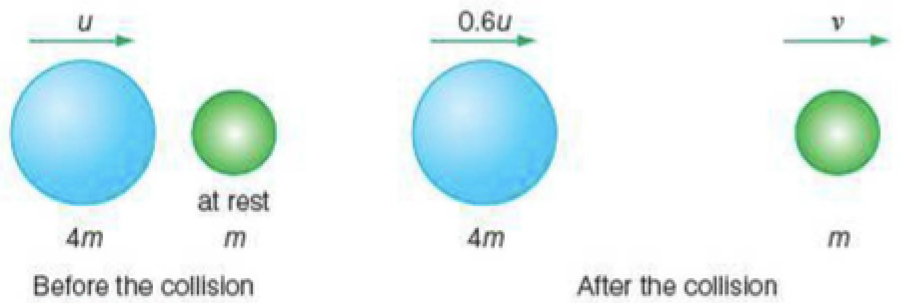


Figure 11.11

First, we can use conservation of momentum to find the velocity of the proton after collision:

4mu + m(0) = 4m(0.6u) + mv

4mu - 2.4mu = mv

1.6u = v.

We can then find the kinetic energy before and after the collisions.

It is important to consider that energy is *not a vector*, so when substituting **v** into **1/2mv^2**, we do not have to consider if the velocity is positive or negative; we just substitute the magnitude of the velocity.

$E_{k\ before}$ = 1/2(4m)$(u)^2$ = 2mu^2.

$E_{k\ after}$ = (1/2 * (4m) * $(0.6u)^2$) + (1/2 * m * $(1.6u)^2$) = (2m * 0.36u^2) + (1/2m * 2.56u^2)

= $0.72mu^2$ + $1.28mu^2$ = $2mu^2$.

The total kinetic energy before and after the collisions is equal, so no energy is lost to the surroundings, making the collision elastic.

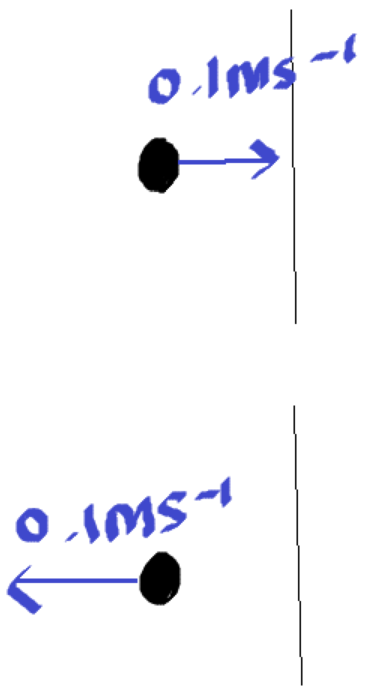
-

In reality, most collisions are not elastic, as a collision usually involves a transfer of energy to the surroundings.

On the macroscopic scale (e.g. car crashes, collision of two billiard balls), kinetic energy is not conserved in collisions - so the collision is *inelastic*. However, in many situations, such little kinetic energy is lost that we assume the collision to be elastic.

On the microscopic scale, collisions between atoms and molecules are almost completely elastic, with very little energy being lost to the surroundings.

For example, when a gas molecule collides with the wall of a container, the collision is assumed to be elastic. The gas molecule retains its kinetic energy, so the magnitude of the velocity before is equal to the magnitude of the velocity after:



If a gas molecule moving with velocity $0.1ms^{-1}$ collides with the boundary of a container, the velocity after collision is $0.1ms^{-1}$ in the opposite direction.

So if a gas particle of mass **m** and velocity **v** collides with a container, its momentum before is **mv**, and its momentum after is **-mv**, thus the change in momentum, or impulse is **-mv - mv = -2mv**.

Inelastic collisions

In an inelastic collision, momentum is conserved, but kinetic energy *is not conserved*. In such a collision, the collision results in energy being transferred to the surroundings. For example, a car crash will involve a large transfer of thermal energy and sound energy to the surroundings, such that the kinetic energy of the cars is not conserved.

Suppose we have a body with a kinetic energy of $\mathbf{E_{ki1}}$ before collision, and a kinetic energy of $\mathbf{E_{kf1}}$ after collision. The body collides with another body with a kinetic energy of $\mathbf{E_{ki2}}$ before collision and $\mathbf{E_{kf2}}$ after collision. If the collision is inelastic, we can say that the total kinetic energy before the collision is the total kinetic energy after the collision plus the energy loss in the collision (let this be $\mathbf{E_L}$).

$$E_{ki1} + E_{ki2} = E_{kf1} + E_{kf2} + E_L.$$

E.g. if the kinetic energy before collision is 30J, and the kinetic energy after collision is 25J, the energy lost to the surroundings is 5J.

-

For example, suppose a **5kg** body travelling at **3ms**$^{-1}$ collides with an **8kg** body travelling at **1ms**$^{-1}$ in the opposite direction. After the collision, the **5kg** body changes its direction of motion, and its velocity is **1.5ms**$^{-1}$. The collision between the bodies is inelastic. **What is the loss in energy during the collision?**

We need to start by finding the velocity of the 8kg particle after collision. Although the collision is inelastic, the principle of conservation of momentum still applies.

Taking the initial direction of travel of the 5kg particle as positive and applying conservation of momentum:

5(3) + 8(-1) = 5(-1.5) + 8v

15 - 8 = -7.5 + 8v

v = (7+7.5) / 8 = 1.81ms^{-1}.

We can now find the total kinetic energy of the particle's before collision and the total energy after collision. Energy is *not a vector*, so we do not have to consider the positive or negative velocities when finding kinetic energy, only the magnitude of the velocities.

$\mathbf{E_k}$ **before = (1/2 * 5 * 3**^{2}**) + (1/2 * 8 * 1**^{2}**) = 26.5J**.

$\mathbf{E_k}$ **after = (1/2 * 5 * 1.5**^{2}**) + (1/2 * 8 * 1.81**^{2}**) = 18.7J**.

The loss in kinetic energy during the collision is therefore **26.5 - 18.7 = 7.8J**.

-

An inelastic collision can be classed as **perfectly inelastic**. This is an inelastic collision where the maximum amount of kinetic energy is lost.

In such a collision, the colliding bodies stick together after collision.

An example of a perfectly inelastic collision is a car crash where the cars stick together.

Example: A 720kg mass car travelling at 45ms^{-1} collides with a 1200kg truck travelling in the same direction at 12ms^{-1}. The car and truck stick together after the collision.

Find the loss in kinetic energy during the collision.

First, we must find the velocity of the vehicles after the collision.

When the car and truck stick together, they are now moving at the same velocity. Applying conservation of momentum:

720(45) + 1200(12) = v(720 + 1200)

46800 = 1920v

v = 24.4ms^{-1}.

The total kinetic energy before collision is: **(1/2 * 720 * 45^2) + (1/2 * 1200 * 12^2) = 8.15*10^{5}J**.

The total kinetic energy after the collision is **(1/2 * (720 + 1200) * 24.4^2) = 5.71*10^{5}J**.

The loss in kinetic energy in the collision is therefore **(8.15*10^5) - (5.71*10^5) = 2.44*10^{5}J**.

-

Another example of a perfectly inelastic collision is firing a bullet that becomes embedded into a block.

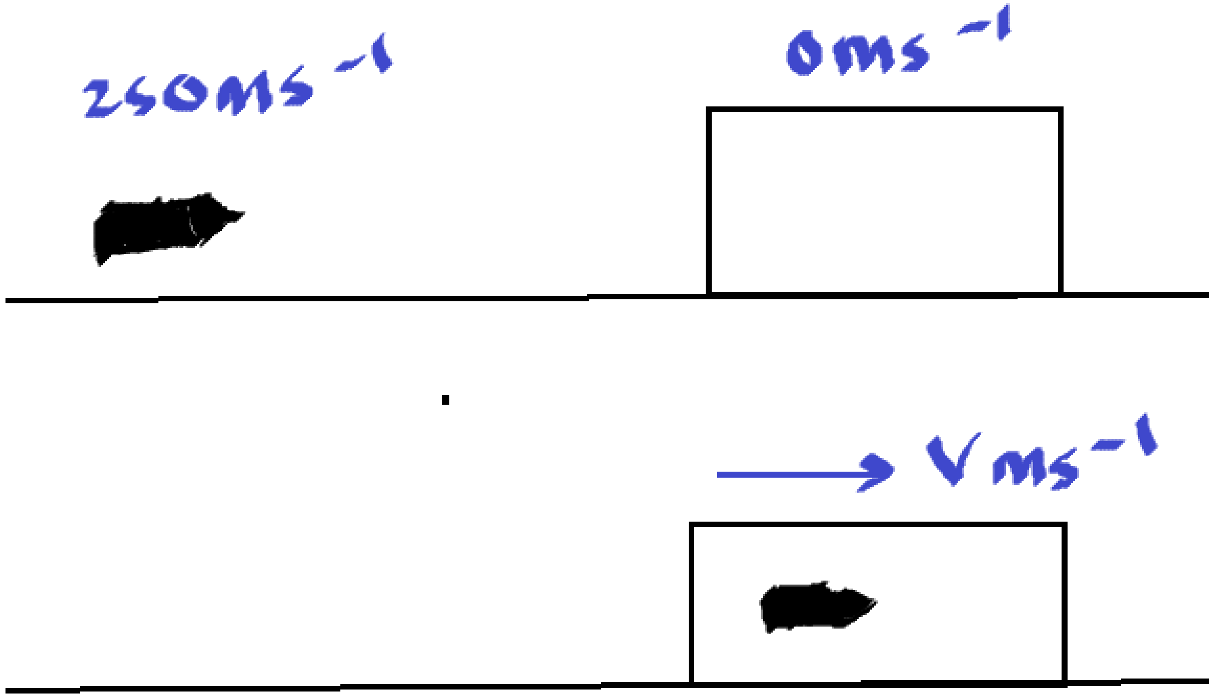
Example:

A **10g** bullet moving at **250ms^{-1}** embeds into a **0.5kg** block that is initially stationary.

a) **What is the loss in kinetic energy during the collision?**

The collision is perfectly inelastic because the bullet and the block are stuck together, leading to the maximum possible loss of kinetic energy.

First, we must find the velocity of the bullet-block system after collision using conservation of momentum.



250(0.010) = v(0.5 + 0.010)

v = 2.5 / (0.5 + 0.010) = 4.9ms^{-1}.

The kinetic energy before the collision is **1/2 * 250^2 * 0.010 = 312.5J**.

The kinetic energy after the collision is **1/2 * 4.9^2 * (0.5 + 0.010) = 6.1J**.

Thus, the change in kinetic energy is **312.5 - 6.1 = 306J** (3sf) => this will be lost to the surroundings as heat / sound energy.

-

The bullet takes 5ms to embed into the block.

b) **Calculate the force exerted on the bullet by the block.**

The initial momentum of the bullet is **250(0.010) = 2.5kgms^{-1}**.

The momentum of the bullet after collision is **4.9(0.010) = 0.049kgms^{-1}**.

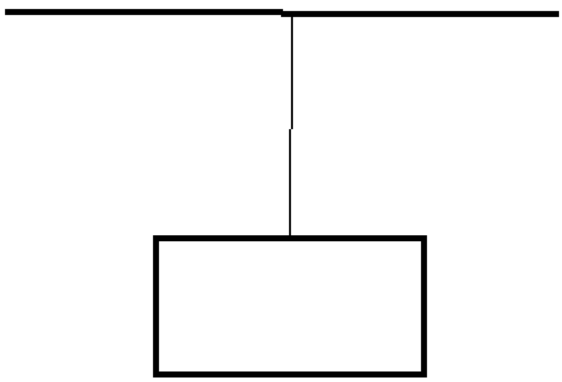
The change in momentum is therefore **2.5 - 0.049 = 2.451kgms^{-1}**.

If a change in momentum of **2.451kgms^{-1}** occurs over a time of **5*10^{-3}s**, the force exerted is **2.451 / (5*10^{-3}) = 490N**.

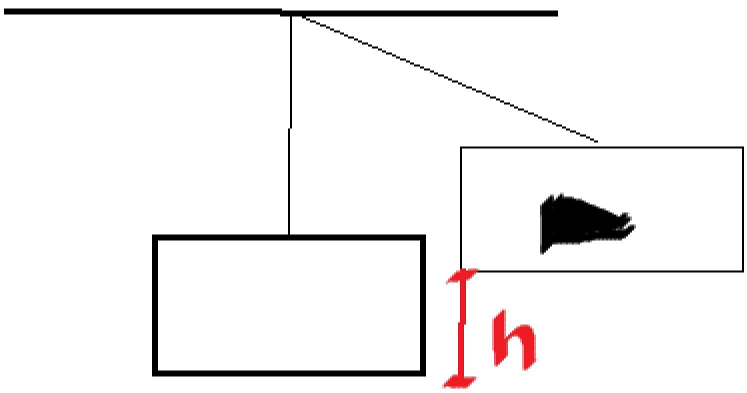
-

This type of problem can be extended to a *ballistic pendulum* situation.

Suppose we have a block of mass **m_A** attached to a string that is attached to the ceiling:



If a bullet of mass **m_B** is fired into this block at velocity **v_A**, and the bullet embeds into the block, it will cause the block to rise a height **h** with an initial velocity of **v_{A+B}**:



When the block rises, assuming no air resistance, all of the kinetic energy gained by the collision will be converted into potential energy, which enables us to find the height gained.

Example:

A bullet of mass 70g moving at 400ms^{-1} collides with a stationary 1.2kg block that is attached to a string that is bound to the ceiling (a pendulum). Upon collision, the bullet embeds into the block. ***Determine how high the bullet-block system will rise after collision.***

We first apply conservation of momentum to determine the velocity of the bullet-block system after collision:

0.070 * 400 = v(1.2 + 0.070) => v = 22.05ms^{-1}.

The kinetic energy of the bullet block system is therefore **1/2 * (0.070 + 1.2) * 22.05^2 = 309J**.

Assuming no air resistance, the kinetic energy of the bullet-block system is converted completely into potential energy upon rising.

Thus, the gain in potential energy is **309J**.

309 = mgh = (0.070 + 1.2)gh

h = 309 / 1.27g = 24.8m.

Example:

A bullet of mass 50g travelling at ***vms^{-1}*** *collides with a stationary block of mass 1.3kg attached by a string to the ceiling. The bullet embeds into the block upon collision, and the bullet-block system rises 4.5m above its initial position.* ***Determine the value of v.***

The gain in potential energy of the bullet-block system after collision is **4.5 * g * (0.050 + 1.3) = 59.6J**.

This gain in potential energy is converted from kinetic energy, so the kinetic energy after collision was also **59.6J**.

This enables us to determine the velocity of the bullet-block system after collision (let this be **v_c**).

59.6 = 1/2 * (0.050 + 1.3) * v_c^2 => v_c = 9.4ms^{-1}.

Applying conservation of momentum, **0.050v = 9.4(1.3 + 0.050)** => **v = 254ms^{-1}**.

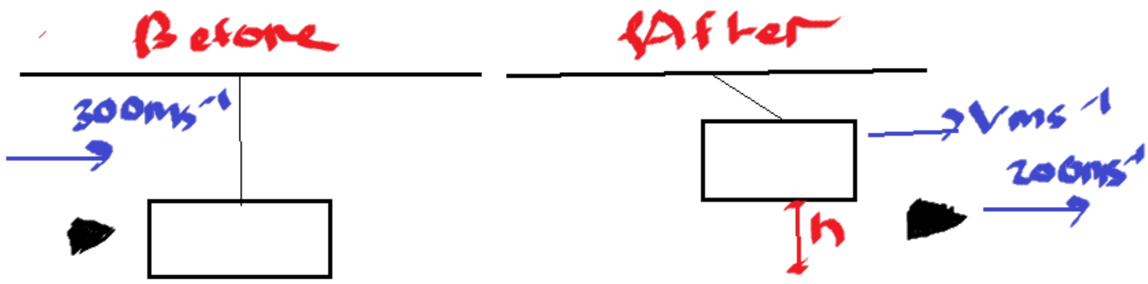
-

Now let us apply this concept to a situation where the inelastic collision between the block and bullet is not *perfectly* inelastic.

Example:

A bullet of mass 90g collides with a stationary block of mass 1.5kg attached to a ceiling by means of a string. The bullet is initially travelling at 300ms^{-1}, and upon collision, it pierces *through the block, and exits at a velocity of 200ms^{-1}.* ***Determine the height the block ascends above its original position after collision.***

The bullet and the block are no longer a joint system, so we only have to consider the kinetic energy of the block after collision.



Considering conservation of momentum:

The total momentum before the collision is **300(0.090) = 27kgms^{-1}**.

The total momentum after the collision is **200(0.090) + 1.5v = 18kgms^{-1} + 1.5v**

Where **v** is the velocity of the block after collision.

Thus, **27 = 18 + 1.5v** => **v = 6ms^{-1}**.

The kinetic energy of the block after collision is therefore **1/2 * 6^2 * 1.5 = 27J**.

This is converted into potential energy, thus **27 = 1.5gh** => **h = 1.83m.**

The time it took for the bullet to pierce through the block was ***4ms****.* ***Calculate the width of the block, assuming the force exerted on the block is constant.***

The block exerts a force on the bullet, which causes the bullet to lose momentum.

Using SUVAT, we know three variables for the movement of the bullet through the block:

u = 300, v = 200, t = 4*10^{-3}.

Thus, **s = 1/2 (200+300) * 4*10^{-3} = 1m**.

This is the width of the block.

Explosions

In an explosion, momentum is conserved. While kinetic energy is lost in inelastic collisions, there must be a kinetic energy *input* during an explosion. This kinetic energy input is generally created by chemical means (e.g. combustion of fuel). Thus, in an explosion, the kinetic energy before the explosion is *less than* the kinetic energy after the explosion.

We can say that the total kinetic energy of a system after an explosion, $\mathbf{E_{kf}}$, is equal to the energy input by the explosion, **X**, + the initial kinetic energy, $\mathbf{E_{ki}}$.

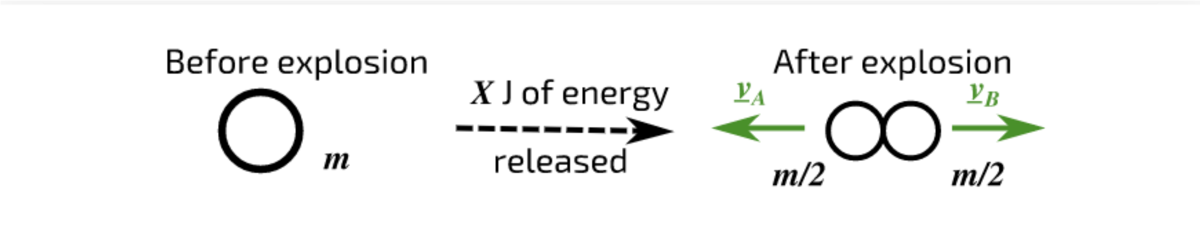
$$\mathbf{X + E_{ki} = E_{kf}}.$$

For example, if the kinetic energy before an explosion is **100J**, and the explosion inputs **500J** of energy, the kinetic energy after the explosion is **600J**.

If the exploding body is initially stationary, we can say $\mathbf{X = E_{kf}}$.

-

Consider the following situation where a ball of mass **m** explodes into two separate pieces of equal mass, **m/2**, with velocities $\mathbf{v_A}$ and $\mathbf{v_B}$. The total energy inputted by the explosion is **X J**.



How can we express V_A in terms of X and m?

Considering conservation of momentum, $\mathbf{0 = m/2(V_A) + m/2(V_B)}$ => $\mathbf{m/2(V_A) = -m/2(V_B)}$ => $\mathbf{V_A = -V_B}$.

So the two fragments are ejected with equal speeds.

The kinetic energy gain of the fragments is $\mathbf{(1/2 * m/2 * v_A^2) + (1/2 * m/2 * v_B^2)}$. Since $\mathbf{v_A}$ has equal magnitude to $\mathbf{v_B}$, we can simply say $\mathbf{v_B = v_A}$ when considering kinetic energy. Thus:

$$\mathbf{\Delta E_k = (1/2 * m/2 * v_A^2) + (1/2 * m/2 * v_A^2) = (m/4 (v_A^2)) + (m/4 (v_A^2)) = 2m/4 (v_A^2) = m/2 (v_A^2)}$$

This change in kinetic energy is caused by the energy inputted by the explosion, **X**. Thus,

$$\mathbf{X = m/2 (v_A^2)}$$

Re-arranging for $\mathbf{v_A}$:

$$\mathbf{2X = mv_A^2}$$

$$\mathbf{v_A^2 = 2X / m} \Rightarrow \mathbf{V_A = \sqrt{(2X / m)}}$$

Example

A musket of mass 24kg fires a bullet of mass 10g. A freeze-frame camera records that the musket recoils a distance of 5cm in a time of 200ms in the explosion.

a) **Calculate the kinetic energy inputted into the bullet-gun system in the explosion.**

The kinetic energy before the explosion is **0J**. To determine the kinetic energy after the explosion, we need to determine the velocity of the bullet and gun after the explosion.

The recoil velocity of the gun is **0.05 / (200*10^{-3}) = 0.25ms^{-1}**.

Now, applying conservation of momentum, we can determine the velocity of the musket ball.

0.25(24) + 0.010(v) = 0 => **v = -600ms^{-1}**.

Thus, the total kinetic energy of the bullet and gun after the explosion is **(1/2 * 600^2 * 0.010) + (1/2 * 0.25^2 * 24) = 1801J**.

This means that **1801J** of kinetic energy must have been transferred to the bullet and gun in the explosion.

-

If the total chemical energy created by the explosion is 5000J,

b) Determine the efficiency of the explosion

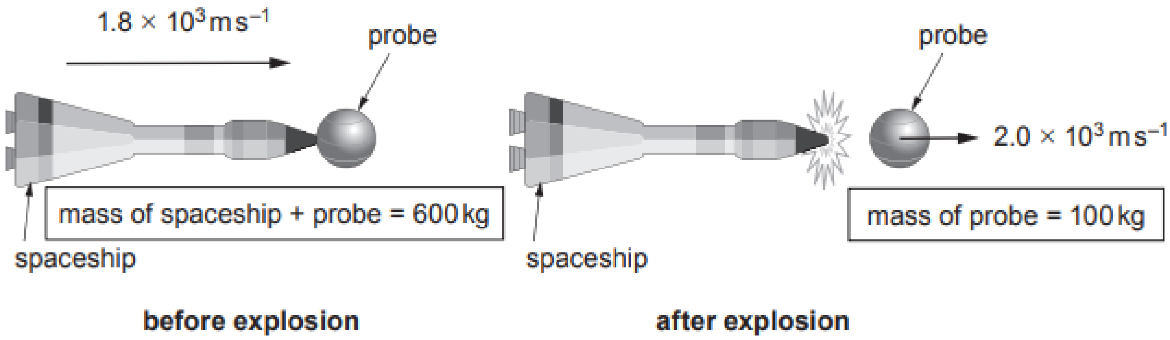
A total of **5000J** of energy was inputted, but only **1801J** was transferred to kinetic energy. The rest of the energy would have been outputted as heat and sound energy.

Thus, the efficiency of the explosion is **(1801 / 5000) * 100 = 36%**.

-

Example

A space probe is attached to a large spaceship as shown. The probe can be ejected from the space ship by an explosion. The spaceship and probe together have a mass of 600 kg and they travel in a straight line through deep space at $1.8 \times 10^3\,m\,s^{-1}$. Explosives are detonated, separating the probe from the spaceship. Immediately after the explosion the probe, of mass 100 kg, continues in the original straight line at $2.0 \times 10^3\,m\,s^{-1}$.



Find the gain in energy of the rocket-probe system after the explosion.

We first apply the principles of conservation of momentum to determine the velocity of the rocket after explosion.

The rocket has mass **500kg** since the probe has mass **100kg** and the total mass of the spaceship + probe is **600kg**.

$1.8*10^3 * 600 = (2*10^3 * 100) + 500v$

$v = 1760ms^{-1}$.

The kinetic energy before the explosion is **$1/2 * 600 * (1.8*10^3)^2 = 9.72*10^8J$**

The kinetic energy after the explosion is **$(1/2 * 500 * (1.76*10^3)^2) + (1/2 * 100 * (2*10^3)^2) = 9.744*10^8J$**.

The kinetic energy inputted by the explosion is therefore **$(9.744*10^8) - (9.72*10^8) = 2.4*10^6J$ = 2.4MJ**.

This kinetic energy input would have been from the combustion of the fuel that created the explosion.

During the explosion, a mean force, F, acts on the probe for 2.0 ms. Calculate the value of F. [2]

The change in momentum of the probe is **$(2*10^3*100) - (1.8*10^3*100) = 20000kgms^{-1}$**.

This occurs over **$2*10^{-3}s$**, so the force exerted is **$20000 / (2*10^{-3}) = 1*10^7N$**.

-

We can prove that the force exerted on the probe is equal and opposite to the force exerted on the spacecraft.

$\Delta p_{spacecraft} = (500 * 1800) - (500 * 1760) = 20000kgms^{-1}$.

The spacecraft loses momentum in the positive direction since the probe gains momentum in the positive direction.

-

Questions

1

3. Air hockey is a game played by two players on a low-friction table using a paddle each and a puck. This question will explore the nature of collisions in one and two dimensions during a game.

Simon and Andrena are practising using two pucks of different masses. They hit their pucks towards each other. The resultant collision is head-on and is illustrated in Figure 1.

Total for Question 3: 12

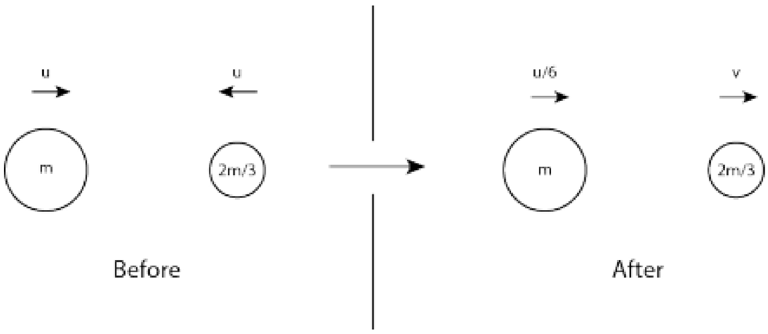


Figure 1: Head-on collision between pucks of different masses. The arrows show the direction of the pucks' motion.

Use the principle of conservation of momentum to express the velocity v in terms of u.

$\mathbf{p_{before} = p_{after}}$

mu - 2m/3(u) = m(u/6) + 2m/3 (v)

mu - 2mu/3 = mu/6 + 2mv/3

mu/3 = mu/6 + 2mv/3

mu/6 = 2mv/3

u/6 = 2v/3

3u/6 = 2v

v = (3u/6) / 2 = 3u/12 = u/4.

Show that the collision is inelastic and calculate the amount of energy converted to forms other than kinetic.

The kinetic energy before the collision is $\mathbf{1/2(m)(u)^2 + 1/2(2m/3)(u^2) = mu^2/2 + 2mu^2/6 = 5mu^2/6}$.

The kinetic energy after the collision is $\mathbf{1/2(m)(u/6)^2 + 1/2(2m/3)(u/4)^2 = mu^2/72 + 2mu^2/96 = 5mu^2/144}$.

The kinetic energy before is not equal to the kinetic energy after, so the collision is inelastic.

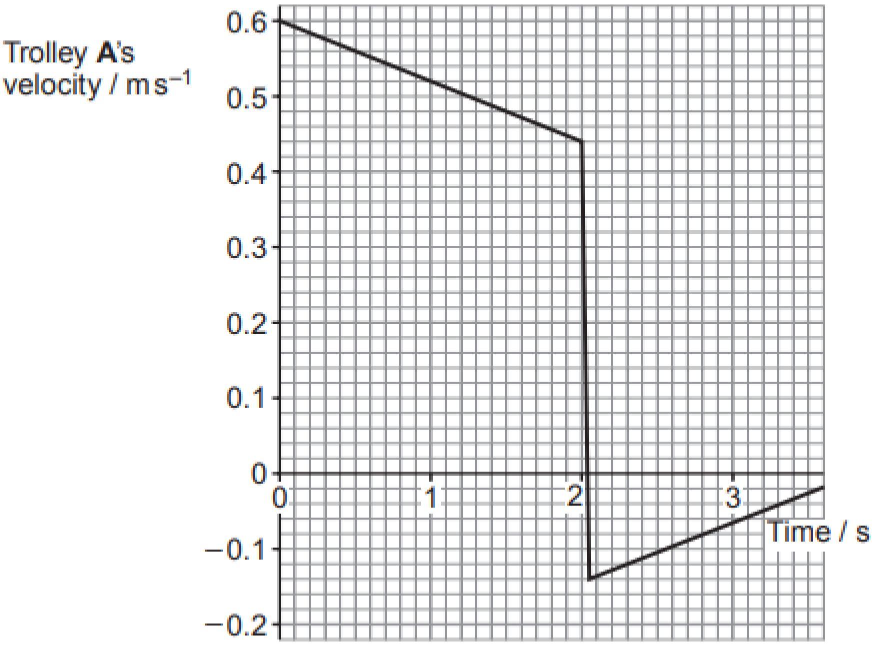
The loss in kinetic energy is $\mathbf{5mu^2/6 - 5mu^2/144 = 115mu^2/144}$ => this is converted into other forms.

-

2

A trolley, **A**, is initially moving on a flat surface towards a stationary trolley, **B**, as in the diagram.

A datalogger is used to produce a velocity-time graph for **A**, starting before the collision and continuing after the collision.



(i) Calculate the resistive force on trolley **A** before the collision. [3]

Trolley A is decelerating before the collision, so there must be a resistive force acting on the trolley.

The change in velocity of the trolley between t=0 and t=2 is $0.16ms^{-1}$. Thus, **a = 0.16 / 2 = $0.08ms^{-2}$**.

Using N2L, **F = 0.08(1.20) = 0.096N**.

Calculate the work done by this resistive force between time t = 0 and time t = 2.0 s. [3]

The velocity before the collision is **$0.44ms^{-1}$**.

Thus, u = 0.60, v = 0.44, t = 2.

S = 1/2 (0.60 + 0.44) * 2 = 1.04m.

If a force of **0.096N** acts over a distance of **1.04m**, the work done by the force is **0.096(1.04) = 0.0998J = 0.010J (2sf)**

Determine the velocity of trolley **B** immediately after the collision. *[Ignore the effects of resistive forces during the collision.]* [4]

The velocity of A before collision is **0.44ms^{-1}**, and the velocity of A after collision is **0.14ms^{-1}** in the opposite direction.

Applying conservation of momentum: **0.44(1.2) = -0.14(1.2) + v(3)**

3v = 0.696 => **v = 0.232ms^{-1} (2sf)**

-

Jasmine suggests that this is an elastic collision. Determine whether or not she is right, showing your working clearly. [3]

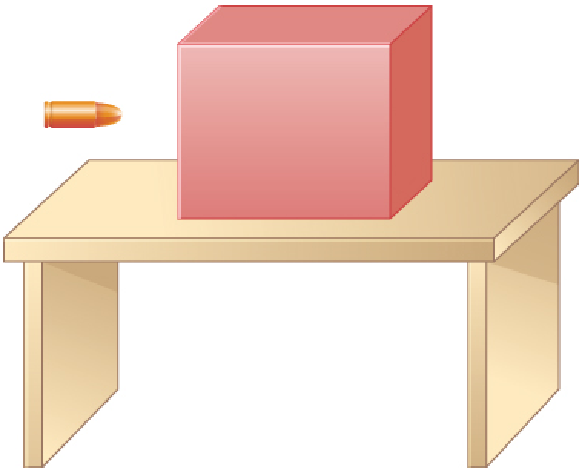
The total kinetic energy before collision is **1/2 * 1.2 * 0.44^2 = 0.12J**.

The total kinetic energy after collision is **1/2(0.14^2)(1.2) + 1/2(0.232^2)(3) = 0.092J**.

The kinetic energy before the collision does not equal the kinetic energy after the collision, so the collision is inelastic.

-

37. The figure below shows a bullet of mass 200 g traveling horizontally towards the east with speed 400 m/s, which strikes a block of mass 1.5 kg that is initially at rest on a frictionless table.



After striking the block, the bullet is embedded in the block and the block and the bullet move together as one unit.

a. What is the magnitude and direction of the velocity of the block/bullet combination immediately after the impact?
b. What is the magnitude and direction of the impulse by the block on the bullet?
c. What is the magnitude and direction of the impulse from the bullet on the block?
d. If it took 3 ms for the bullet to change the speed from 400 m/s to the final speed after impact, what is the average force between the block and the bullet during this time?

A.

Applying conservation of momentum,

0.2(400) = v(1.5+0.2)

80 = 1.7v

v = 47.1ms^{-1}.

The direction must be to the east since the initial vertical momentum was 0kgms^{-1}, so the final vertical momentum must also be 0kgms^{-1}, thus there is no vertical velocity, only horizontal velocity.

B.

The change in momentum of the bullet is **0.2(400) - 0.2(47.1) = 70.6kgms^{-1}**. The direction of this impulse is to the west.

C.

The impulse exerted on the bullet by the block is equal and opposite to the impulse exerted on the block by the bullet.

Thus, the impulse exerted on the block is **70.6kgms^{-1}** to the *east*.

D.

F = Δp / t = 70.6 / (3*10^{-3}) = 23500N to the east.

-

4

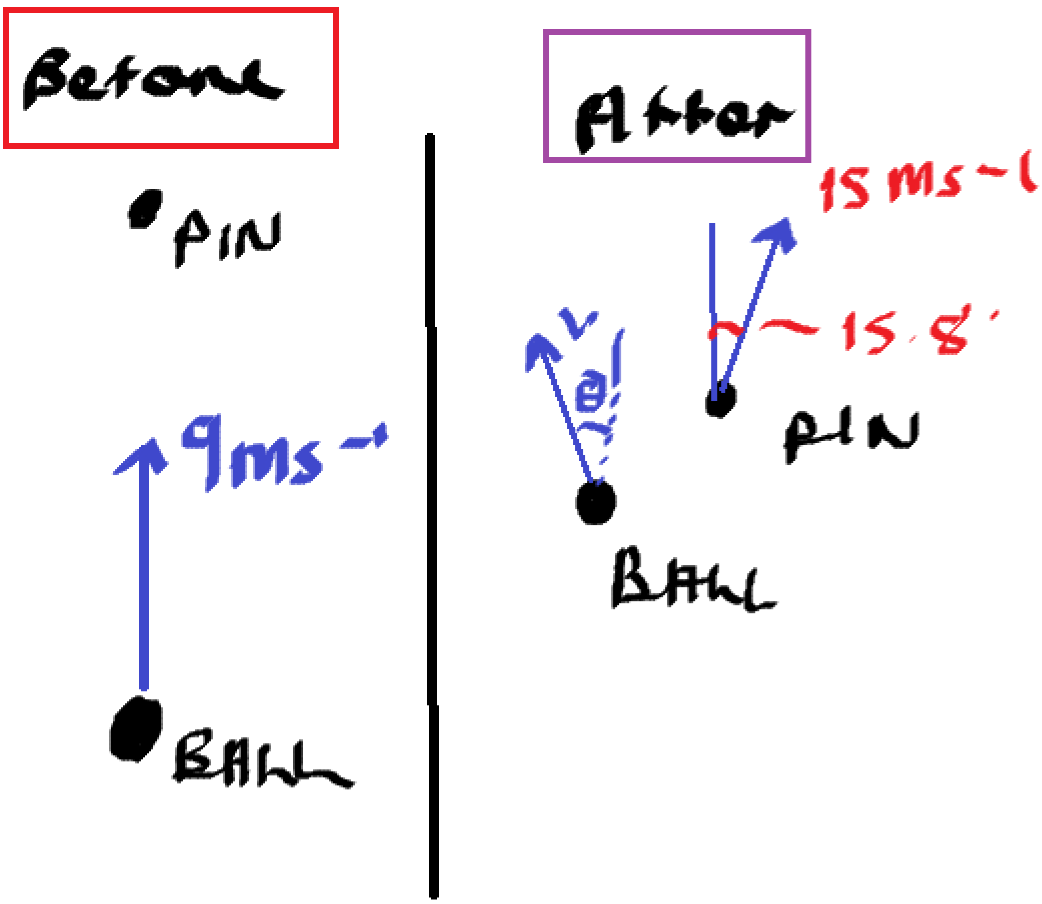
43. A 5.50-kg bowling ball moving at 9.00 m/s collides with a 0.850-kg bowling pin, which is scattered at an angle of 15.8° to the initial direction of the bowling ball and with a speed of 15.0 m/s.

a. Calculate the final velocity (magnitude and direction) of the bowling ball.

b. Is the collision elastic?

a.

We will say that the bowling ball is scattered at an angle of Θ to its original direction with velocity vms^{-1}.



Momentum is conserved independently in the horizontal and vertical planes.

Vertical Plane:

The initial vertical momentum is **9(5.50)** = **49.5**

The final vertical momentum is **vcosΘ(5.50) + 15cos15.8(0.850) = 5.50vcosΘ + 12.27**.

Thus, **49.5 = 5.50vcosΘ + 12.27**.

Making v the subject gives **v = 37.23 / 5.50cosΘ.**

Horizontal Plane:

The initial horizontal momentum is **0kgms^{-1}**, since the ball is moving vertically in the diagram we have drawn.

If we take the final horizontal velocity component of the bowling ball as positive, the final horizontal momentum is **vsinΘ(5.50) - 15sin15.8(0.850).**

Thus, **0 = 5.50vsinΘ - 3.47.**

Making **v** the subject: **v = 0.627/sinΘ**.

Equating our expressions for **v**:

0.627 / sinΘ = 37.23 / 5.50cosΘ

3.45cosΘ = 37.23sinΘ

tanΘ = 3.45 / 37.23 => **Θ = 5.3°**.

Thus, **v = 0.627 / sin5.3 = 6.8ms^{-1}**.

The bowling ball is therefore scattered at an angle of **5.3°** to the left of its original direction with a velocity of **6.8ms^{-1}**.

b.

The initial kinetic energy of the bowling ball is **1/2 * 9^2 * 5.50 = 223J**.

The final kinetic energy of the bowling ball and pin is **(1/2 * 6.8^2 * 5.50) + (1/2 * 0.850 * 15^2) = 223J**.

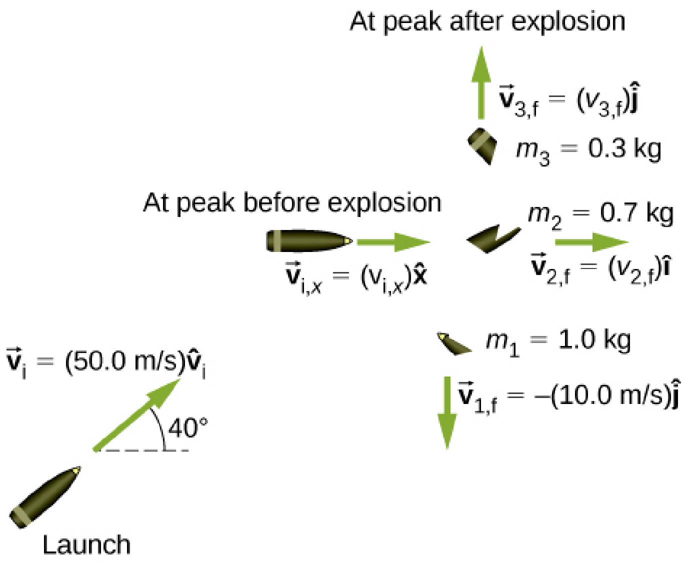
Note: when calculating kinetic energy, we do not have to consider the components of the velocity since energy is not a vector. We just consider the magnitude of the velocities.

The initial kinetic energy equals the final kinetic energy, so the collision is elastic.

--

Example

55. A projectile of mass 2.0 kg is fired in the air at an angle of 40.0° to the horizon at a speed of 50.0 m/s. At the highest point in its flight, the projectile breaks into three parts of mass 1.0 kg, 0.7 kg, and 0.3 kg. The 1.0-kg part falls straight down after breakup with an initial speed of 10.0 m/s, the 0.7-kg part moves in the original forward direction, and the 0.3-kg part goes straight up.



a. Find the speeds of the 0.3-kg and 0.7-kg pieces immediately after the break-up.
b. How high from the break-up point does the 0.3-kg piece go before coming to rest?
c. Where does the 0.7-kg piece land relative to where it was fired from?

A.

To find the speed of the projectiles, we consider that momentum is conserved independently in the horizontal and vertical planes.

Finding velocity of 0.7kg piece:

The horizontal velocity of the bullet when it was initially fired is **50cos40**. This remains constant throughout the motion since there is no acceleration in the horizontal plane.

Thus, the horizontal momentum of the bullet before explosion is **2(50cos40) = 76.6kgms^{-1}**.

The only fragment with horizontal momentum after the explosion is the **0.7kg** piece. The momentum of this piece is **$0.7v_2$.**

Thus, **$76.6 = 0.7v_2$** => **v_2 = 109ms^{-1}**.

Finding velocity of 0.3kg piece:

The momentum of the bullet before explosion is **0kgms^{-1}** because it is at its apex and thus has no vertical velocity.

The 0.7kg piece has no momentum in the vertical plane after explosion since it is fired off in the horizontal plane, so the only pieces with vertical momentum are the 0.3kg piece and the 1kg piece.

Thus, $\mathbf{1(-10) + 0.3v_3 = 0}$ => $\mathbf{0.3v_3 = 10}$ => $\mathbf{v_3 = 33.3ms^{-1}}$.

B.

The 0.3kg piece is projected at $\mathbf{33.3ms^{-1}}$ in the upwards direction. The only force acting on the piece after the explosion is the gravitational force, causing an acceleration of $9.81ms^{-2}$ downwards.

The piece reaches its maximum height when its velocity has been reduced to $\mathbf{0ms^{-1}}$.

Thus, $\mathbf{u = 33.3, v = 0, a = -g}$, $\mathbf{0^2 = 33.3^2 + 2(-g)s}$ => $\mathbf{s = 56.5m}$.

-

C.

The 0.7kg piece is ejected at $\mathbf{109ms^{-1}}$ in the horizontal direction.

If we know the time taken for it to reach the ground, we can determine the horizontal distance travelled by the piece.

First, let us find the height at which the initial explosion occurs.

The 2kg bullet is fired with an initial vertical velocity of **50sin40**.

Thus, **u = 50sin40, a = -g, v = 0**.

$\mathbf{0^2 = (50\sin40)^2 + 2(-g)s}$ => $\mathbf{s = 52.6m}$.

Thus, the 0.7kg piece will travel a vertical distance of **52.6m** to the ground.

We can use this, and the fact that its initial vertical velocity is $\mathbf{0ms^{-1}}$ to find the time taken to reach the ground.

u = 0, s = 52.6, a = g

$\mathbf{52.6 = 1/2(g)t^2}$ => $\mathbf{t = 3.27s}$.

So, the piece travels for **3.27s** at a velocity of $\mathbf{109ms^{-1}}$ in the horizontal plane. The horizontal acceleration is $0ms^{-2}$, so the distance travelled is **109(3.27) = 356m**.

To find the total horizontal distance of the piece from where the bullet was initially fired, we add the horizontal displacement of the bullet before it explodes to the horizontal displacement of the 0.7kg piece after explosion.

The time taken for the 2kg bullet to reach its apex is equal to the time taken for the 0.7kg piece to reach the ground since the vertical displacement is exactly the same.

Thus, since the initial horizontal velocity of the 2kg bullet is **50cos40**, the total distance it travels is **50cos40(3.27) = 125m**.

The total horizontal displacement of the 0.7kg piece from the initial firing position is therefore **356 + 125 = 451m**.

D) Determine the kinetic energy inputted into the system by the explosion.

Before the explosion, the bullet has a kinetic energy of **1/2 * 2 * $(50\cos40)^2$ = 1467J**.

The total kinetic energy of the pieces after the explosion (*remember: we ignore the directions of the velocities; energy is not a vector)* is:

(1/2 * 1 * 10^2) + (1/2 * 0.7 * 109^2) + (1/2 * 0.3 * 33.3^2) = 4375J.

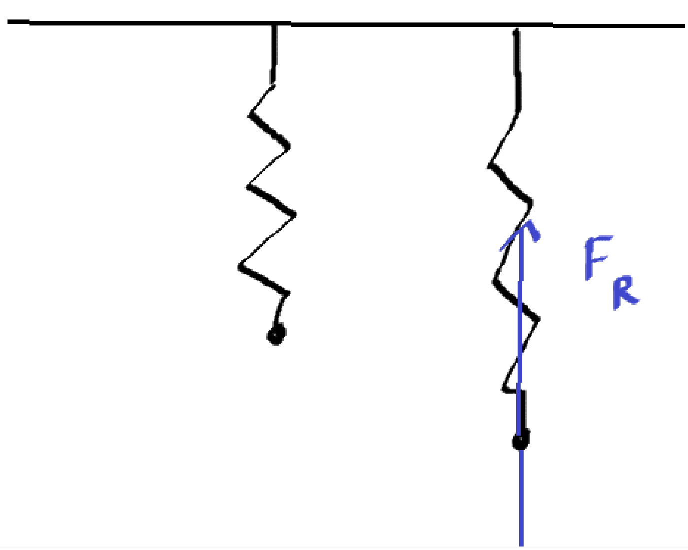
The total energy input is therefore **4375 - 1467 = 2908J**.

1.5 - Deformation of materials

Springs

Restoring force of a spring

If a force is applied to a spring, it will cause the spring to compress or extend depending on the direction of the force.



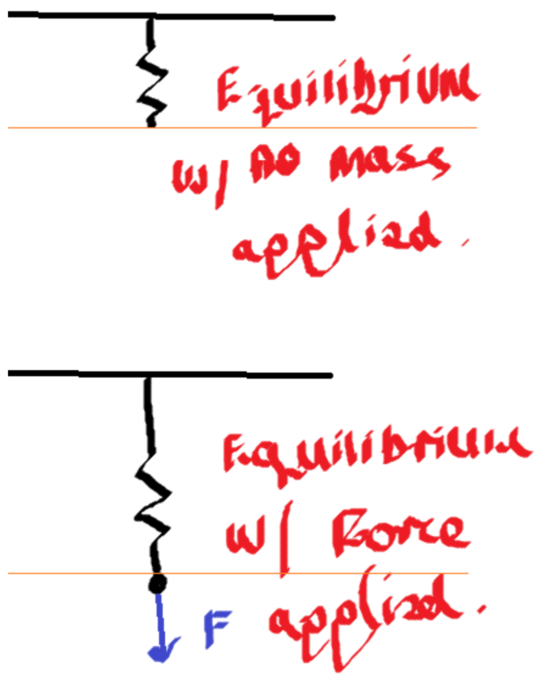
Suppose a force $\mathbf{F_A}$ is applied to a spring attached to a ceiling. If the force $\mathbf{F_A}$ is applied in the downwards direction, it will cause the spring to extend downwards.

When a string is stressed, a force called the **restoring force**, F_R, is exerted on the spring in the opposite direction to the force being applied (*F_A and F_R are not a N3L pair because they act on the same bodies)*.

The restoring force attempts to restore the spring to its original position. The *natural* position (position when no force is applied) of a spring is known as its **equilibrium position**.

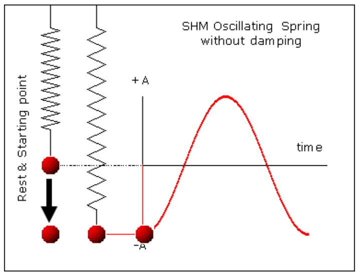
When the spring is accelerating in the direction of the applied force, the applied force is greater than the restoring force, $\mathbf{F_A > F_R}$.

As the spring is accelerated downwards, the restoring force increases. Eventually, the restoring force is equal to the applied force, $\mathbf{F_A = F_R}$. At this point, the spring is in a new equilibrium position:



So, when a force is applied to a spring, the spring obtains a new equilibrium position for that force (the equilibrium position will vary depending on the magnitude of the force exerted).

Once a spring has reached its equilibrium position for a particular force, it does not stop. It in fact *oscillates* about this equilibrium position in *simple harmonic motion*.

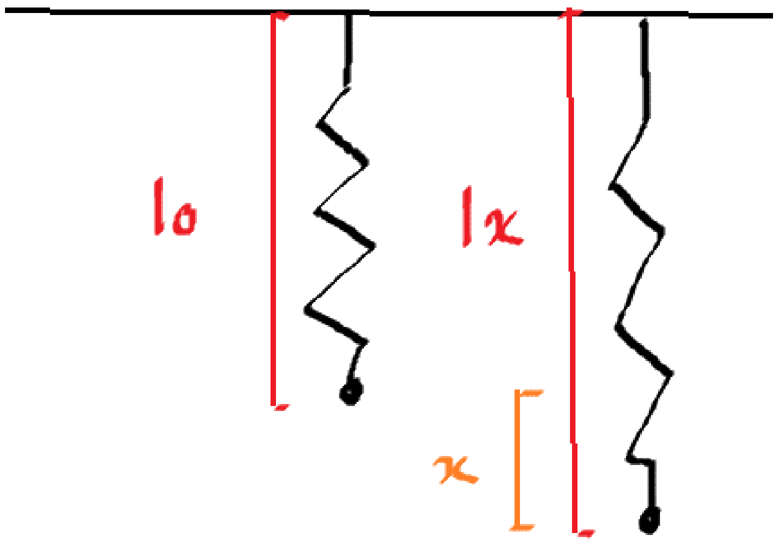


When the spring is moving below the equilibrium position, the restoring force is greater than the applied force, so the spring is accelerating back up to the equilibrium position. The spring has an initial downwards velocity component, so although its acceleration is upwards, it will continue to move downwards until its downwards velocity is reduced to $0ms^{-1}$. It is then accelerated in the upwards direction back to the equilibrium position.

When the spring goes above the equilibrium position, the applied force becomes greater than the restoring force (like when the spring was initially moving down), so the spring accelerates downwards. The spring has a velocity component in the upwards direction, so it is decelerated to a velocity of $0ms^{-1}$, then accelerates back down towards the equilibrium position in the opposite direction.

Hooke's Law

A spring has a natural length. This is the length of the spring when no force is applied to it. We will let this be $\mathbf{l_o}$.



When a force is applied to the spring, it will cause the spring to extend by a certain amount. We will call the extension of the spring **x**, and the length of the spring after extension be $\mathbf{l_x}$.

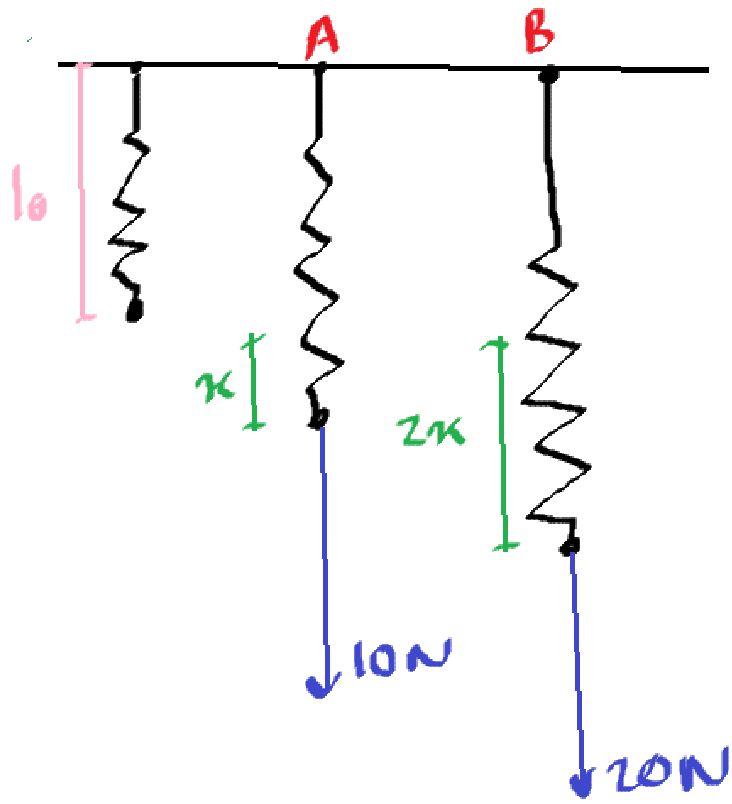
We can say that $\mathbf{x = l_x - l_o}$.

For example, if a spring has a natural length of 10m, and it extends to 15m when a force is applied, **x = 5m**. It is important to note that this extension is the extension *at equilibrium* - the spring can in-fact extend beyond this as explained above.

-

Hooke discovered that the *force applied to a spring is directly proportional to the extension of the spring at equilibrium* (when the spring is behaving elastically). This is known as **Hooke's Law**.

For example, if a 10N force is applied to spring A, and a 20N force is applied to spring B, if spring A and B are identical and behaving elastically, spring B will extend twice as much as spring A.



When a force is applied to the spring, and the spring is at equilibrium position, we know that **restoring force = force applied** (due to the nature of the equilibrium). At this point, **F**, the force applied, and **x**, the extension, are directly proportional:

$$\mathbf{F \propto x}$$

The force applied is directly proportional to the extension.

To make this into an equation, we must insert a constant. We call this constant **k**, the spring constant.

Thus, **$F = kx$**. *The base units of **k** are $N / m = Nm^{-1} = kgms^{-2}m^{-1} = kgs^{-2}$.*

k remains constant for a spring when the spring is behaving elastically.

Example:

A spring of natural length 10cm extends to a length of 15cm when a 12N force is applied.

Determine the spring constant for the spring.

The extension of the spring is **5cm**.

Thus, **$k = F / x = 12 / 0.05 = 240Nm^{-1}$**.

/

A 30N force is applied to the same spring.

Determine the extension of the spring.

The spring constant for this spring is always **240Nm^{-1}**.

Thus, when a **30N** force is applied, **x = F / k = 30 / 240 = 0.125m**.

--

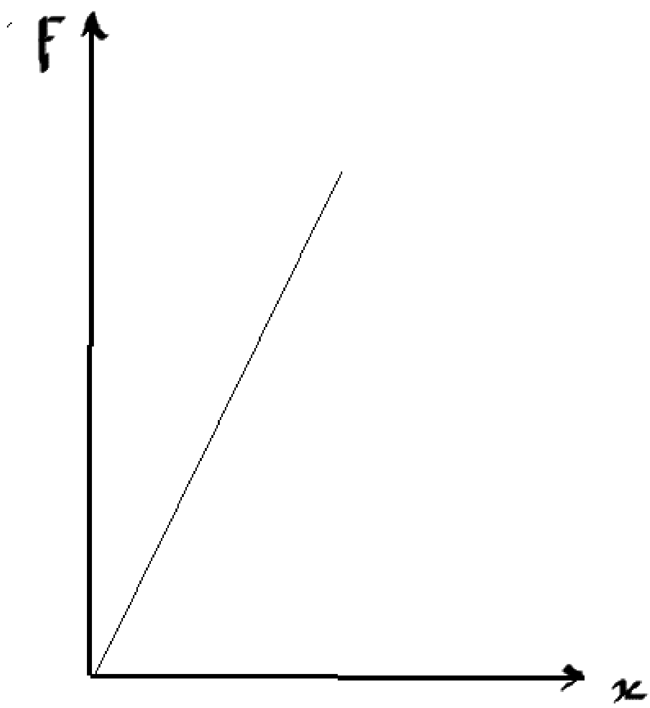
The higher the value of **k** the more 'stiff' a spring is. **k** is often known as the *stiffness constant* as a result.

For example, take a spring (A) with a spring constant of **50Nm^{-1}** and a spring (B) with a spring constant of **100Nm^{-1}**.

The spring constant of A tells us that it takes **50N** of force to extend the spring by **1m**, and the spring constant of **B** tells us it takes **100N** of force to extend B by **1m** - so B is clearly stiffer.

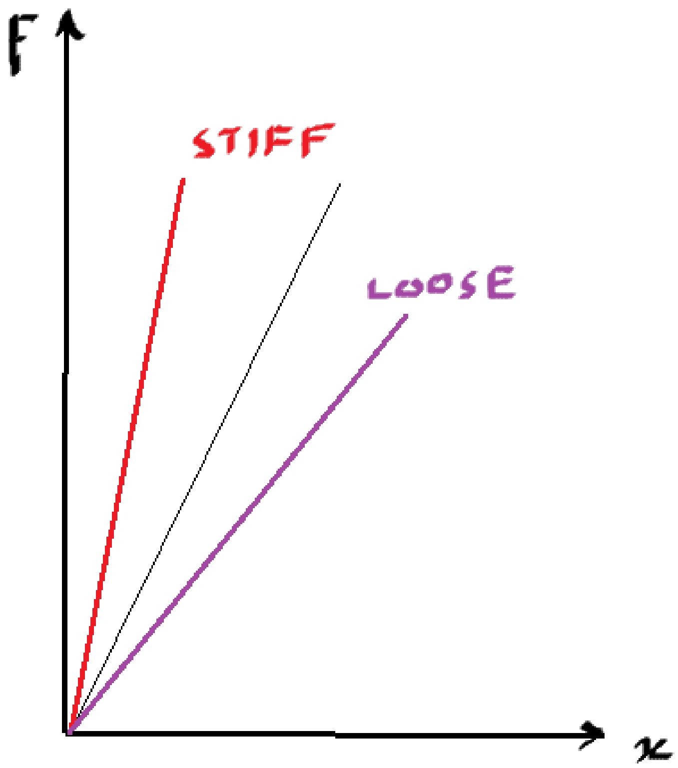
--

Hooke's Law can be proved experimentally by applying different forces to a spring and measuring the extension of the spring. Plotting **F** against **x** will show a linear relationship:



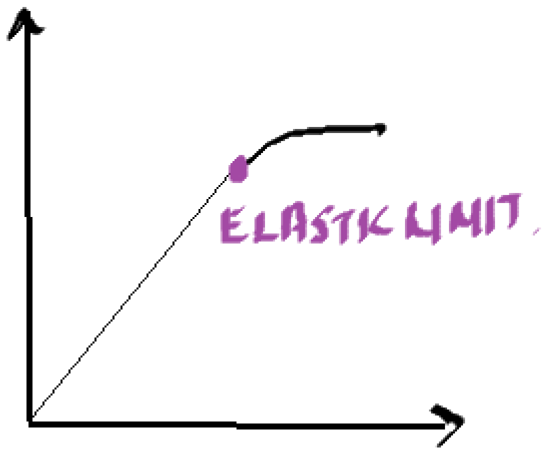
The gradient of this graph is **F / x**, which is equal to **k**.

The steeper the gradient of the graph, the higher the spring constant, and thus the higher the stiffness:



However, it is important to recognise that Hooke's Law only applies when a string is behaving elastically.

All springs (and all materials) have an **elastic limit**, a point beyond which force and extension are no longer directly proportional. This limit is reached when the force applied to the spring exceeds the elastic limit.



Beyond the elastic limit, we can no longer assume that **k = F / x**. The spring now behaves **plastically** and cannot return to its original shape.

-

Hooke's Law also applies to the *compression* of springs:

Example:

A 15N force is applied to a spring, causing it to compress by 5cm. If a 100g mass is applied to the same spring when the spring is hung off a ceiling, **what will the extension of the spring be?**

The spring constant of the spring is **F / x = 15 / 0.05 = 300Nm^{-1}**.

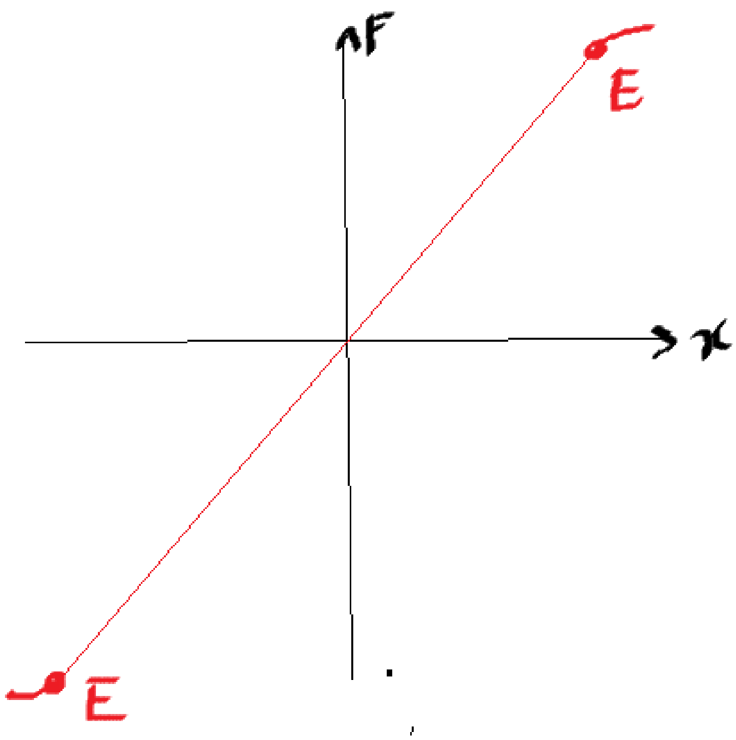
This spring constant applies to either compression or extension.

When a **100g** mass is applied to the spring, the force exerted by this mass is **0.1(9.81) = 0.981N**.

Thus, **x = F / k = 0.981 / 300 = 3.27*10^{-3}m = 3.27mm**.

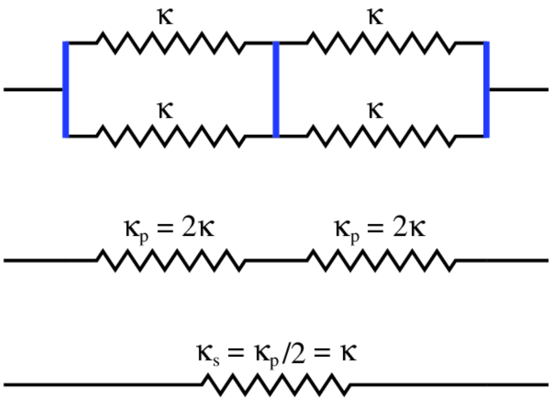
-

We can plot a graph of **F** against **x** where a positive value of **F** represents extension and a negative value of **F** represents compression:



The gradient of the graph remains constant because **k** is the same for compression or extension.

-



Multiple spring system

When multiple springs are placed in *parallel* or *in series*, we can use equations to model all of the springs as one single, or *effective*, spring with one effective spring constant.

Series:

When two springs are in series with each other, it becomes easier for the overall, or **effective**, spring to be extended.

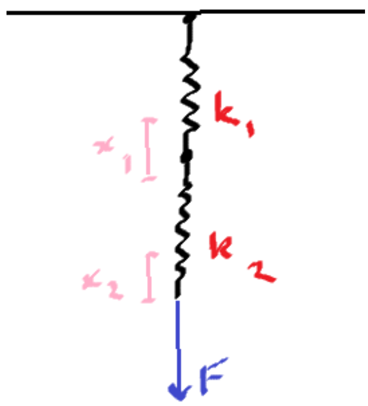
We can model the two springs as a single spring with a spring constant of $\mathbf{(1/k_1 + 1/k_2)^{-1}}$.

For example, two identical springs with spring constant of $\mathbf{25Nm^{-1}}$ that are attached in series have an effective spring constant of $\mathbf{(1/25 + 1/25)^{-1} = 12.5Nm^{-1}}$. In other words, the stiffness of the effective spring is **halved**, so the effective spring extends by double the amount of the extension of each individual spring when the same force is applied.

Proof:

Take two springs of spring constant $\mathbf{k_1}$ and $\mathbf{k_2}$ that are attached in series. Suppose a force **F** is applied to the spring system, causing one spring to extend by $\mathbf{x_1}$ and the other by $\mathbf{x_2}$.

The total extension of the effective spring, $\mathbf{x_T}$, is $\mathbf{x_1 + x_2}$.



$\mathbf{x_T = x_1 + x_2}$

E.g. if one spring extends by 15cm, and the other extends by 25cm, the total extension is 40cm.

Applying Hooke's Law,

$\mathbf{x_1 = F/k_1}$ and $\mathbf{x_2 = F/k_2}$.

Similarly, $\mathbf{x_T = F/k_T}$ where $\mathbf{k_T}$ is the spring constant of the effective spring.

Thus, $\mathbf{F/k_T = F/k_1 + F/k_2}$.

Since the force applied to both springs is the same, we can cancel **F** to give:

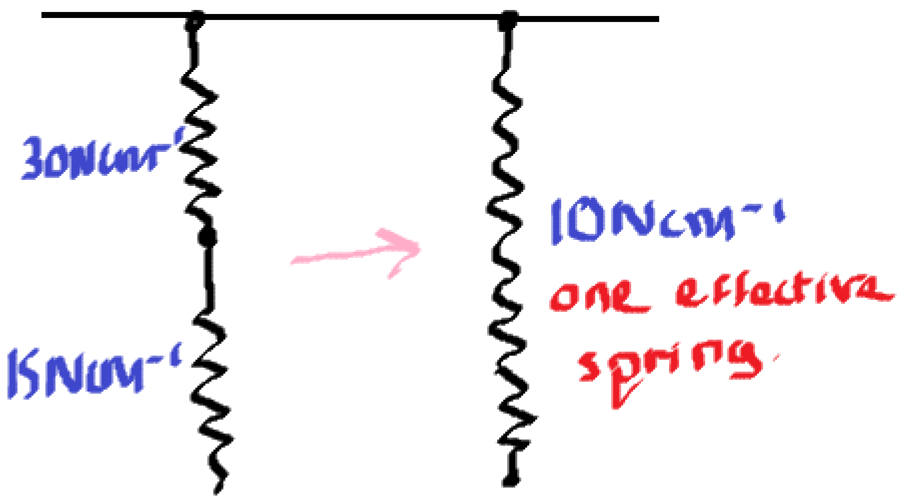
$\mathbf{1/k_T = 1/k_1 + 1/k_2}$, so $\mathbf{k_T = (1/k_1 + 1/k_2)^{-1}}$.

-

Example

Two springs with spring constants of $30Ncm^{-1}$ and $15Ncm^{-1}$ are attached in series. When a 20N force is applied to the spring system, **determine the total extension of the springs**.

The spring constant of the effective spring is $\mathbf{(1/30 + 1/15)^{-1} = 10Ncm^{-1}}$.



When a force of **20N** is applied to a spring with a spring constant of $\mathbf{10Ncm^{-1}}$, the extension of the spring is **20 / 10 = 2cm**.

Example

Two springs are attached in series. One spring has a spring constant of $\mathbf{15Ncm^{-1}}$ and the other spring has a spring constant of $\mathbf{y\ Ncm^{-1}}$. When a force of **20N** is applied to the springs, the effective spring extends by **40cm**.

Determine the value of y.

If a force of **20N** is applied to a spring, and it extends by **40cm**, the spring constant of the spring is **20 / 40 = 0.5Ncm^{-1}**.

This is the spring constant of the *effective spring*.

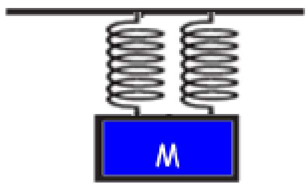
Using our equation **$1/k_1 + 1/k_2 = 1/k_T$**, we can therefore say **1/15 + 1/y = 1 / 0.5**.

1/y = 29/15 => **y = 15/29 Ncm^{-1}**.

-

Parallel

Springs can be placed in parallel with a single board connecting them as shown below:



In this arrangement, the extension of the effective spring is lower than the extension of the individual springs when the same force is applied to those springs. In other words, the effective spring is stiffer than the individual springs.

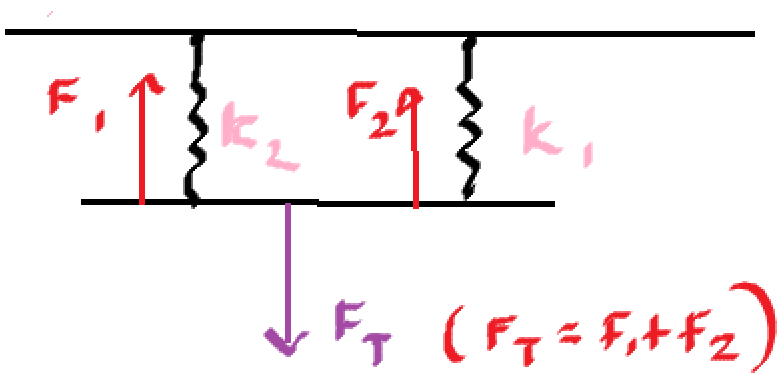
We can say that the effective spring constant of the springs, **k_T**, is **$k_T = k_1 + k_2$**. For example, two springs connected in parallel with spring constant of 10Ncm^{-1} and 15Ncm^{-1} have an overall spring constant of **25Ncm^{-1}**; so the overall spring is stiffer.

Proof:

Suppose two springs with spring constants **k_1** and **k_2** are in parallel with each other.

We can say that the total force exerted on the springs, **F_T**, is equal to the sum of the force exerted on each spring, **$F_T = F_1 + F_2$**.

For example, if there was a restoring force of **4N** in one spring and **5N** in the other spring, the total force exerted in the downwards direction to retain equilibrium must be **9N**.



Using Hooke's Law,

$F_T = k_T x$
$F_1 = k_1 x$

$\mathbf{F_2 = k_2x}$.

Thus, $\mathbf{k_Tx = k_1x + k_2x}$.

Since each spring extends equally when in parallel, we can cancel **x** to say $\mathbf{k_T = k_1 + k_2}$.

Example

Two springs with spring constants **10Nm^{-1}** and **15Nm^{-1}** are in a parallel arrangement. **Determine the extension of the springs when a 10N force is applied**.

The two springs can be modelled as a single spring with a spring constant of **10 + 15 = 25Nm^{-1}**.

Thus, **x = F / k = 10 / 25 = 0.4m**.

Example

Three springs connected in parallel with spring constants 10Ncm^{-1}, 5Ncm^{-1} and **y**cm^{-1} extend by 4cm when a 100N force is applied. **Find y**.

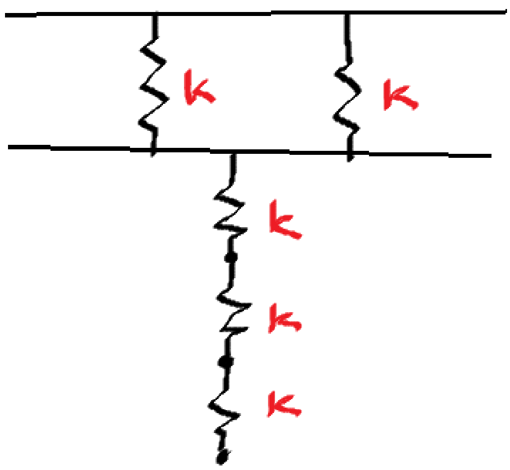
The effective spring constant of the springs is **100 / 4 = 25Ncm^{-1}**.

Thus, **10 + 5 + y = 25** => **y = 10Ncm^{-1}**.

-

Now let us consider situations where there are springs connected in series *and* parallel:

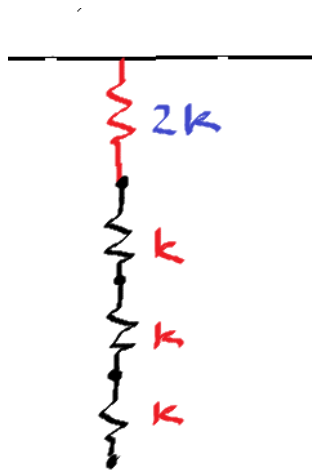
Example



The diagram above shows 5 identical springs with spring constants of **k**. Two of the springs are in parallel, and the remaining three are in series with each other.

What is the effective spring constant of the springs?

First, we can model the two springs in parallel as a single spring with a spring constant of **2k**:

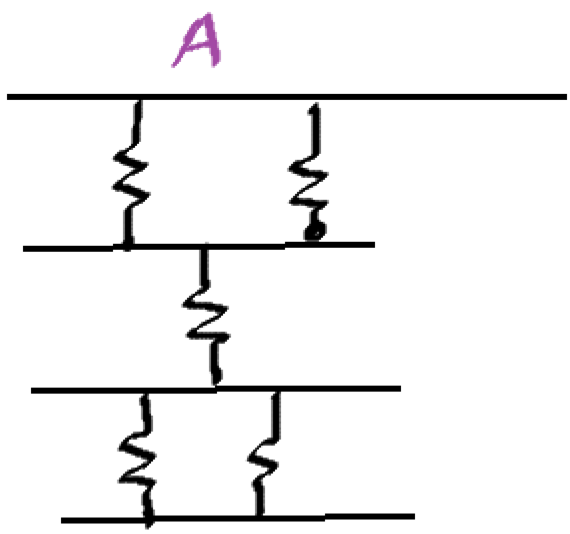


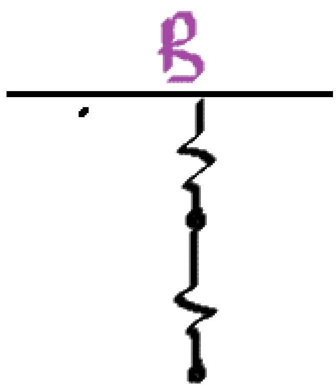
Now, there are effectively 4 springs in series. These have an effective spring constant of:

k_T **= (1/2k + 3(1/k))$^{-1}$ = (1/2k + 3/k)$^{-1}$ = ((k + 6k) / 2k^2)$^{-1}$ = (7k / 2k^2)$^{-1}$ = (7/2k)$^{-1}$ = 2k/7**.

-

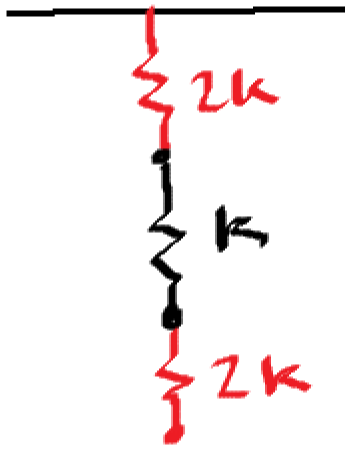
Example





Two combinations of springs are shown above, **A** and **B**. Each spring in the above arrangements has a spring constant of **k**. **Determine the ratio of the effective spring constant of arrangement A, k_A, and the effective spring constant of arrangement B, k_B.**

In arrangement A, the two parallel arrangements can be modelled as single springs with spring constants of **2k**:



Thus, $\mathbf{k_A = (2(1/2k) + 1/k)^{-1} = (1/k + 1/k)^{-1} = (2/k)^{-1} = k/2}$.

The spring constant in arrangement B is $\mathbf{(1/k + 1/k)^{-1} = k/2}$.

Thus, the ratio of $\mathbf{k_A}$ to $\mathbf{k_B}$ is therefore a 1:1 ratio. This means that when the same force is applied to both spring arrangements, the extension of the springs will be equal.

-

Example

Determine the ratio of the extension of two identical springs connected in series and the extension of the same springs when connected in parallel when the same force is applied.

We will let the springs have a spring constant of **k**.

The effective spring constant of the springs in series is $\mathbf{(1/k + 1/k)^{-1} = k/2}$.

The effective spring constant of the springs in parallel is **2k**.

Thus, the springs in series will extend 4 times as much as when they are in parallel.

Mathematical proof:

Series: $x = F / k = F / (k/2) = 2F/k$

Parallel: $x = F / 2k$

Ratio of x in series and parallel = $(2F/k) / (F/2k) = (2/k) / (1/2k) = 4k / k = 4$ => the springs in series extend 4 times as much.

Elastic potential energy

When stress is applied to a material, it causes the material to deform. All materials initially deform *elastically* when stress is applied. This means that when the stress is removed, the material returns to its original shape. However, materials have an *elastic limit*, a point beyond which they can no longer continue to deform elastically. Beyond this elastic limit, they deform *plastically* and cannot return to their original shape.

A material that is deforming elastically has a type of energy stored within it known as **elastic potential energy**. If the material is being stretched, the bonds between the atoms in the material stretch, and energy is stored within these stretched bonds. When the stress is relieved, the bonds return to their original shape, releasing the energy that was stored within them.

Similarly, when a material is compressed, the bonds in the material are compressed, storing energy. When the stress is relieved, the bonds return to their original length, releasing the stored energy.

The stored energy within elastically deforming materials is *elastic potential energy*. The term *potential energy* in elastic potential energy refers to how the deformed material has a *potential* to return to its original shape. When it does so, the energy in the material is released and converted into other forms (kinetic energy, heat energy, sound energy…). This is comparable to *gravitational potential energy*, where a body with a height **h** in a gravitational field has the *potential* to fall to a lower height and lose this potential energy, converting it into other forms.

-

The equation for the elastic potential energy stored in a material, E_e, is **1/2Fx**, where **F** is the force applied to the extending material, and **x** is the extension of the material.

$$E_e = 1/2Fx.$$

Let us understand this in the context of a spring.

We know that **work = force * distance** => energy is the capacity to do work.

When a force, **F** is applied to a spring, it causes it to extend a distance, **x**, so work is being done to the spring. This work done causes the spring to gain elastic potential energy.

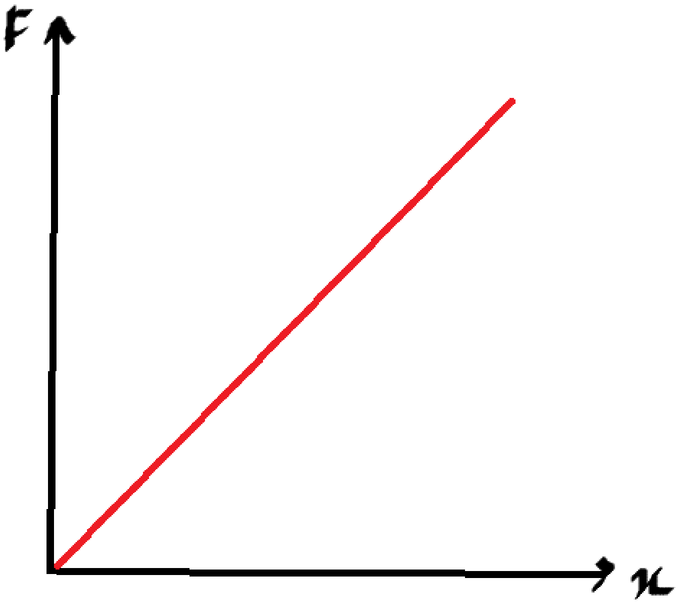
However, the restoring force in the spring, which is the force that induces the elastic potential energy, does not remain constant during the extension process.

When the force is initially applied to the spring, the restoring force is low; but as the spring deforms more, the restoring force increases, leading to an increase in the work done.

The restoring force is therefore not a constant, so to find the work done, we multiply the *average force*, **1/2F**, by the distance over which the force is applied, **x**.

Work done = elastic potential energy = 1/2Fx.

This can also be considered in terms of a force-extension graph:



The area under a force-distance graph (see this section) is equal to the work done.

The area under the graph to the left is **1/2Fx**, so the work done to the spring is equal to **1/2Fx**. This work done is stored as elastic potential energy.

According to Hooke's Law, **F = kx**. If we substitute this into $\mathbf{E_e = 1/2Fx}$, we get:

$\mathbf{E_e = 1/2(kx)x = 1/2kx^2}$.

This is another definition of elastic potential energy.

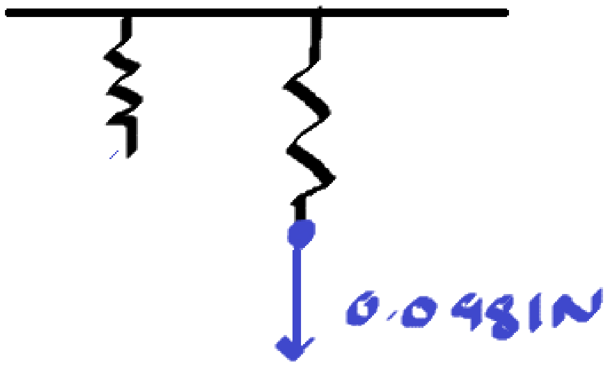
*[The **x** in **1/2Fx** also applies to compression.]*

-

Example:

A spring has a spring constant of **$10Nm^{-1}$**. The spring is attached to the ceiling and a **10g** mass is applied. **Determine the elastic potential energy stored in the spring when the mass has expanded the string such that it is at rest**.

The force exerted by a 10g mass is **0.010 * 9.81 = 0.981N**.



This leads to an extension of **0.981 / 10 = 0.0981m**.

Thus, **$E_e = 1/2kx^2 = 1/2(10)(0.0981^2) = 0.0481J$**.

So the potential energy stored in the spring is **0.0481J**. This has a *potential* to convert into other forms of energy.

-

Example:

A 10N force applied to a spring causes it to extend by 0.4m. When a 32N force is applied to the spring, **determine the elastic potential energy in the spring**.

The spring constant of the spring is **$10 / 0.4 = 25Nm^{-1}$**.

When a **32N** force is applied, **x = 32 / 25 = 1.28m**. Thus, **$E_e = 1/2 * 25 * 1.28^2 = 20.5J$**.

-

Example:

5J of work is used to stretch a spring by 20cm. **Determine the spring constant of the spring**.

If 5J of work is inputted into a spring, the elastic potential energy stored in the spring is 5J when its extension is 20cm.

Using **$E_e = 1/2kx^2$** => **$5 = 1/2k(0.2^2)$** => **$k = 250Nm^{-1}$**.

Determine the work that must be done to compress the spring by 32cm.

The potential energy in the spring with a compression of **0.32m** is **$1/2 * 250 * 0.32^2 = 12.8J$** => this is the work that must be done.

-

Example:

*If **10J** of work is required to compress a spring by **25cm**, determine the additional amount of work that must be put in to compress the spring by a further **5cm**.*

First, we can find the spring constant of the spring:

$10 = 1/2 * k * 0.25^2 \Rightarrow k = 320Nm^{-1}$.

The work that must be done to compress the spring by **30cm** is thus:

$1/2 * 320 * 0.30^2 = 14.4J$.

This means that an additional **14.4 - 10 = 4.4J** of work must be put in.

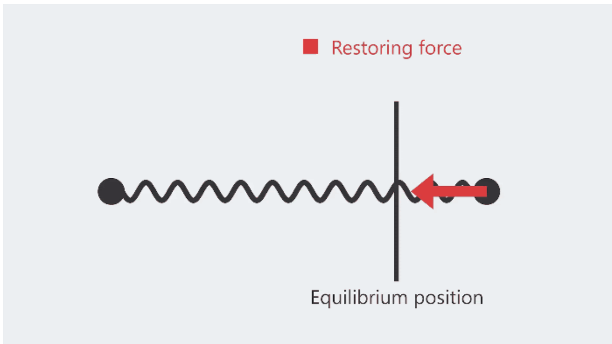
-

Maximum extension of a spring

(may help to read applying conservation of energy to springs first)

When we calculate the extension of a spring for a particular force using Hooke's Law, this *does not* tell us the **maximum** extension - it tells us the extension of the spring at its equilibrium position.

In **F = kx**, the **F** represents the restoring force in the spring. When the spring is at equilibrium for that particular force, the restoring force and the force applied must be in equilibrium, and therefore we can say that the force applied is equal to **kx**.

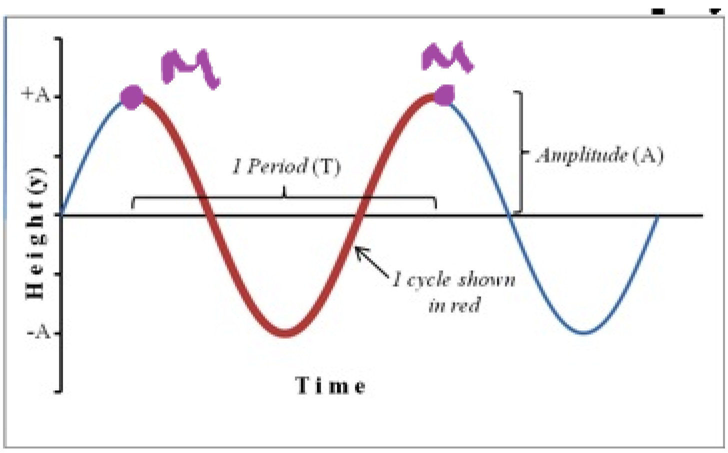


However, as shown to the right, a spring oscillates *about* the equilibrium position. The spring can in-fact extend further than the equilibrium position to a point of maximum extension, and it also can compress beyond the equilibrium position, and return to its natural length that it existed in before the force was applied.

When the spring is extending past the equilibrium position, the restoring force is greater than the force applied. When the spring is compressing past the equilibrium, the force applied is greater than the restoring force.

To determine the *maximum* extension of a spring, it is necessary to consider energy changes.

First, consider a graph of the extension of a spring against time relative to the equilibrium position:



A height of 0 represents the equilibrium position of the spring.

At the maximum displacement of the spring from the equilibrium position (marked as **M**), you can see that the velocity of the spring is equal to 0 (gradient of the graph is 0 as it is a turning point). This means that the spring has no kinetic energy. When the spring is at equilibrium position, its velocity is a maximum.

Velocity of a spring over a single oscillation cycle (starting from equilibrium position and extending beyond equilibrium):

1 - The spring passes the equilibrium position with its maximum velocity. When it begins to extend beyond equilibrium, the restoring force increases above the applied force, so the spring decelerates in the opposite direction.

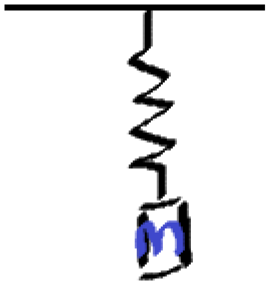
2 - The spring's velocity reduces to $0ms^{-1}$ due to this deceleration. This is the point of maximum extension. At this point, the restoring force is still greater than the applied force, so the spring continues to accelerate upwards back towards equilibrium.

3 - The spring reaches its maximum velocity at equilibrium. It then compresses beyond equilibrium. The applied force now becomes greater than the restoring force, so the spring decelerates to $0ms^{-1}$. At the point of 0 velocity, the spring is at its minimum extension. The force applied is still greater than the restoring force (the restoring force is not high as the spring is not very stressed), so the spring accelerates back to its maximum velocity at equilibrium.

-

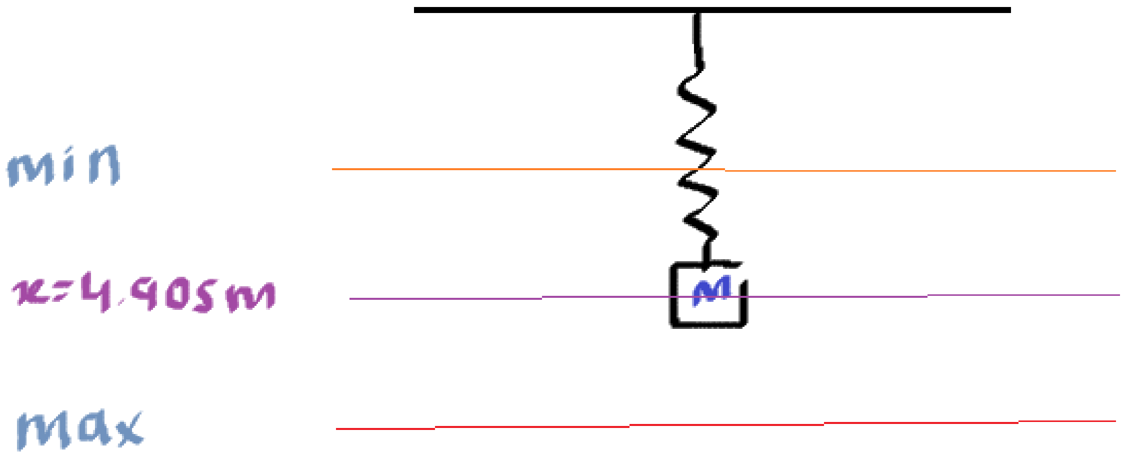
Let us apply this knowledge to a vertical spring system.

Suppose we have a spring with spring constant **10Nm^{-1}** attached vertically to a ceiling, with a mass of **5kg** attached to the spring.



According to Hooke's Law, the extension of the spring at equilibrium is **5g / 10 = 4.905m**.

So the equilibrium position of the spring for that mass is **4.905m**. However, the spring can also oscillate above and below this point, as shown previously:



When the spring passes the equilibrium position, we can prove that the mass will have a velocity component in the downwards direction.

The change in height of the mass from when it was initially released, to the equilibrium position, is 4.905m, so the change in *gravitational* potential energy is **4.905g(5) = 240.6J**.

As the spring falls, this gravitational potential energy is converted into kinetic energy (because the mass is accelerated by gravity) and elastic potential energy (because the extension of the spring is increasing).

When the spring is at an extension of **4.905m**, its elastic potential potential energy is **1/2 * 10 * 4.905^2 = 120.3J**.

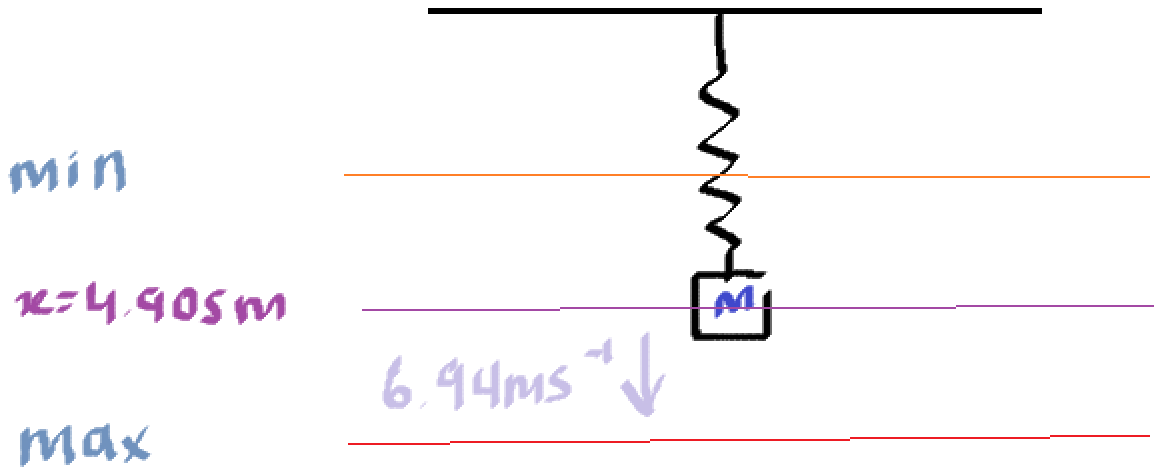
So **240.6J** of gravitational potential energy has converted into **120.3J** of elastic potential energy. The rest of the energy must have been converted into kinetic energy.

Thus, the kinetic energy of the mass at the equilibrium position is **240.6 - 120.3 = 120.3J**.

The mass thus has a velocity of: **120.3 = 1/2 * 5 * v^2** => **v = 6.94ms^{-1}** at equilibrium.

-

When the spring reaches its equilibrium position, this velocity of **6.94ms^{-1}** cannot instantly be depleted, causing the mass to come to rest. The mass still has a downwards velocity component and therefore still continues to move downwards.



When the spring extends beyond the equilibrium position, the restoring force increases above the force exerted by the mass, which causes the kinetic energy of the mass to be depleted. In other words, the applied force does work on the spring to reduce its velocity. The work done to deplete kinetic energy is transferred to elastic potential energy.

At the maximum extension, v = 0, and therefore the kinetic energy is reduced to **0J**.

At the maximum extension, we can therefore say that all of the gravitational potential energy that was initially possessed by the mass is converted into elastic potential energy.

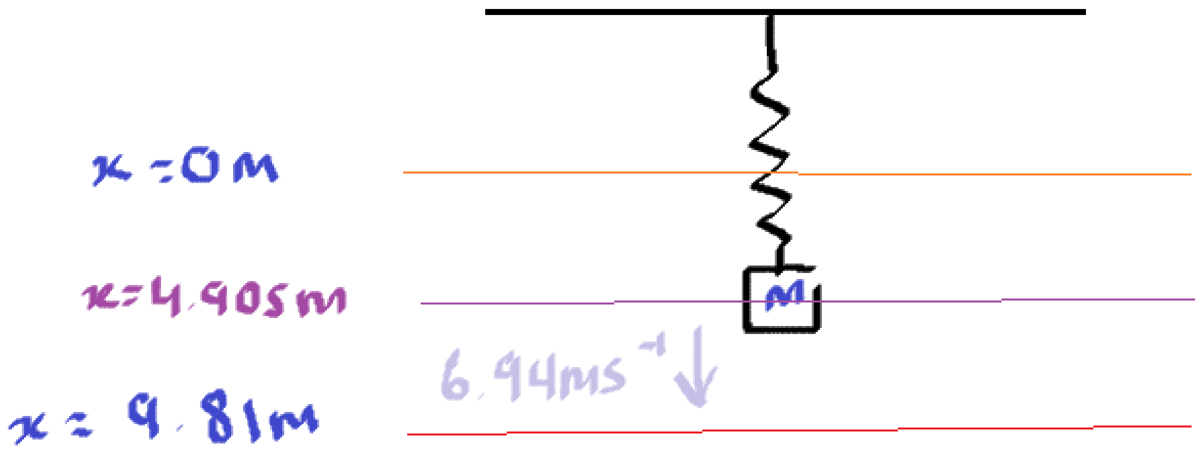
At equilibrium, the loss in gravitational potential energy is converted into kinetic energy and potential energy. When the mass falls below equilibrium, its potential energy continues to convert into elastic potential energy, and the kinetic energy of the mass is converted into elastic potential energy (since the kinetic energy decreases). This means that there is no overall change in the kinetic energy from the initial position to the final position, so $E_g = E_e$.

If we let the maximum extension be $\mathbf{x_{max}}$, we can say that the total change in gravitational potential energy of the block from its initial height to its maximum extension is $\mathbf{5gx_{max}}$.

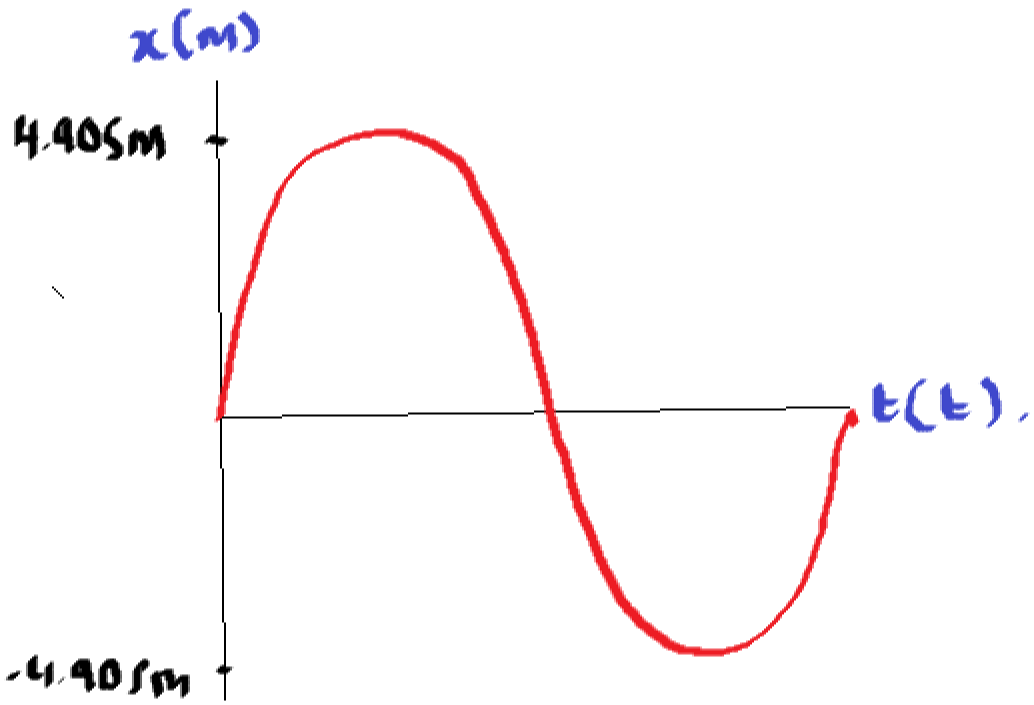
This is converted into elastic potential energy, $\mathbf{1/2(10)x_{max}^2}$ = $\mathbf{5x_{max}^2}$.

Thus, $\mathbf{5gx_{max} = 5x_{max}^2}$.

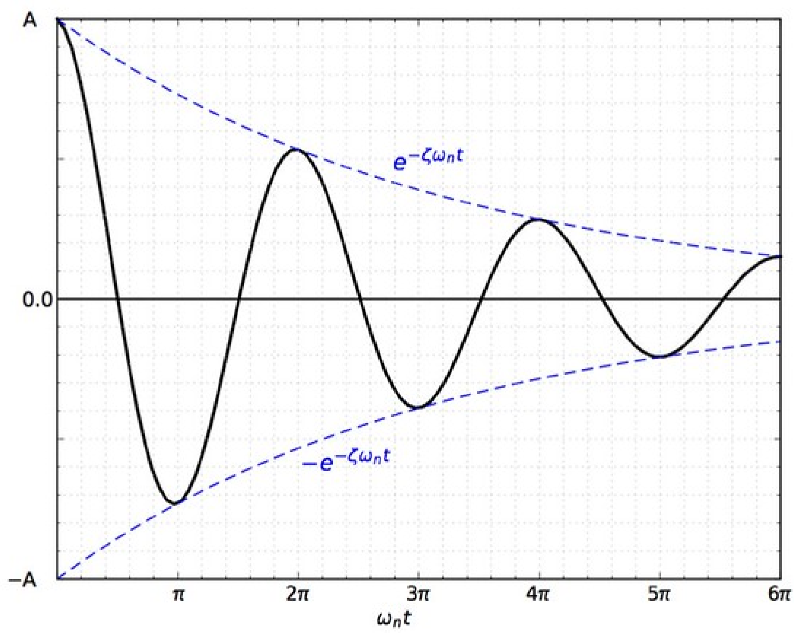
$\mathbf{5g = 5x_{max}}$ => $\mathbf{x_{max} = 9.81m}$. The spring will therefore extend to double its initial extension at equilibrium.



If we were to plot the oscillation of the spring against time, it would look as follows:



[In actuality, the maximum displacement of the spring from the equilibrium position would decrease each cycle because of the friction in the spring / air resistance. The spring would be ***dampened***]:



-

Now let us generalise the maximum extension (**x_{max}**) of a spring with spring constant **k** when a force of **F** is applied to the spring (e.g. F could be **mg**).

As the spring falls from rest to maximum extension, the total work done by the force is **Fx_{max}**. This is converted completely into elastic potential energy, **$1/2kx_{max}^2$**, at maximum extension since v=0. Thus, **$Fx_{max} = 1/2kx_{max}^2$**.

$F = 1/2kx_{max}$ => **$x_{max} = 2F / k$**.

So, when a force **F** is applied to a spring of spring constant **k**, its extension at equilibrium (**x_{eq}**) is **F / k**. However, its maximum extension, **x_{max}**, is **2F/k** => so the spring can actually extend to twice the extension at equilibrium.

We used the force **F** rather than a force of **mg** as the same principles also apply to a horizontally-positioned spring.

Suppose a spring with spring constant **$20Nm^{-1}$** is positioned horizontally and a force of **10N** is applied to the spring.

According to Hooke's Law, the extension of the spring at equilibrium is **10 / 20 = 0.5m**.

However, the spring can extend beyond this position. Suppose the maximum extension is $\mathbf{x_{max}}$. When the spring is extended to a position of $\mathbf{x_{max}}$, the total work done by the **10N** force is $\mathbf{10x_{max}}$. This is converted completely into elastic potential energy since v=0 at maximum extension.

Thus, $\mathbf{10x_{max} = 1/2(20)x_{max}^2}$ => $\mathbf{10 = 10x_{max}}$ => $\mathbf{x_{max} = 1m}$. The spring can therefore extend to twice its extension at equilibrium.

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Example question:

A block weighing $1.2\,\mathrm{N}$ is hung from a spring with a spring constant of $6.0\,\mathrm{N/m}$, as shown in Figure 8.4. (a) What is the maximum expansion of the spring, as seen at point C? (b) What is the total potential energy at point B, halfway between A and C? (c) What is the speed of the block at point B?

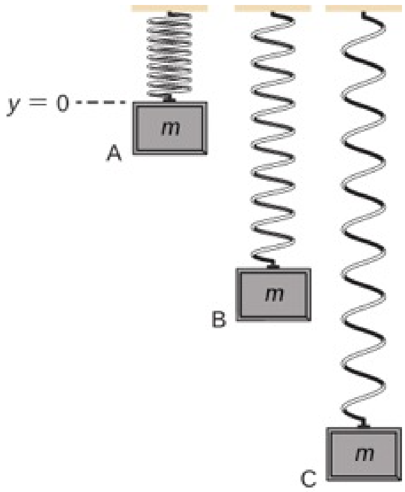


Figure 8.4 A vertical mass-spring system, with the positive *y*-axis pointing upward. The mass is initially at an unstretched spring length, point A. Then it is released, expanding past point B to point C, where it comes to a stop.

a)

The maximum extension of the spring is **2F/k = 2(1.2) / 6 = 0.4m**.

b)

[B is the equilibrium position, half-way between A and C]

The total potential energy is a sum of both the gravitational and elastic potential energy.

The gravitational potential energy relative to **A** is **-0.2(1.2) = -0.24J**.

The elastic potential energy is $\mathbf{1/2(6)(0.2^2) = 0.12J}$.

Thus, the total potential energy is **-0.12J**.

c)

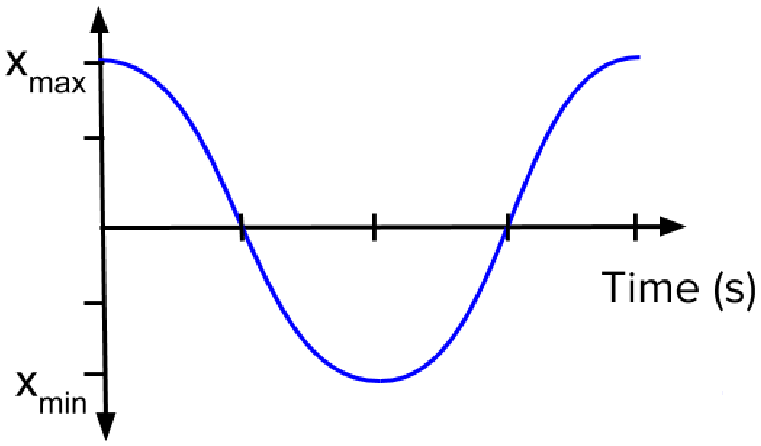
If the spring has gained **0.12J** of elastic potential energy at B, and there has been a loss in **0.24J** of gravitational potential energy, the total kinetic energy at B is **0.12J**.

Thus, $\mathbf{0.12 = 1/2 * (1.2/9.81) * v^2}$ => $\mathbf{v = 1.4ms^{-1}}$.

--

CALCULATING VELOCITY AT OTHER POINTS IN THE OSCILLATION CYCLE

The maximum velocity of a spring occurs when it is at its equilibrium position. When the spring extends beyond the equilibrium position, the restoring force increases above the weight, so the spring decelerates to $0ms^{-1}$. When it returns from maximum extension to equilibrium, the velocity increases back to its maximum as the restoring force accelerates the spring. When it compresses beyond the equilibrium position, the weight is greater than the restoring force, so the spring decelerates to $0ms^{-1}$. When returning from the minimum extension to the equilibrium position, the weight is greater than the restoring force, so the velocity increases back to maximum.



We can apply our knowledge of the maximum extension of a spring and our knowledge of energy changes to determine the velocity of a spring at different points in its oscillation cycle.

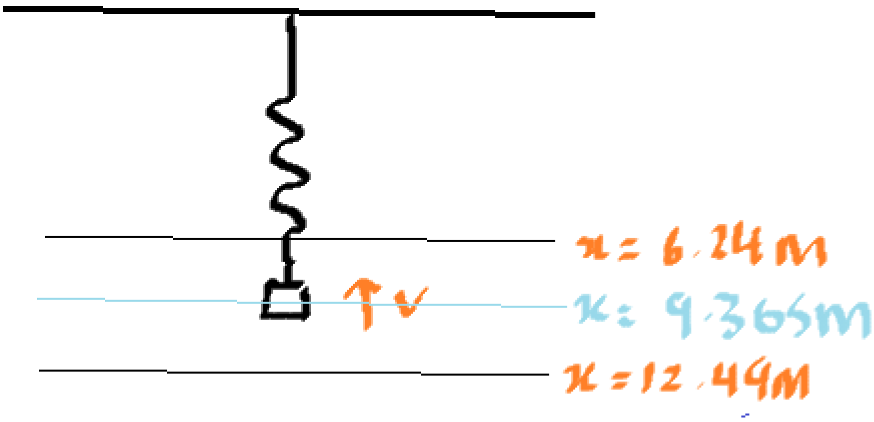
Example question:

A spring with a spring constant of **$22Nm^{-1}$** is attached to the ceiling and positioned vertically downwards. A mass of **14kg** is applied to the spring, and the mass is released. **Find the velocity of the mass at the halfway point between the point of maximum extension and the equilibrium position.**

The maximum extension of the spring is **2(14g) / 22 = 12.49m.**

The extension at equilibrium is **14g / 22 = 6.24m**.

The halfway point between these points is therefore **6.24 + ((12.49-6.24) / 2) = 9.365m**.



When the spring is at its maximum extension, it has an elastic potential energy of **$1/2(22)(12.49^2)$ = 1716J**.

As the mass rises, the extension of the spring decreases, and therefore the elastic potential energy decreases.

At an extension of **9.365m**, the elastic potential energy is **$1/2(22)(9.365^2)$ = 965J**.

Therefore, the change in elastic potential energy between maximum extension and **x=9.365m** is **1716-965 = 751J**.

The mass is accelerating between the point of maximum extension and equilibrium due to the restoring force exerted by the spring (which is greater than the weight of the mass). The spring is therefore doing work such that the mass gains kinetic energy. The mass is also gaining gravitational potential energy.

Thus, **$751 = (1/2 * 14 * v^2) + (14g * (12.49-9.365))$**

$751 = 7v^2 + 429.2$

$v = 6.78ms^{-1}$.

We would expect the velocity of the mass at equilibrium to be higher than this, since the mass continues to accelerate.

The gain in kinetic energy from the initial rest point to the equilibrium position is the loss in gravitational potential energy subtracted by the gain in elastic potential energy:

E_k = **$(14g(6.24)) - (1/2 * 22 * 6.24^2)$ = 428J**.

Thus, **$v = 7.82ms^{-1}$**, which is indeed higher than the velocity of **$6.78ms^{-1}$** calculated.

-

Example question:

A mass of **0.2kg** is applied to a vertically-positioned spring with spring constant **0.05Ncm^{-1}**. The spring is initially in equilibrium, and then the mass is released. **Determine the velocity of the mass when it has descended by 0.21m**.

The spring constant of the spring in metres is **0.05(100) = 5Nm^{-1}**.

The extension of the spring at equilibrium for this mass is **0.2g / 5 = 0.392m**.

We are therefore finding the velocity of the spring *before* it has reached equilibrium.

The change in gravitational potential energy of the mass by descending **0.21m** is **0.21g(0.2) = 0.412J**.

The elastic potential energy at this extension is **1/2 * 5 * 0.21^2 = 0.11025J**.

Thus, the kinetic energy of the mass at this extension is **0.412 - 0.11025 = 0.302J.**

0.302 = 1/2 * 0.2 * v^2 => **v = 1.73ms^{-1}**.

-

From the concepts described, we can now:

- Calculate the velocity of a spring at its equilibrium position (the maximum velocity).
- Calculate the maximum extension of a spring.
- Calculate the velocity of the spring at other points in its oscillation.

--

Applying conservation of energy to springs

The laws of conservation of energy apply to springs. The overall *mechanical energy* of a spring at one point in time must equal the overall mechanical energy of the spring at another point in time provided that there are no external forces.

Firstly, when a spring is compressed or extended, the overall work done to compress or extend the spring must equal the overall elastic potential energy that is stored in the spring.

We can say that the energy input, **W**, is equal to the elastic potential energy, **E_e:**

$$W = E_e.$$

When the force that was exerted to compress or extend the spring is withdrawn, this elastic potential energy is released in other forms.

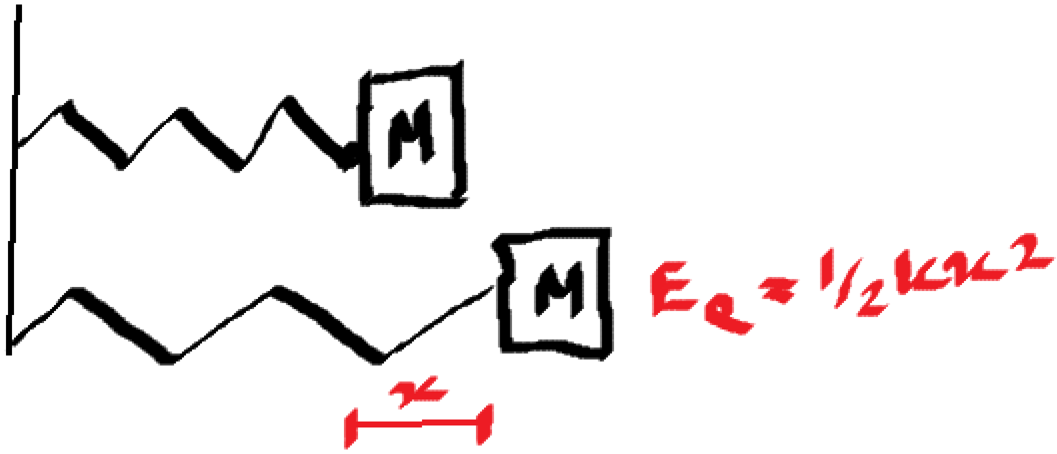
Since **work done = kinetic energy** according to the work-energy principle, and **work = E_e**, the released elastic potential energy can be converted into kinetic energy - so the spring can transfer kinetic energy to masses that are attached to it.

Similarly, **gravitational potential energy** is another form of work. When a spring is rising or falling, we have to consider that the gravitational potential energy of any mass attached to the spring is changing, so the elastic potential energy stored in the spring is converted into both kinetic energy and gravitational potential energy.

*Situation 1: **Mass attached to a horizontal spring**.*

Consider a situation where a spring with a spring constant of **k Nm^{-1}** is attached to a wall, with a mass **M** attached to the spring. Initially, the spring is in its equilibrium position:

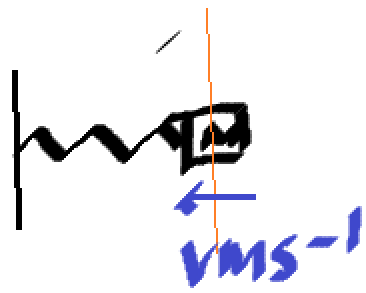
If a force is applied to the mass such that the spring extends a distance of **x**, the elastic potential energy in the spring increases from 0 to **$1/2kx^2$**:



If the force applied to the mass is withdrawn, the mass will snap back to its original position. This will cause the mass to move - so it gains kinetic energy.

When the spring has returned to the equilibrium position, its extension is **0m**, so its elastic potential energy is **0J**; so its elastic potential energy has decreased from **1/2kx²** to **0J**.

According to conservation of energy, this loss in elastic potential energy cannot simply be destroyed - it must be transferred to another form. The elastic potential energy that was in the spring is transferred as kinetic energy to the mass. The force exerted by the spring on the mass does work on the mass, causing the elastic potential energy in the spring to be transferred as kinetic energy to the mass.



When the mass M has returned to the equilibrium position of the spring, we can say that all of the elastic potential energy in the spring is converted into kinetic energy in the mass, assuming no loss in energy due to friction. The mass will be moving with a certain velocity **v**.

Thus, **$1/2kx^2 = 1/2mv^2$** => **$kx^2 = mv^2$** => **$v = \sqrt{(kx^2/m)}$**.

Example

A 2kg block is attached to a spring with spring constant **$10Nm^{-1}$** that is attached horizontally to the wall. The spring is extended a distance of **0.5m** from its equilibrium position, and then released, so that it returns to its equilibrium position.

Determine the velocity of the block when the spring has returned to its equilibrium position.

By extending the spring a distance of **0.5m**, the spring gains an elastic potential energy of **$1/2 * 10 * 0.5^2 = 1.25J$**.

When the spring returns to its equilibrium position, all of this elastic potential energy is converted into kinetic energy in the block, as the extension of the spring has reduced to 0m.

Thus, considering the kinetic energy of the block is **1.25J**, **$1.25 = 1/2 * 2 * v^2$** =>
$v = 1.12ms^{-1}$.

-

Suppose the mass attached to the spring is on a rough surface. We now need to consider the effects of friction.

The elastic potential energy in the spring will not be converted completely into kinetic energy when the mass is released back to equilibrium position, as some of this energy will be lost as heat / sound energy.

We can therefore say:

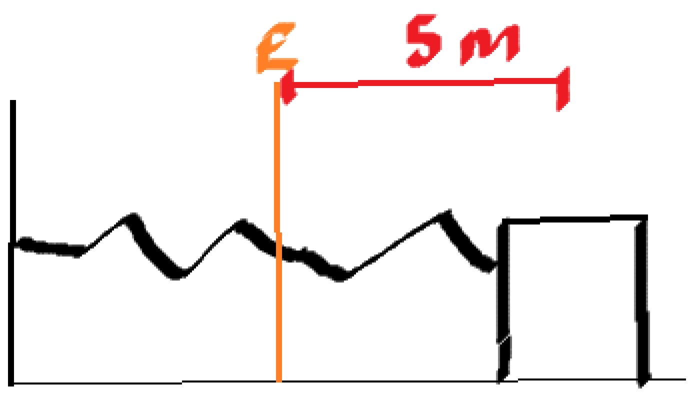
$$E_e = E_K + W_f \text{ where } W_f \text{ is the work done by friction.}$$

Example

A spring attached horizontally to a wall has a 4kg block attached to it. The spring is extended from its equilibrium position by a distance of 5m such that its elastic potential energy increases by **32J**. The spring is then released, causing the attached block to snap back. The frictional force exerted on the mass during this motion is a constant **4N**.

Determine the velocity of the mass when it has passed the original equilibrium position of the spring.

The distance between the point of release and the equilibrium position is 5m:



This means that the block will experience a resistive force of **4N** over a distance of **5m** when returning to the equilibrium position, so the work done by this resistive force is **4(5) = 20J**.

So **20J** of the **32J** of elastic potential energy is converted into work done by friction, meaning that the remaining **12J** is converted into kinetic energy.

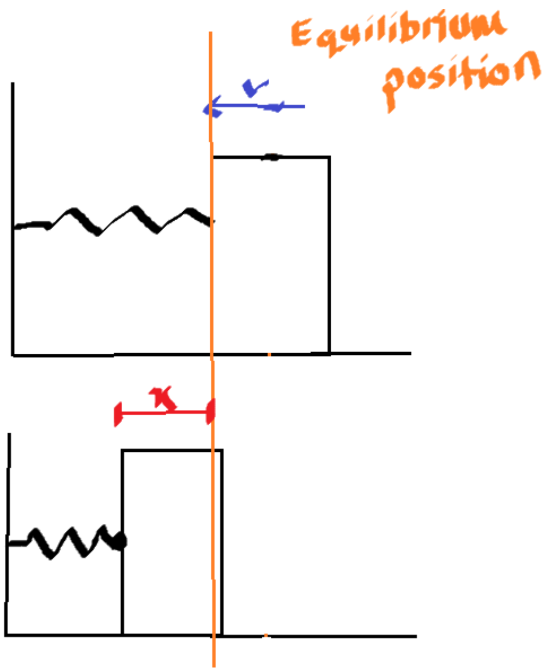
Thus, **12 = 1/2 * 4 * v^2** => **v = 2.45ms^{-1}**.

--

Compression beyond the equilibrium position

Once the mass has returned to the equilibrium position, because it has velocity in the direction of the wall, it will continue moving in that direction.

This means that the spring now starts to become compressed, so it starts gaining elastic potential energy again.



The block will start to slow down due to the resistive force exerted by the spring. The result is that the resistive force exerted by the spring will do work on the attached mass, causing its kinetic energy to deplete. The kinetic energy lost is converted into elastic potential energy in the spring.

Returning to the previous example, the kinetic energy of the 4kg block when it has returned to the equilibrium position is **12J**.

When the block surpasses the equilibrium position, the spring begins to compress, and this kinetic energy is converted into potential energy in the spring.

If there was no friction, we would assume that all of this kinetic energy is converted into elastic potential energy, so the spring gains **12J** of elastic potential energy. This would enable us to find how far it becomes compressed.

Firstly, we need to use the initial data given to determine the spring constant of the spring. When the spring is extended by 5m, its elastic potential energy increases by **32J**.

Thus, $32 = 1/2\ k * 5^2 \Rightarrow k = 2.56\text{Nm}^{-1}$.

Thus, if the spring gains **12J** of elastic potential energy, its compression is:

$12 = 1/2 * 2.56 * x^2 \Rightarrow x = 3.06\text{m}$.

<u>However, what if the frictional force was also involved?</u>

If the block still has the frictional force of **4N** exerted on it, not all of its kinetic energy is converted into elastic potential energy. There will be a certain work done by friction. This means that the gain in elastic potential energy is less than **12J**, so the compression is less than **3.06m**.

Applying conservation of energy, $E_k = E_e + W_f$.

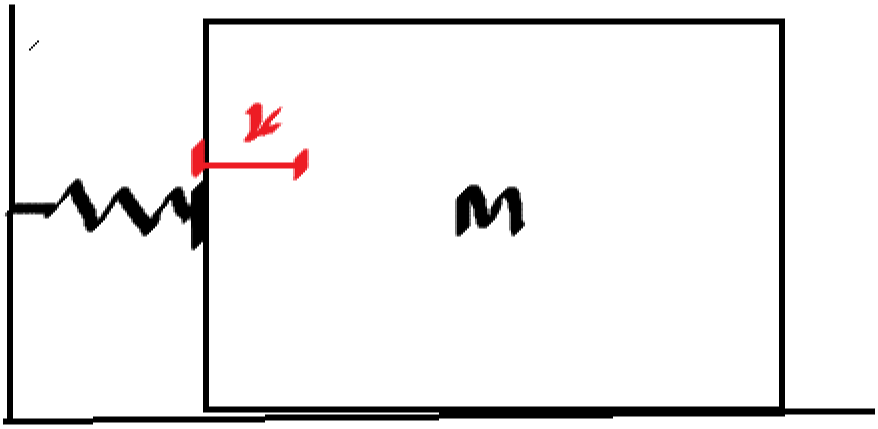
If we let the compression be **x**, the work done by friction is **4x.**

So, $12 = (1/2 * 2.56 * x^2) + 4x$

Solving for **x** => $12 = 1.28x^2 + 4x \Rightarrow 1.28x^2 + 4x - 12 = 0$ => quadratic formula… $x = 1.875\text{m}$.

Situation 2: ***releasing a mass from a compressed spring***

Suppose a toy car of mass **m** is pushed into a spring of spring constant **k**, such that the compression of the spring is **x**.



When the spring is compressed, it has an elastic potential energy of **$1/2kx^2$**.

If the spring is then released, the stored elastic potential energy is transferred to the toy car as kinetic energy. This is because the spring exerts a force on the car over a certain distance, causing an energy transfer to the car. Consequently, the car fires off in the direction of the spring's extension.

If there is no friction, we can assume that all of the elastic potential energy in the spring is transferred as kinetic energy to the toy car once the toy car has surpassed the equilibrium position.

Example:

A spring with spring constant **$50Nm^{-1}$** is attached horizontally to a wall. A toy car of mass **0.5kg** is compressed into the spring a distance of **0.8m**, and then the car is released. **Determine the velocity of the car upon release.**

The elastic potential energy transferred to the spring by compression is **$1/2 * 50 * 0.8^2 = 16J$**.

When the car is released, it gains a kinetic energy of **16J** upon passing the equilibrium position of the spring.

Thus, **$16 = 1/2 * 0.5 * v^2$** => **$v = 8ms^{-1}$**.

-

The car is initially moving on a smooth plane, but it then enters onto a rough plane. The rough plane exerts a frictional force of **0.5N** on the car. **Determine the time taken for the car to be brought to rest.**

When the car is on a smooth plane, there are no forces acting on it to reduce its velocity, so it maintains a velocity of **$8ms^{-1}$**. However, on the rough plane, the frictional force causes an acceleration of **$0.5 / 0.5 = 1ms^{-2}$**.

Thus, **u = 8, a = -1, v = 0**.

t = (0 - 8) / -1 = 8s.

--

Example:

A block of mass 5kg on a horizontal plane is compressed into a spring with a spring constant of **1000Nm^{-1}**, causing the spring to compress by **1m**. The spring is released, causing the block to move off along a horizontal plane. The work done by friction when the spring releases until the point where it reaches its equilibrium position is **100J**. The block moves along the horizontal plane with a frictional force of **FN** acting against it after it separates from the spring. This frictional force acts over a distance of **5m** to bring the block to rest.

Determine the value of F.

The elastic potential energy created by the compression of the spring is **1/2 * 1000 * 1^2 = 500J**.

When the block is released, this elastic potential energy is transferred as kinetic energy to the block. However, during the release, **100J** of work is done by friction, so only **400J** of the elastic potential energy is transferred as kinetic energy.

The block is therefore released from the spring with a kinetic energy of **400J**. After release, the only force acting on the block is the frictional force. This frictional force depletes the kinetic energy of the block in **5m**.

For a force to deplete **400J** of energy over a distance of **5m**, the force exerted must be **400 / 5 = 80N**.

This situation can also be applied in reverse.

Suppose a block with a kinetic energy of **E_k** collides with a spring attached to the wall head-on.

As the block collides with the spring, it will cause the spring to be compressed beyond its equilibrium position, such that the spring gains elastic potential energy. The kinetic energy of the block is converted into elastic potential energy in the spring as the force exerted by the spring does work on the block. We can say **E_e = E_k** assuming no friction.

Example

A 4kg block is moving with a velocity of **8ms^{-1}** when it collides with a horizontal spring head-on. The spring has a spring constant of **40Nm^{-1}** and is compressed by a distance of **xm**. *Assume no frictional forces are acting*.

Determine x.

The kinetic energy of the block upon collision is **1/2 * 64 * 4 = 128J**.

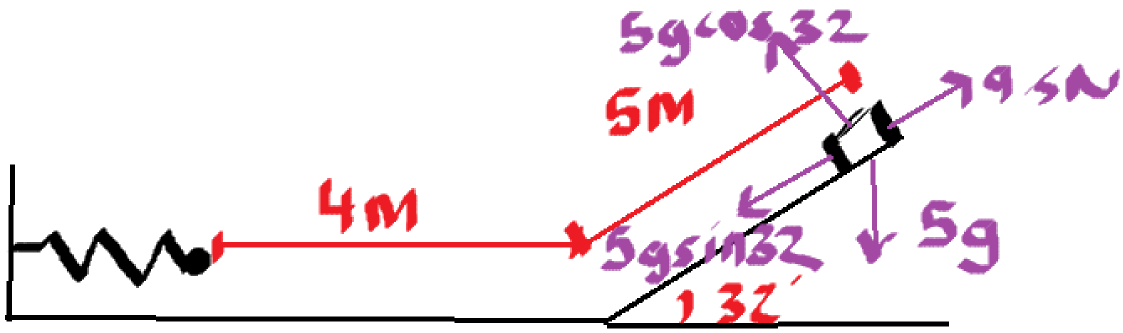
This is converted into elastic potential energy in the spring.

The spring therefore gains **128J** of elastic potential energy.

Thus, **128 = 1/2 * 40 * x^2** => **x = 2.53m**.

Example

A block of mass 5kg is positioned 5m up a slope inclined at 32° to the horizontal. The block is released, and moves down the slope. The block then hits the bottom of the slope, and moves along a horizontal plane. The block moves a horizontal distance of 4m before hitting a spring attached horizontally to the wall. The spring constant of the spring is **40Nm^{-1}**. Given that the block has a constant frictional force of **9.5N** exerted against it throughout the entire motion, **determine how far the spring is compressed upon the block colliding with it.** ***Ignore the effects of any frictional forces during the compression.***



We need to determine the kinetic energy of the block when it hits the spring.

At the top of the slope, the block has a gravitational potential energy of **5sin32 * g * 5 = 130J**.

When the block slides down the slope, this potential energy is converted into kinetic energy + the work done by friction.

The work done by friction when sliding down the slope is **9.5(5) = 47.5J**. This means that the kinetic energy of the block at the bottom of the slope is **130 - 47.5 = 82.5J**.

The block then travels a distance of **4m**, with a frictional force of **9.5N** opposing it. The work done to deplete kinetic energy is thus **4(9.5) = 38J**.

Thus, the kinetic energy of the block upon hitting the spring is **82.5 - 38 = 44.5J**.

This is converted completely into elastic potential energy in the spring upon compression (assuming no friction in the spring itself).

Thus, $44.5 = 1/2 * 40 * x^2$ => **x = 1.49m**.

--

Example

A 0.4kg toy car is on top of a slope of height **h** inclined at 30° to the horizontal. The toy car is released from rest, and moves down the slope. A mechanism in the toy car acts like an engine. The 'engine' has a power output of **20W**. The car hits the bottom of the slope in 1.4s, then continues to move along a horizontal plane. Its engine is turned off during the motion along the horizontal plane. It then collides with a spring of spring constant **$30Nm^{-1}$**, causing the spring to compress by **2m**.

Ignoring the effects of frictional forces, determine the height h.

The gain in elastic potential energy of the spring upon compression is $1/2 * 30 * 2^2 = 60J$.

This means that the kinetic energy of the toy car upon hitting the spring must have been **60J**. Since no frictional forces or driving forces are acting on the car during its horizontal motion, it must have had a kinetic energy of **60J** at the bottom of the slope.

When the car was at the top of the slope, it had a potential energy of **0.4gh**. This potential energy is transferred into kinetic energy at the bottom of the slope. However, there is also an input of energy due to the work done by the engine.

Thus, we can say that the total energy transfer to kinetic energy is the loss in potential energy plus the work done by the car engine.

$E_k = E_g + W_E$.

If the car engine operates at **20W** for **1.4s**, the total work done by the engine is **20(1.4) = 28J**.

Thus, **60 = 0.4gh + 28** => **32 / 0.4g = h = 8.15m**.

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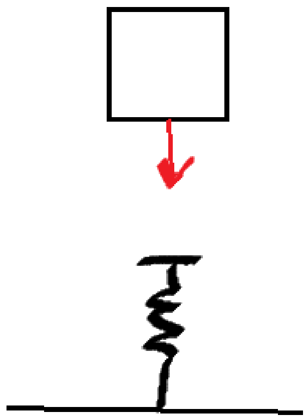
Situation 3: **Mass falling onto vertical spring**

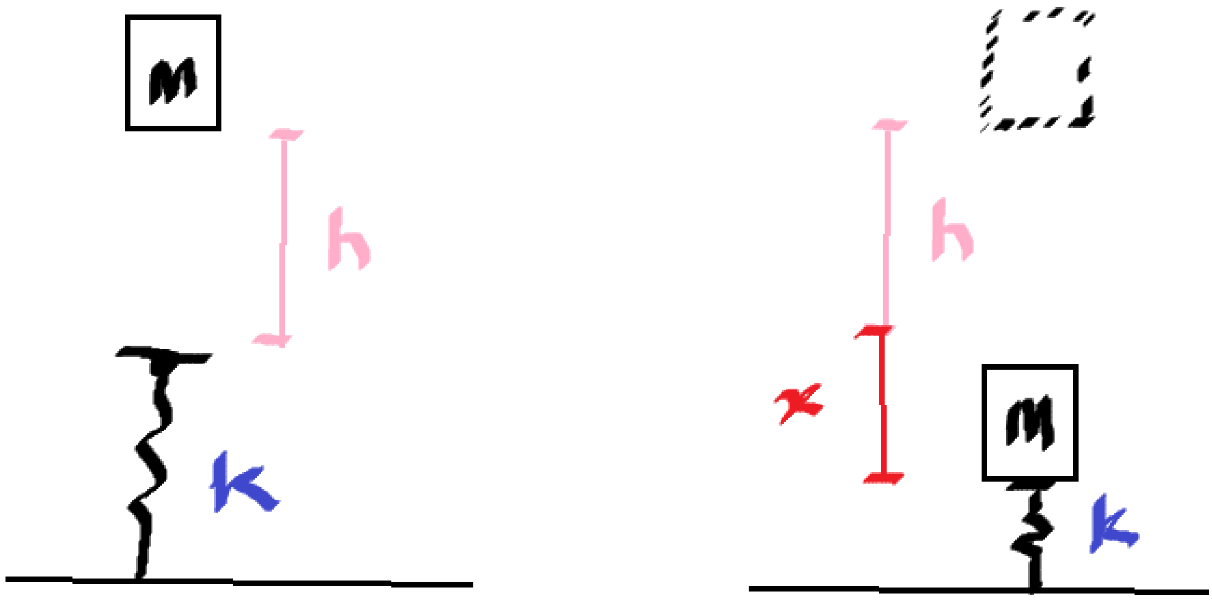
Suppose we have a mass being dropped vertically onto a spring:



As the mass falls, its potential energy is converted into kinetic energy. Upon hitting the spring, the spring compresses, and the kinetic energy in the mass is converted into elastic potential energy in the spring.

However, we cannot assume that the kinetic energy of the mass upon hitting the spring is equal to the elastic potential energy that the spring will gain. This is because when the spring is compressed, the mass continues to lose height and thus continues to lose gravitational potential energy. This means that even more kinetic energy is gained by the loss in potential energy.

Suppose a mass **m** is dropped from a height **h** onto a spring with spring constant **k**. The spring compresses by a distance **x**.



The total loss in height of the block is **h+x**. Thus, the change in gravitational potential energy is **mg(h+x)**.

This change in gravitational potential energy is converted completely into kinetic energy assuming no air resistance, and this kinetic energy is converted completely into elastic potential energy in the spring.

Thus, we can say **$mg(h+x) = 1/2kx^2$**.

Example

A 5kg block is released from 3m above a vertically-positioned spring. The spring has a spring constant of **$400Nm^{-1}$**. **Determine the maximum compression of the spring**.

If we let the maximum compression of the spring be **x**, the change in potential energy of the block up to the maximum compression is **5g(3+x) = 15g + 5gx**.

This is converted into elastic potential energy in the spring.

Thus, **$15g + 5gx = 1/2 * 400 * x^2$**

$15g + 5gx = 200x^2$

$200x^2 - 5gx - 15g = 0$.

Using the quadratic formula, **x = 0.99m**.

-

Situation 4: ***releasing a mass vertically upwards from a spring***

Suppose a vertically-positioned spring has a mass compressed onto it, and the mass is released.

When the mass is released, it jumps up. The force exerted by the spring on the mass causes the elastic potential energy in the spring to be transferred as kinetic energy to the mass. However, the mass also has a gravitational force acting against it, which depletes some of the gained kinetic energy - in other words, the mass is also gaining gravitational potential energy.

Therefore, assuming no friction, we can say that the stored elastic potential energy in the spring is converted into potential energy and kinetic energy.

$\mathbf{E_e = E_k + E_g}$.

Example

A spring facing vertically upwards has a spring constant $100Nm^{-1}$. The spring is compressed from an initial position A by a distance of 0.5m. A mass of **0.5kg** is placed on the spring, and the spring is released.

Determine the velocity of the mass upon reaching point A, assuming no frictional forces.

The elastic potential energy stored in the compressed spring is $\mathbf{1/2 * 100 * 0.5^2 = 12.5J}$.

Once the spring has returned to point A (equilibrium position) after release, its extension is 0m, and thus it has no elastic potential energy. All of the spring's elastic potential energy has now been converted into kinetic and gravitational energy assuming no friction.

At point A, the mass has gained a potential energy relative to its initial position of **0.5g*0.5 = 2.45J**.

This means that its gain in kinetic energy is **12.5 - 2.45 = 10.05J**.

Thus, $\mathbf{10.05 = 1/2 * 0.5 * v^2}$ => $\mathbf{v = 6.34ms^{-1}}$.

Determine the height that the mass will jump from point A [Assume no air resistance]

The mass is released from the spring with a kinetic energy of **10.05J**. As the mass jumps up, this kinetic energy is converted completely into potential energy. The mass no longer continues to gain kinetic energy because there is no force acting in the upwards direction to do work.

Thus, **10.05 = 0.5gh** => **h = 2.05m**.

Example

A vertical spring attached to the ground with spring constant **150Nm^{-1}** is compressed from its equilibrium position, point A, by a distance of **x m**. A mass of **2kg** is placed on the spring when it is compressed. The spring is released. When the spring has returned to point A, the mass has a velocity of **4ms^{-1}**. Assuming no frictional forces act, **determine the value of x**.

The elastic potential energy stored in the spring is **1/2 * 150 * x^2 = 75x^2**.

This is converted into potential energy and kinetic energy.

The gain in kinetic energy of the mass is **1/2 * 4^2 * 2 = 16J**.

The gain in potential energy of the mass is **2g * x**.

Thus, **75x^2 = 16 + 2gx**.

75x^2 - 2gx - 16 = 0

Using the quadratic formula, **x = 0.61m**.

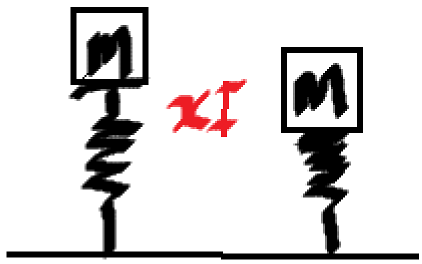
Example

A 10kg mass is placed on an unstressed vertically-positioned string that is attached to the ground. The spring constant of the spring is 800Nm^{-1}.

a) **Determine the extension of the spring at its new equilibrium position**

This question can be answered by simply considering Hooke's Law.

If a force of **10g** acts on a spring with spring constant **800Nm^{-1}**, the compression of the spring is **10g / 800 = 0.122m**.



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The spring is compressed a *further distance* of 30cm. It is then released.

b) **Determine the velocity of the 10kg mass when it has returned to the initial equilibrium position found in part a).**

The spring has now compressed a total distance of **0.3 + 0.122 = 0.422m**. This means it has an elastic potential energy of **1/2 * 800 * 0.422^2 = 71.2J**.

When the spring has returned to its equilibrium position for that mass, which is at a compression of **0.122m**, its elastic potential energy is **1/2 * 800 * 0.122^2 = 5.95J**.

This means that the change in potential energy between the compressions of **0.422m** and **0.122m** is **71.2 - 5.95 = 65.25J**.

This loss in elastic potential energy must be converted into other forms according to conservation of energy.

The mass is gaining velocity, and it is gaining height, so the elastic potential energy is converted into kinetic energy and gravitational potential energy.

$E_e = E_K + E_g$.

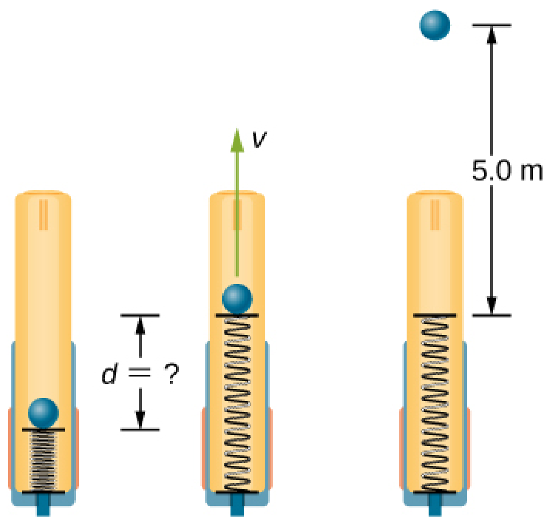
The gain in potential energy between the two points is **(10 * g * 0.3) = 29.43J**.

Thus, the gain in kinetic energy is **65.25 - 29.43 = 35.8J**.

Thus, **35.8 = 1/2 * 10 * v^2** => **$v = 2.68ms^{-1}$**.

Example

43. The massless spring of a spring gun has a force constant $k = 12\,\mathrm{N/cm}$. When the gun is aimed vertically, a 15-g projectile is shot to a height of 5.0 m above the end of the expanded spring. (See below.) How much was the spring compressed initially?



The projectile gains a gravitational potential energy of **5(g)(0.015) = 0.73575J** upon being released from the spring.

If we assume no air resistance, this means that the kinetic energy of the projectile after being released is **0.73575J**.

The spring constant of the spring should be converted to Nm^{-1}: **$12Ncm^{-1} = 1200Nm^{-1}$** (if 12N is required to stretch by 1cm, 100 times this is required to stretch by one metre).

The elastic potential energy stored in the spring at maximum compression is **$1/2(1200)d^2$** = **$600d^2$** where **d** is the compression.

This is converted into a gravitational potential energy of **0.015gd** and a kinetic energy of **0.73575J**.

Thus, **$600d^2 = 0.015gd + 0.73575$** => **$600d^2 - 0.015gd - 0.73575 = 0$**

Solving for **d** using the quadratic formula gives **d = 0.035m**.

(the gain in gravitational potential energy from release to equilibrium, 0.00515J, is negligible in this situation).

Situation 5: ***mass on a freely-hanging spring***

Take an unstretched spring (spring constant **k**) attached to a ceiling. If a mass of **m** is attached to the spring, the gravitational force created by the mass, **mg**, will cause the spring to accelerate downwards, extending by a distance **x**.

This will cause the spring to extend from its natural position, so it gains elastic potential energy. It also gains kinetic energy due to the acceleration. When the spring is at its equilibrium, it will oscillate about that point in *simple harmonic motion* due to the velocity component that it still has in the downwards direction.

Conservation of energy states that energy cannot be created from nowhere. Thus, the gain in elastic and kinetic energy must have been converted from another form. This form is gravitational potential energy. As the spring descends a distance **x**, the force exerted by gravity is **mg**, so the total work done is **mgx**. This work done is converted into elastic potential energy and kinetic energy.

If the velocity of the spring at its new equilibrium position is **v**, we can say:

$mgx = 1/2kx^2 + 1/2mv^2$.

$mgx - 1/2kx^2 = 1/2mv^2$

$2mgx - kx^2 = mv^2$

$v = \sqrt{(2mgx - kx^2) / m}$

-

Example

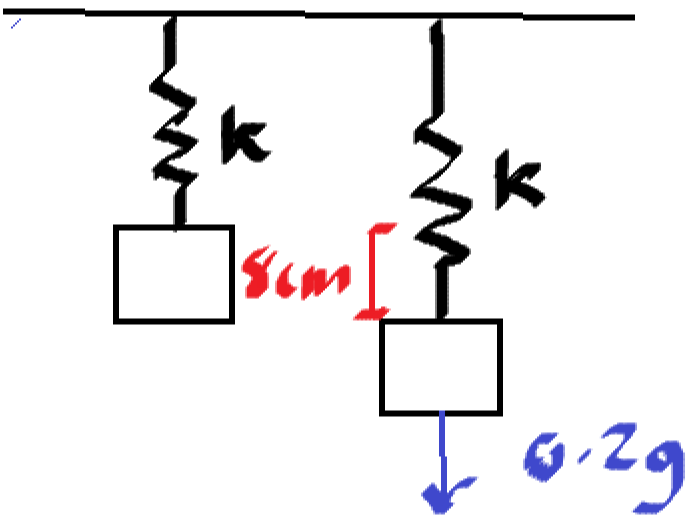
A spring extends by 8.0cm when a load of 200g is hung from it. When the extension is 8.0cm, **determine the velocity of the load.**

When the spring is extended by a distance of 8.0cm, the elastic potential energy in the spring is **1/2 * F * x = 1/2 * 0.2g * 0.08 = 0.07848J**.

The load has a kinetic energy of **1/2 * 0.2 * v^2 = 0.1v^2**.

When descending a distance of **0.08m**, the change in potential energy of the load is **0.2g*0.08 = 0.15696J**.

Thus, **0.15696 = 0.1v^2 + 0.07848** => **v = 0.886ms^{-1}**.



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Example

A block weighing 1.2N is hung from a spring with spring constant 6Nm^{-1}. **What is the velocity of the block at one quarter of the spring's maximum extension?**

The maximum extension of the spring is **2(1.2) / 6 = 0.4m**.

One quarter of this is **0.1m**.

When the block descends a distance of **0.1m**, its loss in gravitational potential energy is **0.1(1.2) = 0.12J**.

The elastic potential energy of the block at this point is **1/2 * 6 * 0.1^2 = 0.03J**.

Thus, the kinetic energy of the block is **0.09J**.

0.09 = 1/2 * (1.2/9.81) * v^2 => **v = 1.21ms^{-1}**.

-

Now let us consider an example where a mass is thrown upwards into a vertically-hanging spring.

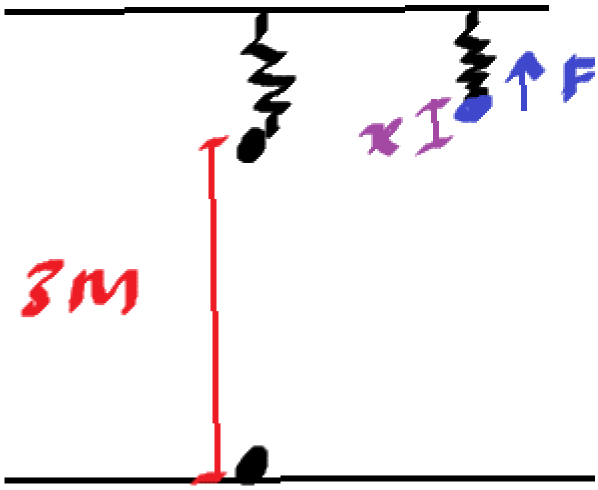
Suppose the mass collides with the spring with a kinetic energy of $\mathbf{E_K}$. This kinetic energy is transferred as elastic potential energy to the spring. However, as the mass rises, it gains gravitational potential energy, so not all of the kinetic energy present in the mass is converted into elastic potential energy.

Thus, $\mathbf{E_k = E_g + E_e}$.

Example

A 5kg mass is thrown vertically upwards with a velocity of $18ms^{-1}$. The mass collides head-on with a spring (spring constant $400Nm^{-1}$) that is attached to the ceiling, causing it to compress by xm. The height of the spring relative to the initial position of the mass before it is thrown is 3m. **Determine the maximum value of x**, *assuming no frictional forces and that the mass and spring are in contact throughout the compression. Also assume that the spring does not compress all the way back to the ceiling.*

First, we need to determine the kinetic energy of the mass before it hits the spring.



It has an initial kinetic energy of $\mathbf{1/2 * 18^2 * 5 = 810J}$.

Its potential energy at a height of **3m** is **5 * 9.81 * 3 = 147J**.

This must have been converted from kinetic energy, so the kinetic energy at this height is **810 - 147 = 663J**.

When the ball and spring collide, the compression of the spring is **x**. Thus, the gain in elastic potential energy during the compression is $\mathbf{1/2 * 400 * x^2 = 200x^2}$.

However, there is not a complete conversion of kinetic energy to elastic potential energy because the ball also gains potential energy as the spring is compressed. If the spring is compressed a distance **x**, the gain in potential energy of the mass is **5gx**.

Thus, $\mathbf{663 = 5gx + 200x^2}$ => $\mathbf{200x^2 + 5gx - 663 = 0}$.

Using the quadratic formula, **x = 1.7m**.

Elastic potential energy in other elastic materials

An elastic material has the ability to return to its original shape after stress is applied. All elastic materials - not only the typical spring - obey Hooke's Law, **F = kx**, and can therefore possess elastic potential energy when they are extended or compressed.

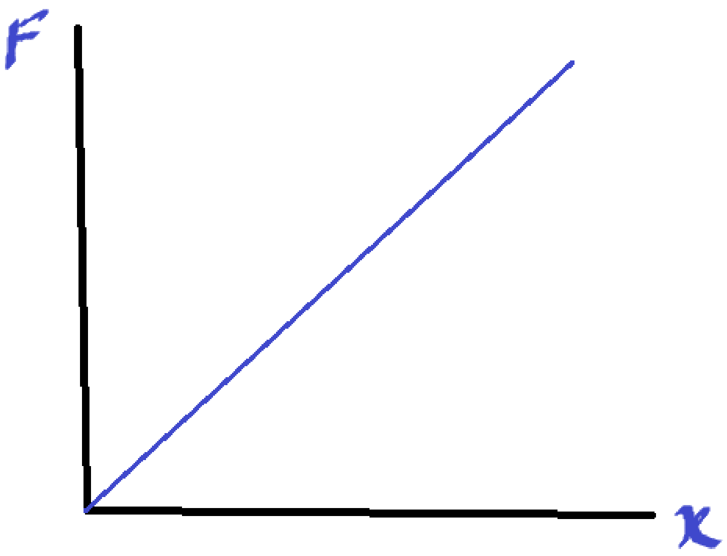
Example 1: energy in a bowstring

Bowstrings are classed as elastic materials. If a bowstring is pulled back a distance **d** by a force **F**, the gain in elastic potential energy of the bowstring is therefore **1/2Fx**.

When an arrow is released from the bowstring, the bowstring exerts a force on the arrow. This force causes the elastic potential energy in the string to be transferred as kinetic energy to the arrow.

Assuming no friction in the bowstring / air resistance, we can say that when the bowstring has returned to its rest position, **1/2Fx = 1/2mv^2** => **Fx = mv^2**.

The elastic potential energy stored in a bowstring can also be found with a graph.



The graph on the left shows the force exerted on a bowstring, and the distance it is pulled back.

The elastic potential energy stored in the bowstring at a certain distance is equal to the area under the graph, **1/2Fx**.

Q:

A force of **800N** is exerted on a bowstring, causing it to extend back by **0.6m**. An arrow of mass **50g** is placed in this position, and the bowstring is released. **Determine the velocity of the arrow.**

The elastic potential energy in the bowstring is **1/2 * 800 * 0.6 = 240J**.

This is transferred as kinetic energy to the arrow. Thus, **240 = 1/2 * 0.050 * v^2** => **v = 98ms^{-1}**.

Q:

An arrow of mass **40g** is released from a bowstring at an angle of **30°** to the horizontal at a height of 1.2m. The force exerted in the bowstring was **300N**. If the arrow travels a horizontal distance of 150m after release, **determine the distance the bowstring was pulled back to achieve this.**

We can first use SUVAT to determine the velocity at which the arrow leaves the bow.

If we let this velocity be **v**, the initial horizontal velocity is **vcos30** and the vertical velocity is **vsin30**.

The variables that relate both components are **v** and the time taken to reach the ground, **t**.

First, consider the vertical motion:

u = vsin30, s = -1.5, a = -g, t = t

$-1.5 = v\sin30t + 1/2(-g)t^2$.

Now consider the horizontal motion,

150 = vcos30t => **v = 150 / cos30t**.

Substituting **v = 150 / cos30t** into **$-1.5 = v\sin30t + 1/2(-g)t^2$** gives:

$-1.5 = \sin30t(150/\cos30t) - gt^2/2$

$-1.5 = 150\tan30 - gt^2 / 2$

$gt^2 / 2 = 150\tan30 + 1.5$ => **t = 4.23s**.

Thus, **v = 150 / cos30(4.23) = 40.9ms^{-1}**.

So the arrow is released with a velocity of **40.9ms^{-1}**, and thus a kinetic energy of **$1/2 * 0.040 * 40.9^2 = 33.5$J**.

This kinetic energy is transferred from elastic potential energy.

Thus, **33.5 = 1/2 * 300 * x** => **x = 0.223m**.

-

Q:

A person has two bows. One bow has a bowstring with a spring constant of $\mathbf{k_1}$ and the other bow has a bowstring with a spring constant of $\mathbf{k_2}$. An arrow of mass **50g** is placed in the bows, and the arrow is drawn back **dm** before being fired. The arrow is released from a height of **0.8m** above the ground in both instances in the horizontal direction. The arrow in the bow with spring constant $\mathbf{k_1}$ travels a distance of **80m** after being fired, and the arrow in the bow with spring constant $\mathbf{k_2}$ travels a distance of **92m**. **Find the ratio of $\mathbf{k_1}$ to $\mathbf{k_2}$**.

Let us determine the horizontal velocity of the arrow as it leaves each bow (let this be $\mathbf{v_1}$ for the k_1 bow and $\mathbf{v_2}$ for the k_2 bow) . In both instances, the arrow has an initial vertical velocity of $\mathbf{0ms^{-1}}$ since it is fired horizontally.

K_1 bow:

In the horizontal plane, $\mathbf{80 = v_1t}$ => $\mathbf{t = 80 / v_1}$.

In the vertical plane, u = 0, a = g, $\mathbf{t = 80/v_1}$, s = 0.8.

$\mathbf{0.8 = 1/2(g)(80/v_1)^2}$.

$\mathbf{0.163 = 6400 / v_1^2}$.

$\mathbf{v_1 = 198ms^{-1}}$.

K_2 bow:

$\mathbf{t = 92 / v_2}$.

$\mathbf{0.8 = 1/2(g)(92/v_2)^2}$.

$\mathbf{V_2 = 227ms^{-1}}$.

-

For the k_1 bow, the kinetic energy upon release of the arrow is $\mathbf{1/2 * 198^2 * 0.050 = 980J}$. This kinetic energy transfer is from elastic potential energy.

Thus, $\mathbf{980 = 1/2 * k_1 * d^2}$ => $\mathbf{k_1 = 1960 / d^2}$.

For the k_2 bow, the kinetic energy upon release of the arrow is $\mathbf{1/2 * 227^2 * 0.050 = 1288J}$.

Thus, $\mathbf{1288 = 1/2 * k_2 * d^2}$ => $\mathbf{k_2 = 2576 / d^2}$.

Thus, $\mathbf{k_1 / k_2}$ = $\mathbf{(1960/d^2) / (2576 / d^2) = 1960 / 2576 = 35/46}$.

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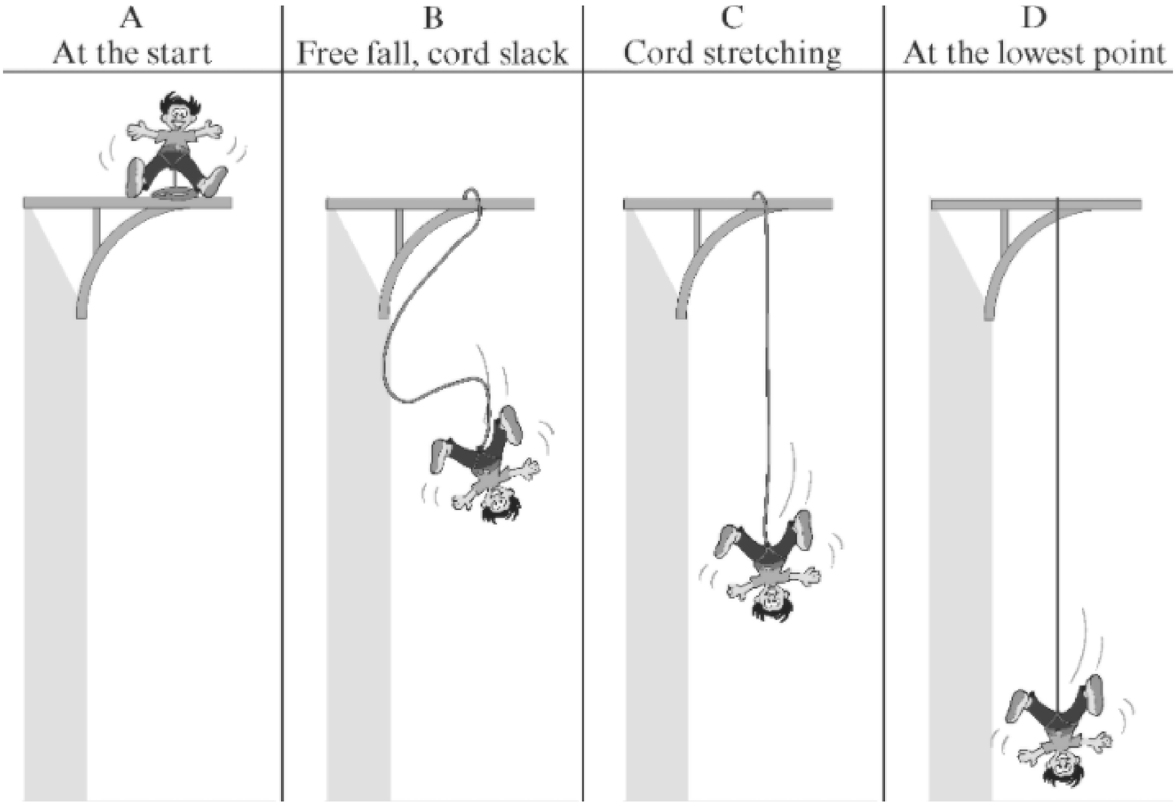
Example 2: **Bungee jump**

A bungee is an elastic material, so when it extends, it gains elastic potential energy.



Consider a bungee jump situation, where a person starts at point **A** and jumps downwards. Initially, the bungee cord is slack, so the bungee cord exerts no force - the person is free falling. This is shown in part **B**. In part **C**, the bungee cord begins to extend beyond its natural length, and at point **D**, the bungee cord is at its maximum extension, and the person is stationary.

Let us consider the energy changes involved (ignoring air resistance):

A	At point **A**, the person has a certain amount of gravitational potential energy.
B	In section **B**, the person jumps, and they are in a free fall state. The height of the person decreases, so their gravitational potential energy is converted into kinetic energy. In section **B**, the cord is slack - it has not extended beyond its natural length and therefore no force is being exerted on the cord by the person. This means that the cord has no extension and therefore gains no elastic potential energy.
C	At point **C**, the change in height of the person is greater than the length of the bungee cord, so the bungee cord now begins to extend. Stress is being applied to the bungee cord. The bungee cord initially exerted no force on the person, but now that it is extending, a restoring force is being exerted on the person. This means that a certain amount of work is being done against the

	person to deplete their kinetic energy. This work done is transferred as elastic potential energy to the cord.
D	At point D, the cord is at its maximum possible extension. At this point, the kinetic energy of the person reduces to 0J, and the person stops falling. All of the gravitational potential energy at the start has now been converted into elastic potential energy.

Consider the following question:

A bungee jumper of mass 70 kg jumps from a high bridge using a bungee cord of natural length 80 m. When he reaches the lowest point for the first time the length of the cord is 130 m. Calculate

(i) the loss of gravitational potential energy from his position on the bridge to the lowest point for the first time, [2]

(ii) the stiffness constant (k) of the bungee cord assuming the cord obeys Hooke's law and that there are no losses due to air resistance, [3]

(iii) the extension of the cord when he finally comes to rest (after having 'bounced' a few times). [2]

i)

The total loss in gravitational potential energy is **70g(130) = 89300J**.

ii)

This loss in gravitational potential energy is converted completely into elastic potential energy assuming no air resistance. The extension of the cord is **50m**.

Thus, **$89300 = 1/2 * k * 50^2$** => **$k = 71.4Nm^{-1}$**.

iii)

The maximum extension of a spring is greater than the actual extension of the spring when the mass attached to the spring is at equilibrium.

The weight of the person is **70g**. If **$k = 71.4Nm^{-1}$**, applying Hooke's Law, **x = F / k = 70g / 71.4 = 9.62m**.

--

In the maximum extension of a spring section, we showed that the maximum extension of a spring for a given force is **2F/k**.

In this question, we found the extension of the bungee cord at equilibrium for this person is **9.62m**. We would therefore expect the maximum extension to be **2(9.62) = 19.24m**, but the cord extends to **50m?** This is because when we carry out the calculation **2F/k**, we are assuming that the spring starts from rest, such that there is a conversion of only gravitational potential energy into elastic potential energy. In the case of the bungee jump, when the cord starts to extend, the person isn't starting from rest - they have an initial amount of kinetic energy. This kinetic energy and the change in gravitational potential energy is converted into elastic potential energy, so the maximum extension is greater than when the person is simply released from rest. When the person is at rest, the cord has an equilibrium extension of **9.62m** but oscillates down to a maximum extension of **19.24m**.

-

Tensile and compressive deformation

When a force is applied to a material, it causes the material to *deform*. Different materials have different levels of deformation when a force is applied.

Materials have a property known as *elasticity*. When the force applied to a material is below its *elastic limit*, the material deforms elastically. This means that when the force is withdrawn, the material will return to its original shape.

Like with springs, when a material deforms elastically, there is a **restoring force** that attempts to restore the material to its original shape. We say that the material is under **stress** - and the stress is attempting to return the material to its original shape. The greater the force exerted, the greater the restoring force and thus the greater the stress.

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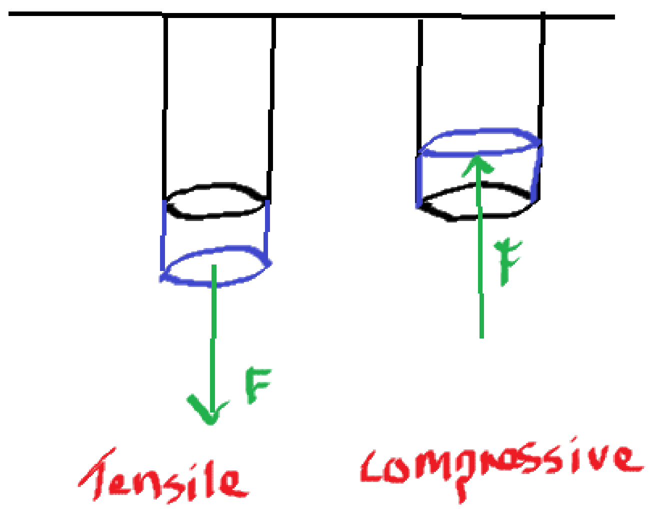
We will be discussing **tensile** and **compressive** deformation. Tensile deformation is when a material is stretched, and compressive deformation is when a material is compressed:



Initially, we will mainly focus on tensile deformation - but the same principles apply to compressive deformation.

*Issues with using **F = kx** to model elastic deformation*

When a material behaves elastically, it obeys *Hooke's Law*. This states that the force applied to an elastic material is directly proportional to the extension of the material.

$$\mathbf{F} \propto \mathbf{x}.$$

This enables us to obtain the equation **F = kx**, where **k** is the **stiffness constant** of the material.

This equation is useful for modelling a particular material of particular dimensions. However, the deformation of a material is dependent upon many other factors other than force.

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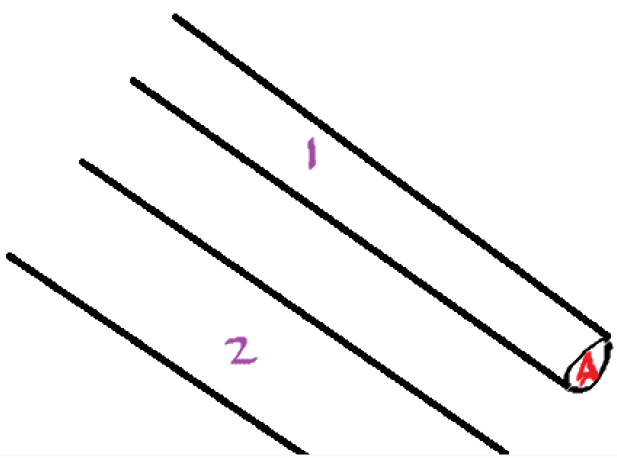
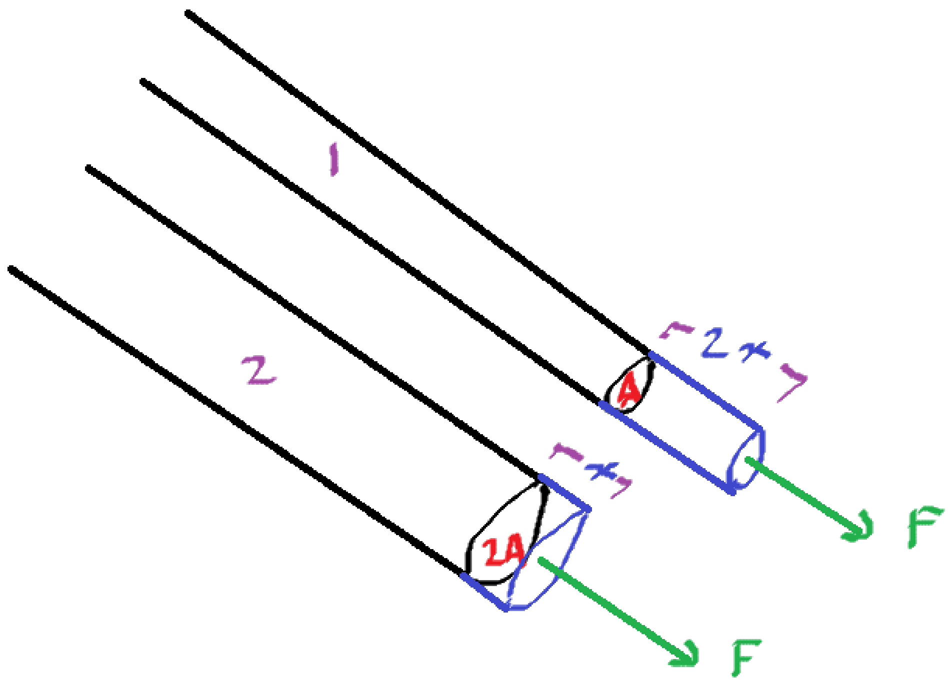
Suppose we take two steel rods, **1** and **2**. Both rods are composed of the same type of steel and have the same length, but rod **2** has double the cross-sectional area of rod **1**.



If we applied the same force **F** to both rods, rod **2** will clearly extend less than rod **1**, because it is thicker.



Experiments show that the extension of a material is inversely proportional to the cross-sectional area; so the greater the cross-sectional area, the less the extension.

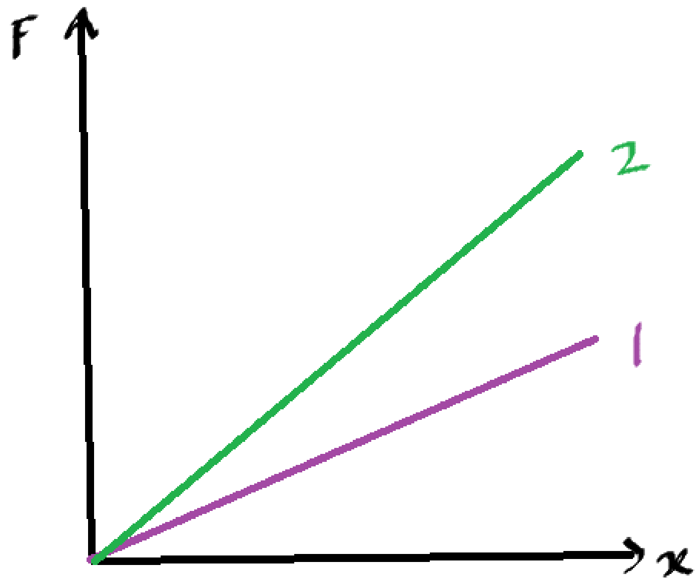
If rod **2** extends by a distance of **x**, rod **1** extends by a distance of **2x**. Rod **2** has double the stiffness of rod **1**.

If we were to find **k** for both rods:

Rod 1: $k_1 = F / 2x$
Rod 2: $k_2 = F / x$

The ratio of **k** for rod **1** and **2** is **(F/2x) / (F/x) = 1/2**. So rod **1** has half the stiffness constant of rod **2**.

If we were to plot the **force-extension** graph for both rods, they would look as follows:



The issue with the use of **k**, the stiffness constant, in an engineering context is that it does not allow us to generalise the strength of a particular type of material. The stiffness constant does not apply to the material as a whole - it only applies to a material of particular dimensions. Both rods in the example above are composed of the same type of steel, but one rod deforms more than the other due to the different cross-sectional areas, and therefore they have different values of **k**.

Using **F = kx** alone does not allow an engineer to determine how much a particular type of material (e.g. steel, concrete, glass) deforms under stress - we would have to consider other factors that affect the deformation, such as cross-sectional area.

A more useful quantity than **k** is the **modulus of elasticity**. This modulus is a constant for a particular material. For example, copper has a **Young's Modulus** (a type of modulus of elasticity) of **117GPa**. This quantity is useful as it measures a material's resistance to deformation based on the *type* of material it is. This enables us to consider how an object will deform based on its material type, cross-sectional area and length.

Stress and strain

There are four variables that influence how much an elastic material will deform when a force is applied to it:

- The strength of the force
- The cross-sectional area of the material
- The length of the material
- The type of material

Strength of force:

Hooke's Law states that the extension of an elastic material is directly proportional to the force applied.

This law applies to all materials that are behaving elastically - no matter the dimensions or type of material.

If double the force is applied to an elastic material, it will extend to double the value.

$$\mathbf{F \propto x}.$$

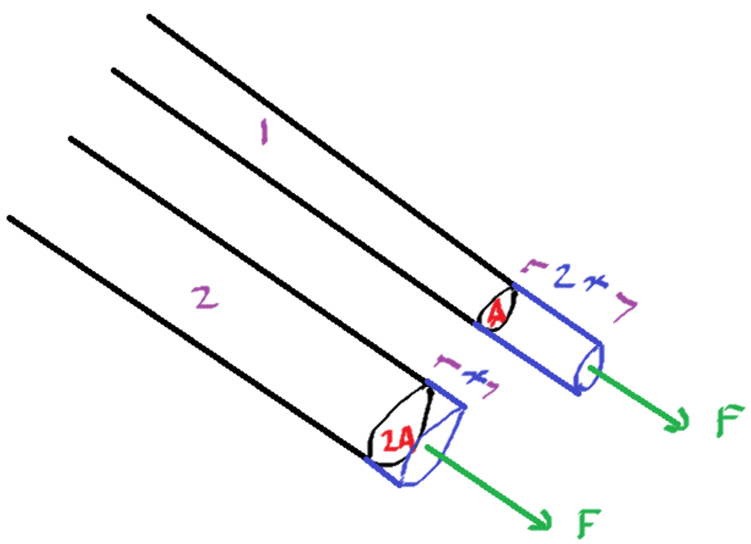
Cross-sectional area:

As shown previously, the extension of a material is inversely proportional to its cross-sectional area. The greater the cross-sectional area, the more force that is required to cause the same extension. This is because when the cross-sectional area is greater, the force exerted is spread out over a larger area - so each section of the material extends less.

$$\mathbf{x \propto 1/A}$$

*where **A = cross-sectional area**.*

Suppose you extend a copper wire with cross-sectional area **A** a distance of **10cm** with a force **F**. If you had another copper wire with cross-sectional area **2A**, it would extend to **5cm** if the same force **F** was applied. It would take a force of **2F** to extend the wire with cross-sectional area **2A** to **10cm**.



The same principle applies to compression. A wire with a cross-sectional area that is double the cross-sectional area of another wire of the same material and length requires double the amount of force to *compress* by the same amount.

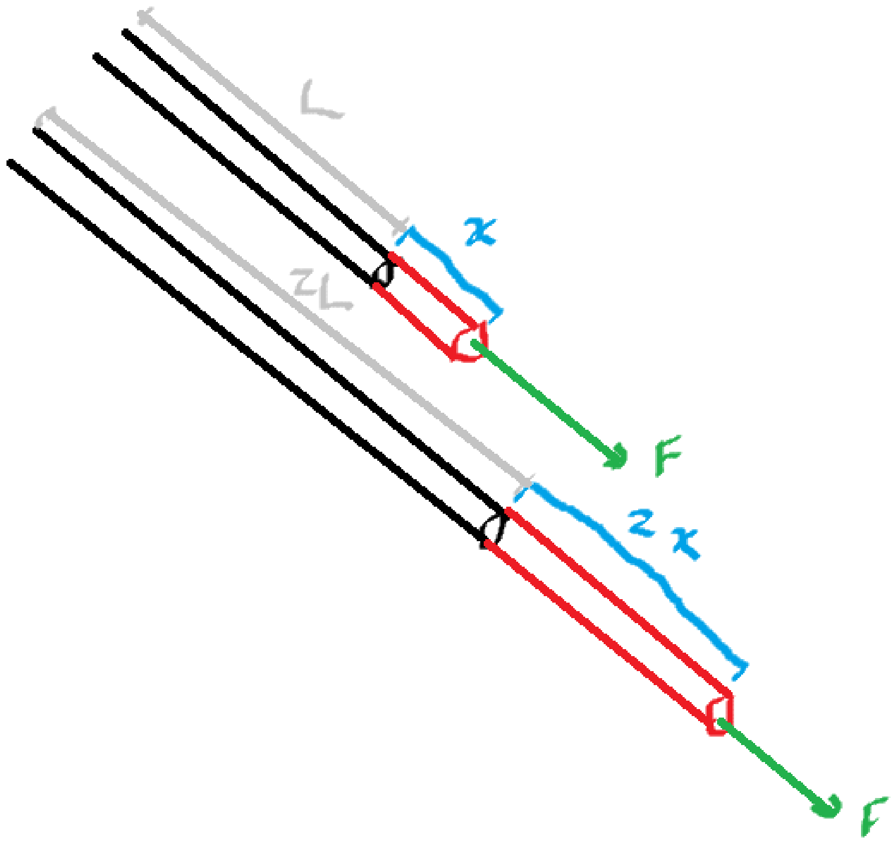
Length:

The longer a material, the greater it will extend when a force is applied.

$$\mathbf{L \propto x}$$

*Where **L = length of material.***

If you have a copper wire of length **L** and cross-sectional area **A**, and another copper wire of length **2L** and cross-sectional area **A**, the copper wire with length **2L** will extend double the amount of the wire with length **L** when a force **F** is applied.



This makes sense because a material with twice the length has twice the amount of matter to potentially be extended.

Similarly, a material with double the length will also *compress* by double the amount when a force is applied.

Let us consider some examples of how length and cross-sectional area affect extension:

Example 1:
A force **F** is applied to two cylindrical copper wires of the same length. Wire A has a diameter of **d**, and wire B has a diameter of **2d**. **What is the ratio of the extension of the wires?**

Because **cross-sectional area** is π**(d/2)**^{2}, if wire B has a diameter of **2d**, it has a cross-sectional area **4 times** greater than wire A. Since extension is inversely proportional to cross-sectional area, the extension of wire B is therefore 1/4 the extension of wire A.

Example 2:
A force **F** is applied to a copper wire, A. Wire A has a length of **L** and a diameter of **3d**. The same force **F** is applied to a copper wire, B. Wire B has a length of **5L** and a diameter of **d**. **What is the ratio of the extension of the wires?**

If we considered length only, wire B would extend **five times** as much as wire A. However, because wire A has three times the diameter of wire B, its cross-sectional area is **9 times** greater than that of wire B. If we considered cross-sectional area only, wire B would therefore extend **9 times** as much as wire A.

If we take the product of these values, **9(5) = 45**, wire B extends **45** times more than wire A.

--

Now let us consider how we can relate these variables that influence the extension of a material. We use two quantities: **stress (σ)** and **strain (ε)**.

Stress

Stress, **σ**, is a measure of the force applied per unit of area.

$$\sigma = F / A.$$

The units of stress are **Nm^{-2}**, or **pascals**, **Pa**. 1 **Pa** is defined as a force of **1N** exerted over an area of **1m^2**.

Stress measures how strongly a material is trying to return to its original shape. A material under more stress experiences more extension, and there is a greater restoring force attempting to restore the original shape.

The deformation of a material is influenced by the cross-sectional area of the material, **$x \propto 1/A$**.

When a material has a greater cross-sectional area, the force exerted on the material is spread out over a greater area. This means that the material is under *less stress* - the restoring force is weaker as each point in the material is experiencing less force.

-

Suppose we have a cylindrical copper wire, A, with a cross-sectional area of **2m^2** and a force of **10N** is applied to the wire.

The force applied to each **m^2** of wire is **5N** (10 / 2 = 5). This is the stress in wire A.

Now suppose we have another cylindrical copper wire, B, with a cross-sectional area of **4m^2**. If a **10N** force is applied to the wire, this force is now spread out over double the area. The force exerted per **m^2** is **2.5N**. Wire B therefore has a stress of **2.5Nm^{-2}**.

Wire B is clearly under less stress, as the force is spread out over a larger area. For wire B to have the same force exerted per **m^2** as wire A, a force of **20N** would have to be exerted. When a force of **20N** is exerted on wire B, the force exerted per **m^2** is **5N**. This means that each point on wire B is now under the same stress as in wire A, which leads to the same amount of extension **($F \propto x$)**.

-

Example 1:

A cylindrical nylon wire, wire A, has a radius of **0.2m**. A force of **28N** is applied to the wire. Another cylindrical nylon wire, wire B, has a radius of **0.7m** and the same length as wire A. **Determine the force that would have to be exerted on wire B to have the same amount of stress as wire A**.

The cross-sectional area of wire A is $\pi(0.2^2) = 0.125$**m^2**. This means that the force exerted **per m^2** on the wire is **28 / 0.125 = 224Nm^{-2}** => this is the stress.

The cross-sectional area of wire B is $\pi(0.7^2) = 1.54$**m^2**. Clearly, because wire B has a greater cross-sectional area, the force exerted on the wire is spread out over a greater area. This means that a greater force must be exerted for the same stress to be exerted.

Since **$\sigma = F / A$**, to produce a stress of **224Nm^{-2}** in a wire of cross-sectional area **1.54m^2**, a force of **224 * 1.54 = 345N** must be exerted.

This is a force that is **345 / 28 = 12.3x** stronger than the force exerted on wire A.

Example 2:

Wire A has a cuboidal shape, and has a force of **200N** exerted on it. The rectangle that forms the cross-sectional area of wire A has a height of **0.2m** and a width of **0.3m**. Wire B has the same length of wire A and is composed of the same material, but is cylindrical. **If the same 200N force is exerted on wire B and it has double the stress of wire A, what must the radius of wire B be?**

The cross-sectional area of wire A is **0.2*0.3 = 0.06m²**.

The force exerted per **m²** is therefore **200 / 0.06 = 3330Nm⁻²**.

If wire B has double the stress when a 200N force is applied, it must have an area of **200 / 2(3330) = 0.03m²** using **σ = F / A**.

Wire B is a cylinder, so its cross-sectional area is π**r²**. Thus, π**r² = 0.03** => **r = 0.098m**.

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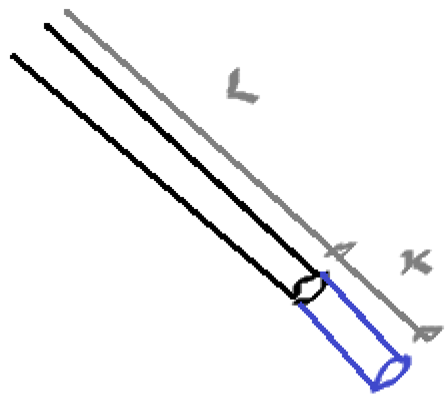
Strain

Another important quantity of deformation is **strain** (ε). Strain is a measurement of how much a material has extended.

All materials have a natural length, **L**. When a material is deformed elastically, however, the material extends beyond its natural length to an extension **x**.



Strain is the ratio of the extension of a material to its natural length:

$$\varepsilon = x / L.$$

For example, if a material has a natural length of **0.2m** and it extends to a length **0.35m** when a force is applied (so the extension is **0.15m**), **ε = 0.15 / 0.2 = 0.75**.

Strain has no units since we are dividing two length values, so the units cancel out.

Strain can also apply to the *compression* of a material. For example, if a material is compressed by **0.5m** from its natural length of **2m**, the strain is **0.5 / 2 = 0.25**. We are only concerned with the *magnitude* of strain - so we do not have to consider positive or negative values.

A material that has a greater extension due to deformation has a greater strain.

Relating Stress and Strain

We have established that the extension of a material is influenced by the **force applied**, the **length** of the material, and the **cross-sectional area** of the material.

$\mathbf{x \propto F}$

$\mathbf{x \propto 1 / A}$

$\mathbf{x \propto L}$

If we combine all of these variables, we get:

$\mathbf{x \propto FL / A}$.

To make a directly proportional relationship into an equation, we have to insert a constant.

The constant that we insert in the case of compressive and tensile deformation is called **Young's Modulus** - with the symbol **E**. This modulus is a type of *modulus of elasticity*. There are other moduli for other types of deformation. Young's Modulus is the same for a given material (e.g. copper always has a Young's Modulus of **117GPa**).

By definition, if a material has a greater Young's Modulus, it is stronger and more difficult to deform. A greater Young's Modulus therefore means a lower extension for the same force.

Thus, $\mathbf{x = FL / AE}$.

Now let us rearrange this equation for Young's Modulus.

xAE = FL => **E = FL / xA**.

This is equal to the **stress / strain**.

Why?

σ = F / A

ε = x / L.

Thus, **σ / ε = (F/A) / (X/L) = FL / xA**.

-

So, we can say that **E = FL / xA = σ / ε**.

The units of Young's Modulus are **pascals**, or **Nm^{-2}**, like for stress. This is because **E = stress / strain**, and strain has no units, so the units of **E** are equal to the units of stress.

-

We can prove that the Young's Modulus is constant for a given material:

Suppose that you have two copper wires:

Wire A: **Length = L**, **Area = 3A**.

Wire B: **Length = 2L**, **Area = A**.

If the same force, **F**, is applied to both wires, because wire **B** has twice the length and 1/3 the cross-sectional area of **A**, and wire A and B are of the same material, wire **B** must extend **6 times** as much as wire **A** [we could not make these assumptions if the materials were different].

If wire **A** has an extension of **x**, wire **B** has an extension of **6x**.

For wire **A**, **E = FL / 3Ax = 1/3(FL/Ax)**.

For wire **B**, **E = F(2L) / A(6x) = 2FL / 6Ax = 1/3(FL/Ax)**.

Both wires have the same Young's Modulus.

This means that there will always be the same ratio of stress over strain for a given material.

-

The Young's Modulus of different materials is shown below:

Material	Young's modulus $\times 10^{10}\text{Pa}$
Aluminum	7.0
Bone (tension)	1.6

-

Another way of viewing Young's Modulus is that **stress** and **strain** are directly proportional. If a material is under more stress, more force is being applied per unit area. This means it will extend to a greater length, such that its strain is greater (since **strain = extension / length**).

Thus, $\sigma \propto \varepsilon$.

We then insert a constant to make this into an equation. This constant is Young's Modulus, **E**.

Thus, $\sigma = E * \varepsilon$. Re-arranging for **E** gives $E = \sigma / \varepsilon$.

-

Young's Modulus is very useful to engineers. By taking the Young's Modulus of a particular material, it is possible to determine how much an object of a particular length and area that is made of that material will deform when a particular force is applied.

Example 1:

A copper wire is clamped to the ceiling. A mass of **5kg** is hung from the copper wire. The wire has a length of **2m**, and a diameter of **4.2mm**. Given that the Young's Modulus of copper is **110GPa**, **find how much the copper wire will extend by**.

*We could not use Hooke's Law in this calculation because the value of **k** in Hooke's Law is specific to this wire and this wire only. If the length or cross-sectional area of the wire was changed, **k** would change. Young's Modulus is useful as it enables us to calculate the extension no matter the area or length, because Young's Modulus is constant for a given material.*

Firstly, we can find the stress in the copper wire.

$\sigma = F / A$.

$F = 5g$ since the weight of the mass is exerted downwards on the wire.

$A = \pi(2.1*10^{-3})^2 = 1.39*10^{-5}m^2$.

Thus, $\sigma = 5g / (1.39*10^{-5}) = 3.53*10^6Pa$.

We know σ and **E**, so we can use $E = \sigma / \varepsilon$ to find ε.

$\varepsilon = (3.53*10^6) / (110*10^9) = 3.21*10^{-5}$.

So the ratio of the extension of the wire to its length is $3.21*10^{-5}$.

Thus, **$3.21*10^{-5}$ = x / 2** => **x = $6.42*10^{-5}$m = 0.0642mm**.

Example 2:

A 65-kg mountain climber is hanging 35m below a rock outcropping. She is attached to the rock outcropping by a Nylon rope of natural length **L** and diameter 0.800cm. The Young's Modulus of Nylon is **$1.35*10^{9}$Pa**. **Find the value of L**.

The Nylon wire is currently in an extended state - it is under stress, and a restoring force is trying to revert it to its original shape.

If the length of the wire at this extension is 35m, and its natural length is L, its extension is **35-L**.

We can therefore say that **ε = (35-L) / L**.

If we can find the value of ε, we can find **L**.

The cross-sectional area of the rope is **$\pi(0.400/100)^2 = 5.03*10^{-5}m^2$**.

This means that **$\sigma = 65g / (5.03*10^{-5}) = 1.27*10^{7}$Pa**.

Thus, **$\varepsilon = (1.27*10^{7}) / (1.35*10^{9}) = 9.41*10^{-3}$**.

$9.41 * 10^{-3}$ = (35-L)/L.

0.00941L = 35-L

1.00941L = 35 => **L = 34.67m**.

-

Example 3:

57. A 90-kg mountain climber hangs from a nylon rope and stretches it by 25.0 cm. If the rope was originally 30.0 m long and its diameter is 1.0 cm, what is Young's modulus for the nylon?

The extension of the rope is **0.25m**. Thus, **ε = x / L = 0.25 / 30.0 = $8.33*10^{-3}$**.

The stress in the wire is **$90g / (\pi(0.5/100)^2) = 1.124*10^{7}$Pa**.

Thus, **$E = \sigma / \varepsilon = (1.124*10^{7}) / (8.33*10^{-3}) = 1.35*10^{9}$Pa = 1.35GPa.**

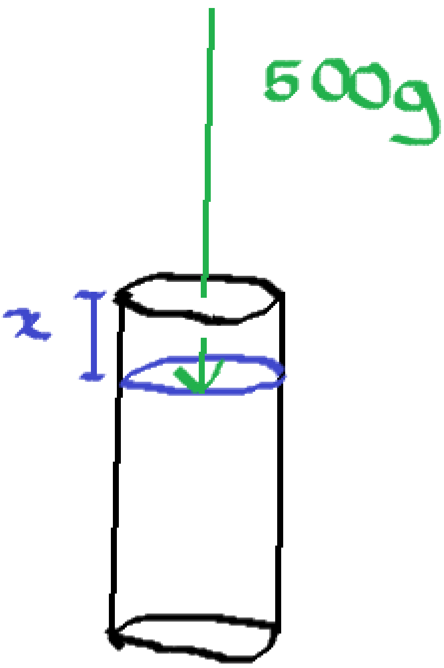
-

Example 4:

Now let us consider an example of compression. The same rules apply for compression as they do for compression.

-

A cylindrical steel pillar sits on the ground. The pillar has a diameter of 0.7m and length 2m. A mass of 500kg is placed on the pillar. Given that the Young's Modulus of the steel is 200GPa, **determine by how much the pillar will be compressed by the mass**.



A force of **500g** is being exerted on the pillar. The stress in the pillar is therefore **$500g / \pi(0.35)^2 = 12745Nm^{-2}$**.

Thus, **$\varepsilon = 12745 / (200*10^9) = 6.37*10^{-8}$**.

Thus, **$x = 6.37*10^{-8} * 2 = 1.27*10^{-7}m = 127.4nm$**.

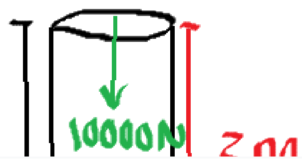
This shows how strong steel is.

-

Example 5:

A statue consists of a large pillar made of granite (Young's modulus $4.5*10^{10}$Pa) with a natural height of 6.0m of cross-sectional area $0.20m^2$ and density $2700kgm^{-3}$, with a sculpture weighing 10000N on top of the pillar [the natural height of the pillar is the height without the statue sitting on the pillar]. **Find the compression at the cross-section 3m below the top of the pillar**.

Let us consider a diagram of this situation:



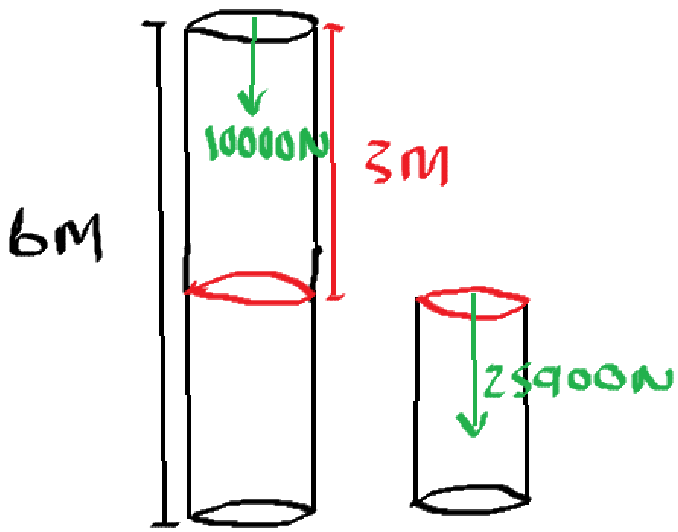
We need to find the stress at half the height of the pillar. We can solve this by determining the weight of the pillar in the top-half. We can then add this to the weight of the statue, and model this is a single weight exerted on the cross-section at the half-way point.

The volume of the top-half of the pillar is **3(0.20) = 0.60m^3**. Thus, the mass is **2700(0.60) = 1620kg**.

The weight of the top half of the pillar is therefore **1620g = 15900N**.

The total force exerted at the half-way point is the weight of the statue + the weight of the pillar, which is **15900 + 10000 = 25900N**.

Thus, **σ = 25900 / 0.20 = 129500Nm^{-2}**.



This means that **ε = 129500 / (4.5*10^{10}) = 2.88*10^{-6}**.

Thus, if the height of the pillar at this section is **3m**, **x = 2.88*10^{-6} * 3 = 8.64*10^{-6}m.**

--

Example 6:

A length of steel wire and a length of brass wire are joined together. This combination is suspended from a fixed support and a force of 80 N is applied at the bottom end, as shown in **Figure 5**.

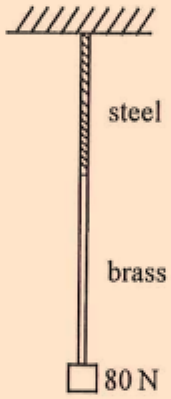


Figure 5

Each wire has a cross-sectional area of $2.4 \times 10^{-6}\,m^2$.

length of the steel wire = 0.80 m
length of the brass wire = 1.40 m
the Young modulus for steel = 2.0×10^{11} Pa
the Young modulus for brass = 1.0×10^{11} Pa

(i) Calculate the total extension produced when the force of 80 N is applied.

The force exerted on both wires must be equal. This is similar to when two springs are hanging in series with each other - both springs have an equal force exerted on them when a force is applied to the bottom of the spring system.

-

Since **E = (F/A)/(x/L) = FL/Ax** => **x = FL/AE**.

For the steel wire, **$x = 80(0.80) / (2.4*10^{-6} * 2*10^{11}) = 1.33*10^{-4}$m**.

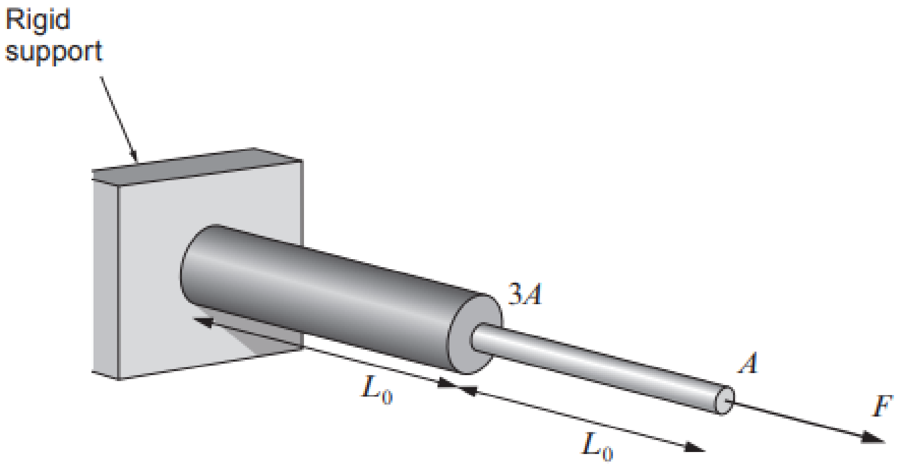
For the brass wire, **$x = 80(1.40) / (2.4*10^{-6} * 1*10^{11}) = 4.67*10^{-4}$m**.

Thus, **$x_{total} = 6.00*10^{-4}$m = 0.6mm**.

-

Example 7:

(a) The bar in the figure below is made from a **single piece of metal**. It consists of two parts of equal length L_0 and cross-sectional area A and $3A$. The diagram is not drawn to scale.



(i) Show that the **total** extension, Δx_{total}, of the bar under the action of an applied force, F, as shown in the diagram, can be given by:

$$\Delta x_{total} = \frac{4FL_0}{3AE}$$

where E represents the Young modulus of the metal in the bar. [3]

Both bars have an equal force exerted on them.

We can assume the Young's moduli of both bars are the same since they are made of the same metal. We will let this modulus be **E**.

We need to find the extension of both bars, and add them together to find $\mathbf{x_{total}}$.

We know that **E = FL/Ax** => **x = FL / AE**.

First, we will determine the extension of the thicker bar (let this be $\mathbf{x_1}$).
$\mathbf{x_1 = FL_0 / 3AE}$.

For the thinner bar, we will let the extension be $\mathbf{x_2}$.

$x_2 = FL_0 / AE$.

$x_{total} = x_1 + x_2 = FL_0 / 3AE + FL_0 / AE$

We can cross-multiply as follows to combine the fractions:

$$x_{total} = \frac{FL_0}{3AE} + \frac{FL_0}{AE}$$

$$= \frac{FL_0AE + 3FL_0AE}{3A^2E^2}$$

$$= \frac{4FL_0AE}{3A^2E^2}$$

cancel AE

$$x_{total} = \frac{4FL_0}{3AE}$$

Example 8:

A 2.00 m light rigid rod is suspended from the ceiling by two vertical wires, A and B, each having a natural length of $\ell = 1.00$ m, attached to each end of the rod. A is a copper wire with a Young's modulus $Y_A = 12.4 \times 10^{10}$ Pa, diameter 1.60 mm, and B is a brass wire with a Young's modulus $Y_B = 9.00 \times 10^{10}$ Pa, diameter 1.00 mm.

a) An 80 kg mass is attached to the midpoint of the rod, calculate:

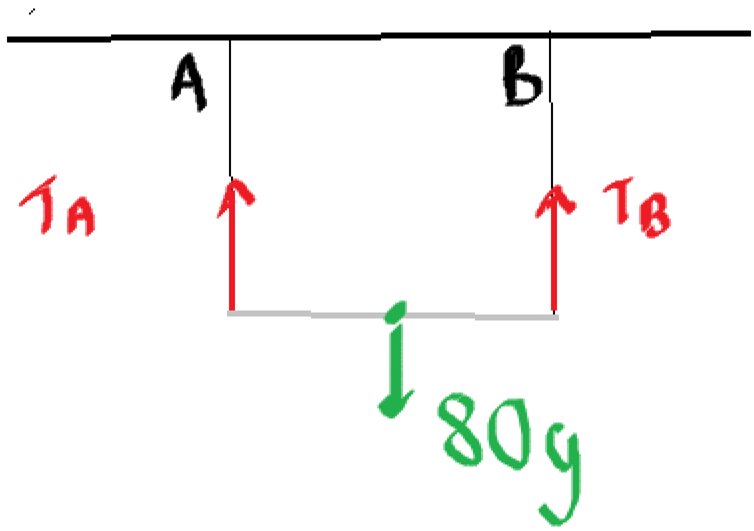
(i) the tension in each wire, assuming the rod is horizontal
(ii) the consequent extension of A
(iii) the consequent extension of B
(iv) the angle the rod makes with the horizontal

[8]

a)

I.

First let us consider a diagram for the situation:



We will let the tension in A be $\mathbf{T_A}$ and the tension in B be $\mathbf{T_B}$.

The weight force should be evenly distributed in each wire, since the weight is applied at the centre of the rod. This means that $\mathbf{T_A = T_B = 80g/2}$.

To prove this, we can use the principle of moments.

If we take moments about the pivot point of A, and consider that the length of the rod is 2m.

Moment of weight force (clockwise) $\mathbf{= 80g(1) = 80g}$.

Moment of $\mathbf{T_B}$ (anticlockwise) $\mathbf{= 2(T_B)}$.

Thus, $\mathbf{2T_B = 80g \Rightarrow T_B = 80g/2}$.

Because the rod is in equilibrium, $\mathbf{T_A + T_B = 80g}$

Thus, $\mathbf{80g/2 + T_A = 80g} \Rightarrow \mathbf{T_A = 80g/2}$.

II.

The force exerted on A is **80g/2** since we can assume that the tension force is equal to the downwards force that causes extension due to the equilibrium.

This means that the stress exerted on A is **(80g/2) / π(0.80*10^{-3})^2 = 1.952*10^{8}Pa**.

Thus, **ε = (1.952*10^8) / (12.4*10^{10}) = 1.57*10^{-3}**.

Thus, **x = 1.57*10^{-3} * 1 = 1.57*10^{-3}m**.

III.

Applying the same process to B, we get **x = 5.55*10^{-3}m**.

The greater extension of B is to be expected because B has a lower cross-sectional area than A, and a lower Young's modulus (so a lower stiffness).

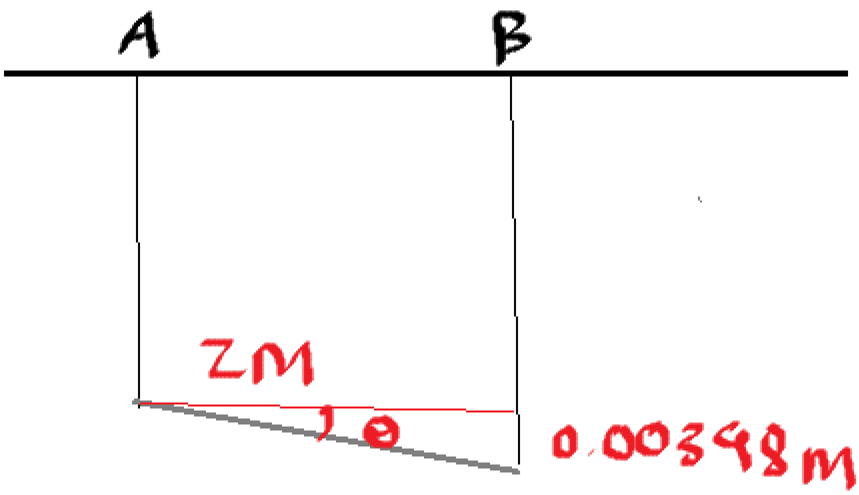
IIII.

Because B is extended slightly further than A, the rod is at an angle to the horizontal.

The length of B increases to **1 + (5.55*10^{-3}) = 1.00555m.**

The length of A increases to **1 + (1.57*10^{-3}) = 1.00157m**.

This means that B extends a distance of **1.00555-1.00157 = 0.00398m** below A:



This means that the angle made with the horizontal is **tan^{-1}(0.00398 / 2) = 0.114°**.

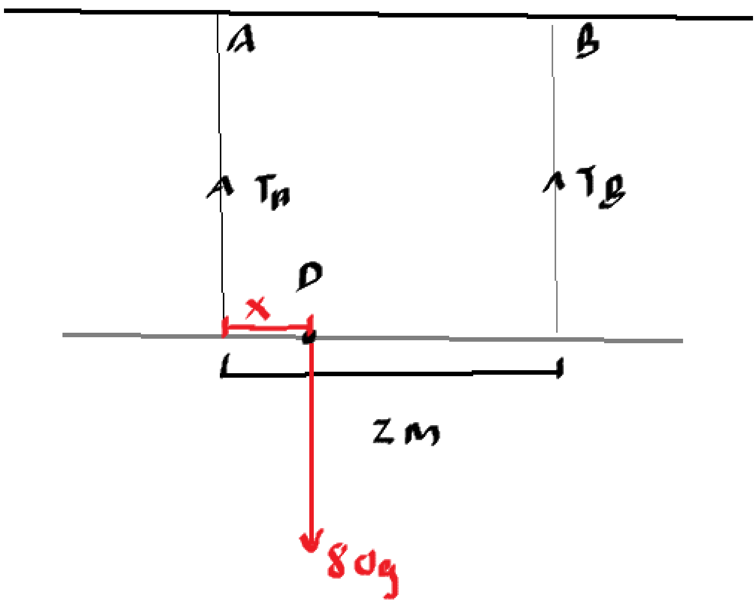
b) The attachment of the 80 kg mass is moved to a point D, a distance x from A, along the rod.

Calculate:

(i) the extension of A
(ii) the extension of B
(iii) the distance x for the rod to be horizontal.

[12]

First, let us consider a diagram for this situation:



If we take moments about A, we can say that $\mathbf{2T_B = 80gx}$ => $\mathbf{T_B = 40gx}$.

This means that $\mathbf{T_A = 80g - 40gx = 40g(2-x)}$.

Now we need to express the extension of wire A (let this be $\mathbf{x_A}$) and wire B (let this be $\mathbf{x_B}$) in terms of $\mathbf{T_A}$ and $\mathbf{T_B}$.

We know that $\mathbf{E = \sigma/\varepsilon = (F/A) / (x/L) = FL/Ax}$.

Thus, $\mathbf{x = FL/AE}$.

Wire A:

The length of **A** is 1m, so **x = F(1) / AE = F/AE**.

The cross-sectional area of **A** in terms of pi is $(0.80*10^{-3})^2\pi = 6.4*10^{-7}\pi\ m^2$

The force applied to A is **40g(2-x)** N.

The Young's Modulus of A is $12.4*10^{10}Pa$.

Thus, $x_A = (40g(2-x)) / (6.4*10^{-7}\pi * 12.4*10^{10}) = (40g(2-x)) / 79360\pi$.

Wire B:

The cross-sectional area of wire **B** is $\pi(0.50*10^{-3})^2 = 2.5*10^{-7}\pi\ m^2$.

The force applied to B is **40gx**.

The Young's Modulus of B is $9.00*10^{10}Pa$.

Thus, $x_B = 40gx / (9.00*10^{10}*2.5*10^{-7}\pi) = 40gx / 22500\pi$.

-

If the rod is completely horizontal, this means that A and B must be extended by the same length.

Therefore, $x_A = x_B$.

(40g(2-x)) / 79360π = 40gx / 22500π

Now we solve for **x**.

80g(22500π) - 40gx(22500π) = 40gx(79360π)

80g(22500π) - 900000πgx = 3174400πgx

80g(22500π) = 4074400πgx

x = (80g(22500π)) / (4074400πg) = 0.442m.

The wires must both extend by **0.442m**.

-

Example 9

An elastic cord of unstretched length 160 mm has a cross-sectional area of 0.64 mm^2. The cord is stretched to a length of 190 mm. Assume that Hooke's law is obeyed for this range and that the cross-sectional area remains constant.

the Young modulus for the material of the cord = 2.0×10^7 Pa

(i) Calculate the tension in the cord at this extension.

(ii) Calculate the energy stored in the cord at this extension.

i)

We know that **σ = F / A**. If we can find **σ** and **A**, we can find **F**.

To find **σ**, we consider the strain and the Young's modulus.

ε = x / L = 30 / 160 = 0.1875.

Thus, **σ = E * ε = $2*10^7$ * 0.1875 = $3.75*10^6 Nm^{-2}$**.

$3.75*10^6$ = F / ($0.64/100^2$)

F = 240N.

ii)

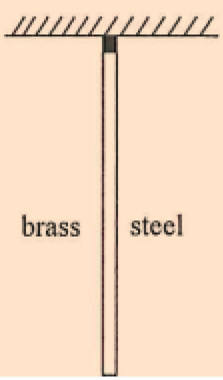
We know that **elastic potential energy = 1/2Fx**.

Thus, **E_e = 1/2(240)($30*10^{-3}$) = 3.6J**.

-

Example 10

Figure 7 shows two wires, one made of steel and the other of brass, firmly clamped together at their ends. The wires have the same unstretched length and the same cross-sectional area. One of the clamped ends is fixed to a horizontal support and a mass M is suspended from the other end, so that the wires hang vertically.



Since the wires are clamped together the extension of each wire will be the same. If E_S is the Young modulus for steel and E_B the Young modulus for brass, show that

$$\frac{E_S}{E_B} = \frac{F_S}{F_B},$$

where F_S and F_B are the respective forces in the steel and brass wire.

We will let the cross-sectional areas of both wires be **A**, the lengths **L**, and the extensions be **x**.

We know that **E = FL/Ax**.

Thus, $\mathbf{E_s = F_sL / Ax}$.

$\mathbf{E_B = F_BL / Ax}$.

This means that $\mathbf{E_s / E_B}$ is $\mathbf{(F_sL/Ax) / (F_BL/Ax)}$.

Since the areas, lengths, and extensions of both wires are equal, we can cancel **A**, **L** and **x** to give:

$\mathbf{E_s / E_B = F_s / F_B}$.

(ii) The mass M produces a total force of 15N. Show that the magnitude of the force $F_S = 10\,N$.

the Young modulus for steel $= 2.0 \times 10^{11}$ Pa
the Young modulus for brass $= 1.0 \times 10^{11}$ Pa

The force is shared evenly between each mass, such that $\mathbf{F_{Total} = F_s + F_B}$.

Thus, $\mathbf{15 = F_s + F_B}$ => $\mathbf{F_B = 15 - F_s}$.

Plugging values into $\mathbf{E_s / E_B = F_s / F_B}$ gives $\mathbf{(2*10^{11}) / (1*10^{11}) = F_s / (15-F_s)}$.

$\mathbf{2 = F_s / (15-F_s)}$

$\mathbf{30 - 2F_s = F_s}$

$\mathbf{3F_s = 30}$ => $\mathbf{F_s = 10N}$.

Therefore $\mathbf{F_B = 5N}$.

-

iii) Determine the total elastic potential energy in the wires when the mass M produces a force of 15N. *The cross sectional-area of each wire is $8*10^{-8}m^2$ and the length of each wire is 1.6m.*

First, we can find the extension of each wire.

x = FL / AE.

For the steel wire, **x = 10(1.6) / ($8*10^{-8}$ * $2*10^{11}$) = $1*10^{-3}$m**.

For the brass wire, we can prove **x** must be the same. **x = 5(1.6) / ($8*10^{-8}$ * $1*10^{11}$) = $1*10^{-3}$m**.

The elastic potential energy in each wire is given by **1/2Fx**.

For the steel wire, **E_e = 1/2 (10)($1*10^{-3}$) = 0.005J**.

For the brass wire, **E_e = 1/2(5)($1*10^{-3}$) = 0.0025J**.

Thus, the total elastic potential energy in the wires is **0.0075J**.

-

Linking elastic potential energy and stress/strain

As seen with many of the previous examples, in any type of elastically-deforming material, we can apply the elastic potential energy equation **1/2Fx** to find the elastic potential energy stored in the material at a certain deformation.

Example:

A steel wire (E=200GPa) has a force of 100N exerted on it. The cross-sectional area of the wire is $0.00020m^2$, and its length is 2m. **Determine the elastic potential energy stored in the steel wire at its maximum extension**.

We know **x = FL/AE = 100(2)/($200*10^9$*0.00020) = $5*10^{-6}$m**.

Thus, **E_e = 1/2Fx = 1/2(100)($5*10^{-6}$) = $2.5*10^{-4}$J**.

Example:

A steel wire (E=200GPa) has a force **F** exerted on it such that its elastic potential energy at its maximum extension for that force is **$3*10^{-3}$J**. The steel wire has a natural length of 4.0m, and its cross-sectional area is $0.0050m^2$.

Determine the value of F.

We can say that **$3*10^{-3} = 1/2Fx$**.

We can use the data given to find a value for **x** in terms of **F**.

$x = FL/AE = F(4)/(0.0050(200*10^9)) = 4F / 10^9$.

Thus, **$3*10^{-3} = 1/2F(4F/10^9)$**

$6*10^{-3} = 4F^2 / 10^9$.

$4F^2 = 6*10^6$.

F = 1225N.

Example:

A steel wire (E=200Gpa) is connected in series with a wire with an unknown Young's Modulus. When a force of 100N is applied to the wires, the total elastic potential energy stored in both wires is **$3.5*10^{-3}$J**. The steel wire has a length of 2m, but is extended by 50μm when the force is applied. The wire of unknown Young's Modulus has a length of 3m, and a cross-sectional area of $0.0030m^2$. **Find the ratio of the Young's Modulus of steel to the unknown Young's Modulus**.

The total elastic potential energy stored in the wires is a sum of the elastic potential energy in the steel wire (**E_s**) and the elastic potential energy in the unknown wire (let this be **E_u**).

$3.5 * 10^{-3} = E_s + E_u$.

If we find **E_s**, this enables us to find **E_u**.

To find **E_s**, we need to know the extension of the steel wire. A force of 100N is applied to both wires since they are in series.

Thus, **$E_s = 1/2(100)(50*10^{-6}) = 2.5*10^{-3}$J**.

Thus, **$E_u = (3.5*10^{-3})-(2.5*10^{-3}) = 1*10^{-3}$J**.

We can use this to find the extension of the unknown wire:

$E_u = 1/2Fx$ => **$1*10^{-3} = 1/2(100)x$** => **$x = 2*10^{-5}$m**.

$E = FL/Ax = 100(3) / (0.0030*2*10^{-5}) = 5*10^9$Pa = 5Gpa.

The ratio of the Young's Modulus of steel to the unknown Young's Modulus is therefore **200 / 5 = 40**.

-

Example:

*Two wires - one steel wire (Young's Modulus **200GPa**) and one aluminium wire (Young's Modulus **69GPa**) are connected in parallel with each other. A force of **FN** is applied to the wires, such that the total elastic potential energy stored in the wires is **9J**. The force exerted in the steel wire is **10kN**. The steel wire has a length of **2.0m**, and a cross-sectional area of* $\boldsymbol{1*10^{-4}m^2}$*. The aluminium wire extends by **2mm** when the force is applied. **Calculate F**.*

Because the wires are connected in parallel, the total force exerted in the wires, **F**, is a sum of the forces in each wire. Thus, $\mathbf{F = F_s + F_A}$ where $\mathbf{F_s}$ and $\mathbf{F_A}$ are the forces exerted in the steel and aluminium wires respectively. Since $\mathbf{F_s = 10000N}$, $\mathbf{F = 10000 + F_A}$. To find **F**, we therefore need to find the force exerted in the aluminium wire.

Firstly, we have enough data to find the extension in the steel wire.

$\mathbf{x_s = FL / AE = 10000(2) / (1*10^{-4}*200*10^9) = 1*10^{-3}m}$.

This means that the elastic potential energy stored in the steel wire is $\mathbf{1/2(1*10^{-3})*10000 = 5J}$.

The total elastic potential energy in the wires is a sum of the elastic potential energy in both wires.

So, if the total energy stored in the wires is **9J**, and **5J** is stored in the steel wire, the aluminium wire has **4J** stored in it.

Thus, $\mathbf{4 = 1/2 * F_A * 2*10^{-3}} \Rightarrow \mathbf{F_A = 4000N}$.

The total force exerted in the wires is therefore **10000 + 4000 = 14000N = 14kN**.

-

Example:

*Wire A and wire B have the same force, **F**, applied to them. The wires are connected in parallel. The force is applied such that the force in each wire is **not** equal. Wire A has a Young's modulus of* $\boldsymbol{E_A}$*, and wire B has a Young's Modulus of* $\boldsymbol{E_B}$*. When the force is applied, the elastic potential energy stored in wire A is **3 times** as much as the elastic potential energy in wire B, and the extension of A is two times that of B. Wire A has a diameter that is four times less than wire B, and a length that is half that of wire B. **Find the ratio of*** $\boldsymbol{E_A}$ ***to*** $\boldsymbol{E_B}$*.*

Firstly, we can consider the force in each wire (let this be $\mathbf{F_A}$ and $\mathbf{F_B}$ for wire A and wire B respectively).

*The total force, **F**, exerted on the wires must be shared between A and B, but not equally, as stated in the question:* $F = F_A + F_B$.

We will let the length of wire A be **L** and its area **A**. The extension of wire A will be **x**.

Since **E = FL/Ax**, we can say $E_A = F_AL / Ax$.

For B, the length is double - **2L**, and since its diameter is four times greater than A, its cross sectional area is **16 times greater** - **16A**. The extension of B must be **1/2x**.

Thus, $E_B = F_B2L / 16A(1/2x) = F_B2L / 8Ax = F_BL / 4Ax$.

Currently, it is difficult to find the ratio of E_A and E_B as the equation has 2 unknowns - F_A and F_B. We can use the elastic potential energy to express F_A in terms of F_B.

We will let the elastic potential energy stored in A be **e**.

$e = 1/2\ F_Ax$.

If the elastic potential energy in B is 3 times less than A, the elastic potential energy in B is **1/3e**.

Thus, $1/3e = 1/2F_B(1/2x) = 1/4F_Bx$.

If we multiply this equation by **3**, we get $e = 3/4F_Bx$.

We now have two expressions for **e** that we can equate: $3/4F_Bx = 1/2F_Ax \Rightarrow 3/4F_B = 1/2F_A$.

Thus, $F_A = 3/2F_B$.

We can now substitute this into our equation for E_A, $E_A = F_AL / Ax$, to give $E_A = 3/2F_BL / Ax = 3F_BL / 2Ax$.

Thus, we can now find the ratio of E_A to E_B:

$$E_A = \frac{3F_B L}{2Ax}$$

$$E_B = \frac{F_B L}{4Ax}$$

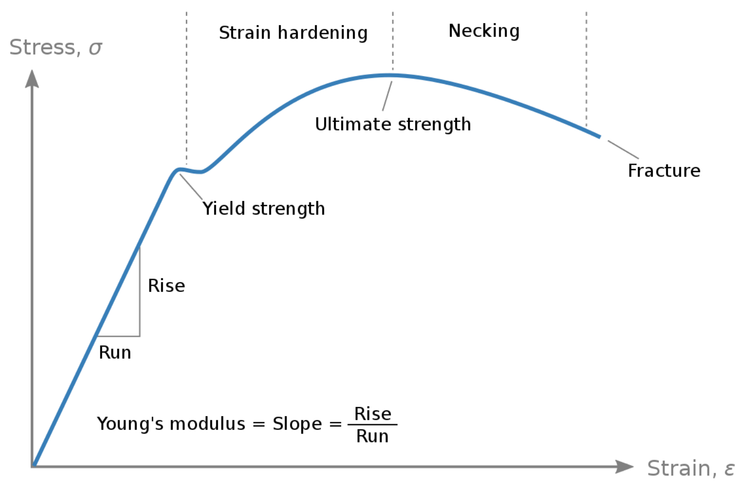
$$\Rightarrow \quad E_A / E_B = \frac{\left(\frac{3F_B L}{2Ax}\right)}{\left(\frac{F_B L}{4Ax}\right)} = \frac{(3/2)}{(1/4)} = 12/2 = 6.$$

$$\therefore E_A = 6E_B$$

--

Stress-strain curves

A material can be deformed, and the stress applied (**force/area**) can be plotted against the strain (**extension/length**). This gives a **stress-strain curve** for the material:



A stress-strain curve will always have a linear region where stress and strain are directly proportional. This region represents when the material is deforming **elastically**.

When a material deforms elastically, it deforms according to Hooke's Law: **F** ∝ **x**. Because **stress** and **strain** are in terms of force and extension (and cross-sectional area / length remain constant), we can say that for an elastic material, **stress** ∝ **strain** - hence explaining why the linear region represents elastic deformation.

When a material deforms elastically, it snaps back to its original shape when stress is relieved. All materials have an **elastic limit** - a point beyond which they no longer continue to deform elastically, and therefore no longer obey Hooke's Law. A material that has exceeded its elastic limit deforms **plastically**. Plastic deformation is when a material *will not* return to its original shape when the force is relieved.

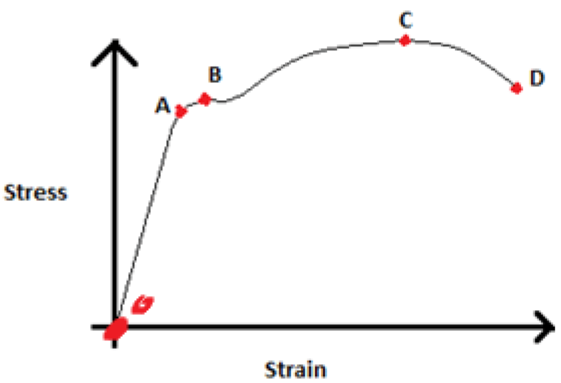
Different types of material behave differently during plastic deformation. Link to material science document.

-

Analysis of stress-strain curves

The gradient of a stress-strain curve is equal to **stress / strain**. This is equal to Young's Modulus. Therefore, the gradient of a stress-strain curve gives the Young's Modulus of a material.

However, Young's Modulus only applies to elastic deformation, so only the gradient of the straight line representing elastic deformation can be used to find the Young's Modulus of a material.



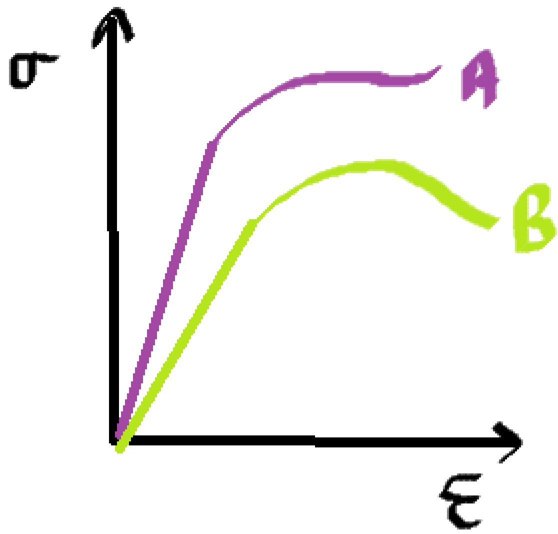
*Only the gradient in the region **OA** can be used to find the Young's Modulus of a material.*

Because the Young's Modulus is constant for a given material, no matter the dimensions of an object, the gradient of the stress-strain curve will always be the same.

For example, if you measure the stress and strain of a steel bar of diameter 0.2m, and a steel bar (same type of steel) of diameter 0.5m, the gradient of the stress-strain curve in the elastic region will be the same since it is the same material.

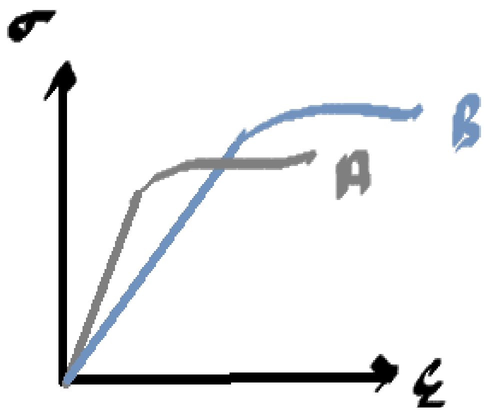
-

The greater the Young's Modulus of a material, the steeper the stress-strain curve.



The above graph shows how material A requires a greater stress to produce the same strain as material B. This means that A is stiffer - i.e. it has a higher Young's Modulus.

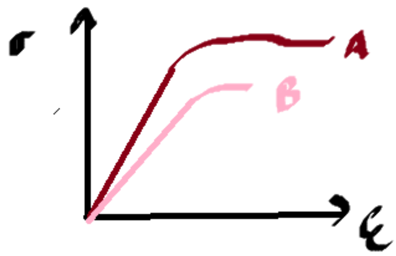
Although the Young's Modulus of one material may be higher than the other, it does not have to be *stronger* by definition.



In the above graph, material **A** has a higher Young's Modulus than material **B**. However, you can see that it requires a higher amount of stress for material **B** to reach its elastic limit. So, although material **A** is stiffer and extends less under the same stress, it cannot withstand as much stress as material **B** before deforming plastically. We say that material B is **more elastic** than material A, because it deforms elastically at higher stresses.

Ductility is a measure of how easily a material stretches, or how much a material stretches. The ductile properties of a metal are primarily shown in the plastic deformation region of the stress-strain graph. Metals have a **yield point** in the plastic deformation region, where a small amount of stress causes a significant increase in strain (i.e. a significant increase in extension).

If a metal is more ductile, its strain will increase to a higher value during the yield point (higher strain = more extension):



Material A is more ductile than material B as its strain increases very readily in the plastic deformation region. B has low ductility because a small increase in stress beyond the elastic limit causes the material to fracture.

-

Now let us consider the area under a stress-strain curve.

The area under a stress-strain curve tells us the **work done per unit volume** up to that point in the deformation.

Why?

The area under a stress-strain curve is the product of the stress and the strain: **stress * strain**.

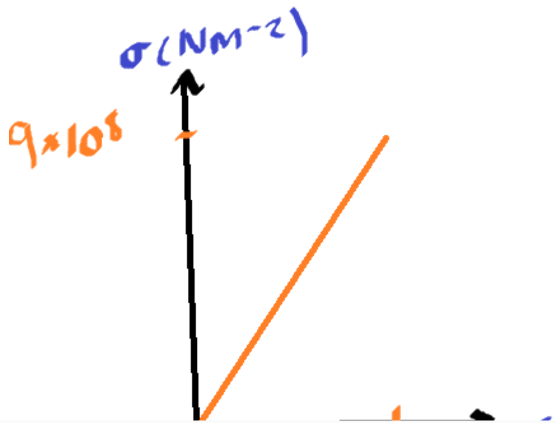
Since **stress = F/A** and **strain = x/L**, **area = F/A * x/L = Fx/AL**.

When a force, **F**, is applied over an extension, **x**, this is work done. Thus, **Fx = work**.

An area, **A**, multiplied by a length, **L**, is volume, **V**. **V = AL**.

Thus, **area = work / volume = work per unit volume**.

-



For example, in the above graph, the total stress applied to increase the strain (the ratio of extension and length) of a material to **0.02** is **$9*10^8Nm^{-2}$**.

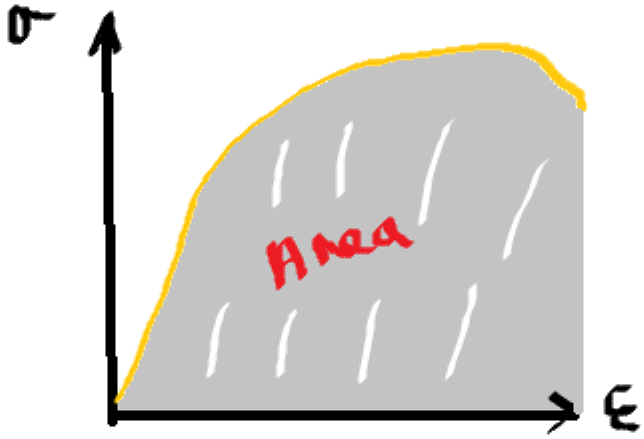
The area under the graph up to this point is **$1/2 * 9 * 10^8 * 0.02 = 9*10^6Jm^{-3}$**.

This means that **9MJ** of work is done to each **m^3** of volume in the material to cause this deformation.

Suppose we knew the volume of the material to be **$12.5m^3$**. If **9MJ** of work is done to each **m^3**, the total work done to the material is **12.5(9) = 112.5MJ**.

-

Now suppose we consider the area under the graph up to the fracture point:



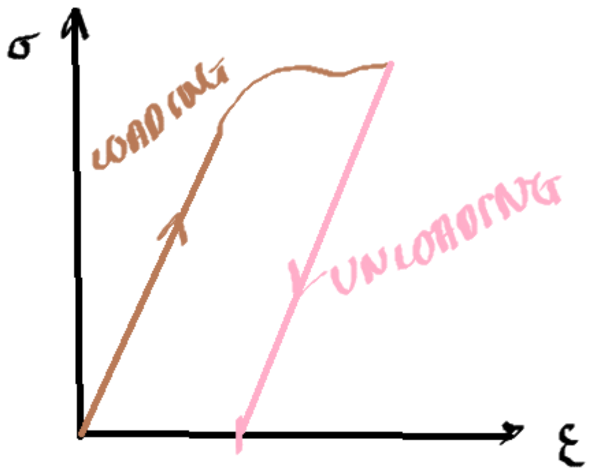
This tells us the total work that must be inputted to each **m^3** on the material to cause it to break. This is a quantity known as **toughness**.

The higher the area under the graph, the greater the energy that can be stored during deformation. The material therefore has a higher toughness.

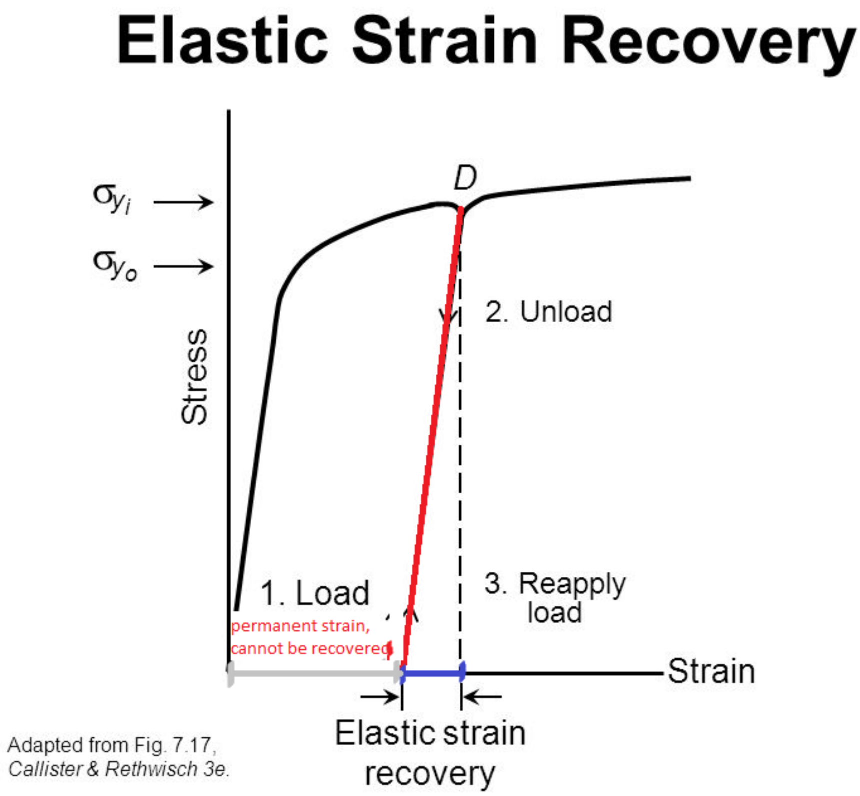
Unloading

Suppose we deform a material to its elastic limit, and beyond this point. We then unload the force being applied to the material incrementally, and measure the strain of the material.

Because the material has exceeded its elastic limit, it will never return to its original shape. When the stress is reduced to **0**, the strain (ratio of extension to length) will not be **0**, because the material is not in its original shape:

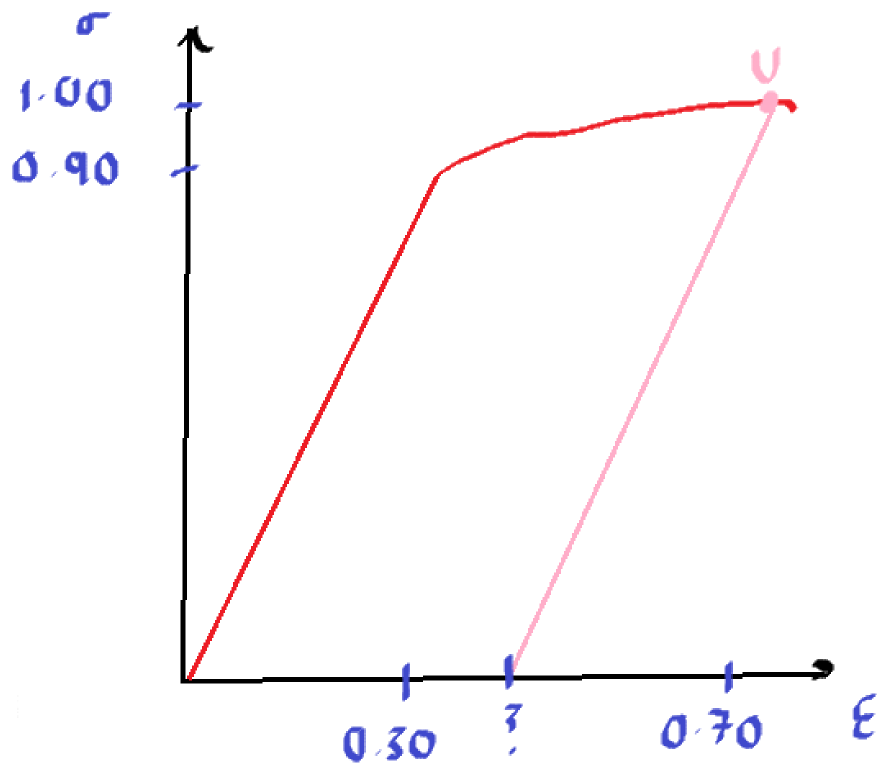


The material 'recovers' the elastic part of its deformation. The material has been deformed beyond its elastic limit, and therefore it has some *permanent strain* in it that cannot be removed. It can, however, recover the elastic part of its deformation.



The gradient of the unloading line must be equal to the gradient of the loading line during elastic deformation.

Take the following graph:



The material is deformed beyond its elastic limit, and then at the point **U**, where strain = 0.70, the material is unloaded.

Suppose we want to find the strain of the material when it is fully unloaded.

We know that the material recovers the strain that it had during the elastic deformation period, and therefore the gradient of the unloading line is equal to the gradient of the loading line.

The gradient of the loading line up to the elastic limit is **0.90 / 0.30 = 3**.

Using the point **(0.70,1.00)** on the loading line, we can form an equation for the straight line in the form **y=mx+c**.

1.00 = 3(0.70)+c => **c = 1 - 2.10 = -1.10**.

y = 3x - 1.10.

When the material is fully unloaded, the unloading line returns to the y-axis, so stress = 0.

Thus, **0 = 3x-1.10** => **x = 1.10 / 3 = 0.367**.

The strain when fully unloaded is **0.367**.

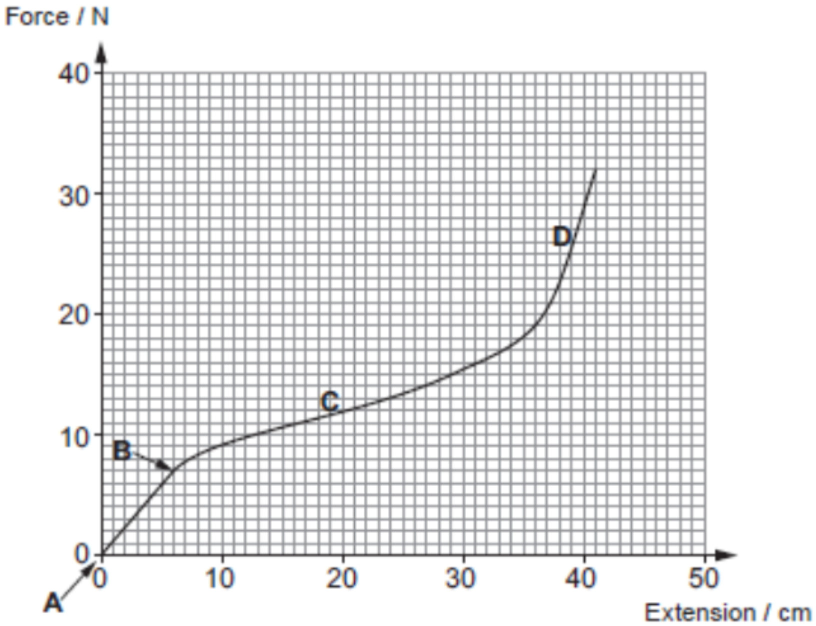
-

Suppose a material is loaded beyond its elastic limit, then unloaded such that it does not return to its original shape. If the material is loaded again, the new limit of proportionality is the unloading point from the first loading. For example, referring to the graph above, if the material is re-loaded from a strain of **0.367**, its new limit of proportionality will be the point **U**. This *does not mean* that the material deforms elastically up to the point U, then starts deforming plastically, because it has already exceeded the elastic limit. The limit of proportionality and the elastic limit are different measurements; a plastic material can still deform proportionally.

Stress-strain curve and general deformation questions

1

The results from such an experiment for a rubber band of unstretched length 8.0 cm are plotted in a graph.



(i) Calculate the strain in the rubber at point B. [1]

Strain = x / L.

At point B, extension = **6cm**.

If **L = 8cm**, **strain = 6 / 8 = 0.75**.

(ii) Determine the Young modulus of the rubber in the region AB. Assume the band has a total cross-sectional area of 0.050 cm^2. [3]

Young's Modulus is given by **E = FL / Ax**.

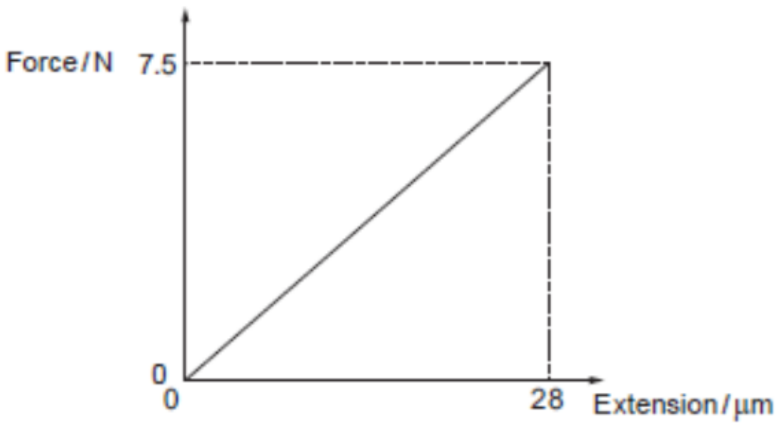
The gradient of this graph is **F / x**. Thus, **E = gradient * L/A**.

The gradient of the graph in the region **AB** is **7/6Ncm^{-1}**.

Thus, **E = 7/6 * (8/0.050) = 187Ncm^{-2}** or **18700Nm^{-2}**.

2

Emily carries out an experiment to obtain a force-extension graph for a thin glass fibre. She loads the thin glass fibre until it breaks. The force-extension graph obtained is shown below.



(a) (i) Emily measures the length and diameter of the glass fibre and finds them to be 19.8 cm and 1.01 mm respectively. Suggest what measuring instruments she uses. [1]

The measurement of length is likely to have been made with a ruler, and the measurement of diameter with Vernier Calipers.

(ii) Determine the Young modulus of glass. [3]

Again, **E = FL / Ax = gradient * L/A**.

If we convert all measurements to metres:

Length = 19.8cm = 0.198m.

Diameter = 1.01mm = $1.01*10^{-3}$m => **area = $\pi(1/2(1.01*10^{-3}))^2 = 8.01*10^{-7}m^2$**

28μm = $28*10^{-6}$m => **gradient = $7.5 / (28*10^{-6}) = 2.68*10^5 Nm^{-1}$**.

Thus, **E = $2.68*10^5 * (0.198 / (8.01*10^{-7}m^2)) = 6.6*10^{10}$Pa**.

-

(iv) Determine the energy stored in the glass just before breaking point is reached. [2]

The energy in the glass is **$1/2Fx = 1/2(7.5)(28*10^{-6}) = 1.05*10^{-4}$J**.

-

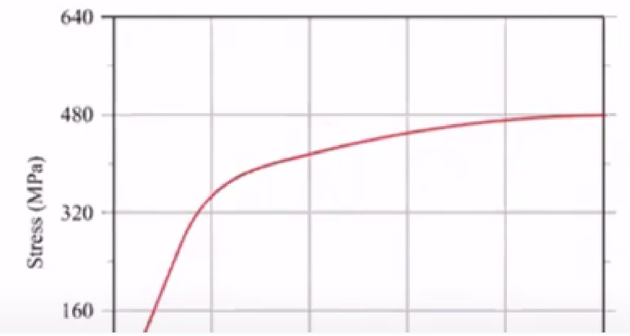
(b) Glass is a brittle material. Describe the process by which glass fractures. [3]

When glass is under tension, due to its low tensile strength and surface imperfections, cracks can form. A lot of stress is concentrated in the cracked region, so the crack propagates and fractures the glass.

-

3

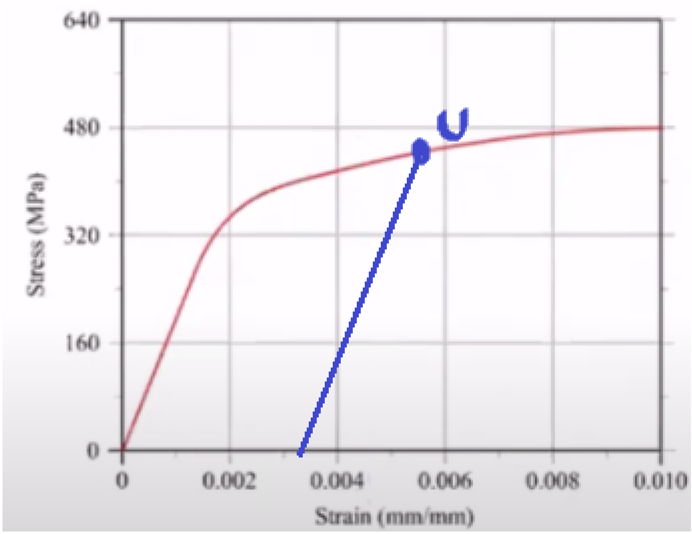
A portion of the stress-strain curve for a stainless steel alloy is shown below. A 350-mm-long bar is loaded in tension until it elongates 2.0 mm and then the load is removed.



What will the length be (approximately) when the bar is fully unloaded after being loaded as described above?

The bar is loaded until it elongates by **2.0mm**. We can find the strain in the bar at this length by considering that **strain = x / L = 2 / 350 = 5.71*10^{-3}**.

The point when the material is unloaded is marked as **U** on the graph:



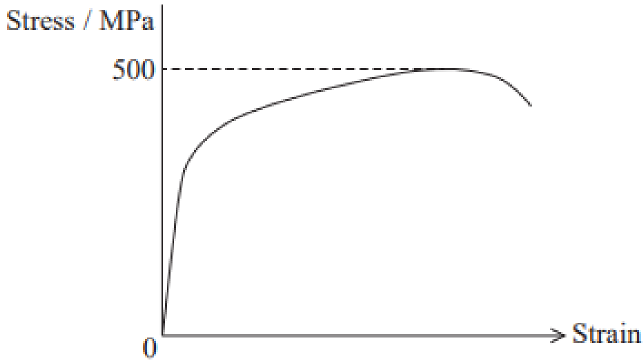
When the material is unloaded, it recovers its elastic deformation, such that the gradient of the unloading line is equal to the gradient of the loading line. We can draw a line from **U** to the y-axis with the same gradient as the loading line.

As an approximation, this means that the stress of the bar when fully unloaded is around **0.0033**.

Thus, if its natural length is **350mm**, **0.0033 = x / 350** => **x = 1.2mm**. The length of the bar when fully unloaded is therefore **350 + 1.2 = 351.2mm**.

We can also calculate the gradient of the graph to find a slightly more accurate value, but it will not be more accurate in this context because of the nature of the graph representation.

(b) The stress-strain graph for a sample of brass is shown.



The typical stress when turning a key in a lock is about 10 MPa.

Use information from the graph to suggest why brass is a suitable material for use in keys.

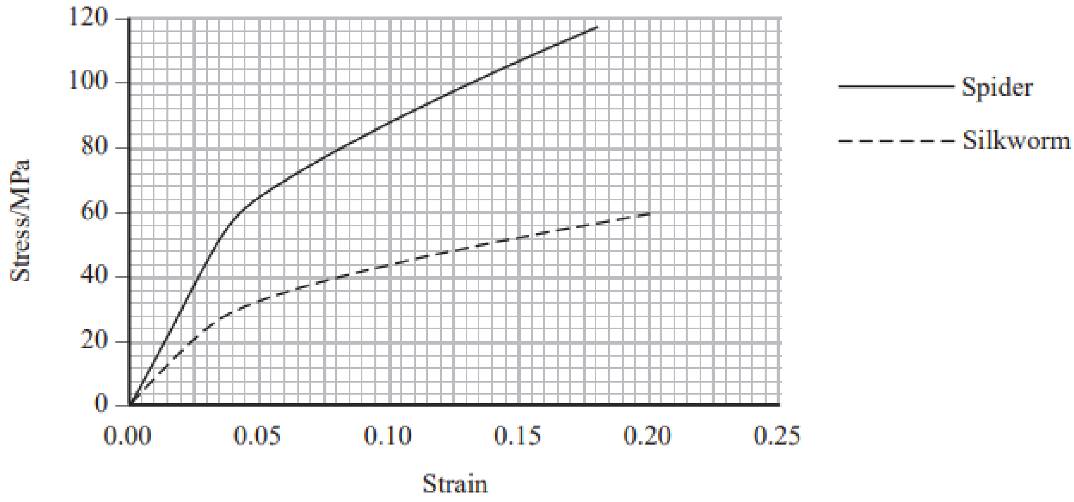
(4)

-It is important that a key only deforms elastically when placed into a lock, as it must return to its original shape to be re-usable. The stress applied when turning a key in the lock is far within the elastic limit of brass, so there will be no plastic deformation when using the key. The graph also has a high gradient in the elastic region, indicating a high Young's Modulus. This is important for keys as it means that they are stiff; a large stress is required to produce a small strain.

-

5

Silk is a natural protein fibre produced by spiders and silkworms. It is a material of high tensile strength. The graph gives the stress-strain curves, up to the point of fracture, for silk produced by spiders and by silkworms.



Spiders use silk to build webs to catch insects. Use the graph to explain how the properties of spider silk make it more suitable than silkworm silk for building webs to catch insects.

(4)

Firstly, the spider silk has a higher elastic limit than silkworm silk (i.e. it is more elastic). This is useful because it means that a web made of spider silk will require a greater amount of stress to plastically deform, so it will retain its shape when an insect collides with it. Spider silk also has a higher Young's Modulus than silkworm - so it is stiffer. This is useful as an insect colliding with a spider silk net will decelerate in less time, leading to a greater and more damaging force being exerted. Spider silk has a higher UTS than silkworm silk, so it will require more stress to completely fracture a spider silk net. The area under the graph for spider silk is also greater, so it can absorb more energy before fracturing - which is useful for reducing the kinetic energy of insects.

-

(b) (i) Use the graph to determine the Young modulus of spider silk for small stresses.

(2)

E = gradient in elastic region = **52 / 0.03 = 1730MPa = 1.73GPa**.

ii) An insect flies into a spider's web and becomes attached to a single thread. This creates a tension in the thread of 580 μN. The thread extends by approximately 3% of the original length.

Calculate the radius of a single thread of spider silk.

(4)

Strain is the ratio of extension to length, which is the percentage by which a material extends relative to its initial length. In the case of the spider silk, if the extension increases by **3%** above the natural length, the strain in the silk must be **0.03**.

E.g. if extension = 3m, and original length = 100m, strain = 3 / 100 = 0.03. An extension of 3m is 3% of the original length.

From the graph, when **strain = 0.03**, **stress = 44MPa**.

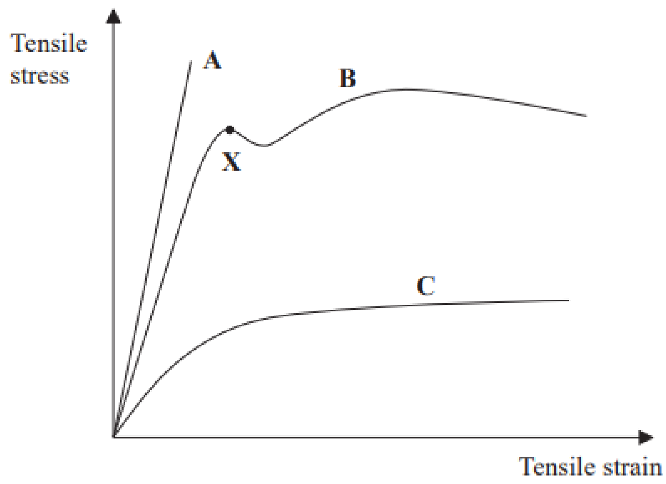
We know that **stress = force / area**. Thus, **area = force / stress = $(580 \times 10^{-6})/(44 \times 10^{6})$ = $1.32 \times 10^{-11} m^2$**.

Assuming the thread is cylindrical, **$1.32 \times 10^{-11} = \pi r^2$** => **$r = 2.05 \times 10^{-6} m$ = 2.05μm**.

-

6

7 The graph shows the stress-strain curves for three materials A, B, and C up to the point of fracture.



(a) (i) Identify which of the materials A, B or C is

(4)

a brittle material

a ductile material

the strongest material

the least stiff material

A is brittle because it does not have a yield point like B or C; if stress is applied beyond the elastic limit, it fractures.

B is the most ductile because its maximum tensile strain is the greatest.

A is the strongest material because it fractures at the highest tensile stress.

C is the least stiff material due to the low gradient of the graph in the elastic deformation region.

-

(b) Explain why steel is a suitable material for making paper clips.

(3)

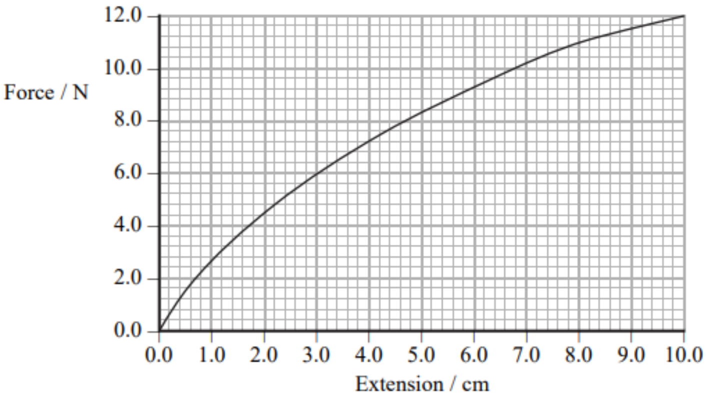
It is important that paper clips retain their shape. Steel is therefore a good material for paper clips as it has a high elastic limit; a lot of stress is required to induce a permanent deformation. This means that a steel paper clip will return to its original shape after being opened. Steel also has a high Young's Modulus; it is very stiff, which will allow a steel paper clip to firmly hold paper in-place. Moreover, steel is ductile and non-brittle, so it can easily be drawn out into a paper clip shape.

-

7

***3** The photograph shows a long rubber band being used to launch a model aeroplane.

The following graph shows force against extension for the rubber band.



(a) Explain whether the rubber band obeys Hooke's law.

(2)

The rubber band does not obey Hooke's Law because force and extension are not directly proportional to each other.

The rubber band is extended by 10.0 cm before being released to launch the aeroplane. Calculate the maximum possible initial speed of the aeroplane.

Mass of aeroplane = 0.027 kg

(3)

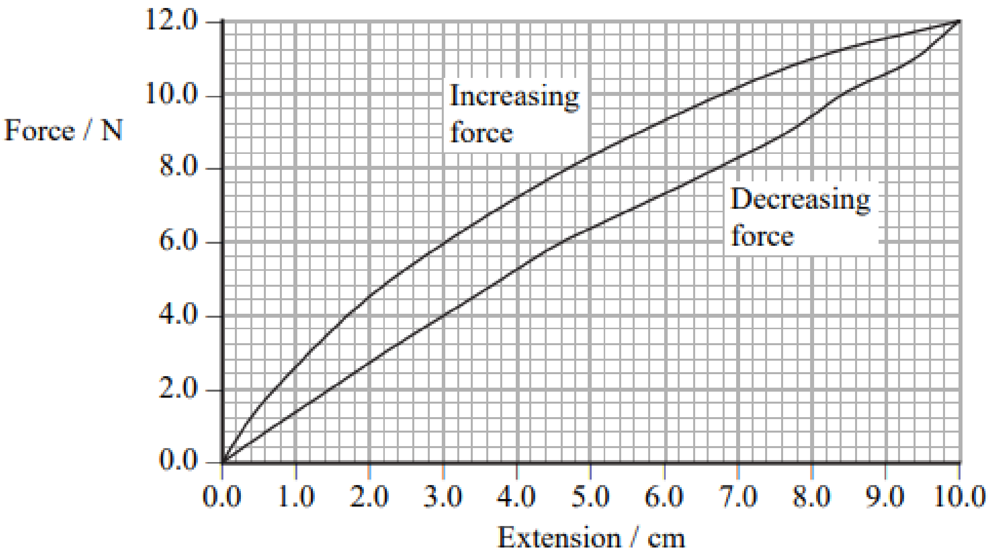
The elastic potential energy in the rubber band is equal to the area under the force-extension curve. When the aeroplane is released, this elastic potential energy is transferred to kinetic energy in the aeroplane.

The are under the curve is approximately **1/2 * 10 * 0.12 = 0.6J** (remember to convert cm to m).

Thus, **0.6 = 1/2 * 0.027 *** $\mathbf{v^2}$ => **v = 6.7ms**$^{-1}$

-

(d) The following graph shows two lines. Measurements were obtained by increasing the force on the band to 12 N and then decreasing the force.



(ii) The maximum speed of the aeroplane will be less than that calculated in (c). Without further calculation use the graph to explain this.

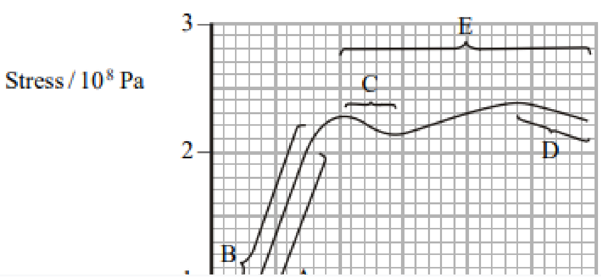
(3)

The total elastic potential energy stored in the rubber band during extension is equal to the area under the loading curve. The total elastic potential energy released when the rubber band is released is equal to the area under the unloading curve. The area under the unloading curve is lower than the area under the loading curve (hysteresis), so we cannot assume that all of the elastic potential energy gain during loading is transferred as kinetic energy to the aeroplane; some energy is transferred as heat energy.

-

8

The graph shows the behaviour of a copper alloy when it is stressed.



Elastic deformation - region **B** (region **A** is proportional deformation; a material can continue to extend elastically beyond this).

Necking - region **D**.

Plastic flow - region **E**.

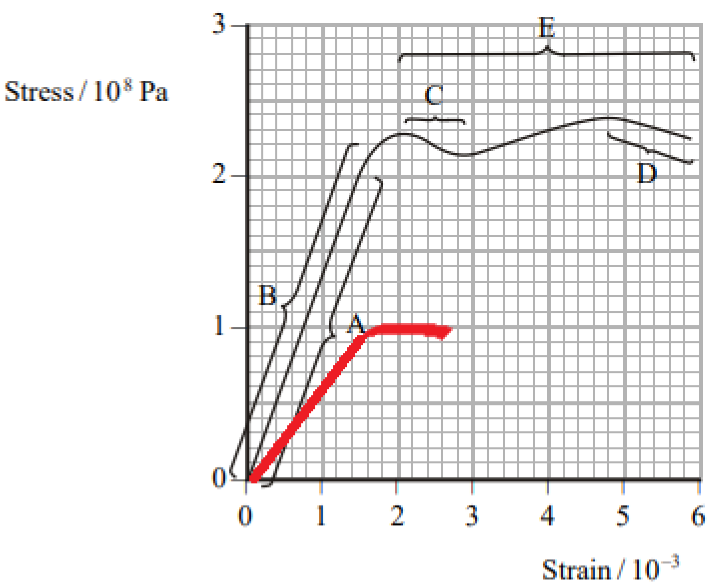
-

(ii) Calculate the Young modulus of the copper alloy.

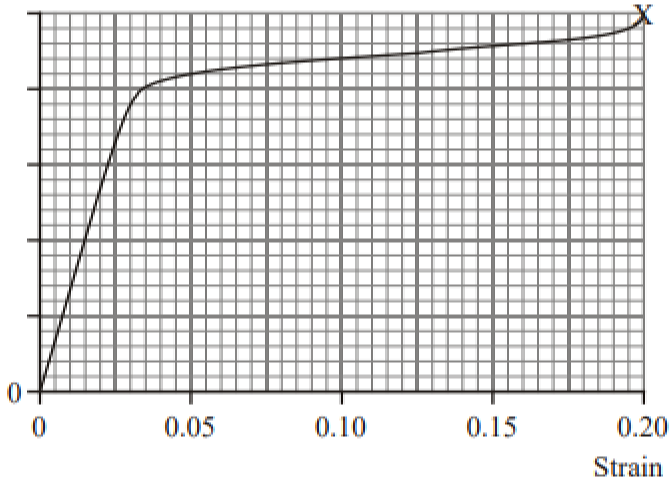
Young modulus = **gradient in region A = $(2*10^8)$ / $(1.6*10^{-3})$ = $1.25*10^{11}$Pa**.

Add a second graph to the axes above to show the behaviour of a brittle material with a Young modulus lower than that of the copper alloy and which fractures at a strain of 2.6×10^{-3}.

(3)



A copper wire is stretched in an experiment. The graph shows the behaviour of the copper until it breaks at point X.

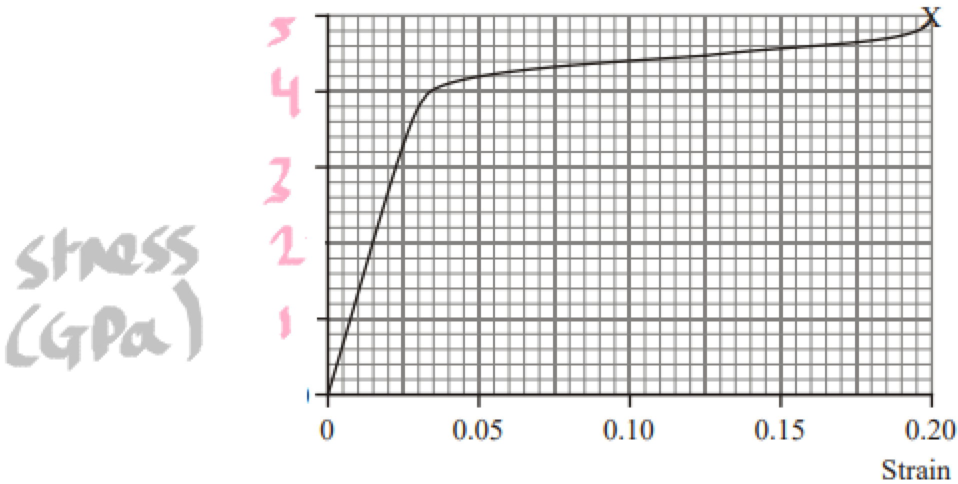


The Young modulus of copper is 130 GPa. By using an appropriate calculation add a suitable scale to the *y*-axis.

The gradient of the graph in the elastic region gives us the Young's Modulus.

When **strain = 0.025**, **stress / 0.025 = 130** => **stress = 3.25GPa**.

This is the 16th square up, so each square represents **3.25 / 16 = 0.203**, which is approximately **0.2GPa**.



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The volume of the copper wire is 3.8×10^{-7} m^3. Calculate the work done on this wire in the experiment.

The total area under the graph is the work done per unit volume.

The area under the graph is approximately **(1/2 * 0.05 * 4.2) + (1/2(4.2 + 4.4) * 0.05) + (1/2(4.3+5) * 0.1) = 0.785GJm^{-3}**.

Thus, the overall work is **0.785 * 3.8 * 10^{-7} = 2.98*10^{-7}GJ = 298J**.

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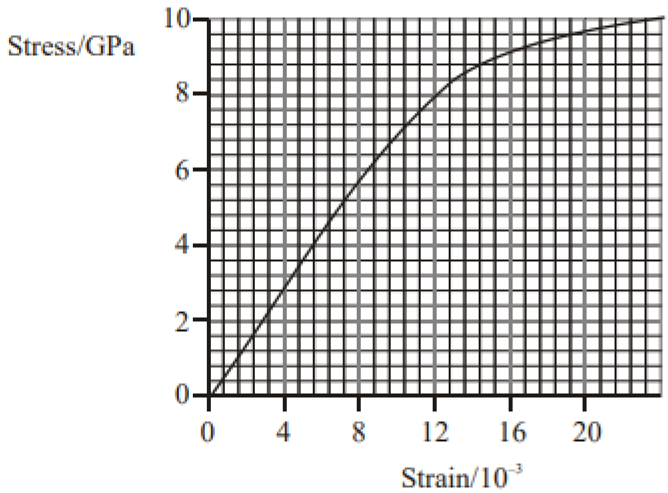
The graph shows the stress-strain relationship for a material from which car seat belts can be made.



A car seat belt is **2m long**, **6cm wide** and **1.5mm thick**. A car is driving at a velocity of **vms^{-1}** with a passenger of mass **65kg** sitting in the car seat. The car suddenly breaks, reducing its velocity to **0ms^{-1}**. This causes the stress in the seat belt to increase to **7.2GPa**. **Calculate the value of v, stating any assumptions made**.

The person and the car are moving at the same velocity, so when the car brakes, the person's kinetic energy reduces to **0J**. Energy cannot be transferred without a force being exerted. This force is exerted by the seat belt. The force exerted causes the seatbelt to extend, so that its elastic potential energy increases. We can assume that all of the loss in kinetic energy of the person is transferred as elastic potential energy to the seat belt.

We can calculate the area under the graph when **stress = 7GPa**, and multiply it by the volume. This tells us the elastic potential energy gain in the seatbelt when the person comes to rest.

The volume of the seatbelt is **2(0.06)(1.5*10^{-3}) = 1.8*10^{-4}m^3**.

When **stress = 7.2Gpa**, **strain = 10.4*10^{-3}**.

The area under the graph is therefore **1/2 * 7.2*10^9 * 10.4*10^{-3} = 3.744*10^{7}Jm^{-3}**.

The total work done is therefore **3.744*10^7*1.8*10^{-4} = 6740J**.

This is the gain in elastic potential energy of the seat belt, which must equal the loss in kinetic energy of the person assuming no frictional forces.

Thus, **6740 = 1/2 * 65 * v^2** => **v = 14.4ms^{-1}**.

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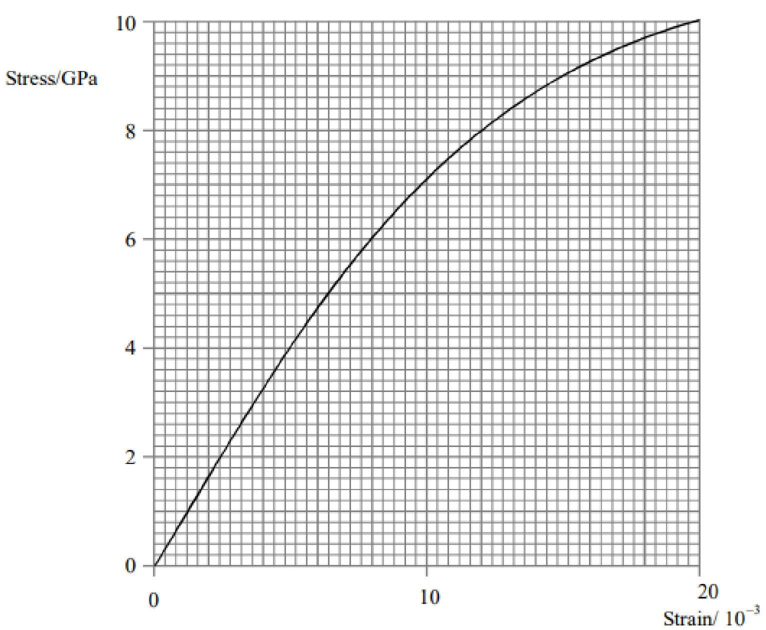
The graph shows how stress varies with strain for the seat belt material.



A car is moving at **20ms^{-1}** and there is a passenger of mass **60kg** in the car. The car suddenly stops and the strain in the seat belt increases to the maximum value shown on the graph. **If the cross-sectional area of the seatbelt is 5*10^{-5}m^2, calculate the length of seat belt required to bring the passenger to rest**.

The passenger has a kinetic energy of **1/2 * 20^2 * 60 = 12000J**. This means that the total work done by the seat belt to stop the passenger must be **12000J** - that is, it must gain **12000J** of elastic potential energy.

The area under a stress-strain graph tells us the work per unit volume.

Thus, **area = work / volume** => **volume = work / area**.

If we find the **area** under the curve when the seat belt is at its maximum strain (**$20*10^{-3}$**), we can then find the required volume of seat belt.

When strain = **$20*10^{-3}$**, **stress = 10GPa**. The approximate area under the graph up to this strain is **$1/2 * 20*10^{-3} * (10*10^{9}) = 1*10^{8}$J**.

Thus, **volume = $12000 / (1*10^{8}) = 1.2*10^{-4}m^{3}$**.

Volume = CSA * **length** => **$1.2 * 10^{-4} = 0.00005 * L$** => **L = 2.4m**.

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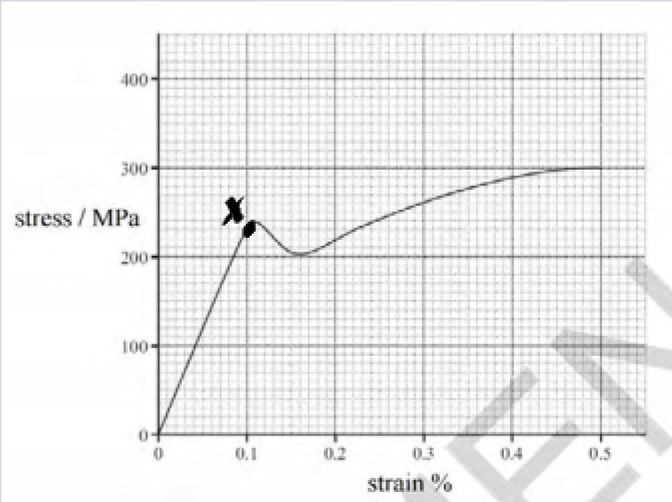
Fig.5.2 shows a stress-strain graph of the same material, obtained from a tensile testing machine.



(d)* Steel is an alloy. Its main constituent is iron. Using ideas about dislocations and metallic structure explain why the steel first shows the elastic behaviour (up to point X) and then shows plastic behaviour (beyond point X). Elaine how the presence of atoms other than iron makes the resulting metal harder and less plastic than pure iron.

The structure of steel is crystalline, and consists of positive ions surrounded by a sea of delocalised electrons. The introduction of foreign atoms into steel reduces the regularity of the bond lengths in steel, so the structure becomes less uniform.

When steel deforms elastically, the extension of the bonds between the atoms is proportional to the force applied (it behaves as a Hookean material). When the force applied is relieved, the bond length increases to its initial value - there is no permanent strain.

When steel forms, the atoms are deposited in planes. However, some planes can be deposited as half-planes. Such planes are known as edge dislocations. The bonds around the edge dislocation are under increased stress, so when the stress in the steel is increased to a certain value, the bonds break, which causes the edge dislocation to migrate. This permanently alters the structure of the steel - so it now deforms plastically. As the edge dislocation migrates, it relieves the stress applied to the structure and allows the steel to have a large increase in strain with a small increase in stress. The presence of foreign atoms in steel reduces the ability of edge dislocations to migrate (edge dislocations are ‘pinned down’), which consequently reduces the plastic deformation of the steel.

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