Taylor Series of Sine Without Derivatives

Noah Feinberg

December 2018

1 Introduction

Trying to find a method to calculate the sine of an arbitrary angle was an important problem in mathematics prior to the advent of calculus. With calculus, it became relatively easy to derive a method to approximate arbitrary functions as polynomials.

However, prior to the discovery of calculus, Madhava had discovered a method of calculating trigonometric functions in a similar manner. How exactly he derived his formulae is unknown, which provided the motivation for trying to derive the Taylor series of sine without a derivative.

This is accomplished roughly in 3 steps. First, a formula for $\sin((2n+1)\theta)$, $n \in \mathbb{N}$, in terms of powers of $\sin(\theta)$ is derived. After a slight change of variable, the small angle approximation will be substituted in to reduce the formula to a polynomial. Finally a limiting process will be taken to derive the Taylor series of sine.

2 Proof

2.1 Power series of the Multi-Angle Expansion

Using De Moivre's Theorem and equating the imaginary component with sine one acquires

$$\sin((2n+1)\theta) = \sum_{m=0}^{n} (-1)^m \binom{2n+1}{2m+1} \sin^{2m+1}(\theta) \cos^{2n-2m}(\theta). \tag{1}$$

We chose the argument of sine to be $2n+1$ so that in 1, all powers of cosine are even. Doing so allows us to use the Pythagorean identity to obtain

$$\cos^{2n-2m}(\theta) = (1-\sin^2(\theta))^{n-m}. \tag{2}$$

Our next step is to rewrite 1 as a power series of $\sin(\theta)$ using 2 to eliminate cosine. In expanding 2 we acquire

$$(1-\sin^2(\theta))^{n-k} = \sum_{i=0}^{n} (-1)^i \binom{n-m}{i} \sin^{2i}(\theta). \tag{3}$$

Note that the upper bound is n not $n-m$, this is because $\forall a > b$ we have that $\binom{a}{b} = 0$, so all terms after $i = n-m$ are 0. Now, substituting 3 into 1 we obtain

$$\sum_{m=0}^{n} \sum_{i=0}^{n} (-1)^{i+m} \binom{2n+1}{2m+1} \binom{n-m}{i} \sin^{2i+2m+1}(\theta). \tag{4}$$

Now, all that remains is to reorganize into the proper form of a power series. For an arbitrary power, call it $2k+1$, to find the coefficient, we need to select the terms from 4 that have the correct power. All such terms from 4 will satisfy $i + m = k$ which we use to eliminate i

$$\sum_{k=0}^{n} (-1)^k \sin^{2k+1}(\theta) \sum_{m=0}^{n} \binom{2n+1}{2m+1} \binom{n-m}{k-m}. \tag{5}$$

2.2 Small Angle Approximation

It is well know that for small enough values of x

$$\sin(\theta) \approx \theta$$

and therefore, by substituting $\frac{x}{2n+1}$ for θ we get

$$\sin(\frac{x}{2n+1}) \approx \frac{x}{2n+1}. \tag{6}$$

By making the previous substitution into 5 we obtain the equation

$$\sin(x) \approx \sum_{k=0}^{n} (-1)^k x^{2k+1} \sum_{m=0}^{n} \frac{\binom{2n+1}{2m+1}\binom{n-m}{k-m}}{(2n+1)^{2k+1}}. \tag{7}$$

Since we replaced $\sin(x)$ with an approximation that is only accurate for small values, to make the error as small as possible we shoot make n as large as possible.

2.3 Limiting Process

While the equation

$$\sum_{m=0}^{n} \frac{\binom{2n+1}{2m+1}\binom{n-m}{k-m}}{(2n+1)^{2k+1}}$$

may look intimidating, it is in fact only the sum of rational expressions, meaning taking the limit should be relatively easy. Let us start by rewriting each summand using the definition of binomial coefficients

$$\frac{\binom{2n+1}{2m+1}\binom{n-m}{k-m}}{(2n+1)^{2k+1}} = \frac{1}{(2m+1)!(k-m)!} * \frac{(2n+1)!(n-m)!}{(2n+1)^{2k+1}(2n-2m)!(n-k)!}.$$

The left fraction on the RHS is a coefficient independent of n, while the right is a rational expression. The numerator has degree $(2n+1) + (n-m) = 3n + 1 - m$

and the denominator $(2k + 1) + (2n - 2m) + (n - k) = 3n + k + 1 - 2m$. Therefore the limit as goes to infinity of a given term will be non zero only if $3n+1-m \geq 3n+1+k-2m$ or $m \geq k$. We also have $k \geq m$, otherwise $\binom{n-m}{k-m}$ would be 0. This means $m = k$. Putting all that together we may simplify to say

$$\lim_{n\to\infty} \sum_{m=0}^{n} \frac{\binom{2n+1}{2m+1}\binom{n-m}{k-m}}{(2n+1)^{2k+1}} = \frac{1}{(2k+1)!} \lim_{n\to\infty} \frac{(2n+1)!}{(2n+1)^{2k+1}(2n-2k)!}.$$

Only one limit left to evaluate. By expanding out the numerator and denominator of

$$\frac{(2n+1)!}{(2n+1)^{2k+1}(2n-2k)!}$$

we see that the leading coefficient of both the numerator and denominator are 2^{2k+1} so the limit of the rational expression is 1. Therefore we may conclude that

$$\lim_{n\to\infty} \sum_{m=0}^{n} \frac{\binom{2n+1}{2m+1}\binom{n-m}{k-m}}{(2n+1)^{2k+1}} = \frac{1}{(2k+1)!}.$$

Given that our approximation of sine becomes more accurate the smaller the angle, we may therefore conclude with confidence that

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \square$$