# Multidimensional Integration - Short

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There are many ways to integrate in multiple dimensions, first we look at line integrals.

A line integral finds the integral through a path in a vector field. We define the path by parameterising it.

$$I = \int_C \boldsymbol{G}(\boldsymbol{r}) \cdot d\boldsymbol{r}$$

Where $\boldsymbol{G}$ is the vector field, C is the whole path, $\boldsymbol{r}$ is the path, and $d\boldsymbol{r}$ is each infinitesimal segment of the path.

When a path is given, as well as the start and end point, we parameterise it such that:

$$\boldsymbol{r}(x, y) \rightarrow x(t), y(t) \rightarrow \boldsymbol{r}(t)$$

We can therefore rewrite the integral:

$$I = \int_{t_0}^{t_1} \boldsymbol{G}(t) \frac{d\boldsymbol{r}}{dt} \cdot dt$$

An area integral finds the sum of all infinitesimal areas of the total area.

An area integral is defined by:

$$I = \int_A f(x, y) dA$$

Each area $dA$ is equal to $dx\, dy$ and so:

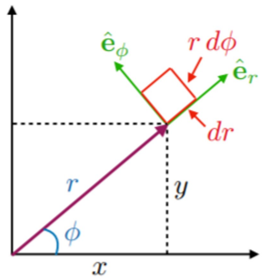
$$I = \int_{x_0}^{x_1} dx \int_{y_0}^{y_1} dy\, f(x, y)$$

Which is found by integrating the function over one variable while the other stays constant, and repeating till all variables have been integrated over.

If the area isn't a rectangular area, then the bounds of each integral will have to be defined in terms of the other.

This can get complicated quickly, and so other coordinates systems can be used for ease.

Polar coordinates are another coordinate system used. Integrating in polar or cartesian will give the same result, but it is easier for some examples to use polar coordinates




In polar coordinates we define $f(r, \phi)$

Where r is the distance from the origin, and $\phi$, is the angle that the straight line r makes with the x axis.

From polar to cartesian:

$$x = r\cos(\phi), y = r\sin(\phi)$$

In polar coordinates, an infinitesimal area $dA$ is equal to $dr * r d\phi$

This means our area integral can be found:

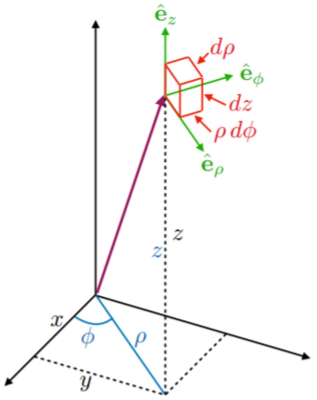
$$I = \int_A f(r, \phi) dA \rightarrow I = \int_{r_0}^{r_1} dr \int_{\phi_0}^{\phi_1} d\phi \;\; r \cdot f(r, \phi)$$

We can also integrate over 3 dimensions, allowing us to find volumes.

$$I = \int_V f(x, y, z) dV = \int_{x_0}^{x_1} dx \int_{y_0}^{y_1} dy \int_{z_0}^{z_1} dz\;\, f(x, y, z)$$

As with area integrals, for any volume that isn't a cuboid, the boundaries will be defined in terms of each other. We can simplify by using other coordinates systems.

In 3 dimensional space we can define cylindrical coordinates




Cylindrical coordinates are a 3D representation, which is basically a polar coordinates system in the $x, y$ plane with a z value. We define $f(\rho, \phi, z)$

where $\rho$ is the distance between the point and the z axis ($\rho = \sqrt{x^2 + y^2}$), $\phi$ is the angle between the line from the origin to the $x, y$ position of the point and the x axis, and z is simply the same z coordinate as used in cartesian

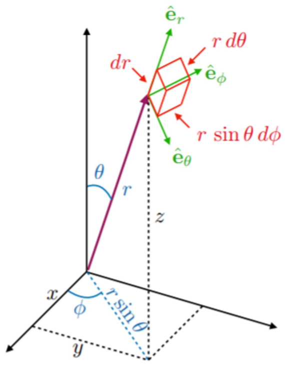
From cylindrical to cartesian:

$$x = \rho\cos(\phi), y = \rho\sin(\phi), z = z$$

In this system, an infinitesimal volume $dV$ can be written as:

$$dV = \rho d\phi \cdot d\rho \cdot dz$$

We can also use spherical coordinates:




In spherical coordinates we define $f(r, \theta, \phi)$, where $r$ is the distance between the point and the origin $\left(r = \sqrt{x^2 + y^2 + z^2}\right)$, $\theta$ is the angle between the point and the z axis, and $\phi$ is the angle between the $x, y$ coordinate of the point and the x axis.

From spherical to cartesian:

$$x = rsin(\theta)\cos(\phi), y = rsin(\theta)\sin(\phi), z = rcos(\theta)$$

In this system, an infinitesimal volume $dV$ can be written as:

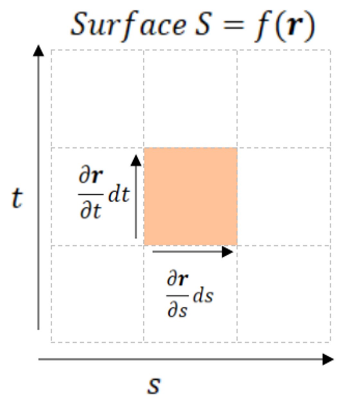
$$dV = dr \cdot rd\theta \cdot rsin(\theta)d\phi = r^2 dr \sin(\theta)\, d\theta\, d\phi$$

In 3 dimensions we often wish to find the area of a surface, we do this using surface integrals.

We define $dS$ as the infinesimal devisions of area of the surface $S$ such that:

$$I = \int_S f(\boldsymbol{r})\, dS$$

To find $dS$, we paramaterise $\boldsymbol{r} = \boldsymbol{r}(s, t)$




Because the surface is in 3 dimensional space and is often curved, the area $dS$ may be a parallelogram. This means the area $dS$ can be found by:

$$dS = \left|\frac{\partial \boldsymbol{r}}{\partial t} \times \frac{\partial \boldsymbol{r}}{\partial s}\right| dt\, ds$$

And so:

$$I = \int_{s_0}^{s_1} ds \int_{t_0}^{t_1} dt\, f(s,t) \left|\frac{\partial \boldsymbol{r}}{\partial t} \times \frac{\partial \boldsymbol{r}}{\partial s}\right|$$

We may want to find only the component of each surface perpendicular to the field, such as when finding flux of a vector field through a surface.

$$I = \int_S (\boldsymbol{G}(\boldsymbol{r}) \cdot \hat{\boldsymbol{n}}) dS = \int_S \boldsymbol{G}(\boldsymbol{r}) \cdot d\boldsymbol{S}$$

As before, we parametrise the surface, and find the area of $dS$ by using the cross product. However, because we only want the component normal to the surface, we do not take the modulus of the area, leaving it as a vector:

$$d\boldsymbol{S} = \hat{\boldsymbol{n}}\, dS = \hat{\boldsymbol{n}} \left|\frac{\partial \boldsymbol{r}}{\partial t} \times \frac{\partial \boldsymbol{r}}{\partial s}\right| dt\, ds = \left(\frac{\partial \boldsymbol{r}}{\partial t} \times \frac{\partial \boldsymbol{r}}{\partial s}\right) dt\, ds$$

And so:

$$I = \int_{s_0}^{s_1} ds \int_{t_0}^{t_1} dt\, \boldsymbol{f}(s,t) \left(\frac{\partial \boldsymbol{r}}{\partial t} \times \frac{\partial \boldsymbol{r}}{\partial s}\right)$$

We can use other coordinate systems for surface integrals.

If the surface is part of a sphere with a fixed radius R, spherical coordinates would be wise to use. We parametrise the surface by angles:

$$\boldsymbol{r} = \boldsymbol{r}(\theta, \phi) = R \begin{pmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix}$$

And so find $d\boldsymbol{S}$:

$$d\boldsymbol{S} = \left(\frac{\partial r}{\partial \theta} \times \frac{\partial r}{\partial \phi}\right) d\theta d\phi = R^2 \begin{pmatrix} \sin^2(\theta)\cos(\phi) \\ \sin^2(\theta)\sin(\phi) \\ sin(\theta)\cos(\theta) \end{pmatrix} = \mathrm{R}^2 \sin(\theta)\, d\theta d\phi\, \widehat{\mathbf{e_r}}$$

If a surface is defined by $z = f(x,y)$ we can find the area by paramaterising:

$$\boldsymbol{r} = \boldsymbol{r}(x,y) = \begin{pmatrix} x \\ y \\ f(x,y) \end{pmatrix}$$

$$\therefore dS = \left|\frac{\partial \boldsymbol{r}}{\partial x} \times \frac{\partial \boldsymbol{r}}{\partial y}\right| dx\, dy$$

In some cases we can simplify our surface integrals.

One way is using Gauss's divergence theorem.

If a surface $S$ enclosed a volume $V$, we write $S = \partial V$

If we wish to find the flux of this surface in a field $\boldsymbol{G}(\boldsymbol{r})$, Gauss's divergence theorem states that:

$$\int_{S=\partial V} \boldsymbol{G}(\boldsymbol{r}) \cdot d\boldsymbol{S} = \int_V div(\boldsymbol{G}) \cdot dV$$

Another way is using Stokes theorem.

If we have an open surface $S$ with a boundary $C$ we write $C = \partial S$

The loop integral of the field through this boundary is equal to the surface integral of the curl of the vector field:

$$\oint_{C=\partial S} \boldsymbol{G}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_S curl\ \boldsymbol{G} \cdot d\boldsymbol{S}$$