Wolfgang Nolting

Theoretical Physics 1

Classical Mechanics

Springer

Theoretical Physics 1

Wolfgang Nolting

Theoretical Physics 1

Classical Mechanics

Springer

Wolfgang Nolting
Inst. Physik
Humboldt-Universität zu Berlin
Berlin, Germany

ISBN 978-3-319-40107-2 ISBN 978-3-319-40108-9 (eBook)
DOI 10.1007/978-3-319-40108-9

Library of Congress Control Number: 2016943655

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer International Publishing AG Switzerland

General Preface

The seven volumes of the series *Basic Course: Theoretical Physics* are thought to be textbook material for the study of university-level physics. They are aimed to impart, in a compact form, the most important skills of theoretical physics which can be used as basis for handling more sophisticated topics and problems in the advanced study of physics as well as in the subsequent physics research. The conceptual design of the presentation is organized in such a way that

Classical Mechanics (volume 1)
Analytical Mechanics (volume 2)
Electrodynamics (volume 3)
Special Theory of Relativity (volume 4)
Thermodynamics (volume 5)

are considered as the theory part of an *integrated course* of experimental and theoretical physics as is being offered at many universities starting from the first semester. Therefore, the presentation is consciously chosen to be very elaborate and self-contained, sometimes surely at the cost of certain elegance, so that the course is suitable even for self-study, at first without any need of secondary literature. At any stage, no material is used which has not been dealt with earlier in the text. This holds in particular for the mathematical tools, which have been comprehensively developed starting from the school level, of course more or less in the form of recipes, such that right from the beginning of the study, one can solve problems in theoretical physics. The mathematical insertions are always then plugged in when they become indispensable to proceed further in the program of theoretical physics. It goes without saying that in such a context, not all the mathematical statements can be proved and derived with absolute rigour. Instead, sometimes a reference must be made to an appropriate course in mathematics or to an advanced textbook in mathematics. Nevertheless, I have tried for a reasonably balanced representation so that the mathematical tools are not only applicable but also appear at least 'plausible'.

The mathematical interludes are of course necessary only in the first volumes of this series, which incorporate more or less the material of a bachelor program. In the second part of the series which comprises the modern aspects of theoretical physics,

Quantum Mechanics: Basics (volume 6)
Quantum Mechanics: Methods and Applications (volume 7)
Statistical Physics (volume 8)
Many-Body Theory (volume 9),

mathematical insertions are no longer necessary. This is partly because, by the time one comes to this stage, the obligatory mathematics courses one has to take in order to study physics would have provided the required tools. The fact that training in theory has already started in the first semester itself permits inclusion of parts of quantum mechanics and statistical physics in the bachelor program itself. It is clear that the content of the last three volumes cannot be part of an *integrated course* but rather the subject matter of pure theory lectures. This holds in particular for *Many-Body Theory* which is offered, sometimes under different names as, e.g., *Advanced Quantum Mechanics*, in the eighth or so semester of study. In this part, new methods and concepts beyond basic studies are introduced and discussed which are developed in particular for correlated many particle systems which in the meantime have become indispensable for a student pursuing master's or a higher degree and for being able to read current research literature.

In all the volumes of the series *Basic Course: Theoretical Physics,* numerous exercises are included to deepen the understanding and to help correctly apply the abstractly acquired knowledge. It is obligatory for a student to attempt on his own to adapt and apply the abstract concepts of theoretical physics to solve realistic problems. Detailed solutions to the exercises are given at the end of each volume. The idea is to help a student to overcome any difficulty at a particular step of the solution or to check one's own effort. Importantly these solutions should not seduce the student to follow the *easy way out* as a substitute for his own effort. At the end of each bigger chapter, I have added self-examination questions which shall serve as a self-test and may be useful while preparing for examinations.

I should not forget to thank all the people who have contributed one way or an other to the success of the book series. The single volumes arose mainly from lectures which I gave at the universities of Muenster, Wuerzburg, Osnabrueck, and Berlin in Germany, Valladolid in Spain and Warangal in India. The interest and constructive criticism of the students provided me the decisive motivation for preparing the rather extensive manuscripts. After the publication of the German version, I received a lot of suggestions from numerous colleagues for improvement, and this helped to further develop and enhance the concept and the performance of the series. In particular I appreciate very much the support by Prof. Dr. A. Ramakanth, a long-standing scientific partner and friend, who helped me in many respects, e.g. what concerns the checking of the translation of the German text into the present English version.

Special thanks are due to the Springer company, in particular to Dr. Th. Schneider and his team. I remember many useful motivations and stimulations. I have the feeling that my books are well taken care of.

Berlin, Germany Wolfgang Nolting
May 2015

Preface to Volume 1

The first volume of the series *Basic Course: Theoretical Physics* presented here deals with *Classical Mechanics*, a topic which may be described as

> *analysis of the laws and rules according to which physical bodies move in space and time under the influence of forces.*

This formulation already contains certain fundamental concepts whose rigorous definitions appear rather non-trivial and therefore have to be worked out with sufficient care. In the case of a few of these fundamental concepts, we have to even accept them, to start with, as more or less plausible facts of everyday experience without going into the exact physical definitions. We assume a *material body* to be an object which is localized in space and time and possesses an *(inertial) mass*. The concept is still to be discussed. This is also valid for the concept of *force*. The forces are causing changes of the shape and/or in the state of motion of the body under consideration. What we mean by space is the three-dimensional Euclidean space being unrestricted in all the three directions, being homogeneous and isotropic, i.e. translations or rotations of our world as a whole in this space have no consequences. The *time* is also a fact of experience from which we only know that it does exist flowing uniformly and unidirectionally. It is also homogeneous which means no point in time is a priori superior in any manner to any other point in time.

In order to describe natural phenomena, a physicist needs mathematics as *language*. But the dilemma lies in the fact that *theoretical mechanics* can be imparted in a proper way only when the necessary mathematical tools are available. If theoretical physics is started right in the first semester, the student is not yet equipped with these tools. That is why the first volume of the *Basic Course: Theoretical Physics* begins with a concise mathematical introduction which is presented in a concentrated and focused form including all the material which is absolutely necessary for the development of *theoretical classical mechanics*. It goes without saying that in such a context not all mathematical theories can be proved or derived with absolute stringency and exactness. Nevertheless, I have tried for a reasonably balanced representation so that mathematical theories are not only readily applicable but also at least appear *plausible*. Thereby only that much mathematics is offered which is necessary to proceed with the presentation of theoretical physics. Whenever in the presentation one meets new *mathematical barriers,* a corresponding mathematical insertion appears in the text. Therefore, mathematical discourses are found only at the positions where they are directly needed. In this connection, the numerous exercises provided are of special importance and should be worked without fail in order to evaluate oneself in self-examination.

This volume on *classical mechanics* arose from respective lectures I gave at the German Universities in Muenster and Berlin. The animating interest of the students in my lecture notes has induced me to prepare the text with special care. This volume as well as the subsequent volumes is thought to be a textbook material for the study of basic physics, primarily intended for the students rather than for the teachers. It is presented in such a way that it enables self-study without the need for a demanding and laborious reference to secondary literature. I had to focus on the essentials, presenting them in a detailed and elaborate form, sometimes consciously sacrificing certain elegance. It goes without saying that after the basic course, secondary literature is needed to deepen the understanding of physics and mathematics.

I am thankful to the Springer company, especially to Dr. Th. Schneider, for accepting and supporting the concept of my proposal. The collaboration was always delightful and very professional. A decisive contribution to the book was provided by Prof. Dr. A. Ramakanth from the Kakatiya University of Warangal (India). Many thanks for it!

Berlin, Germany Wolfgang Nolting
May 2015

Contents

Chapter 1
Mathematical Preparations

The basic differential and integral calculus are normally part of the content of curriculum in secondary school. However, experience has shown that the knowledge of basic mathematics has a large variation from student to student. The things which are completely clear or even trivial to one can pose high barriers to another. Therefore, in this introductory chapter the most important elements of differential and integral calculus will e recapitulated which are vital for the following course of *Theoretical Physics*. It is clear that this cannot replace the precise representation of a mathematics course. It is to understand only as an *'auxiliary program'* to provide the basic tools for starting *Theoretical Physics*. The reader who is familiar with elementary differential and integral calculus may either use Sects. 1.1 and 1.2 as a revision for a kind of self-examination or simply skip them.

1.1 Elements of Differential Calculus

1.1.1 Set of Numbers

One defines the following types of numbers:

$$\begin{aligned}
&\mathbb{N} = \{1, 2, 3, \ldots\} && \textit{natural numbers}\\
&\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\} && \textit{integer numbers}\\
&\mathbb{Q} = \left\{x;\ x = \tfrac{p}{q};\ p \in \mathbb{Z},\ q \in \mathbb{N}\right\} && \textit{rational numbers}\\
&\mathbb{R} = \{x;\ \text{continuous number line}\} && \textit{real numbers}\ .
\end{aligned}$$

Therefore

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}\ .$$

W. Nolting, *Theoretical Physics 1*, DOI 10.1007/978-3-319-40108-9_1

The body of *complex numbers* $\mathbb{C}$ will be introduced and discussed later in Sect. 2.3.5. For the above-mentioned set of numbers the basic operations *addition* and *multiplication* are defined in the well-known manner. We will remind here only shortly to the process of **raising to a power**.

For an arbitrary real number a the n-th power is defined as:

$$a^n = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{n\text{-fold}} \qquad n \in \mathbb{N} \, . \tag{1.1}$$

There are the following *rules*:

1.

$$(a \cdot b)^n = \underbrace{(a \cdot b) \cdot (a \cdot b) \cdot \ldots \cdot (a \cdot b)}_{n\text{-fold}} = a^n \cdot b^n \tag{1.2}$$

2.

$$a^k \cdot a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{k\text{-fold}} \cdot \underbrace{a \cdot a \cdot \ldots \cdot a}_{n\text{-fold}} = a^{k+n} \tag{1.3}$$

3.

$$(a^n)^k = \underbrace{a^n \cdot a^n \cdot \ldots \cdot a^n}_{k\text{-fold}} = a^{n \cdot k} \, . \tag{1.4}$$

Even *negative exponents* are defined as can be seen by the following consideration:

$$a^n = a^{n+k-k} = a^n \cdot a^{-k} \cdot a^k \quad \curvearrowright \quad a^{-k} \cdot a^k = 1 \, .$$

Therefore we have:

$$a^{-k} \equiv \frac{1}{a^k} \quad \forall a \in \mathbb{R} \;\; (a \neq 0) \, . \tag{1.5}$$

Furthermore, we recognize the important special case:

$$a^{k-k} \equiv a^0 = 1 \quad \forall a \in \mathbb{R} \, . \tag{1.6}$$

This relation is valid also for $a = 0$.

Analogously and as an extension of (1.4) *split exponents* can be defined:

$$b^n = a = \left(a^{\frac{1}{n}}\right)^n \quad \curvearrowright \quad b = a^{\frac{1}{n}} \, .$$

One denotes

$$a^{\frac{1}{n}} \equiv \sqrt[n]{a} \ : \qquad \textbf{\textit{n}-th root of } a \tag{1.7}$$

Thus it is a number the n-th power of which is just a.

Examples

$$\begin{aligned} \sqrt[2]{4} \equiv 4^{\frac{1}{2}} &= 2 \quad \text{because: } 2^2 = 2 \cdot 2 = 4 \\ \sqrt[3]{27} \equiv 27^{\frac{1}{3}} &= 3 \quad \text{because: } 3^3 = 3 \cdot 3 \cdot 3 = 27 \\ \sqrt[4]{0.0001} \equiv 0.0001^{\frac{1}{4}} &= 0.1 \quad \text{because: } 0.1^4 = 0.1 \cdot 0.1 \cdot 0.1 \cdot 0.1 = 0.0001 \ . \end{aligned}$$

Eventually we can accept also *rational exponents*:

$$a^{\frac{p}{q}} \equiv \sqrt[q]{a^p} \equiv \left(\sqrt[q]{a}\right)^p \ . \tag{1.8}$$

The final generalization to arbitrary real numbers will be done at a later stage.

1.1.2 Sequence of Numbers and Limiting Values

By a *sequence of numbers* we will understand a sequence of (indexed) real numbers:

$$a_1, a_2, a_3, \cdots, a_n, \cdots \qquad a_n \in \mathbb{R} \ . \tag{1.9}$$

We have finite and infinite sequences of numbers. In case of a finite sequence the index n is restricted to a finite subset of $\mathbb{N}$. The sequence is formally denoted by the symbol

$$\{a_n\}$$

and represents a mapping of the natural numbers $\mathbb{N}$ on the body of real numbers $\mathbb{R}$:

$$f \ : \ n \in \mathbb{N} \longrightarrow a_n \in \mathbb{R} \qquad (n \longrightarrow a_n) \ .$$

Examples

1.

$$a_n = \frac{1}{n} \quad \longrightarrow a_1 = 1, \ a_2 = \frac{1}{2}, \ a_3 = \frac{1}{3}, \ a_4 = \frac{1}{4} \ \cdots \tag{1.10}$$

2.

$$a_n = \frac{1}{n(n+1)} \quad \longrightarrow a_1 = \frac{1}{1 \cdot 2},\, a_2 = \frac{1}{2 \cdot 3},\, a_3 = \frac{1}{3 \cdot 4},\, \cdots \tag{1.11}$$

3.

$$a_n = 1 + \frac{1}{n} \quad \longrightarrow a_1 = 2,\, a_2 = \frac{3}{2},\, a_3 = \frac{4}{3},\, a_4 = \frac{5}{4},\, \cdots \tag{1.12}$$

Now we define the

Limiting value (limit) of a sequence of numbers

If a_n approaches for $n \to \infty$ a single finite number a, then a is the limiting value (limes) of the sequence $\{a_n\}$:

$$\lim_{n \to \infty} a_n = a \ ; \ a_n \stackrel{n \to \infty}{\longrightarrow} a \, . \tag{1.13}$$

The mathematical definition reads:

$$\begin{gathered} \{a_n\} \ \textit{converges to } a \\ \Longleftrightarrow \forall \varepsilon > 0 \ \ \exists\, n_\varepsilon \in \mathbb{N} \ \text{ so that } \ |a_n - a| < \varepsilon \quad \ \forall n > n_\varepsilon \, . \end{gathered} \tag{1.14}$$

Does such an *a not* exist then the sequence is called *divergent*. In case $\{a_n\}$ converges to a, then for each $\varepsilon > 0$ only a finite number of sequence elements has a distance greater than ε to a.

Examples

1.

$$\{a_n\} = \left\{ \frac{1}{n} \right\} \longrightarrow 0 \qquad (\textit{null sequence}) \tag{1.15}$$

2.

$$\{a_n\} = \left\{ \frac{n}{n+1} \right\} \longrightarrow 1 \tag{1.16}$$

because:

$$\frac{n}{n+1} = \frac{1}{1 + \frac{1}{n}} \longrightarrow \frac{1}{1+0} = 1 \, .$$

In anticipation, we have here already used the rule (1.22).

3.

$$\{a_n\} = \{q^n\} \longrightarrow 0 \quad , \quad \text{if} \quad |q| < 1 \; . \tag{1.17}$$

The proof of this statement is provided elegantly by the use of the special function *logarithm*, which, however, will be introduced only with Eq. (1.65). Thus we present the justification of (1.17) after the derivation of (1.70).

4.

$$a_n = \left(1 + \frac{1}{n}\right)^n \longrightarrow e = 2.71828\ldots \quad \textit{Euler number} \; . \tag{1.18}$$

The limiting value of this sequence, which is very important for applications, is given here without proof. For details the reader is referred to special textbooks on mathematics.

Again without proof we list up the following

rules for sequences of numbers

the explicit, rather straightforward derivation of which shall be left to the reader. Assuming the convergence of the two sequences $\{a_n\}$ and $\{b_n\}$:

$$\lim_{n\to\infty} a_n = a \;\; ; \;\; \lim_{n\to\infty} b_n = b \; .$$

we get:

$$\lim_{n\to\infty} (a_n \pm b_n) = a \pm b \tag{1.19}$$

$$\lim_{n\to\infty} (c \cdot a_n) = c \cdot a \qquad (c \in \mathbb{R}) \tag{1.20}$$

$$\lim_{n\to\infty} (a_n \cdot b_n) = a \cdot b \tag{1.21}$$

$$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{a}{b} \qquad (b, b_n \neq 0 \; \forall n) \; . \tag{1.22}$$

1.1.3 Series and Limiting Values

Adding up the terms of an infinite sequence of numbers leads to what is called a **series**:

$$a_1, a_2, a_3, \cdots, a_n, \cdots \curvearrowright a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{m=1}^{\infty} a_m \; . \tag{1.23}$$

Strictly, the series is defined as limiting value of a sequence of (finite) *partial sums*:

$$S_r = \sum_{m=1}^{r} a_m \ . \tag{1.24}$$

The series *converges* to S if

$$\lim_{r\to\infty} S_r = S \tag{1.25}$$

does exist. If not then it is called **divergent**.

A *necessary* condition for the series $\sum_{m=1}^{\infty} a_m$ to be convergent is

$$\lim_{m\to\infty} a_m = 0 \tag{1.26}$$

For, if $\sum_{m=1}^{\infty} a_m$ is indeed convergent then it must hold:

$$\lim_{m\to\infty} a_m = \lim_{m\to\infty} (S_m - S_{m-1}) = \lim_{m\to\infty} S_m - \lim_{m\to\infty} S_{m-1} = S - S = 0 \ .$$

However, Eq. (1.26) is not a sufficient condition. A prominent counter-example represents the **harmonic series**:

$$\sum_{m=1}^{\infty} \frac{1}{m} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots \ . \tag{1.27}$$

It is divergent, although $\lim_{m\to\infty} \frac{1}{m} = 0$! The proof of this is given as an Exercise 1.1.3. In mathematics (analysis) one learns of different necessary and sufficient conditions of convergence for infinite series:

comparison criterion ,
ratio test ,
root test

In the course of this book we do not need these criteria explicitly and thus restrict ourselves to only making a remark.

The **geometric series** turns out to be an important special case of an infinite series being defined as

$$q^0 + q^1 + q^2 + \cdots + q^m + \cdots = \sum_{m=1}^{\infty} q^{m-1} \ . \tag{1.28}$$

The partial sums

$$S_r = q^0 + q^1 + \cdots + q^{r-1}$$

can easily be calculated analytically. For this purpose we multiply the last equation by q,

$$q\,S_r = q^1 + q^2 + \cdots + q^r$$

and build the difference:

$$S_r - q\,S_r = S_r(1-q) = q^0 - q^r = 1 - q^r \; .$$

Then we get the important result:

$$S_r = \frac{1-q^r}{1-q} \; . \tag{1.29}$$

Interesting is the limit:

$$\lim_{r\to\infty} S_r = \frac{1 - \lim_{r\to\infty} q^r}{1-q} \; .$$

For this, Eqs. (1.19) and (1.20) have been exploited. Because of (1.17) we arrive at:

$$S = \lim_{r\to\infty} S_r = \begin{cases} \frac{1}{1-q} \, , \text{ if } |q| < 1 \\ \\ \text{not existent, if } |q| \geq 1 \end{cases} \; . \tag{1.30}$$

1.1.4 Functions and Limits

By the term **function** $f(x)$ one understands the unique attribution of a *dependent* variable y from the **co-domain** W to an *independent* variable x from the **domain of definition** D of the function $f(x)$:

$$y = f(x) \quad ; \quad D \subset \mathbb{R} \xrightarrow{f} W \subset \mathbb{R} \; . \tag{1.31}$$

We ask ourselves how $f(x)$ changes with x. All elements of the sequence

$$\{x_n\} = x_1, x_2, x_3, \cdots, x_n, \cdots$$

shall be from the domain of definition of the function f. Then for each x_n there exists a

$$y_n = f(x_n)$$

and therewith a *'new'* sequence $\{f(x_n)\}$.

Definition $f(x)$ possesses at x_0 a *limiting value* f_0, if for *each* sequence $\{x_n\} \to x_0$ holds:

$$\lim_{n\to\infty} f(x_n) = f_0 \ . \tag{1.32}$$

That is written as:

$$\lim_{x\to x_0} f(x) = f_0 \ . \tag{1.33}$$

Examples

1.

$$f(x) = \frac{x^3}{x^3 + x - 1} \qquad ; \qquad \lim_{x\to\infty} f(x) = \ ? \tag{1.34}$$

This expression can be reformulated for all $x \neq 0$:

$$f(x) = \frac{1}{1 + \frac{1}{x^2} - \frac{1}{x^3}} \ .$$

For all sequences $\{x_n\}$, which tend to ∞, $\frac{1}{x^2}$ and $\frac{1}{x^3}$ become null sequences. That means:

$$\lim_{x\to\infty} \frac{x^3}{x^3 + x - 1} = 1 \ .$$

2.

$$f(x) = (1 + x)^{\frac{1}{x}} \qquad ; \qquad \lim_{x\to 0} f(x) = \ ? \tag{1.35}$$

For the special null sequence $\{x_n\} = \{\frac{1}{n}\}$ according to (1.18) we know the limit of this function. It can be shown, however, that the same is true for arbitrary null sequences:

$$\lim_{x\to 0} (1 + x)^{\frac{1}{x}} = e \ . \tag{1.36}$$

In case of a one-to-one mapping

$$x \overset{f}{\longleftrightarrow} y \tag{1.37}$$

one can define the so-called

'inverse function' f^{-1}

belonging to f which comes out by solving $y = f(x)$ with respect to x:

$$f^{-1}(f(x)) = x \; . \tag{1.38}$$

Example

$$y = f(x) = ax + b \qquad a, b \in \mathbb{R}$$

$$\curvearrowright \quad x = f^{-1}(y) = \frac{1}{a}y - \frac{b}{a} \; .$$

Later we will encounter some further examples. Note that in general

$$f^{-1}(x) \not\equiv \frac{1}{f(x)} \; .$$

It is important to stress once more the uniqueness of f^{-1}, because only then f^{-1} can be defined as *'function'*. In this respect the 'inverse' of $y = x^2$ is not unique: $x = \pm\sqrt{y}$. However, if the domain of definition for f is restricted, e.g., to non-negative x, then the inverse does exist.

1.1.5 Continuity

We are now coming to the very important concept

continuity

$y = f(x)$ is called **continuous** at x_0 from the domain of definition of f if *for all* $\varepsilon > 0$ a $\delta > 0$ exists so that *for each* x with

$$|x - x_0| < \delta$$

holds:

$$|f(x) - f(x_0)| < \varepsilon \; .$$

Alternative formulation:

$y = f(x)$ is **continuous** at x_0 from the domain of definition of f if *for each* sequence $\{x_n\} \to x_0$ follows:

$$\lim_{x \to x_0} f(x) = f(x_0) = f_0 \, .$$

The limiting value f_0 is therefore just the function value $f(x_0)$. We elucidate the term of continuity by two examples:

$$f(x) = \begin{cases} x & : \; x \geq 1 \\ 1 & : \; x < 1 \end{cases} \, . \tag{1.39}$$

The function (1.39), represented in Fig. 1.1, is obviously continuous, in contrary to the function from Fig. 1.2:

$$f(x) = \begin{cases} x - 1 & : \; x \geq 1 \\ \quad 1 & : \; x < 1 \end{cases} \, . \tag{1.40}$$

which is apparently discontinuous at $x = 1$:

$$\lim_{x \to 1^-} f(x) = +1 \neq \lim_{x \to 1^+} f(x) = 0 \, .$$

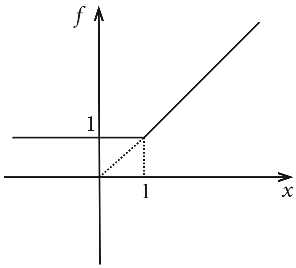


Fig. 1.1 Example of a continuous function

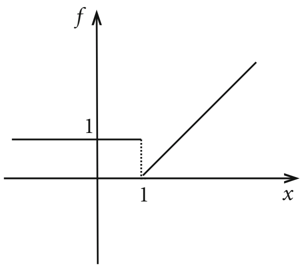


Fig. 1.2 Example of a discontinuous function

1.1.6 *Trigonometric Functions*

It can be assumed that the trigonometric functions are well-known from school-mathematics. Therefore, only the most important relations shall be compiled in this subsection.

- **Radian measure**

  Figure 1.3 illustrates that the angle φ can be expressed not only by angular degrees $^\circ$, but equally uniquely also via the arc of the circle s:

$$s = s(\varphi) \quad : \quad s(360^\circ) = 2\pi r\,; \quad s(180^\circ) = \pi r\,; \quad s(90^\circ) = \frac{\pi}{2}\,r\,;\ldots$$

  One introduces the dimensionless quantity

$$\varphi = \frac{s}{r} \qquad \textbf{`radian'} \tag{1.41}$$

$$\varphi(^\circ) = 360(180, 90, 45, 1) \longrightarrow 2\pi\,\left(\pi, \frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{180}\right) \text{ rad}\,. \tag{1.42}$$

- **Trigonometric functions**

  In the right-angled triangle in Fig. 1.4 a and b, adjacent and opposite to angle α, respectively, are called the leg (side, cathetus) and c the hypothenuse. With these terms one defines:

$$\sin\alpha = \frac{b}{c} \tag{1.43}$$

$$\cos\alpha = \frac{a}{c} \tag{1.44}$$

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{b}{a} \tag{1.45}$$

$$\cot\alpha = \frac{\cos\alpha}{\sin\alpha} = \frac{1}{\tan\alpha} = \frac{a}{b}\,. \tag{1.46}$$

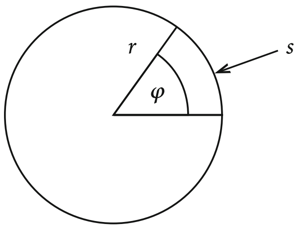


Fig. 1.3 To the definition of radian measure

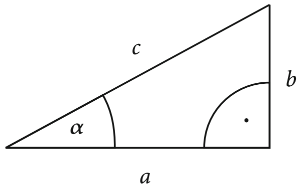


Fig. 1.4 To the definition of trigonometric functions

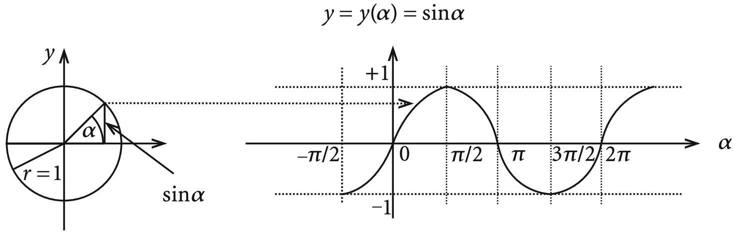


Fig. 1.5 Graphical representation of the sine function

According to *Pythagoras' theorem* it holds:

$$a^2 + b^2 = c^2 \quad \curvearrowright \quad \frac{a^2}{c^2} + \frac{b^2}{c^2} = 1 \; .$$

That leads to the important and frequently used formula:

$$\sin^2\alpha + \cos^2\alpha = 1 \; . \tag{1.47}$$

- **Sine function**

  The **sine function** can be graphically illustrated as in Fig. 1.5. Thereby one should notice that the angle α has to be counted in the mathematically positive sense, i.e. counterclockwise. The sine is periodic with the period 2π. It is an odd function of the angle α:

$$\sin(-\alpha) = -\sin(\alpha) \; . \tag{1.48}$$

As an insertion, let us investigate a special limiting case in connection with the sine:

$$f(x) = \frac{\sin x}{x} \qquad ; \qquad \lim_{x\to 0} f(x) = \; ? \tag{1.49}$$

At first glance the limit appears to be undefined ($'0/0'$). We try a graphic solution by use of Fig. 1.6. x shall be a piece (from B to C) of a circle with radius $R = 1$ around the center $\mathcal{O}$ (radian measure). Then it holds for the segment fixed by the

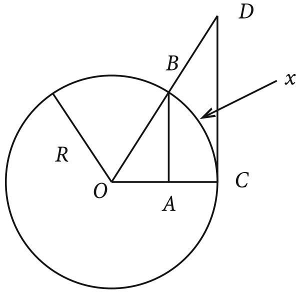


Fig. 1.6 For the calculation of $\lim_{x \to 0} \sin x / x$

points $\mathcal{O}$, B and C:

$$F(\mathcal{O}BC) = \pi R^2 \cdot \frac{x}{2\pi R} = \frac{x R}{2} = \frac{x}{2} .$$

Furthermore, one reads from the sketch:

$$\overline{\mathcal{O}B} = \overline{\mathcal{O}C} = 1 \ ; \ \overline{\mathcal{O}A} = \cos x \ ; \ \overline{BA} = \sin x .$$

In addition the intercept theorem yields:

$$\frac{\overline{DC}}{\overline{BA}} = \frac{\overline{\mathcal{O}C}}{\overline{\mathcal{O}A}} \quad \curvearrowright \quad \overline{DC} = \sin x \cdot \frac{1}{\cos x} = \tan x .$$

Obviously, the following estimation for the areas holds:

$$F(\mathcal{O}BA) < F(\mathcal{O}BC) < F(\mathcal{O}DC) .$$

That means:

$$\begin{aligned} &\frac{1}{2} \cos x \sin x < \frac{x}{2} < \frac{1}{2} \tan x \\ \curvearrowright \quad &\cos x < \frac{x}{\sin x} < \frac{1}{\cos x} \qquad\qquad (\sin x > 0) \\ \curvearrowright \quad &\frac{1}{\cos x} > \frac{\sin x}{x} > \cos x . \end{aligned}$$

Eventually we can exploit that for the limiting process $x \to 0$ it follows $\cos x \to 1$ and $\frac{1}{\cos x} \to 1$ leading therewith to:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 . \tag{1.50}$$

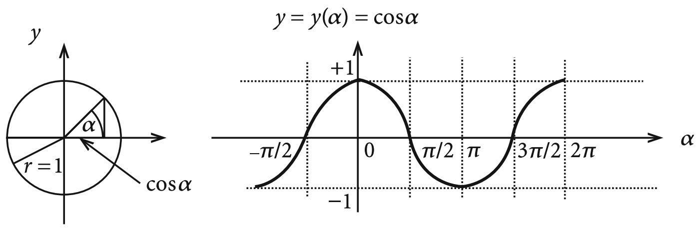


Fig. 1.7 Graphical representation of the cosine function

In (1.94) we shall derive a series expansion for the sine:

$$\sin\alpha = \alpha - \frac{1}{3!}\alpha^3 + \frac{1}{5!}\alpha^5 + \ldots = \sum_{n=0}^{\infty}(-1)^n\frac{\alpha^{2n+1}}{(2n+1)!}\,. \tag{1.51}$$

Here we used the term

$$n! = 1\cdot 2\cdot 3\cdot\ldots\cdot n\;\;;\;\;0! = 1! = 1 \qquad (n\text{-}factorial)\,. \tag{1.52}$$

In particular, the series expansion makes clear that for small angles α (radian measure!) it is approximately

$$\sin\alpha \approx \alpha\,. \tag{1.53}$$

This once more confirms the limit (1.50).

If the angle α is restricted to the interval $[-\pi/2, +\pi/2]$, then the sine function has a unique inverse which is denoted as **'arc sine'**:

$$\alpha = \sin^{-1}(y) = \arcsin(y)\,. \tag{1.54}$$

This function maps the interval $[-1, +1]$ for y onto the interval $[-\pi/2, +\pi/2]$ for α. This inverse function delivers the value of the angle α in radian measure, whose sine-value is just y.

- **Cosine-function**
While, according to Fig. 1.5, the sine is fixed by the side opposite to the angle in the right-angled triangle the *cosine*-function is determined in a analogous manner by the adjacent side (Fig. 1.7). One recognizes from the right-angled triangles in the Figs. 1.5 and 1.7 that the cosine is nothing else but the $\pi/2$-shifted sine:

$$\cos(\alpha) = \sin\left(\alpha + \frac{\pi}{2}\right)\,. \tag{1.55}$$

If the angle α is restricted to the interval $0 \leq \alpha \leq \pi$ a unique inverse function does exist which is called the **'arc cosine'**:

$$\alpha = \cos^{-1}(y) = \arccos(y) . \tag{1.56}$$

The cosine is an even function of α:

$$\cos(-\alpha) = \cos(\alpha) . \tag{1.57}$$

As Exercise 1.1.12 we derive the series expansion of the cosine:

$$\cos(\alpha) = 1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \frac{\alpha^6}{6!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{\alpha^{2n}}{(2n)!} . \tag{1.58}$$

From this expansion we conclude that for small angles α (radian measure!) approximately holds:

$$\cos\alpha \approx 1 \tag{1.59}$$

Extremely useful are the **'addition theorems'** for trigonometric functions, the relatively simple proofs of which are provided in a subsequent section (Exercise 2.3.9) with the aid of Euler's formula for complex numbers:

$$\sin(\alpha \pm \beta) = \sin\alpha\ \cos\beta \pm \sin\beta\ \cos\alpha \tag{1.60}$$

$$\cos(\alpha \pm \beta) = \cos\alpha\ \cos\beta \mp \sin\alpha\ \sin\beta \tag{1.61}$$

1.1.7 Exponential Function and Logarithm

- **Exponential function**

  By this one understands the following function:

$$y = a^x . \tag{1.62}$$

a is called the *'basis'* and x the *'exponent'*. Here a may be an arbitrary real number. Very often one uses *Euler's number e* (1.18) writing:

$$y = y_0\, e^{\alpha x} \equiv y_0\ \exp(\alpha x) . \tag{1.63}$$

This function is of great importance in theoretical physics and appears often in a variety of contexts (rate of growth, increase of population, law of radioactive decay, capacitor charge and discharge, ...) (Fig. 1.8).

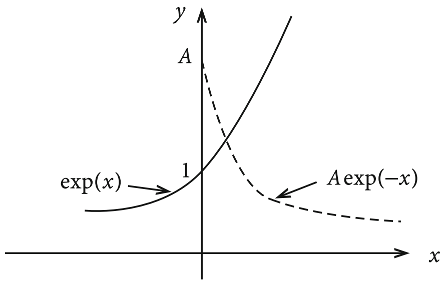


Fig. 1.8 Schematic behavior of the exponential function

In Sect. 1.1.10 we will be able to prove, by using the Taylor expansion, the following important series expansion of the exponential function:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \,. \tag{1.64}$$

- **Logarithm**
  It is just the inverse function of $y = a^x$ being defined only for $y > 0$:

Logarithm to the base a

$$x = \log_a y \,. \tag{1.65}$$

Thus, if a is raised to the power of $\log_a y$ one gets y. Rather often one uses $a = 10$ and calls it then *'common (decimal) logarithm'*:

$$\log_{10} 100 = 2 \,; \quad \log_{10} 1000 = 3 \,; \ldots$$

However, in physics we use most frequently the **'natural logarithm'** with base $a = e$ denoted by the symbol $\log_e \equiv \ln$. In this case the explicit indication of the base is left out:

$$\ln(e^x) = x \quad \Longleftrightarrow \quad e^{\ln x} = x \,. \tag{1.66}$$

With $y = e^x$ and $y' = e^{x'}$ as well as $a, c \in \mathbb{R}$ we can derive some important rules for the logarithm:

$$\begin{aligned} \ln\left(y \cdot y'\right) &= \ln\left(e^x \cdot e^{x'}\right) = \ln\left(e^{x+x'}\right) = x + x' \\ &= \ln y + \ln y' \end{aligned} \tag{1.67}$$

$$\begin{aligned} \ln(c \cdot y) &= \ln(c \cdot e^x) = \ln\left(e^{\ln c} \cdot e^x\right) = \ln\left(e^{\ln c + x}\right) = \ln c + x \\ &= \ln c + \ln y \end{aligned} \tag{1.68}$$

$$\ln(y^a) = \ln((e^x)^a) = \ln(e^{ax}) = a\,x$$
$$= a\,\ln y\,. \tag{1.69}$$

One still recognizes the special cases:

$$\ln(1) = \ln(e^0) = 0 \qquad ; \qquad \ln x < 0 \quad \text{if} \;\; 0 < x < 1\,. \tag{1.70}$$

Finally, let us still work out the proof of (1.17) which we had to postpone because it exploits properties of the logarithm. Equation (1.17) is concerned with the following statement about the limit of the sequence

$$\{a_n\} = \{q^n\} \longrightarrow 0 \;\;,\;\; \text{if} \;\; |q| < 1\,.$$

We assume

$$|a_n - 0| < \varepsilon < 1\,.$$

That means (Fig. 1.9):

$$|q^n| = |q|^n < \varepsilon < 1 \quad \Leftrightarrow \quad \ln|q|^n < \ln\varepsilon < 0$$
$$\Leftrightarrow \quad n\,\underbrace{\ln|q|}_{<0} < \ln\varepsilon < 0 \quad \Rightarrow \quad n > \frac{\ln\varepsilon}{\ln|q|} > 0\,.$$

If n_ε is the smallest natural number (integer) with

$$n_\varepsilon \geq \frac{\ln\varepsilon}{\ln|q|}\;,$$

then the starting inequality is fulfilled for all $n \geq n_\varepsilon$ and 0 is indeed the limit of the sequence for all $|q| < 1$.

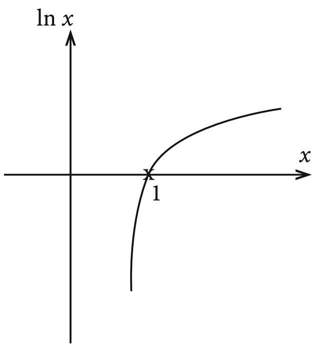


Fig. 1.9 Schematic behavior of the natural logarithm

In addition one sees:

$$|q| > 1 \ \Rightarrow \ n < \frac{\ln \varepsilon}{\ln |q|} < 0 \ \Rightarrow \ \textit{sequence divergent}$$
$$q = 1 \ \Rightarrow \ \lim_{n \to \infty} a_n = 1 \ \Rightarrow \ \textit{sequence convergent}$$
$$q = -1 \ \Rightarrow \ -1, +1, -1, +1, \cdots \ \Rightarrow \ \textit{sequence divergent (but bounded)} .$$

1.1.8 Differential Quotient

The *'slope (gradient)'* of a straight line is the quotient of *'height difference'* Δy and *'base line'* Δx (see Fig. 1.10). For the gradient angle α we obviously have:

$$\tan \alpha = \frac{\Delta y}{\Delta x} . \tag{1.71}$$

Analogously one defines the slope (gradient) of an arbitrary function $f(x)$ at a point P (see Fig. 1.11). The secant $\overline{PQ}$ has the increase

$$\frac{\Delta y}{\Delta x} = \tan \alpha' .$$

One denotes

$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{1.72}$$

as **'difference quotient'**. If we now shift the point Q along the curve towards the point P then the increase of the secant becomes the increase of the *tangent* on the

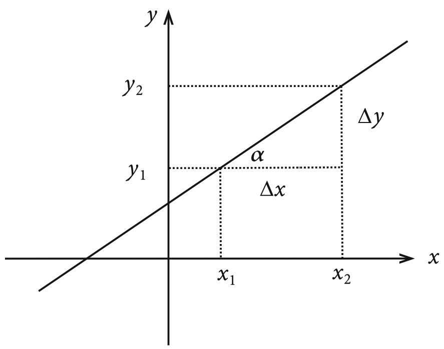


Fig. 1.10 Slope of a straight line

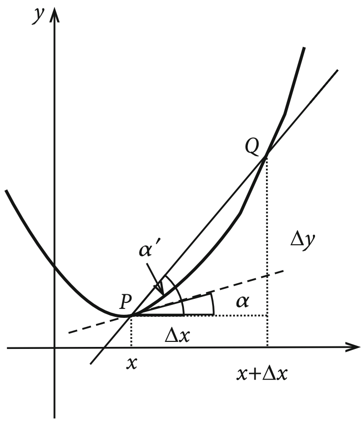


Fig. 1.11 To the definition of the derivative of a function $y = f(x)$

curve $f(x)$ at P (broken line in Fig. 1.11),

$$\tan\alpha = \lim_{\alpha' \to \alpha} \tan\alpha' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

and one arrives at the **'differential quotient'**

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \equiv \frac{dy}{dx} \; . \tag{1.73}$$

which is called the **'first derivative of the function $f(x)$ with respect to x at the point x'**:

$$\frac{dy}{dx} \equiv \frac{d}{dx} f(x) \equiv f'(x) \; . \tag{1.74}$$

Example

$$f(x) = x^2$$

Difference quotient:

$$\frac{\Delta y}{\Delta x} = \frac{(x + \Delta x)^2 - x^2}{\Delta x} = \frac{2x\Delta x + (\Delta x)^2}{\Delta x} = 2x + \Delta x \; .$$

Thus the first derivative is:

$$f'(x) = 2x \; .$$

All the differential quotients do not exhibit a unique limit everywhere! The curve in Fig. 1.12 is continuous at P, but has there different slopes if we come, respectively,

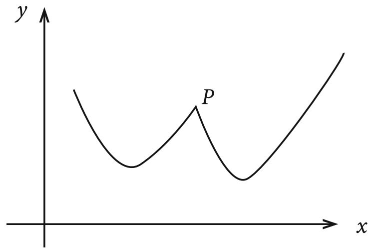


Fig. 1.12 Example of a function $y = f(x)$ being not differentiable in the point P

from the left and the right hand side. One says that $f(x)$ is **'not differentiable'** at the point P.

Definition

- $y = f(x)$ is **differentiable** at x_0 if and only if $f(x_0)$ is defined and a unique limiting value of the difference quotient exists:

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

- The function $y = f(x)$ is differentiable in the interval $[a, b]$ if it is differentiable for all $x \in [a, b]$!

From a graphic view, one denotes $f'(x)$ as the **'slope'** of the curve $f(x)$ in x.

If we look at the change of the value of the function between the two points P and Q (Fig. 1.11),

$$\Delta y = f(x + \Delta x) - f(x) = \frac{f(x + \Delta x) - f(x)}{\Delta x} \, \Delta x \, ,$$

then we realize that for $\Delta x \to 0$ the prefactor becomes the tangent in x. That leads to the

'differential' of the function $y = f(x)$

$$dy = f'(x) \, dx \, . \tag{1.75}$$

In general it holds $dy \neq \Delta y$.

Examples

1.

$$y = f(x) = c \cdot x^n \quad ; \; n \in \mathbb{N} \quad ; \; c \in \mathbb{R} \, . \tag{1.76}$$

This function is differentiable for all x yielding:

$$f'(x) = n \, c \cdot x^{n-1} \tag{1.77}$$

That can be seen as follows:

$$(x + \Delta x)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} \Delta x + \ldots + \binom{n}{n} \Delta x^n$$

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

$$\curvearrowright \quad \frac{\Delta y}{\Delta x} = c\, \frac{(x + \Delta x)^n - x^n}{\Delta x} \qquad (n \geq 2)$$

$$= \frac{c}{\Delta x} \left(\binom{n}{1} x^{n-1} \Delta x + \binom{n}{2} x^{n-2} \Delta x^2 + \ldots \right.$$

$$\left. \ldots + \binom{n}{n} \Delta x^n \right)$$

$$= c \left(n\, x^{n-1} + \binom{n}{2} x^{n-2} \Delta x + \ldots + \Delta x^{n-1} \right)$$

$$\curvearrowright \quad \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = c\, n\, x^{n-1} \, .$$

For $n = 0$ (or $n = 1$) the difference quotient is already identical to zero (or c), i.e. independent of Δx, so that the assertion is immediately fulfilled.

2.

$$y = f(x) = c \;\; ; \;\; c \in \mathbb{R} \quad \Longrightarrow \quad f'(x) \equiv 0 \tag{1.78}$$

because:

$$\frac{\Delta y}{\Delta x} = \frac{c - c}{\Delta x} = 0 \, .$$

This is of course simply the $n = 0$-special case of the first example.

3.

$$y = f(x) = e^x \quad \Longrightarrow \quad f'(x) = e^x \, . \tag{1.79}$$

The exponential function is differentiable for all x as can be seen as follows:

$$\frac{\Delta y}{\Delta x} = \frac{e^{x+\Delta x} - e^x}{\Delta x} = e^x\, \frac{e^{\Delta x} - 1}{\Delta x}$$

$$= e^x\, \frac{1 + \Delta x + \frac{1}{2}\Delta x^2 + \ldots - 1}{\Delta x}$$

$$= e^x \left(1 + \frac{1}{2}\Delta x + \frac{1}{6}\Delta x^2 + \dots\right)$$

$$\curvearrowright \quad \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = e^x \, .$$

Here we have used the anticipated series expansion (1.64) of the exponential function, which will be explicitly derived in (1.95).

4.

$$y = f(x) = \sin x \quad \Longrightarrow \quad f'(x) = \cos x \, . \tag{1.80}$$

$\sin x$ is differentiable for all real x, because:

$$\begin{aligned} \frac{\Delta y}{\Delta x} &= \frac{\sin(x + \Delta x) - \sin x}{\Delta x} \\ &= \frac{\sin x \, \cos \Delta x + \cos x \, \sin \Delta x - \sin x}{\Delta x} \\ &= \frac{\sin x(\cos \Delta x - 1)}{\Delta x} + \cos x \, \frac{\sin \Delta x}{\Delta x} \, . \end{aligned}$$

In the second step we have applied the addition theorem (1.60). If we furthermore use the relation proved as Exercise 1.1.5,

$$1 - \cos \Delta x = 2 \sin^2 \frac{\Delta x}{2} \, ,$$

then it remains to calculate:

$$f'(x) = \lim_{\Delta x \to 0} \left(- \sin x \, \sin \frac{\Delta x}{2} \, \frac{\sin \frac{\Delta x}{2}}{\frac{\Delta x}{2}} + \cos x \, \frac{\sin \Delta x}{\Delta x} \right) = \cos x \, .$$

At the end we exploited (1.50) for the terms in the parenthesis.

5.

$$y = f(x) = \cos x \quad \Longrightarrow \quad f'(x) = -\sin x \, . \tag{1.81}$$

The cosine, too, is differentiable for all real x. The calculation of the first derivative is performed in a completely analogous manner as that for the sine in the preceding example and will be explicitly done as Exercise 1.1.6.

The derivative of a function $f(x)$ is in general again a function of x and can possibly also be further differentiated. That leads to the concept of

'higher' derivatives

In case the respective limits exists, one writes:

$$
\begin{aligned}
y &= f(x) = f^{(0)}(x) \\
y' &= f'(x) = \frac{d}{dx} f(x) \\
y'' &= f''(x) = \frac{d^2}{dx^2} f(x) \\
&\dots \qquad \dots \\
y^{(n+1)} &= f^{(n+1)}(x) = \frac{d^{n+1}}{dx^{n+1}} f(x) = \frac{d}{dx}\left(f^{(n)}(x)\right) \equiv \left(y^{(n)}\right)'
\end{aligned}
$$

Examples

$$
\begin{aligned}
f(x) = x^3 \ \curvearrowright f'(x) = 3x^2 \ \curvearrowright f''(x) = 6x \\
\curvearrowright f^{(3)}(x) = 6 \ \curvearrowright f^{(4)} = 0 \ \curvearrowright f^{(n)}(x) \equiv 0 \ \ \forall\, n \geq 4
\end{aligned}
$$

Functions which are differentiable to arbitrary order are called **'smooth'**.

1.1.9 Rules of Differentiation

We list some of the central rules for differentiating functions of one independent variable:

1. **constant factor:**

$$
y = c \cdot f(x) \implies y' = c \cdot f'(x) \ , \tag{1.82}
$$

proof:

$$
\begin{aligned}
y' &= \lim_{\Delta x \to 0} \frac{c \cdot f(x + \Delta x) - c \cdot f(x)}{\Delta x} \\
&= c \cdot \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = c \cdot f'(x) \ .
\end{aligned}
$$

2. **sum:**

$$
y = f(x) + g(x) \implies y' = f'(x) + g'(x) \ . \tag{1.83}
$$

This can directly be read off from the definition.

3. **product:**

$$y = f(x) \cdot g(x) \implies y' = f'(x) \cdot g(x) + f(x) \cdot g'(x) \; , \tag{1.84}$$

proof:

$$\begin{aligned} y' =& \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left(f(x+\Delta x) \cdot g(x+\Delta x) - f(x) \cdot g(x) \right) \\ =& \lim_{\Delta x \to 0} \frac{1}{\Delta x} \Big(\left(f(x+\Delta x) - f(x) \right) \cdot g(x+\Delta x) \\ &+ g(x+\Delta x) \cdot f(x) - f(x) \cdot g(x) \Big) \\ =& \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \cdot g(x+\Delta x) \\ &+ \lim_{\Delta x \to 0} f(x) \cdot \frac{g(x+\Delta x) - g(x)}{\Delta x} \\ =& f'(x) \cdot g(x) + f(x) \cdot g'(x) \; . \end{aligned}$$

In the last step we have exploited the fact that the functions g and f of course have to be continuous since otherwise the derivatives would not exist.

Example Suppose $n \in \mathbb{N}$, then:

$$\begin{aligned} x^n \cdot \frac{1}{x^n} = 1 \quad \curvearrowright \quad & (x^n)' \cdot \frac{1}{x^n} + x^n \cdot \left(\frac{1}{x^n} \right)' = 0 \\ \curvearrowright \quad & n x^{n-1} \cdot \frac{1}{x^n} = -x^n \cdot (x^{-n})' \; . \end{aligned}$$

As an extension to (1.77) we now have a code for how to differentiate a power of x with *negative* exponent:

$$(x^{-n})' = -n \, x^{-(n+1)} \; . \tag{1.85}$$

4. **quotient**

$$y = \frac{f(x)}{g(x)} \; ; \; g(x) \neq 0 \implies y' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)} \; , \tag{1.86}$$

proof:
First we investigate the derivative of

$$h(x) = \frac{1}{g(x)} \; ,$$

where we can again presume the continuity of $g(x)$:

$$\begin{aligned} h'(x) &= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\frac{1}{g(x+\Delta x)} - \frac{1}{g(x)}\right) \\ &= -\lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \cdot \frac{1}{g(x+\Delta x)\cdot g(x)} \\ &= -g'(x) \cdot \frac{1}{g^2(x)} \ . \end{aligned}$$

With the product rule (1.84) the assertion is then proven.

5. **chain rule:**

$$y = f\,(g(x)) \implies y' = \frac{df}{dg} \cdot g'(x) \ , \tag{1.87}$$

Proof Let $u = g(x)$ be differentiable in x and $y = f(u)$ differentiable in $u = g(x)$, then it can be written with $g(x+\Delta x) = u + \Delta u$ (continuity!):

$$\frac{f\,(g(x+\Delta x)) - f\,(g(x))}{\Delta x} = \frac{f(u+\Delta u) - f(u)}{\Delta u} \cdot \frac{g(x+\Delta x) - g(x)}{\Delta x} \ .$$

Utilizing once more the continuity of $u = g(x)$ ($\Delta x \to 0 \curvearrowright \Delta u \to 0$) we get:

$$\lim_{\Delta x \to 0} \frac{f\,(g(x+\Delta x)) - f\,(g(x))}{\Delta x} = \frac{d}{du} f(u) \cdot \frac{d}{dx} g(x) \ .$$

Formally we thus obtain a result which appears to be taken from *'normal fractional arithmetic'*:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \ .$$

Example We demonstrate the chain rule in connection with an important application. For this purpose we calculate the first derivative of

$$y = f(x) = \ln x \ ,$$

which exists for all positive x. We use the chain rule together with (1.79) to differentiate the expression $x = e^{\ln x}$ with respect to x:

$$1 = e^{\ln x} \frac{d}{dx} \ln x \ .$$

Obviously this yields:

$$\frac{d}{dx}\ln x = \frac{1}{e^{\ln x}} = \frac{1}{x} \; , \tag{1.88}$$

Now we can generalize once more the rules of differentiation (1.77) and (1.85), respectively. Suppose that α is now an arbitrary real number. Then we have:

$$x^\alpha = e^{\ln x^\alpha} = e^{\alpha \ln x}$$

$$\curvearrowright \quad \frac{dx^\alpha}{dx} = \left.\frac{de^u}{du}\right|_{u=\alpha \ln x} \cdot \frac{d(\alpha \ln x)}{dx} = e^{\alpha \ln x} \cdot \alpha \frac{1}{x} = x^\alpha \alpha \frac{1}{x} \; .$$

That yields the generalization of (1.77) and (1.85), respectively,

$$\frac{dx^\alpha}{dx} = \alpha\, x^{\alpha - 1} \; , \tag{1.89}$$

which is thus proven now for arbitrary real numbers α.

6. Finally, we will consider the inverse function (1.38):

$$f^{-1}\left(f(x)\right) = x \; .$$

With the chain rule we have:

$$\frac{d}{df}\left(f^{-1}\right)(f) \cdot f'(x) = 1 \; .$$

That means:

$$\frac{d}{df}\left(f^{-1}\right)(f) = \frac{1}{f'(x)} \; . \tag{1.90}$$

With

$$y = f(x) \quad \curvearrowright \quad x = f^{-1}(y) \quad \curvearrowright \quad \frac{d}{dy}\left(f^{-1}(y)\right) = \frac{dx}{dy}$$

we get an expression which again seems to stem from elementary fractional arithmetic:

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} \; . \tag{1.91}$$

At the end, to demonstrate the above-derived rules, let us inspect the following

Examples

- to 1.:

$$f(x) = a \sin x \;\; ; \; a \in \mathbb{R} \implies f'(x) = a \cos x$$

- to 2.:

$$f(x) = x^5 - 3 \ln x \implies f'(x) = 5x^4 - \frac{3}{x}$$

- to 3.:

$$f(x) = x^3 \cos x \implies f'(x) = 3x^2 \cos x - x^3 \sin x$$

- to 4.:

$$f(x) = \frac{x^2}{\sin x} \implies f'(x) = \frac{2x \sin x - x^2 \cos x}{\sin^2 x}$$

- to 5.:

$$f(x) = 3 \sin(x^3) \implies f'(x) = 3 \cos(x^3) \cdot 3x^2 = 9x^2 \cos(x^3)$$

1.1.10 Taylor Expansion

Occasionally it is unavoidable for a physicist to digress from rigorous mathematical exactness in order to come by adopting some *'reasonable'* mathematical simplifications to concrete physical results. In this respect, the so-called *'Taylor expansion (series)'* of a mathematical function $y = f(x)$ represents a very important and frequently used auxiliary means. We assume that this function possesses arbitrarily many continuous derivatives at $x = x_0$. Then the following power series expansion is valid what is explicitly proved as Exercise 1.1.9:

$$\begin{aligned} f(x) &= f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \ldots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \qquad (1.92) \\ f^{(n)}(x_0) &= f^{(n)}(x)\big|_{x = x_0} \;. \end{aligned}$$

The assumption $|x - x_0| < 1$ guarantees the convergence of the series. Then one can assume that the terms of the series become smaller and smaller with increasing index n, so that it should be allowed, in the sense of a controlled approximation, to cut the series after a finite number of summands. The error can strictly be estimated as will be demonstrated in Sect. 1.2 of volume 3.

However, the Taylor expansion can also be used for the derivation of *exact* series as is shown by the following examples:

1.

$$f(x) = \frac{1}{1+x} \;\; ; \; x_0 = 0 \;\; ; \; |x| < 1 \; .$$

We use

$$f(0) = 1 \;\; ; \; f'(0) = -1(1+0)^{-2} = -1 \;\; ; \; f''(0) = 2(1+0)^{-3} = 2 \; \ldots$$
$$\curvearrowright \;\; f^{(n)}(0) = (-1)^n \, n! \;\; ; \; x - x_0 = x \; .$$

That means

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-x)^n \; . \tag{1.93}$$

Compare this result with (1.30)!

2.

$$f(x) = \sin x \;\; ; \; x_0 = 0 \; .$$

Now we apply the following terms in the Taylor series (1.92):

$$f(0) = 0 \;\; ; \; f'(0) = \cos(0) = 1 \;\; ; \; f''(0) = -\sin(0) = 0 \;\; ;$$
$$f'''(0) = -\cos(0) = -1 \;\; ; \; \ldots$$
$$\curvearrowright \;\; f^{(2n)}(0) = 0 \;\; ; \; f^{(2n+1)}(0) = (-1)^n \; .$$

Thus we find in this case:

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \; . \tag{1.94}$$

This expansion has already been anticipated in (1.51).

3.

$$f(x) = e^x \;\; ; \; x_0 = 0 \; .$$

With (1.79) it holds:

$$e^0 = 1 \;\; ; \;\; \frac{d}{dx}\, e^x = e^x \;\; \curvearrowright \;\; \frac{d^n}{dx^n}\, e^x = e^x \;\; \curvearrowright \;\; \frac{d^n}{dx^n}\, e^x \bigg|_{x\,=\,0} = 1 \; .$$

Therewith we get:

$$e^x = \sum_{n\,=\,0}^{\infty} \frac{x^n}{n!} \; . \tag{1.95}$$

We have already used this result in (1.64).

1.1.11 Limiting Values of Indeterminate Expressions

We now consider expressions of limiting values of type $0/0$ and $\pm\infty/\infty$, respectively, which, of course, are not defined as well as in the following special examples:

- $$\frac{\ln(1+x)}{x} \xrightarrow{x\to 0} \frac{0}{0}$$
- $$\frac{\sin x}{x} \xrightarrow{x\to 0} \frac{0}{0}$$
- $$\frac{\ln x}{\frac{1}{x}} \xrightarrow{x\to 0} \frac{-\infty}{\infty}$$

For expressions of this kind we have the very useful *l'Hospital's rule*, which, however, has to be presented here without proof. If the function

$$f(x) = \frac{\varphi(x)}{\psi(x)}$$

gives for $x \to a$ an indetermined expression of the above kind then one can use

$$\lim_{x\,\to\,a} f(x) = \lim_{x\,\to\,a} \frac{\varphi'(x)}{\psi'(x)} \; . \tag{1.96}$$

If the right-hand side persists to be not defined one replaces the first by the second derivatives. If even then the quotient on the right-hand side continues to be undetermined one takes the third derivatives, and so on. Hence, the above examples are calculated as follows:

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} = \lim_{x \to 0} \frac{\frac{1}{1+x}}{1} = 1 \tag{1.97}$$

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1 \tag{1.98}$$

$$\lim_{x \to 0} \frac{\ln x}{\frac{1}{x}} = \lim_{x \to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0} (-x) = 0 \tag{1.99}$$

1.1.12 Extreme Values

For an actual sketching of a curve it is useful and necessary to know the (local, global) minima and maxima of the corresponding function $f(x)$. We establish:

$f(x)$ has a local maximum (minimum) at x_0,

if there exists a $\delta > 0$ so that it holds for all $x \in U_\delta(x_0)$:

$$f(x) \le f(x_0) \implies \textit{maximum}$$

$$f(x) \ge f(x_0) \implies \textit{minimum}$$

Here we understand by $U_\delta(x_0)$ the *δ-neighbourhood of* x_0:

$$U_\delta(x_0) = \{x;\ |x - x_0| < \delta\}\ . \tag{1.100}$$

Proposition

If $f(x)$ is differentiable at x_0

having there a (local) extremum, then it must hold:

$$f'(x_0) = 0$$

We demonstrate the **proof** for the case of a minimum (Fig. 1.13). In this case it holds if only $|x - x_0|$ is *sufficiently small*:

$$\frac{f(x) - f(x_0)}{x - x_0} \longrightarrow \begin{cases} \ge 0 & \text{for} \quad x > x_0 \\ \le 0 & \text{for} \quad x < x_0 \end{cases}$$

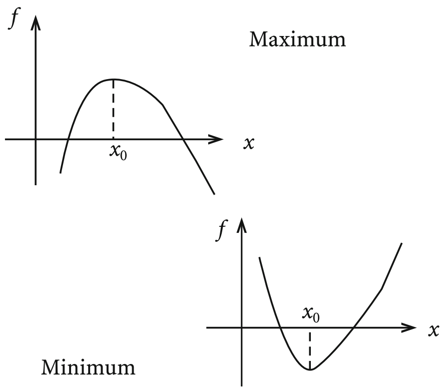


Fig. 1.13 Example of a function $y = f(x)$ with, respectively, a (local) maximum and minimum at x_0

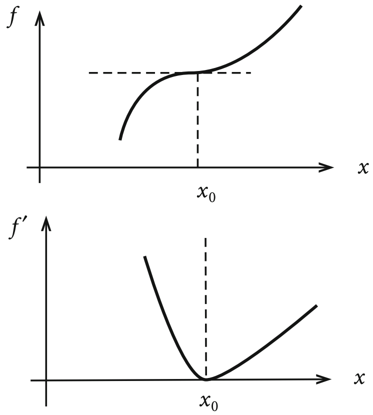


Fig. 1.14 Inflection point of a function $f(x)$ at $x = x_0$

Then it must necessarily be concluded:

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) = 0 \, .$$

However, one has to bear in mind that $f'(x_0) = 0$ turns out to be only a necessary but **not a sufficient** condition for an extremum. It could also be an **'inflection point'**! For the example in Fig. 1.14 the slope $f'(x)$ is monotonically decreasing if $x < x_0$ and monotonically increasing if $x > x_0$. That means:

$$f''(x) \begin{cases} \leq 0 & \text{for} \quad x < x_0 \\ \geq 0 & \text{for} \quad x > x_0 \end{cases}$$

and therewith:

$$f''(x_0) = 0 \qquad \text{(inflection point)} \tag{1.101}$$

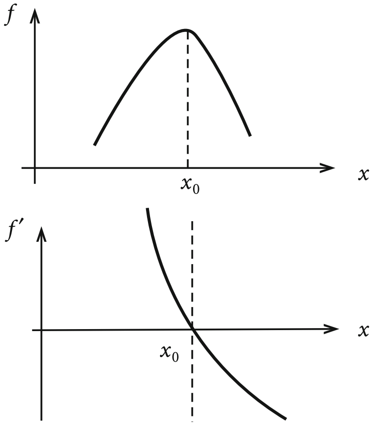


Fig. 1.15 Function $f(x)$ with a maximum at $x = x_0$ and its derivative $f'(x)$

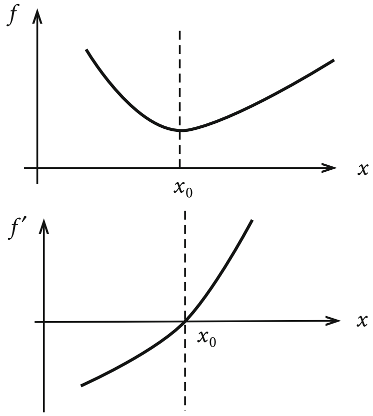


Fig. 1.16 Function $f(x)$ with a minimum at $x = x_0$ and its derivative $f'(x)$

A **sufficient criterion** for an extremum at the point $x = x_0$ can easily be read off from Figs. 1.15 and 1.16:

$$f'(x_0) = 0 \quad \text{and} \quad f''(x_0) \begin{cases} > 0 & : \quad \text{minimum} \\ < 0 & : \quad \text{maximum} \end{cases} . \tag{1.102}$$

Regarding (1.101) it is to be noted that it is also only a **necessary** condition for an inflection point, while a **sufficient** condition would be:

$$f'(x_0) = f''(x_0) = 0 \quad \text{and} \quad f'''(x_0) \neq 0 . \tag{1.103}$$

The general case, however, must be accepted here again without proof:

Let us assume a sufficiently often differentiable function $f(x)$ with the following properties at $x = x_0$:

$$f'(x_0) = f''(x_0) = \ldots = f^{(n)}(x_0) = 0 \text{ with } f^{(n+1)}(x_0) \neq 0 \quad (n \geq 3) , \tag{1.104}$$

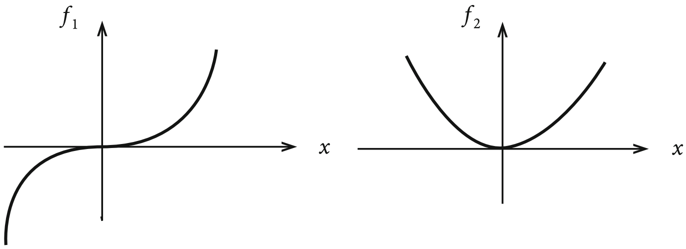


Fig. 1.17 Schematic behavior of the functions $f_1(x) = x^3$ and $f_2(x) = x^4$ in the vicinity of $x = 0$

then $f(x)$ exhibits at $x = x_0$

- *a maximum if n is an odd integer and $f^{(n+1)}(x_0) < 0$,*
- *a minimum if n is an odd integer and $f^{(n+1)}(x_0) > 0$,*
- *an inflection point (with horizontal tangent) if n is an even integer.*

The above discussed special cases are obviously contained herein.

Let us consider two examples to visualize (1.104) (see Fig. 1.17)

1.
$$f_1(x) = x^3 \quad \text{at} \quad x = 0$$

One immediately finds:

$$f_1'(0) = f_1''(0) = 0 \quad ; \quad f_1'''(0) = 6 > 0 \; .$$

Thus the function has an inflection point at $x = 0$.

2.
$$f_2(x) = x^4 \quad \text{at} \quad x = 0$$

In this case holds:

$$f_2'(0) = f_2''(0) = f_2'''(0) = 0 \quad ; \quad f_2^{(4)}(0) = 24 > 0 \; .$$

This function exhibits a minimum at $x = 0$.

1.1.13 Exercises

Exercise 1.1.1 Determine the limiting values of the sequences $\{a_n\}$ for $n \to \infty$ $(n \in \mathbb{N})$

1.

$$a_n = \frac{\sqrt{n}}{n}$$

2.

$$a_n = \frac{n^3 + 1}{2n^3 + n^2 + n}$$

3.

$$a_n = \frac{n^2 - 1}{(n + 1)^2} + 5$$

Exercise 1.1.2

1. Calculate the following sums:

$$S_3 = \sum_{m=1}^{3} 3 \left(\frac{1}{2}\right)^m \qquad ; \qquad S = \sum_{m=1}^{\infty} 3 \left(\frac{1}{2}\right)^m .$$

2. Is 1.111 ... a rational number? If yes, which one?

Exercise 1.1.3 Show that the harmonic series (1.27) does not converge in spite of $\lim_{m \to \infty} \frac{1}{m} = 0$!

Exercise 1.1.4 Try to simplify the following expressions for trigonometric functions:

-
$$\cos^2 \varphi \cdot \tan^2 \varphi + \cos^2 \varphi$$

-
$$\frac{1 - \cos^2 \varphi}{\sin \varphi \cdot \cos \varphi}$$

-
$$1 - \frac{1}{\cos^2 \varphi}$$

-
$$\frac{1}{1 - \sin \varphi} + \frac{1}{1 + \sin \varphi}$$

•

$$\frac{\sin(\varphi_1 + \varphi_2) + \sin(\varphi_1 - \varphi_2)}{\cos(\varphi_1 + \varphi_2) + \cos(\varphi_1 - \varphi_2)}$$

•

$$\frac{\cos^2 \varphi}{\sin 2\varphi}$$

Exercise 1.1.5 Prove the formula

$$1 - \cos\varphi = 2 \sin^2 \frac{\varphi}{2} \ .$$

which has been used for the derivation of (1.80).

Exercise 1.1.6 Verify the following relations for the first derivatives of the trigonometric functions:

1.

$$\frac{d}{dx} \cos x = -\sin x$$

2.

$$\frac{d}{dx} \tan x = \frac{1}{\cos^2 x}$$

3.

$$\frac{d}{dx} \cot x = -\frac{1}{\sin^2 x} \ .$$

Exercise 1.1.7 Find the first derivatives of the following functions:

1.

$$f_1(x) = 3x^5$$

2.

$$f_2(x) = 7x^3 - 4x^{\frac{3}{2}}$$

3.

$$f_3(x) = \frac{x^3 - 2x}{5x^2}$$

4.

$$f_4(x) = \sqrt[3]{x}$$

5.

$$f_5(x) = \sqrt{1 + x^2}$$

6.

$$f_6(x) = 3\cos(6x)$$

7.

$$f_7(x) = \sin(x^2)$$

8.

$$f_8(x) = \exp(2x^3 - 4)$$

9.

$$f_9(x) = \ln(2x + 1) \ .$$

Exercise 1.1.8 Use the rule of differentiation for the inverse function (1.90) in order to find the derivatives of the arc functions (inverses of the trigonometric functions):

1.

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1 - x^2}}$$

2.

$$\frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1 - x^2}}$$

3.

$$\frac{d}{dx} \arctan x = \frac{1}{1 + x^2}$$

4.

$$\frac{d}{dx} \mathrm{arccot} x = -\frac{1}{1 + x^2} \ .$$

Exercise 1.1.9 Assume that the function $f(x)$ is arbitrarily often differentiable. In addition, an expansion into a power series shall exist:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \, .$$

All x for which the series converges constitute the so-called *region of convergence* of the function $f(x)$.

1. Determine the coefficients a_n from the behavior of the function f and its derivatives at $x = 0$.
2. Verify Eq. (1.92):

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} \, (x - x_0)^n \, .$$

Exercise 1.1.10 Why can the function

$$f(x) = (1 + x)^n$$

for $x \ll 1$ be replaced to a good approximation by

$$f(x) \approx 1 + n\,x + \frac{n(n-1)}{2}\,x^2 \text{ ?}$$

Exercise 1.1.11 Verify the series expansion of the logarithm ($|x| < 1$):

$$\ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$$

Exercise 1.1.12 Verify the series expansion (1.58) of the cosine!

Exercise 1.1.13 Given the function

$$f(x) = \frac{x - \sin x}{e^x + e^{-x} - 2} \, .$$

Find the value $f(0)$, on the one hand by use of the series expansions for the exponential function and the sine, on the other hand by applying l'Hospital's rule (1.96).

Exercise 1.1.14 Find the zeros and the extreme values of the following functions:

1.

$$f(x) = 2x^4 - 8x^2$$

2.

$$g(x) = \sin\left(\frac{1}{2}x\right) .$$

1.2 Elements of Integral Calculus

1.2.1 Notions

The technique of *'differentiation'*, which we discussed in the previous section, follows the scope of work:

$$\begin{aligned} &\text{given:} & y &= f(x) \\ &\text{finding:} & f'(x) &= \tfrac{df}{dx} \ : \ \textit{`derivation'} , \end{aligned}$$

The reverse program, namely

$$\begin{aligned} &\text{given:} & f'(x) &= \tfrac{df}{dx} \\ &\text{finding:} & y &= f(x) \end{aligned}$$

leads to the technique of *'integration'*. Consider for example

$$f'(x) = c = \text{const} ,$$

then we remember according to (1.77) that

$$y = f(x) = c \cdot x$$

fulfills the condition $f'(x) = c$.

Definition $F(x)$ is the *'antiderivative (primitive function)'* of $f(x)$, if it holds:

$$F'(x) = f(x) \qquad \forall x . \tag{1.105}$$

In this connection the above example means:

$$f(x) \equiv c \quad \curvearrowright \quad F(x) = c \cdot x + d . \tag{1.106}$$

Because of the constant d the result comes out as a full family of curves. Fixing d needs the introduction of *‘boundary conditions’*. We accept that:

‘Integration’ : Searching for the antiderivative (primitive function)

To generate a graphic image, the integral can be interpreted as the area under the curve $y = f(x)$. If the curve $y = f(x)$ is given then we ask ourselves how we can determine the area F in Fig. 1.18 under the curve between the limits $x = a$ and $x = b$. This can easily be done for the special case that $f(x)$ represents a straight line. However, how can we calculate the area under an arbitrary (continuous) function $f(x)$?

In a first step, we approach the calculation of the area by decomposing the interval $[a, b]$ in n equal sub-intervals Δx_n,

$$\Delta x_n = \frac{b-a}{n} \qquad n \in \mathbb{N}/0 \,, \tag{1.107}$$

where x_i is the center of the i-th partial interval:

$$x_i = a + \left(i - \frac{1}{2}\right) \Delta x_n \;\; ; \;\; i = 1, 2 \ldots, n \,. \tag{1.108}$$

Then $f(x_i)\, \Delta x_n$ is the area of the i-th *pillar* in Fig. 1.19. Hence it holds approximately for the area F:

$$F \approx \sum_{i=1}^{n} f(x_i)\, \Delta x_n \,. \tag{1.109}$$

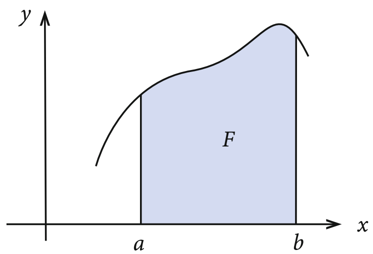


Fig. 1.18 Interpretation of the integral as area under the curve $y = f(x)$

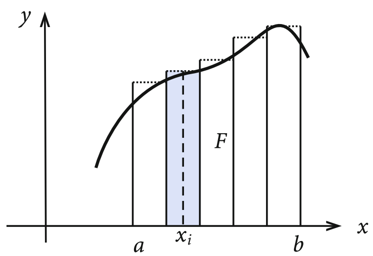


Fig. 1.19 Riemann sum for calculating the integral

For $n \to \infty$ the sub-intervals become arbitrarily small ($\Delta x_n \to 0$), and it appears obvious that the mistake which results from approximating F by the sum of the *'pillar areas'* also becomes arbitrarily small. The limiting value for $n \to \infty$ comes out as a real number and is called:

'definite (Riemann) integral'

$$F = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\, \Delta x_n \equiv \int_a^b f(x)\, dx\,. \tag{1.110}$$

One identifies a as the *lower* and b as the *upper limit of integration*. $f(x)$ is the *integrand* and x the *integration variable*. The equivalence of the definition (1.105) of $F(x)$ as antiderivative of $f(x)$ and the above definition as definite integral, however, has still to be demonstrated.

1.2.2 First Rules of Integration

Some important rules follow directly from the definition of the integral:

- *Identical bounds of integration:*

$$\int_a^a f(x)\, dx = 0 \tag{1.111}$$

- *The 'area' in the sense of an integral has a sign because of* $f(x) \gtrless 0$!
  For the example in Fig. 1.20 one recognizes:

$$F_1 = \int_a^{x_1} f(x)\, dx > 0$$

$$F_2 = \int_{x_1}^{x_2} f(x)\, dx < 0$$

$$F_3 = \int_{x_2}^{x_3} f(x)\, dx > 0$$

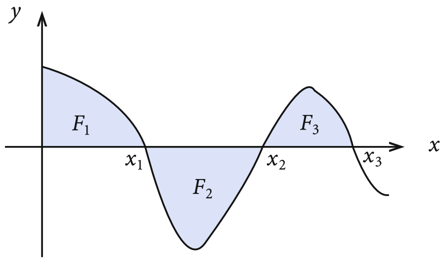


Fig. 1.20 Illustration of the sign fixing of the definite integral

- *Constant factor $c \in \mathbb{R}$:*

$$\int_a^b c \cdot f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} c \cdot f(x_i)\Delta x_n = c \cdot \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x_n \;.$$

Hence it holds:

$$\int_a^b c \cdot f(x)\,dx = c \cdot \int_a^b f(x)\,dx \;. \tag{1.112}$$

- *Sum:*
Assume

$$f(x) \equiv g(x) + h(x) \;.$$

then it follows from the definition of the Riemann integral:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(g(x_i) + h(x_i)\right)\Delta x_n$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} g(x_i)\Delta x_n + \lim_{n\to\infty} \sum_{i=1}^{n} h(x_i)\Delta x_n \;.$$

That means:

$$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx + \int_a^b h(x)\,dx \;. \tag{1.113}$$

The last two rules of integration (1.112) and (1.113) demonstrate the linearity of the integral.

- *Partitioning the interval of integration:*

$$\Delta x_n = \frac{b-a}{n} = \Delta x_n^{(1)} + \Delta x_n^{(2)} = \frac{x_0 - a}{n} + \frac{b - x_0}{n}$$

Therewith we can write:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^{(1)}\right)\Delta x_n^{(1)} + \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^{(2)}\right)\Delta x_n^{(2)} \;.$$

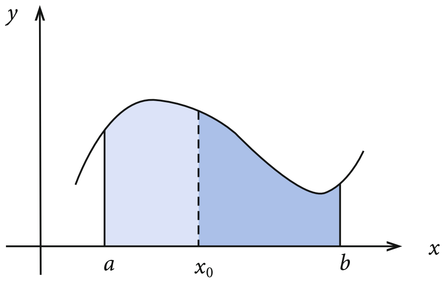


Fig. 1.21 Partitioning the interval of integration

$x_i^{(1,2)}$ are defined as in (1.108) with corresponding $\Delta x_n^{(1,2)}$ $\left(x_i^{(1)} = a + (i - \frac{1}{2})\frac{x_0 - a}{n}\right.$, $\left. x_i^{(2)} = x_0 + (i - \frac{1}{2})\frac{b - x_0}{n}\right)$ (Fig. 1.21). Thus it holds:

$$\int_a^b f(x)\,dx = \int_a^{x_0} f(x)\,dx + \int_{x_0}^b f(x)\,dx \qquad (a \le x_0 \le b)\ . \tag{1.114}$$

- *Interchanged bounds of integration:*
  Formally (1.111) and (1.114) imply:

$$0 = \int_a^a f(x)\,dx = \int_a^b f(x)\,dx + \int_b^a f(x)\,dx\ .$$

Consequently:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\ . \tag{1.115}$$

One should notice that on the right-hand side it must hold that $dx < 0$ because of $b > a$!

1.2.3 Fundamental Theorem of Calculus

We consider the definite integral over a continuous function $f(t)$, but now with variable upper limit:

$$F(x) = \int_a^x f(t)\,dt \qquad \textbf{‘area function’} \tag{1.116}$$

The area under the curve $f(t)$ in this case is not constant but a function of x (Fig. 1.22). If the upper bound of integration is shifted by Δx the area will change by:

$$\Delta F = F(x+\Delta x) - F(x) = \int_a^{x+\Delta x} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+\Delta x} f(t)\,dt\ .$$

In the last step we have used the rule (1.114). Without explicit proof we accept the important

'mean value theorem of integral calculus'

This theorem implies:

$$\exists\ \hat{x} \in [x, x+\Delta x] \quad \text{with} \quad \Delta F = \Delta x \cdot f(\hat{x})\ . \tag{1.117}$$

Although not exactly proven the theorem appears rather plausible according to Fig. 1.23. So we can further conclude:

$$F'(x) = \lim_{\Delta x \to 0} \frac{\Delta F}{\Delta x} = \lim_{\Delta x \to 0} f(\hat{x}) = f(x)\ .$$

Thus after (1.105), the area function is the antiderivative of $f(x)$! Furthermore, the equivalence of the definitions (1.105) and (1.110) for the antiderivative, which

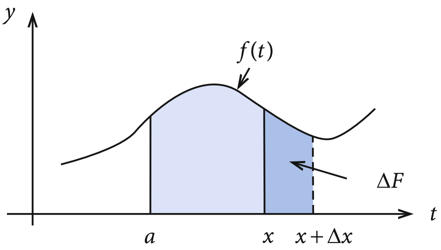


Fig. 1.22 Definition of the area function

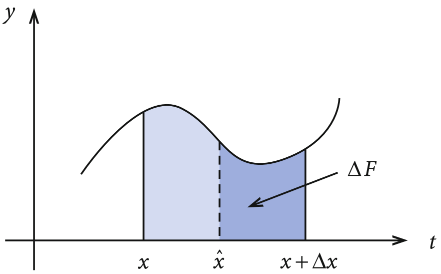


Fig. 1.23 To the mean value theorem (1.117)

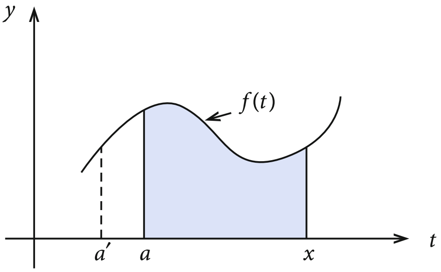


Fig. 1.24 To the influence of the lower bound of integration on fixing the antiderivative

remained unsettled in Sect. 1.2.1, is now settled.

'fundamental theorem of calculus'

$$\frac{d}{dx}F(x) \equiv \frac{d}{dx}\int_a^x f(t)\,dt = f(x)\;. \tag{1.118}$$

The successive performing of integration and differentiation obviously cancel each other!

integration $\cong$ inversion of differentiation

The influence of the lower limit of integration in (1.118) still appears unsettled (Fig. 1.24). To clarify this we therefore investigate:

$$\tilde{F}(x) = \int_{a'}^x f(t)\,dt = \underbrace{\int_{a'}^a f(t)\,dt}_{=A,\text{ independent of }x} + \underbrace{\int_a^x f(t)\,dt}_{F(x)}\;.$$

Therewith it follows that both $F(x)$ and $\tilde{F}(x)$ are antiderivatives of $f(x)$:

$$\tilde{F}(x) = F(x) + A \quad \curvearrowright \quad \frac{d}{dx}\tilde{F}(x) = \frac{d}{dx}F(x) = f(x)\;.$$

The lower limit of integration is therefore in a certain sense dummy, the antiderivative is uniquely fixed except for an additive constant:

$$F(x) \quad \Longleftrightarrow \quad \tilde{F}(x) = F(x) + A\;. \tag{1.119}$$

Therefore one introduces the

$$\textbf{'indefinite integral':} \qquad F(x) = \int f(x)\,dx\;. \tag{1.120}$$

defining therewith the set of all antiderivatives of $f(x)$!

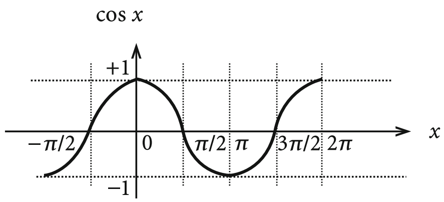


Fig. 1.25 To the definite integral of the cosine

The *'definite integral'*, already known to us, can also be expressed by the antiderivative:

$$F(x) + \alpha = \int_a^x f(t)\,dt \quad \curvearrowright \quad F(a) + \alpha = 0 \quad \curvearrowright \quad F(a) = -\alpha\ .$$

Therewith it follows that when we take $x = b$:

$$\int_a^b f(x)\,dx = F(b) + \alpha = F(b) - F(a) \equiv F(x)\Big|_a^b\ . \tag{1.121}$$

At the extreme we have introduced the usual symbol for the definite integral.

Example

$$f(x) = \cos x \quad \curvearrowright \quad F(x) = \sin x + c \quad (c \in \mathbb{R})\ . \tag{1.122}$$

The antiderivative can easily be *guessed* with (1.80) (Fig. 1.25). Hence we obtain the following definite integrals for the cosine:

- $$\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \cos x\,dx = \sin x\Big|_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} = 1 - (-1) = 2$$

- $$\int_0^{+\frac{\pi}{2}} \cos x\,dx = \sin x\Big|_0^{\frac{\pi}{2}} = 1 - 0 = 1$$

- $$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos x\,dx = \sin x\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = (-1) - 1 = -2 \qquad \text{(sign of the area!)}$$

- $$\int_{\frac{3\pi}{2}}^{2\pi} \cos x\,dx = \sin x\Big|_{\frac{3\pi}{2}}^{2\pi} = 0 - (-1) = 1$$

•

$$\int_0^{\pi} \cos x \, dx = \sin x \Big|_0^{\pi} = 0 - 0 = 0 \qquad \text{(sign of the area!)}$$

1.2.4 *The Technique of Integration*

The goal is to find the antiderivative $F(x)$ of a given function $f(x)$ such that $F'(x) = f(x)$! Firstly one has to realize that there does not exist a generally valid algorithmic procedure of integration. Instead of this one has to act heuristically.

1. 'Guess' and 'Verify'

Let the function $f(x)$ be given, then the correct form of $F(x)$ can be *'guessed'* and subsequently be verified by differentiation: $F'(x) = f(x)$! An important help in this respect are of course integral tables. We list here some examples:

•

$$f_1(x) = x^n \quad (n \neq -1) \quad \curvearrowright \quad F_1(x) = \frac{x^{n+1}}{n+1} + c_1$$

•

$$f_2(x) = x^{-2.3} + x \quad \curvearrowright \quad F_2(x) = \frac{x^{-1.3}}{-1.3} + \frac{x^2}{2} + c_2$$

•

$$f_3(x) = \frac{1}{x} \quad (x > 0) \quad \curvearrowright \quad F_3(x) = \ln x + c_3$$

•

$$f_4(x) = \sin x \quad \curvearrowright \quad F_4(x) = -\cos x + c_4$$

•

$$f_5(x) = \cos x \quad \curvearrowright \quad F_5(x) = \sin x + c_5$$

•

$$f_6(x) = e^x \quad \curvearrowright \quad F_6(x) = e^x + c_6$$

The last three relations can of course be proven also directly via corresponding series expansions. We briefly demonstrate them:

$$
\begin{aligned}
\int \sin x \, dx &= \sum_{n=0}^{\infty} (-1)^n \int \frac{x^{2n+1}}{(2n+1)!} \, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n+2)!} + c \\
&= \sum_{n'=1}^{\infty} (-1)^{n'-1} \frac{x^{2n'}}{(2n')!} + c = -\sum_{n'=0}^{\infty} (-1)^{n'} \frac{x^{2n'}}{(2n')!} + 1 + c \\
&= -\cos x + c_4 \\
\int \cos x \, dx &= \sum_{n=0}^{\infty} (-1)^n \int \frac{x^{2n}}{(2n)!} \, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} + c_5 \\
&= \sin x + c_5 \\
\int e^x \, dx &= \sum_{n=0}^{\infty} \int \frac{x^n}{n!} \, dx = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} + c \\
&= \sum_{n'=1}^{\infty} \frac{x^{n'}}{(n')!} + c = \sum_{n'=0}^{\infty} \frac{x^{n'}}{(n')!} - 1 + c \\
&= e^x + c_6 \, .
\end{aligned}
$$

2. Substitution of the Variable

One tries to modify the integration variable in such a way that the integral becomes a well-known standard integral. That is done according to the following steps:

- Replace

$$
x \longrightarrow u = u(x) \quad \curvearrowright \quad dx \longrightarrow du = \frac{du}{dx} dx \quad \curvearrowright \quad dx = \left(\frac{du}{dx}\right)^{-1} (u) \, du \, .
$$

  In case of a definite integral we have to notice that the limits of integration are also usually changed with the substitution ($x_i \longrightarrow u_i = u(x_i)$).
- It is integrated now with respect to u. The integrand changes accordingly:

$$
f(x) \longrightarrow \tilde{f}(u) = f(x(u)) \, .
$$

- Antiderivative is now $\tilde{F}(u)$:

$$
\tilde{F}(u) = \int^{u} \tilde{f}(u') \left(\frac{du'}{dx}\right)^{-1} (u') \, du' \, .
$$

- Back transformation:

$$\tilde{F}(u) \longrightarrow \tilde{F}(u(x)) \equiv F(x) \; .$$

We demonstrate the procedure by two examples:

(a)

$$F(x) = \int e^{ax}\, dx \qquad (a \in \mathbb{R}) \, .$$

We substitute advantageously $u = ax \curvearrowright du = adx$; $\tilde{f}(u) = e^u$. Therewith follows:

$$\int e^{ax}\, dx = \frac{1}{a} \int e^u\, du = \frac{1}{a}\, e^u + c = \tilde{F}(u) \; .$$

Hence we have found:

$$\int e^{ax}\, dx = \frac{1}{a}\, e^{ax} + c \; . \tag{1.123}$$

c is a real constant.

(b)

$$F(x) = \int (3 + 4x)^5\, dx \, .$$

In this case we substitute $u = 3 + 4x \curvearrowright du = 4dx$; $\tilde{f}(u) = u^5$. That leads to:

$$\int (3 + 4x)^5\, dx = \frac{1}{4} \int u^5\, du = \frac{u^6}{24} + c = \tilde{F}(u) \; .$$

So we are left with:

$$\int (3 + 4x)^5\, dx = \frac{1}{24} (3 + 4x)^6 + c \; . \tag{1.124}$$

c is again an arbitrary real constant.

3. Integration by Parts

Starting point is the product rule of differentiation (1.84)

$$\frac{d}{dx} \left(f_1(x) \cdot f_2(x) \right) = \frac{df_1(x)}{dx} f_2(x) + f_1(x)\, \frac{df_2(x)}{dx} \; .$$

which also means

$$f_1(x)\,\frac{df_2(x)}{dx} = \frac{d}{dx}\,(f_1(x)\cdot f_2(x)) - \frac{df_1(x)}{dx} f_2(x)$$

and therewith

$$\int f_1(x)\,\frac{df_2(x)}{dx}\,dx = f_1(x)\cdot f_2(x) - \int \frac{df_1(x)}{dx} f_2(x)\,dx + c\;. \tag{1.125}$$

The method thus consists in splitting the integrand $f(x) = f_1(x)f_2'(x)$ into $f_1(x)$ and $f_2(x)$ in such a way that the resulting $g(x) = f_1'(x)f_2(x)$ is easier to integrate than $f(x)$. We demonstrate this again with two examples:

(a)

$$F_1(x) = \int x\,e^{\alpha x}\,dx\,.$$

We take

$$f_1(x) = x \qquad \text{and} \qquad f_2'(x) = e^{\alpha x}\;.$$

That means

$$f_1'(x) = 1 \qquad \text{and} \qquad f_2(x) = \frac{e^{\alpha x}}{\alpha}\;.$$

With this we find:

$$F_1(x) = \frac{1}{\alpha} x\,e^{\alpha x} - \int 1\cdot\frac{e^{\alpha x}}{\alpha}\,dx + c' = \frac{1}{\alpha} x\,e^{\alpha x} - \frac{1}{\alpha^2}\,e^{\alpha x} + c\;.$$

Consequently, the result is:

$$\int x\,e^{\alpha x}\,dx = \frac{1}{\alpha}\,e^{\alpha x}\left(x - \frac{1}{\alpha}\right) + c\,. \tag{1.126}$$

(b)

$$F_2(x) = \int \sin^2 x\,dx\;.$$

We choose:

$$f_1(x) = \sin x \qquad \text{and} \qquad f_2'(x) = \sin x\,.$$

Then it must hold:

$$f_1'(x) = \cos x \quad \text{and} \quad f_2(x) = -\cos x \,.$$

That can be evaluated as follows:

$$\begin{aligned}\int \sin^2 x \, dx &= -\sin x \, \cos x + c + \int \cos^2 x \, dx \\ &= -\sin x \, \cos x + c + \int (1 - \sin^2 x) \, dx \\ &= -\sin x \, \cos x + \hat{c} + x - \int \sin^2 x \, dx \,.\end{aligned}$$

Therewith the antiderivative is found:

$$\int \sin^2 x \, dx = -\frac{1}{2} \sin x \cos x + \frac{x}{2} + c' \,. \tag{1.127}$$

1.2.5 Multiple Integrals

Multiple integrals as volume or surface integrals are reduced for their calculation to a set of simple one-dimensional integrals of the kind we have inspected up to now in the preceding sections (Fig. 1.26). Let us consider, as an popular example, of the total mass of a sphere with

$$\textit{`mass density'} \qquad \rho(\mathbf{r}) = \rho(x, y, z) = \left.\frac{dm}{dV}\right|_{\mathbf{r}} = \left.\frac{dm}{dxdydz}\right|_{\mathbf{r}} \,.$$

The volume element $dV = dx \, dy \, dz \equiv d^3r$ at $\mathbf{r}$ then contains the (infinitesimal) mass $dm = \rho(\mathbf{r}) \, dV$. Thus the total mass is given by the triple integral:

$$M = \int_V d^3r \, \rho(\mathbf{r}) = \int \int_V \int dx \, dy \, dz \, \rho(x, y, z) \,.$$

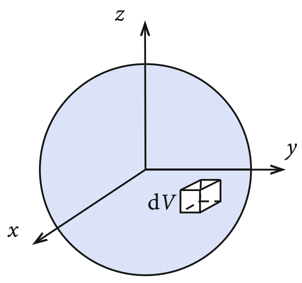


Fig. 1.26 To the calculation of the mass of a sphere by a triple integral

Therefore we have to perform three integrations over integrals which are fixed by the total volume. Here the limits of a particular integration may depend on the variables of the other integrations. For this reason we will distinguish two cases:

1. Constant Bounds of Integration

This is the simpler case. All single integrations are performed one after another according to the rules of the preceding subsections where while performing one the other variables are fixed:

$$\begin{aligned}
M &= \int_V d^3r\,\rho(\mathbf{r}) = \int\int_V\int dx\,dy\,dz\,\rho(x,y,z) \\
&= \int_{c_1}^{c_2} dz \int_{b_1}^{b_2} dy \underbrace{\int_{a_1}^{a_2} dx\rho(x,y,z)}_{\bar{\rho}(y,z;\,a_1,a_2)} \\
&= \int_{c_1}^{c_2} dz \underbrace{\int_{b_1}^{b_2} dy\bar{\rho}(y,z;\,a_1,a_2)}_{\bar{\bar{\rho}}(z;\,a_1,a_2,b_1,b_2)} \\
&= \int_{c_1}^{c_2} dz\bar{\bar{\rho}}\,(z;\,a_1,a_2,b_1,b_2) \\
&= M\,(a_1,a_2,b_1,b_2,c_1,c_2)\ .
\end{aligned}$$

The result is a real number. In the case of constant bounds of integration and a continuous integrand the various integrations are allowed to be interchanged.

Let us calculate as an example the *mass of a rectangular air column above the earth's surface* assuming it to be flat (Figs. 1.27 and 1.28). As a consequence of

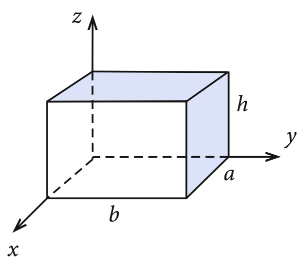


Fig. 1.27 To the calculation of the mass of a rectangular air column above earth's surface

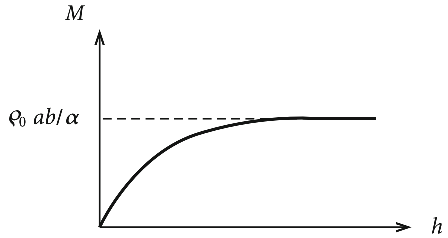


Fig. 1.28 Mass of a rectangular air column of height h above earth's surface as function of h

gravitation the air density decreases exponentially with increasing height:

$$\rho = \rho(z) = \rho_0\, e^{-\alpha z}\,.$$

$$\begin{aligned} M &= \int_0^h dz \int_0^b dy \int_0^a dx\, \rho_0\, e^{-\alpha z} = \rho_0 \cdot a \int_0^h dz \int_0^b dy\, e^{-\alpha z} \\ &= \rho_0 \cdot ab \int_0^h dz\, e^{-\alpha z} = \rho_0 \cdot ab \left(-\frac{1}{\alpha}\right) e^{-\alpha z}\Big|_0^h \\ &= \rho_0 \frac{ab}{\alpha}\left(1 - e^{-\alpha h}\right)\,. \end{aligned}$$

2. Non-constant Bounds of Integration

For at least one of the variables the multiple integral must have fixed bounds of integration and one of the variables must not appear in any of the other bounds of integration. The latter is the first to be integrated. Subsequently, that variable is integrated which after the first integration does not appear in any of the remaining bounds, and so on:

$$M = \int_{c_1}^{c_2} dz \underbrace{\int_{b_1(z)}^{b_2(z)} dy \underbrace{\int_{a_1(y,z)}^{a_2(y,z)} dx\, \rho(x,y,z)}_{\bar{\rho}(y,z)}}_{\bar{\bar{\rho}}(z)}\,. \tag{1.128}$$

Let us practice the procedure by inspecting two special examples.

(a) *surface integral*
As sketched in Fig. 1.29 the two curves $y_1 = 2x^2$ and $y_2 = x^3$ enclose between $x = 0$ and $x = 2$ an area the amount of which shall be calculated. That can be

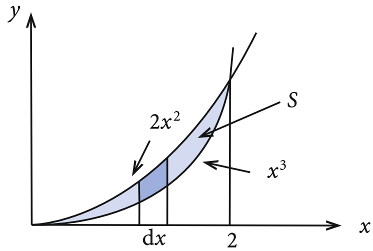


Fig. 1.29 Area S as example for the calculation of a double integral

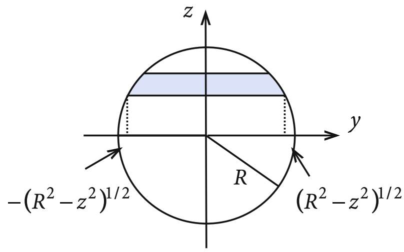


Fig. 1.30 To the calculation of the sphere volume

managed by *'adding stripe by stripe'* the area elements of infinitesimal width dx:

$$S = \int_0^2 dx \int_{x^3}^{2x^2} dy = \int_0^2 dx\,(2x^2 - x^3) = \left(\frac{2}{3}x^3 - \frac{1}{4}x^4\right)\Big|_0^2 = \frac{4}{3}\,.$$

We can verify the result by subtracting the two areas S_1 and S_2 under the two curves in between 0 and 2:

$$S_1 = \int_0^2 dx\,2x^2 = \frac{2}{3}x^3\Big|_0^2 = \frac{16}{3} \quad ; \quad S_2 = \int_0^2 dx\,x^3 = \frac{x^4}{4}\Big|_0^2 = 4\,.$$

It is indeed $S = S_1 - S_2 = \frac{4}{3}$

(b) *volume integral*

We calculate the volume of a sphere of radius R by applying Cartesian coordinates (Fig. 1.30). On the surface we have:

$$R^2 = x^2 + y^2 + z^2\,.$$

That defines the limits of integration for the calculation of the volume V of the sphere:

$$V = \int_{-R}^{+R} dz \int_{-\sqrt{R^2-z^2}}^{+\sqrt{R^2-z^2}} dy \int_{-\sqrt{R^2-y^2-z^2}}^{+\sqrt{R^2-y^2-z^2}} dx\,.$$

This can be evaluated:

$$\begin{aligned}
V &= \int_{-R}^{+R} dz \int_{-\sqrt{R^2-z^2}}^{+\sqrt{R^2-z^2}} dy\, 2\sqrt{R^2-y^2-z^2} \\
&= \int_{-R}^{+R} dz\, 2\cdot \left[\frac{1}{2}\left(R^2-z^2\right) \arcsin\left(\frac{y}{\sqrt{R^2-z^2}}\right) \right. \\
&\quad \left. +\frac{y}{2}\sqrt{R^2-z^2-y^2}\,\right]_{-\sqrt{R^2-z^2}}^{+\sqrt{R^2-z^2}} \\
&= 2\int_{-R}^{+R} dz\, \frac{1}{2}\left(R^2-z^2\right)\cdot\pi \\
&= \pi\left(R^2 z - \frac{z^3}{3}\right)\Bigg|_{-R}^{+R} = \pi R^3 \left(2-\frac{2}{3}\right) \\
&= \frac{4}{3}\pi R^3 \,.
\end{aligned}$$

In the second step, for the y integration, we had to take the help of an appropriate table of integrals.

Later in this course, we will see that very often multiple integrals with not constant bounds can be substantially simplified by a transformation to so-called *'curvilinear coordinates'* which we introduce and inspect in Sect. 1.7. The calculation of the volume of a sphere, e.g., by use of *spherical coordinates* (Sect. 1.7.4) turns out to be much quicker and distinctly more elegant than that with the above used Cartesian coordinates.

1.2.6 Exercises

Exercise 1.2.1 Solve using integration by parts:

1.

$$\int \cos^2 x\, dx$$

2.

$$\int x^2 \cos^2 x\, dx$$

3.

$$\int x \sin x \, dx$$

4.

$$\int x \ln x \, dx$$

Exercise 1.2.2 Calculate the following definite integrals by proper substitution of variables:

1.

$$\int_0^1 (5x - 4)^3 \, dx$$

2.

$$\int_1^{\frac{3}{2}} \sin\left(\pi x + \frac{5\pi}{2}\right) dx$$

3.

$$\int_1^2 \frac{dx}{\sqrt{7 - 3x}}$$

4.

$$\int_{-1}^{+1} x^2 \sqrt{2x^3 + 4} \, dx$$

Exercise 1.2.3 Evaluate the following multiple integrals:

1.

$$\int_{x=0}^1 \int_{y=0}^2 x^2 \, dx \, dy$$

2.

$$\int_{x=0}^{\pi} \int_{y=\frac{1}{2}\pi}^{\pi} \sin x \cdot \sin y \, dx \, dy$$

3.

$$\int_{x=0}^{2} \int_{y=x-1}^{3x} x^2 \, dx \, dy$$

4.

$$\int_{x=0}^{1} \int_{y=0}^{2x} \int_{z=0}^{x+y} dx \, dy \, dz$$

1.3 Vectors

In order to fix a physical quantity one needs three specifications:

dimension, unit of measure, coefficient of measure.

Physical quantities are classified as

scalars, vectors, tensors, …

Tensors will not appear in the first parts of this course. Thus we explain the term tensor at a later stage.

Scalar:
An object which after fixing the dimension and the unit of measure is completely characterized by stating one coefficient of measure
(e.g. *mass, volume, temperature, pressure, wavelength, …*).

Vector:
An object which in addition needs the specification of a direction
(e.g. *displacement, velocity, acceleration, momentum, force, …*)

The conceptually simplest vector is the displacement or

position vector

by which the points of the Euclidean space E_3 can be specified. For this purpose one first defines an

origin of coordinates $\mathcal{O}$

and connects it by a straight line with the considered point A of the E_3 (Fig. 1.31).

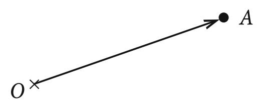


Fig. 1.31 To the definition of the position vector

The connecting line gets a direction by convention to run through the line from the origin of coordinates $\mathcal{O}$ to A. In the following we will mark vectors by bold letters. Each vector $\mathbf{a}$ has a **length**, also called **magnitude**,

$$a = |\mathbf{a}|$$

and a **direction**, the unique fixing of which requires a reference direction, i.e. a reference system. The simplest system of reference is built up by three straight lines, perpendicular to each other and intersecting in one common point, the origin of coordinates $\mathcal{O}$ (six ray star). One assigns directions to the three lines, and that in such a way to build in the sequence $(1, 2, 3)$ and (x, y, z), respectively, a right system (*'right-handed trihedron'*). If one rotates on the shortest way from axis 1 to 2, the axis 3 has the direction into which a right-twisted screw would move (see Fig. 1.32). This is called a

Cartesian coordinate system

Once the reference system is fixed the orientation of a position vector in the E_3 is uniquely determined by two numbers, e.g. two angles what can be demonstrated on the unit sphere (see Fig. 1.33).

One denotes two vectors as **equal** if they have the same lengths and the same directions. Notice, however, that it is **not at all** required that they have the same starting points. Parallel vectors of the same lengths are in this sense 'equal' (see Fig. 1.34).

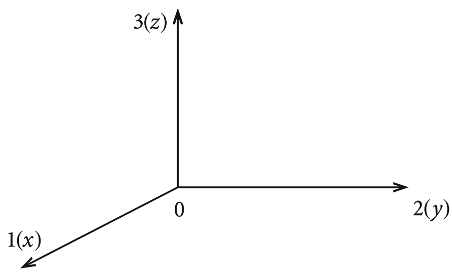


Fig. 1.32 Cartesian system of coordinates as a right system

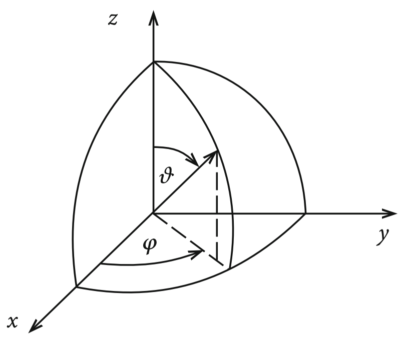


Fig. 1.33 Description of the direction of a vector by quoting two angles

Fig. 1.34 Example of two 'equal' vectors

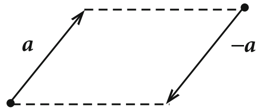


Fig. 1.35 Two 'antiparallel' vectors

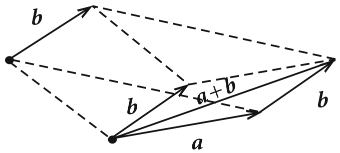


Fig. 1.36 Addition of two vectors

To each vector **a** there does exist an equally long but antiparallel vector (Fig. 1.35) which we denote −**a**.

A *unit vector* is a vector of the magnitude 1.

1.3.1 Elementary Mathematical Operations

(a) Addition

Two vectors **a** and **b** are added by a parallel translation of one of the vectors, say **b**, such that the base point of **b** coincides with the arrowhead of the other vector **a** (Fig. 1.36). The sum vector (**a** + **b**) then starts at the base point of **a** and goes to the arrowhead of **b**. One recognizes that (**a** + **b**) corresponds to the diagonal of the parallelogram spanned by **a** and **b** (*parallelogram law*). We list up some obvious rules for vector sums:

($\boldsymbol{\alpha}$) Commutativity

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a} \,. \tag{1.129}$$

This follows directly from the definition of the sum vector and becomes immediately clear with Fig. 1.37. Decisive for the commutativity is the free parallel mobility of the vectors in the plane.

($\boldsymbol{\beta}$) Associativity

$$(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c}) \,. \tag{1.130}$$

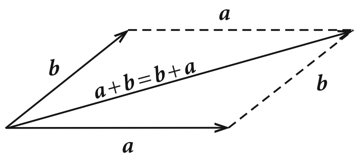


Fig. 1.37 Commutativity of the vector sum

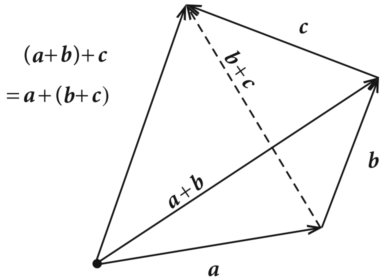


Fig. 1.38 Associativity of the vector summation

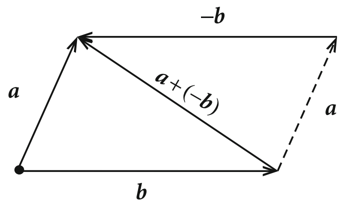


Fig. 1.39 Subtraction of two vectors

The validity of (1.130) can easily be read off from Fig. 1.38.

($\boldsymbol{\gamma}$) Vector Subtraction

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}) \,. \tag{1.131}$$

Subtracting **a** from itself yields the so-called

$$\textbf{zero (null) vector:} \qquad \mathbf{0} = \mathbf{a} - \mathbf{a} \,, \tag{1.132}$$

the only vector which has no definite direction (Fig. 1.39). For **all** vectors holds:

$$\mathbf{a} + \mathbf{0} = \mathbf{a} \,. \tag{1.133}$$

Because of (1.129), (1.130), (1.132) and (1.133) the set of all position vectors build a **(commutative) group**.

(b) Multiplication by a (Real) Number

Let α be a real number ($\alpha \in \mathbb{R}$) and **a** be an arbitrary vector.

Definition $(\alpha\,\mathbf{a})$ is a vector with the following properties:

$$1) \qquad \alpha\,\mathbf{a} = \begin{cases} \uparrow\uparrow \mathbf{a} \text{ if } \alpha > 0 \\ \uparrow\downarrow \mathbf{a} \text{ if } \alpha < 0 \end{cases}$$

$$2) \qquad |\alpha\,\mathbf{a}| = |\alpha|\,a \tag{1.134}$$

Special cases:

$$1\,\mathbf{a} = \mathbf{a}\,, \qquad 0\,\mathbf{a} = \mathbf{0}\,, \qquad (-1)\,\mathbf{a} = -\mathbf{a}\,. \tag{1.135}$$

Calculation rules:
In the following let $\alpha, \beta, \ldots$ be real numbers and $\mathbf{a}, \mathbf{b}, \ldots$ any arbitrary vectors.

($\boldsymbol{\alpha}$) Distributivity

Valid are the following distributive laws:

$$(\alpha + \beta)\mathbf{a} = \alpha\mathbf{a} + \beta\mathbf{a}\,, \tag{1.136}$$

$$\alpha(\mathbf{a} + \mathbf{b}) = \alpha\mathbf{a} + \alpha\mathbf{b}\,. \tag{1.137}$$

The proof of (1.136) immediately results from the definition of the vector. The proof of (1.137) runs as follows:

Proof According to Fig. 1.40 it holds $(\alpha > 0)$:

$$\begin{aligned} \alpha\mathbf{a} + \mathbf{x} &= \mathbf{y}\,, \\ \mathbf{x} &= \hat{\alpha}\mathbf{b} \quad (\hat{\alpha} > 0)\,, \\ \mathbf{y} &= \overline{\alpha}(\mathbf{a} + \mathbf{b}) \quad (\overline{\alpha} > 0)\,. \end{aligned}$$

The assertion is proved if $\hat{\alpha} = \overline{\alpha} = \alpha$:

1. Intercept theorem:

$$\frac{|\mathbf{y}|}{|\mathbf{a} + \mathbf{b}|} = \frac{|\alpha\mathbf{a}|}{|\mathbf{a}|} = \alpha \Longrightarrow \overline{\alpha} = \alpha\,.$$

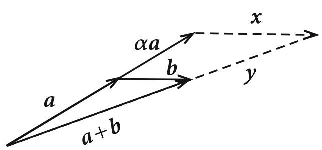


Fig. 1.40 Demonstration of the distributivity of a vector sum with respect to multiplication with a real number

2. Intercept theorem:

$$\frac{|\mathbf{x}|}{|\mathbf{b}|} = \frac{|\alpha\mathbf{a}|}{|\mathbf{a}|} = \alpha \Longrightarrow \hat{\alpha} = \alpha \ .$$

   Insertion into $\alpha\mathbf{a} + \mathbf{x} = \mathbf{y}$ validitates the assertion (1.137). The proof for $\alpha < 0$ is performed analogously (Exercise 1.3.6).

($\boldsymbol{\beta}$) Associativity

$$\alpha(\beta\mathbf{a}) = (\alpha\beta)\mathbf{a} \equiv \alpha\beta\mathbf{a} \ . \tag{1.138}$$

Because of $|\alpha\beta| = |\alpha||\beta|$ the proof is immediately clear.

($\boldsymbol{\gamma}$) Unit Vector

From each vector $\mathbf{a}$ one can construct a unit vector in the direction of $\mathbf{a}$ by multiplying the vector with its reciprocal magnitude $|\mathbf{a}|^{-1}$:

$$\begin{gathered} \mathbf{e}_a = a^{-1}\mathbf{a} \quad \text{with} \quad |\mathbf{e}_a| = a^{-1}a = 1 \\ \mathbf{e}_a \uparrow\uparrow \mathbf{a} \ . \end{gathered} \tag{1.139}$$

Unit vectors are normally denoted by the letters $\mathbf{e}$ or $\mathbf{n}$.

Up to now our considerations have been focussed more or less directly on the position vectors of the E_3. However, we can also interpret the above listed properties of the position vectors as general axioms. *All* objects which fulfill these axioms shall therefore be called in the following as *vectors*. The position vector is only a self-evident special realization of the abstract term vector. The ensemble of all vectors then build a

linear (vector) space V over the body of real numbers $\mathbb{R}$

which, to gather once more, fulfills the following axioms:

Axiom 1.1 Between two elements $\mathbf{a}, \mathbf{b} \in V$ a connection (*'addition'*) is defined

$$\mathbf{a} + \mathbf{b} = \mathbf{d} \in V$$

with

1. $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$ (associativity)
2. zero (null) element $\mathbf{0} \in V$: $\mathbf{a} + \mathbf{0} = \mathbf{a} \ \ \forall \mathbf{a}$

3. (additive) inverse: For all $\mathbf{a} \in V$exists an element $(-\mathbf{a}) \in V$ so that

$$\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$$

4. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ (commutativity)

Axiom 1.2 Multiplication of a vector with elements $\alpha,\ \beta,\ \ldots \in \mathbb{R}$

$$\alpha \in \mathbb{R} \quad \mathbf{a} \in V \Longrightarrow \alpha\mathbf{a} \in V$$

1. $(\alpha + \beta)\mathbf{a} = \alpha\mathbf{a} + \beta\mathbf{a}$

   $\alpha(\mathbf{a} + \mathbf{b}) = \alpha\mathbf{a} + \alpha\mathbf{b}$ (distributivity)
2. $\alpha(\beta\mathbf{a}) = (\alpha\beta)\mathbf{a}$ (associativity)
3. It exists a *unity (identity) element* 1, so that

   $1 \cdot \mathbf{a} = \mathbf{a}$ for all $\mathbf{a} \in V$

We have introduced in this section the multiplication of vectors with scalars. Is it also possible to multiply vectors with vectors? The answer is yes, but the type of multiplication must be specified with care. One knows two types of products built by vectors, the **scalar (inner, dot) product** and the **vector (outer, cross) product**.

1.3.2 Scalar Product

As **scalar (inner, dot) product** of two vectors **a** and **b** is denoted by the following number (scalar):

$$(\mathbf{a}, \mathbf{b}) \equiv \mathbf{a} \cdot \mathbf{b} = ab\cos\vartheta\ , \qquad \vartheta = \sphericalangle(\mathbf{a}, \mathbf{b})\ . \tag{1.140}$$

Illustratively, it is the product of the length of the second vector with the projection of the first vector on the direction of the second (see Fig. 1.41).

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} = 0\ , \text{ if }\ 1)\ a = 0 \quad \text{or/and } b = 0 \\ \text{or } 2)\ \vartheta = \pi/2\ . \end{aligned} \tag{1.141}$$

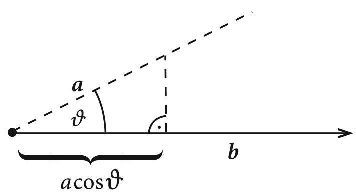


Fig. 1.41 To the definition of the scalar product between two vectors

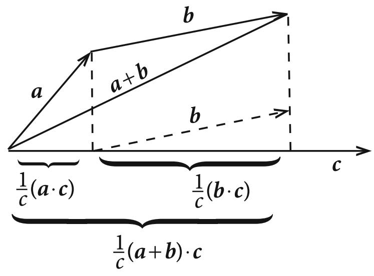


Fig. 1.42 Distributivity of the scalar product

a and **b** are **orthogonal** (**a**⊥**b**) if

$$\mathbf{a} \cdot \mathbf{b} = 0 \quad \text{with} \;\; a \neq 0 \;\; \text{and} \;\; b \neq 0 \, . \tag{1.142}$$

Properties

(a) Commutativity

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \, . \tag{1.143}$$

This relation is directly perceptible from the definition of the scalar product.

(b) Distributivity

$$(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c} \, . \tag{1.144}$$

Figure 1.42 gives immediately the proof, which again exploits the free relocatability of the vectors in the plane.

(c) Bilinearity (*Homogeneity*)

For each real number α holds:

$$(\alpha\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (\alpha\mathbf{b}) = \alpha(\mathbf{a} \cdot \mathbf{b}) \, . \tag{1.145}$$

Proof (Fig. 1.43)

$$\begin{aligned} \alpha > 0: \quad (\alpha\mathbf{a}) \cdot \mathbf{b} &= \alpha a b \cos\vartheta \\ \mathbf{a} \cdot \mathbf{b} &= a b \cos\vartheta \end{aligned}$$

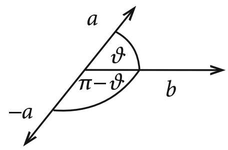


Fig. 1.43 To the proof of the bilinearity of the scalar product of two vectors

$$
\begin{aligned}
\Longrightarrow \quad \alpha(\mathbf{a}\cdot\mathbf{b}) &= (\alpha\mathbf{a})\cdot\mathbf{b} \\
\alpha < 0: \quad \mathbf{a}\cdot\mathbf{b} &= ab\cos\vartheta \\
(\alpha\mathbf{a})\cdot\mathbf{b} &= |\alpha|ab\cos(\pi-\vartheta) = \\
&= -|\alpha|ab\cos\vartheta = \\
&= \alpha ab\cos\vartheta = \\
&= \alpha(\mathbf{a}\cdot\mathbf{b})\ .
\end{aligned}
$$

(d) Magnitude (*Norm*) of a Vector

Because $\cos(0) = 1$ we have:

$$\mathbf{a}\cdot\mathbf{a} = a^2 \geq 0 \Longleftrightarrow a = \sqrt{\mathbf{a}\cdot\mathbf{a}}\ . \tag{1.146}$$

The equality sign is valid only for the zero vector:

$$\mathbf{e}\cdot\mathbf{e} = 1 \quad \Longleftrightarrow \quad \text{unit vector}\ .$$

(e) Schwarz's Inequality

$$|\mathbf{a}\cdot\mathbf{b}| \leq ab\ . \tag{1.147}$$

Since $|\cos\vartheta| \leq 1$ this statement follows directly from the definition (1.140). The latter, however, is related to the *intuitive* position vectors of the E_3. For the elements of an abstract vector space the scalar product is defined by the properties (1.143)–(1.146). More strictly that means:

A connection between two elements **a** and **b** of the vector space V, which assigns to it a real number $\alpha \in \mathbb{R}$

$$\mathbf{a}\cdot\mathbf{b} = \alpha\ ,$$

is denoted as **scalar product** if the axioms (1.143)–(1.146) are fulfilled. A vector space, for which a scalar product is defined is called a **unitary vector space**. Therefore, we want to prove (1.147) by using these properties without referring to the special case of position vectors.

If $a = 0$ or/and $b = 0$ holds, Eq. (1.147) is fulfilled with the equal sign. Therefore we now assume $a \neq 0$ and $b \neq 0$. Then one finds for all real α:

$$
\begin{aligned}
0 &\overset{(1.146)}{\leq} (\mathbf{a} + \alpha\mathbf{b}) \cdot (\mathbf{a} + \alpha\mathbf{b}) = \\
&\overset{(1.144)}{\underset{(1.145)}{=}} a^2 + \alpha^2 b^2 + \alpha\mathbf{b} \cdot \mathbf{a} + \alpha\mathbf{a} \cdot \mathbf{b} = \\
&\overset{(1.143)}{=} a^2 + \alpha^2 b^2 + 2\alpha\mathbf{a} \cdot \mathbf{b} \; .
\end{aligned}
$$

Since α is arbitrary we can specifically choose

$$
\alpha = -\frac{\mathbf{a} \cdot \mathbf{b}}{b^2} \in \mathbb{R}
$$

But therewith follows:

$$
\begin{aligned}
0 &\leq a^2 + \frac{(\mathbf{a} \cdot \mathbf{b})^2 b^2}{b^4} - 2\frac{(\mathbf{a} \cdot \mathbf{b})^2}{b^2} \; , \\
\iff \quad 0 &\leq a^2 b^2 - (\mathbf{a} \cdot \mathbf{b})^2 \Longrightarrow \text{q. e. d.}
\end{aligned}
$$

(f) Triangle Inequality

$$
|a - b| \leq |\mathbf{a} + \mathbf{b}| \leq a + b \; . \tag{1.148}
$$

The proof exploits the Schwarz's inequality:

$$
\begin{aligned}
& -ab \leq \mathbf{a} \cdot \mathbf{b} \leq ab \\
\iff \quad & a^2 + b^2 - 2ab \leq a^2 + b^2 + 2\mathbf{a} \cdot \mathbf{b} \leq a^2 + b^2 + 2ab \\
\iff \quad & (a - b)^2 \leq (\mathbf{a} + \mathbf{b})^2 \leq (a + b)^2 \\
\iff \quad & |a - b| \leq |\mathbf{a} + \mathbf{b}| \leq |a + b| = a + b \; .
\end{aligned}
$$

A special application of the scalar product leads to the **cosine rule** (Fig. 1.44)

$$
\begin{aligned}
& \mathbf{c} = \mathbf{a} - \mathbf{b} \; , \\
& c^2 = (\mathbf{a} - \mathbf{b})^2 = a^2 - 2\mathbf{a} \cdot \mathbf{b} + b^2 \\
\Longrightarrow \quad & c^2 = a^2 + b^2 - 2ab\cos\sphericalangle(\mathbf{a}, \mathbf{b}) \; .
\end{aligned} \tag{1.149}
$$

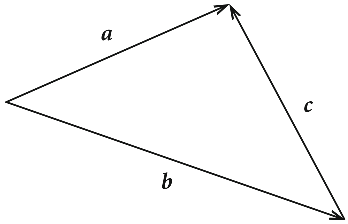


Fig. 1.44 Demonstration of the cosine rule

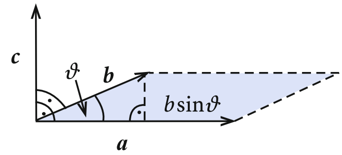


Fig. 1.45 To the definition of the vector product

1.3.3 Vector (Outer, Cross) Product

The product discussed in the last section assigns a number, i.e. a scalar, to the product of two vectors of a vector space. However, there exists a second type of product which addresses to two vectors a third vector from the same vector space. This is known as **vector product, outer product, or cross product**

$$\mathbf{c} = \mathbf{a} \times \mathbf{b}$$

This vector has the following properties:

1.

$$c = a\,b\,\sin\vartheta\ ; \qquad \vartheta = \sphericalangle(\mathbf{a}, \mathbf{b})\ . \tag{1.150}$$

   The magnitude c of the resulting vector corresponds to the area of the parallelogram spanned by the vectors **a** and **b** (Fig. 1.45).
2. **c** is oriented perpendicular to the area defined by **a** and **b** in such a way that **a**, **b**, **c** in this sequence build a right-handed coordinate system.

The second point indicates that the vector product does not simply characterize a direction but more a *'direction of rotation, rotation sense'*. Thus, in various respects the properties of a vector product are different from those of a *'ordinary'* (**polar**) vector. **c** is a so-called **axial** vector (**pseudovector**). The strict distinction becomes clear with the term

Space Inversion

Reflection of all space points (E_3) with respect to a fixed, given point, e.g. the origin of coordinates.

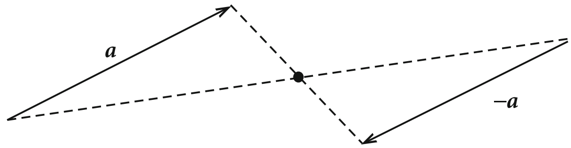


Fig. 1.46 Space inversion of a polar vector

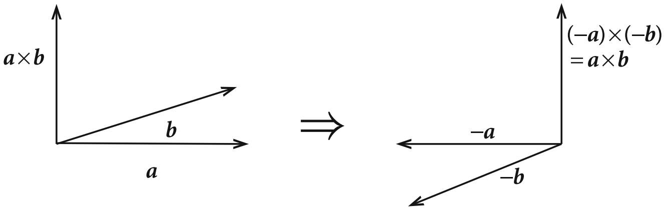


Fig. 1.47 Space inversion of an axial vector

Polar vectors change their signs by inversion (see Fig. 1.46). On the other hand, since the rotation sense does not change after inversion, the axial vector will not change its sign (see Fig. 1.47).

We add a remark. It is clear that the scalar product of either only polar vectors or only axial vectors does not change its sign with inversion, being therefore a genuine scalar. The scalar product of a polar and an axial vector, however, changes into its negative and is for this reason called a **pseudoscalar**.

One has to bear in mind that the scalar product (Sect. 1.3.2) is defined between vectors of an arbitrary-dimensional vector space, while the vector product holds only for three-dimensional vectors.

1.3.3.1 Properties of the Vector Product

(a) Anticommutative

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \,. \tag{1.151}$$

This property becomes immediately evident as consequence of the right-handed cork rule (see Fig. 1.45).

(b)

$$\begin{aligned} \mathbf{a} \times \mathbf{b} \; = 0 \,, \text{ if } \; & 1) \; a = 0 \text{ or/and } b = 0 \,, \\ & 2) \; \mathbf{b} = \alpha \mathbf{a} \,; \;\; \alpha \in \mathbb{R} \,. \end{aligned}$$

Two collinear (equidirectional) vectors cannot span a surface area ($\sin 0 = 0$).

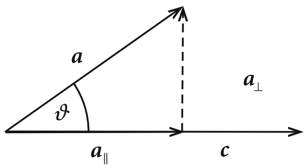


Fig. 1.48 Auxiliary sketch for the proof of the distributivity of the vector product

(c) Distributive

$$(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c} \,. \tag{1.152}$$

Proof The proof is done in two steps:

(α) The vector $\mathbf{c}$ in (1.152) is obviously in some way distinguished. We decompose therefore the three vectors $\mathbf{a}$, $\mathbf{b}$ and $(\mathbf{a} + \mathbf{b})$ into components, respectively, parallel and perpendicular to $\mathbf{c}$ (Fig. 1.48):

$$\begin{array}{l} \mathbf{a} = \mathbf{a}_\parallel + \mathbf{a}_\perp \\ \mathbf{b} = \mathbf{b}_\parallel + \mathbf{b}_\perp \end{array} \; ; \quad \mathbf{a} + \mathbf{b} = (\mathbf{a} + \mathbf{b})_\parallel + (\mathbf{a} + \mathbf{b})_\perp \,. \tag{1.153}$$

Only the components perpendicular to $\mathbf{c}$, however, contribute to the vector product:

$$\mathbf{a} \times \mathbf{c} = \mathbf{a}_\perp \times \mathbf{c} \,. \tag{1.154}$$

For it holds:

$$\begin{aligned} |\mathbf{a}_\perp \times \mathbf{c}| &= a_\perp c \sin\frac{\pi}{2} = \\ &= a_\perp c = a\,c \sin\vartheta = \\ &= |\mathbf{a} \times \mathbf{c}| \,. \end{aligned}$$

Since, in addition, the directions of $\mathbf{a} \times \mathbf{c}$ and $\mathbf{a}_\perp \times \mathbf{c}$ coincide, (1.154) is obviously correct. Thus we can assume, without loss of generality, for the second part of the proof that $\mathbf{a}$ and $\mathbf{b}$ are already orthogonal to $\mathbf{c}$.

(β) By the vector products $\mathbf{a} \times \mathbf{c}$, $\mathbf{b} \times \mathbf{c}$, $(\mathbf{a} + \mathbf{b}) \times \mathbf{c}$ new vectors arise from $\mathbf{a}, \mathbf{b}, (\mathbf{a} + \mathbf{b})$, the magnitudes of which are altered by a factor c. All the three vectors are located in the plane orthogonal to $\mathbf{c}$ which is spanned by $\mathbf{a}$ and $\mathbf{b}$. They are rotated relatively to the original vectors by $\pi/2$. The angles between $\mathbf{a} \times \mathbf{c}$, $\mathbf{b} \times \mathbf{c}$ and $(\mathbf{a} + \mathbf{b}) \times \mathbf{c}$, on the one hand, are thus the same as those between $\mathbf{a}, \mathbf{b}$ and $(\mathbf{a} + \mathbf{b})$, on the other hand (Fig. 1.49):

$$\frac{1}{c}(\mathbf{a} \times \mathbf{c}) + \frac{1}{c}(\mathbf{b} \times \mathbf{c}) = \frac{1}{c}[(\mathbf{a} + \mathbf{b}) \times \mathbf{c}] \,. \tag{1.155}$$

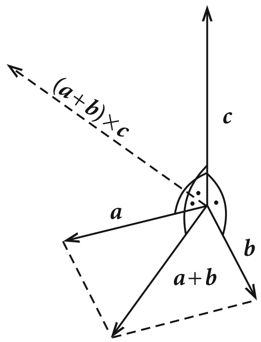


Fig. 1.49 Another auxiliary sketch for the proof of the distributivity of the vector product

With (1.137) follows:

$$\frac{1}{c}\left[(\mathbf{a}\times\mathbf{c})+(\mathbf{b}\times\mathbf{c})\right]=\frac{1}{c}\left[(\mathbf{a}+\mathbf{b})\times\mathbf{c}\right]\ , \tag{1.156}$$

That proves the above statement.

(d) Not Associative

The positions of the brackets in the double vector product are **not** arbitrary. In general it holds:

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\neq(\mathbf{a}\times\mathbf{b})\times\mathbf{c}\ . \tag{1.157}$$

The resulting vector on the left-hand side lies in the $(\mathbf{b},\mathbf{c})$ area, whereas the one on the right-hand side, however, in the $(\mathbf{a},\mathbf{b})$ area.

(e) Bilinear for Real Numbers α

$$(\alpha\mathbf{a})\times\mathbf{b}=\mathbf{a}\times(\alpha\mathbf{b})=\alpha(\mathbf{a}\times\mathbf{b})\ . \tag{1.158}$$

For $\alpha>0$ the proof follows directly from the definition, for $\alpha<0$ one has to take into consideration the right-handed cork screw rule.

Example (Fig. 1.50)

$$\begin{aligned}
&\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{0}\\
\Longrightarrow\ &\mathbf{a}\times\mathbf{b}=\mathbf{a}\times(\mathbf{0}-\mathbf{a}-\mathbf{c})=\\
&\qquad=\mathbf{a}\times(-\mathbf{c})=\\
&\qquad=\mathbf{c}\times\mathbf{a}\ .
\end{aligned}$$

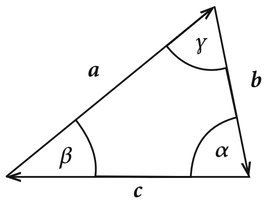


Fig. 1.50 To the derivation of the sine rule

Simultaneously it holds:

$$\mathbf{a} \times \mathbf{b} = (\mathbf{0} - \mathbf{b} - \mathbf{c}) \times \mathbf{b} = (-\mathbf{c}) \times \mathbf{b} = \mathbf{b} \times \mathbf{c} \ .$$

That means:

$$\mathbf{a} \times \mathbf{b} = \mathbf{c} \times \mathbf{a} = \mathbf{b} \times \mathbf{c} \ , \quad \text{if } \mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0} \ . \tag{1.159}$$

For the magnitudes it follows that:

$$ab \sin(\pi - \gamma) = ca \sin(\pi - \beta) = bc \ \sin(\pi - \alpha)$$

or

$$\frac{a}{sin\alpha} = \frac{c}{sin\gamma} = \frac{b}{\sin\beta} \ . \tag{1.160}$$

This is the well-known **sine rule** of trigonometry.

1.3.4 'Higher' Vector Products

We have learned about two possibilities to connect two vectors multiplicatively. Let us now investigate how to build products of more than two vectors. The **scalar product** of two vectors leads to a (real) number, which, as defined in (1.134), can of course be multiplied with a third vector.

$$(\mathbf{a} \cdot \mathbf{b})\, \mathbf{c} = \mathbf{d} \ . \tag{1.161}$$

d has the same direction as **c**.

The **vector product** results in a new vector and can therefore be multiplicatively connected with a further vector in the already discussed two different manners:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} \ ; \ \ (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \ .$$

We discuss at first the **scalar triple product**:

$$V(\mathbf{a},\,\mathbf{b},\,\mathbf{c}) \equiv (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}\;. \tag{1.162}$$

Geometrically the scalar triple product can be understood as the volume of the parallelepiped spanned by the three vectors **a**, **b**, and **c** (see Fig. 1.51).

$$\begin{aligned} \text{volume} &= \text{basal plane } F \cdot \text{height } h = \\ &= |\mathbf{a} \times \mathbf{b}| \cdot c \cdot \cos\varphi = \\ &= (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}\;. \end{aligned}$$

Since it does not matter which side of the parallelepiped is chosen as basal plane F, the scalar triple product will not change by a **cyclic** permutation of the three vectors (Fig. 1.52):

$$V = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}\;. \tag{1.163}$$

One sees that for a fixed (!) sequence of vectors one can interchange the symbols $\times$ and $\cdot$:

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})\;.$$

In case of an **anticyclic** interchange V changes its sign. Therefore one denotes V as a **pseudoscalar**.

Another *'higher'* product of vectors is the **double** vector product:

$$\mathbf{p} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})\;. \tag{1.164}$$

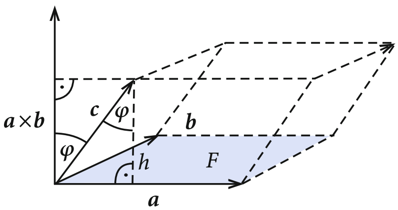


Fig. 1.51 Illustration of the scalar triple product as the volume of a parallelepiped spanned by three vectors

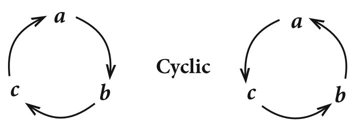


Fig. 1.52 Possible cyclic (!) interchanges in the scalar triple product

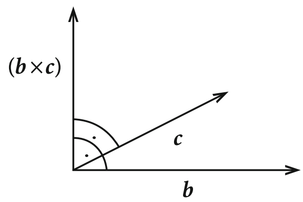


Fig. 1.53 Direction of the vector product of two vectors

The vector $(\mathbf{b} \times \mathbf{c})$ is perpendicular to the $(\mathbf{b},\ \mathbf{c})$-plane, so that $\mathbf{p}$ must lie within this plane. Thus we can start with (Fig. 1.53):

$$\mathbf{p} = \beta\mathbf{b} + \gamma\mathbf{c}\ . \tag{1.165}$$

On the other hand $\mathbf{p}$ is also orthogonal to $\mathbf{a}$:

$$0 = \mathbf{a} \cdot \mathbf{p} = \beta(\mathbf{a} \cdot \mathbf{b}) + \gamma(\mathbf{a} \cdot \mathbf{c})\ .$$

That means:

$$\beta = \alpha(\mathbf{a} \cdot \mathbf{c})\ ; \quad \gamma = -\alpha(\mathbf{a} \cdot \mathbf{b})\ . \tag{1.166}$$

Insertion into (1.165) yields the intermediate result:

$$\mathbf{p} = \alpha\left[\mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})\right]\ . \tag{1.167}$$

Later we will show explicitly that $\alpha = 1$ must be. The result is the **expansion rule** for the double vector product:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})\ . \tag{1.168}$$

By this equation one can easily demonstrate the **non-associativity** of the vector product:

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} &= -\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = -\mathbf{a}(\mathbf{c} \cdot \mathbf{b}) + \mathbf{b}(\mathbf{c} \cdot \mathbf{a}) \\ &\neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})\ .\end{aligned} \tag{1.169}$$

Finally one can prove with the aid of this expansion rule the important **Jacobi identity** (Exercise 1.3.12):

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0\ . \tag{1.170}$$

1.3.5 Basis Vectors

In (1.139) we have defined what are known as *unit vectors*. Since, by definition, their magnitude is equal to 1 they are in particular suitable to identify directions. If one intends to separate statements on direction and magnitude of a vector $\mathbf{a}$, the following representation is recommendable:

$$\mathbf{a} = a\,\mathbf{e}_a\ . \tag{1.171}$$

Two vectors $\mathbf{a}$ and $\mathbf{b}$ with the same direction $\mathbf{e}$ are called **collinear**. For such vectors one can find real numbers $\alpha \neq 0$, $\beta \neq 0$ so that the equation

$$\alpha\mathbf{a} + \beta\mathbf{b} = 0 \tag{1.172}$$

is fulfilled. One says that $\mathbf{a}$ and $\mathbf{b}$ are **linearly dependent**. We generalize this term as follows:

Definition n vectors $\mathbf{a}_1$, $\mathbf{a}_2$, ..., $\mathbf{a}_n$ are called **linearly independent** if the equation

$$\sum_{j=1}^{n} \alpha_j \mathbf{a}_j = 0 \tag{1.173}$$

can be fulfilled only by

$$\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0 \tag{1.174}$$

Otherwise they are called **linearly dependent.**

Definition The **dimension of a vector space** is given by the maximal number of linearly independent vectors required to span the space.

Theorem 1.3.1 *In a d-dimensional vector space each ensemble of d linearly independent vectors build a* ***basis*** *of the space, i.e.* ***any other*** *element of this space can be expressed as linear combination of these d vectors.*

Proof Let $\mathbf{a}_1, \ldots, \mathbf{a}_d$ be linearly independent vectors of the d dimensional space V and $\mathbf{b}$ another arbitrary vector in V. Then $\{\mathbf{b}, \mathbf{a}_1, \ldots, \mathbf{a}_d\}$ are certainly linearly dependent because otherwise V would be at least $(d+1)$-dimensional.

Thus there exist coefficients

$$\{\beta, \alpha_1, \ldots, \alpha_d\} \neq \{0, 0, \ldots, 0\}$$

with

$$\sum_{j=1}^{d} \alpha_j \mathbf{a}_j + \beta \mathbf{b} = \mathbf{0} \; .$$

Moreover $\beta \neq 0$ must hold because otherwise it would be:

$$\sum_{j=1}^{d} \alpha_j \mathbf{a}_j = 0 \quad \text{with} \quad \{\alpha_1, \ldots, \alpha_d\} \neq \{0, \ldots, 0\}$$

Contrary to the initial assumption the $\mathbf{a}_{j, j=1,\ldots,d}$ then would be linearly dependent. With $\beta \neq 0$, however, we can write:

$$\mathbf{b} = -\sum_{j=1}^{d} \frac{\alpha_j}{\beta} \mathbf{a}_j = \sum_{j=1}^{d} \gamma_j \mathbf{a}_j \quad \text{q. e. d.}$$

In many cases especially comfortable as basis vectors are unit vectors which are pairwise orthogonal to each other. Then one speaks of an

orthonormal system $\mathbf{e}_i, \quad i = 1, 2, \ldots, d,$

for which holds:

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} = \begin{cases} 1 & \text{for } i = j \; , \\ 0 & \text{for } i \neq j \; . \end{cases} \tag{1.175}$$

An orthonormal system being simultaneously the basis of the vector space V is denoted as **'complete'**. For an arbitrary vector $\mathbf{a} \in V$ we can then write:

$$\mathbf{a} = \sum_{j=1}^{d} a_j \mathbf{e}_j \; . \tag{1.176}$$

The a_j are the **components** of the respective vector $\mathbf{a}$ with respect to the basis $\mathbf{e}_1, \ldots, \mathbf{e}_d$.

The components a_j are of course dependent on the non-unique choice of the basis. They are nothing but the orthogonal projections of $\mathbf{a}$ onto the basis vectors:

$$\mathbf{e}_i \cdot \mathbf{a} = \sum_{j=1}^{d} a_j \left(\mathbf{e}_i \cdot \mathbf{e}_j\right) = \sum_{j=1}^{d} a_j \delta_{ij} = a_i \; , \quad i = 1, 2, \ldots, d \; . \tag{1.177}$$

For a fixed given basis the vector **a** is uniquely determined by its components. So other representations of the vector may appear reasonable, e.g. as

$$\textbf{column vector}: \quad \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{pmatrix} \quad \text{or}$$

$$\textbf{row vector}: \quad \mathbf{a} = (a_1, a_2, \ \dots \ , a_d)$$

Examples

1. **Plane**

   All pairs of non-collinear vectors **a** and **b** are linearly independent (Fig. 1.54). Each third vector **c** in the plane is then linearly dependent. **a** and **b** thus build a possible basis which of course does not necessarily need to be orthonormal:

$$\mathbf{c} = \alpha\mathbf{a} + \beta\mathbf{b} \equiv (\alpha, \ \beta) \ . \tag{1.178}$$

2. **Euclidean space** E_3

   All sets of three non-complanar vectors (not lying in one and the same plane) are always linearly independent. Each fourth vector is then linearly dependent. So the dimension of the E_3 is $d = 3$. A often used orthonormal basis of the E_3 is the **Cartesian system of coordinates** as plotted in Fig. 1.55 with the basis vectors: $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ (also $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$). For the vector $\mathbf{a} \in E_3$ then holds:

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z \ . \tag{1.179}$$

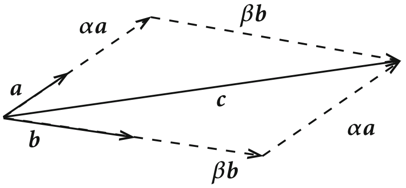


Fig. 1.54 Two non-collinear vectors as basis vectors for the plane

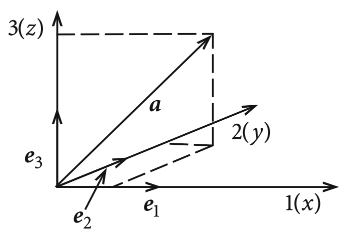


Fig. 1.55 Cartesian system of coordinates

For the **(Cartesian) components** a_i it can be written:

$$a_i = \mathbf{e}_i \cdot \mathbf{a} = a \cos\vartheta_i \ , \qquad \vartheta_i = \sphericalangle(\mathbf{e}_i, \mathbf{a}) \ , \tag{1.180}$$

$$\cos\vartheta_i = \tfrac{a_i}{a} : \quad \textbf{directional cosine} \ . \tag{1.181}$$

The components a_i also fix uniquely the **magnitude (norm) of the vector**:

$$a = \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{\sum_{i,j=1}^{3} a_i a_j \left(\mathbf{e}_i \cdot \mathbf{e}_j\right)} = \sqrt{\sum_{i,j} a_i a_j \delta_{ij}} = \sqrt{a_1^2 + a_2^2 + a_3^2} \ . \tag{1.182}$$

The magnitude (length) of the vector **a** is therefore determined by the square root of the sum of the component squares. Thus it also holds:

$$\cos^2\vartheta_1 + \cos^2\vartheta_2 + \cos^2\vartheta_3 = 1 \ , \tag{1.183}$$

so that by two directional cosines the third is already fixed, at least except for the sign.

1.3.6 Component Representations

In this section we want to rewrite the previously derived calculation rules for vectors by use of components. We restrict our considerations to the E_3:

$$\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3 \ , \quad \text{orthonormal basis of the } E_3,$$

$$\mathbf{a} = (a_1,\ a_2,\ a_3) = \sum_{i=1}^{3} a_i \mathbf{e}_i \qquad \text{vector of the } E_3 \ ,$$

$$\text{analogously : } \ \mathbf{b}, \mathbf{c}, \mathbf{d}, \ \ldots$$

(a) Special Vectors

zero vector:

$$\mathbf{0} \equiv (0,\ 0,\ 0) \ . \tag{1.184}$$

basis vectors:

$$\mathbf{e}_1 = (1,\ 0,\ 0) \ , \quad \mathbf{e}_2 = (0,\ 1,\ 0) \ , \quad \mathbf{e}_3 = (0,\ 0,\ 1) \ . \tag{1.185}$$

(b) Addition

$$\mathbf{c} = \mathbf{a} + \mathbf{b} = \sum_{j=1}^{3} \left(a_j + b_j\right) \mathbf{e}_j = \sum_{j=1}^{3} c_j \mathbf{e}_j$$

$$\Longrightarrow \quad \mathbf{e}_i \cdot (\mathbf{a} + \mathbf{b}) = a_i + b_i = c_i \ , \quad i = 1, 2, 3 \tag{1.186}$$

$$\Longrightarrow \quad \mathbf{c} = (a_1 + b_1, a_2 + b_2, a_3 + b_3) \ . \tag{1.187}$$

One therefore adds two vectors by adding their components using for both the vectors the same basis.

(c) Multiplication by Real Numbers

$$\begin{aligned}
\mathbf{b} &= \alpha \mathbf{a} \ ; \qquad \alpha \in \mathbb{R} \ , \\
\alpha \mathbf{a} &= \sum_{j=1}^{3} \left(\alpha a_j\right) \mathbf{e}_j = \sum_{j=1}^{3} b_j \mathbf{e}_j \ , \\
b_j &= \alpha a_j \ ; \qquad \alpha \mathbf{a} = (\alpha a_1, \alpha a_2, \alpha a_3) \ .
\end{aligned} \tag{1.188}$$

Thus one multiplies a vector by a real number by multiplying each component by this number.

(d) Scalar Product

$$\begin{aligned}
(\mathbf{a} \cdot \mathbf{b}) &= \left(\sum_{i=1}^{3} a_i \mathbf{e}_i\right) \left(\sum_{j=1}^{3} b_j \mathbf{e}_j\right) = \\
&= \sum_{i,j=1}^{3} a_i b_j \left(\mathbf{e}_i \cdot \mathbf{e}_j\right) = \sum_{i,j=1}^{3} a_i b_j \delta_{ij} \\
\Longrightarrow (\mathbf{a} \cdot \mathbf{b}) &= \sum_{j=1}^{3} a_j b_j \ .
\end{aligned} \tag{1.189}$$

We see that the scalar product of two vectors can be written as the sum of the component products. Consider herewith the **projection of a given vector a onto a given direction n**:

$$\mathbf{n} = (n_1, n_2, n_3) \ ; \quad |\mathbf{n}| = 1 \ ; \quad n_i = \cos \sphericalangle (\mathbf{n}, \mathbf{e}_i) \ .$$

According to (1.189) holds:

$$(\mathbf{a} \cdot \mathbf{n}) = \sum_{j=1}^{3} a_j n_j = a \cos \sphericalangle(\mathbf{n}, \mathbf{a}) \ ,$$

where in view of (1.180) we must also have $a_j = a \cos \sphericalangle(\mathbf{a}, \mathbf{e}_j)$. Combining these equations one comes to the useful relation:

$$\cos \sphericalangle(\mathbf{n}, \mathbf{a}) = \sum_{j=1}^{3} \cos \sphericalangle \left(\mathbf{a}, \mathbf{e}_j\right) \cos \sphericalangle \left(\mathbf{n}, \mathbf{e}_j\right) \ . \tag{1.190}$$

(e) Vector Product

We start with the orthonormal basis vectors which are thought to build a right-handed system:

$$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3; \quad \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1; \quad \mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2 \ . \tag{1.191}$$

Together with the anticommutativity of the vector product and the orthonormality relation (1.175) one finds:

$$\mathbf{e}_i \cdot \left(\mathbf{e}_j \times \mathbf{e}_k\right) = \begin{cases} 1 \ , & \text{if } (i,j,k) \textit{ cyclic permutation} \\ & \text{of } (1,2,3) \ , \\ -1 \ , & \text{if } (i,j,k) \textit{ anticyclic permutation} \\ & \text{of } (1,2,3) \ , \\ 0 & \text{in all other cases} \ . \end{cases} \tag{1.192}$$

As an abbreviation one writes:

$$\varepsilon_{ijk} = \mathbf{e}_i \cdot \left(\mathbf{e}_j \times \mathbf{e}_k\right) = \left(\mathbf{e}_i \times \mathbf{e}_j\right) \cdot \mathbf{e}_k \ . \tag{1.193}$$

These are the components of the so-called **fully antisymmetric tensor of third rank**.

Therewith the vector products of the basis vectors can be formulated in a compact manner:

$$\left(\mathbf{e}_i \times \mathbf{e}_j\right) = \sum_{k=1}^{3} \varepsilon_{ijk} \mathbf{e}_k \ . \tag{1.194}$$

For the general vector product we then have:

$$\mathbf{c} = \mathbf{a} \times \mathbf{b} \;=\; \sum_{i,j} a_i b_j \left(\mathbf{e}_i \times \mathbf{e}_j\right) = \sum_{i,j,k} \varepsilon_{ijk} a_i b_j \mathbf{e}_k = \sum_k c_k \mathbf{e}_k$$

$$\Longrightarrow \quad c_k \;=\; \sum_{i,j} \varepsilon_{ijk} a_i b_j \; . \tag{1.195}$$

This is a condensed version of the following three equations:

$$c_1 = a_2 b_3 - a_3 b_2 \; ; \quad c_2 = a_3 b_1 - a_1 b_3 \; ; \quad c_3 = a_1 b_2 - a_2 b_1 \; . \tag{1.196}$$

(f) Scalar Triple Product

With (1.192) and (1.193) this is simply expressible:

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \sum_{i,j,k} a_i b_j c_k \mathbf{e}_i \cdot \left(\mathbf{e}_j \times \mathbf{e}_k\right) = \sum_{i,j,k} \varepsilon_{ijk} a_i b_j c_k \; . \tag{1.197}$$

(g) Double Vector Product

We consider the k-th component of the double vector product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$:

$$\begin{aligned}
[\mathbf{a} \times (\mathbf{b} \times \mathbf{c})]_k &= \sum_{i,j} \varepsilon_{ijk} a_i (\mathbf{b} \times \mathbf{c})_j = \sum_{i,j} \sum_{l,m} \varepsilon_{ijk} \varepsilon_{lmj} a_i b_l c_m = \\
&= -\sum_{i,j} \sum_{l,m} \varepsilon_{ikj} \epsilon_{jlm} a_i b_l c_m \; .
\end{aligned}$$

One can apply here the following formula (proof as exercise!):

$$\sum_j \varepsilon_{ikj} \varepsilon_{jlm} = \delta_{il} \delta_{km} - \delta_{im} \delta_{kl} \; . \tag{1.198}$$

That we use in the above equation:

$$\begin{aligned}
[\mathbf{a} \times (\mathbf{b} \times \mathbf{c})]_k &= \sum_{i,l,m} a_i b_l c_m \left(\delta_{im} \delta_{kl} - \delta_{il} \delta_{km}\right) = \\
&= \sum_i \left(a_i b_k c_i - a_i b_i c_k\right) = b_k (\mathbf{a} \cdot \mathbf{c}) - c_k (\mathbf{a} \cdot \mathbf{b}) = \\
&= [\mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})]_k \; .
\end{aligned}$$

This holds for $k = 1, 2, 3$, so that the **expansion rule for the double vector product** (1.168) is now completely proven:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \tag{1.199}$$

Further, the reader should verify as an exercise the following important relations:

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) \ , \tag{1.200}$$

$$(\mathbf{a} \times \mathbf{b})^2 = a^2 b^2 - (\mathbf{a} \cdot \mathbf{b})^2 \ . \tag{1.201}$$

1.3.7 Exercises

Exercise 1.3.1 $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are orthogonal unit vectors in x, y, z-direction, respectively.

1. Calculate

$$\mathbf{e}_3 \cdot (\mathbf{e}_1 + \mathbf{e}_2) \ ,$$
$$(5\mathbf{e}_1 + 3\mathbf{e}_2) \cdot (7\mathbf{e}_1 - 16\mathbf{e}_3) \ ,$$
$$(\mathbf{e}_1 + 7\mathbf{e}_2 - 3\mathbf{e}_3) \cdot (12\mathbf{e}_1 - 3\mathbf{e}_2 - 4\mathbf{e}_3) \ .$$

2. Determine α so, that the vectors

$$\mathbf{a} = 3\mathbf{e}_1 - 6\mathbf{e}_2 + \alpha\mathbf{e}_3$$

and

$$\mathbf{b} = -\mathbf{e}_1 + 2\mathbf{e}_2 - 3\mathbf{e}_3$$

are orthogonal to each other!

3. How long is the projection of the vector

$$\mathbf{a} = 3\mathbf{e}_1 + \mathbf{e}_2 - 4\mathbf{e}_3$$

onto the direction of

$$\mathbf{b} = 4\mathbf{e}_2 + 3\mathbf{e}_3\,?$$

4. Decompose the vector

$$\mathbf{a} = \mathbf{e}_1 - 2\mathbf{e}_2 + 3\mathbf{e}_3 = \mathbf{a}_{\parallel} + \mathbf{a}_{\perp}$$

into a vector $\mathbf{a}_\perp$ perpendicular and a vector $\mathbf{a}_\parallel$ parallel to the vector

$$\mathbf{b} = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 \ .$$

Verify:

$$\mathbf{a}_\parallel \cdot \mathbf{a}_\perp = 0 \ .$$

5. Determine the angle between the vectors

$$\mathbf{a} = (2 + \sqrt{3})\mathbf{e}_1 + \mathbf{e}_2$$

and

$$\mathbf{b} = \mathbf{e}_1 + (2 + \sqrt{3})\mathbf{e}_2 \ .$$

Exercise 1.3.2

1. Given are two vectors **a** and **b** with the lengths $a = 6\,\text{cm}$, $b = 9\,\text{cm}$ enclosing the following angles: $\alpha = \sphericalangle(\mathbf{a}, \mathbf{b}) = 0,\ 60°,\ 90°,\ 150°,\ 180°$. Determine the length of the vector sum $\mathbf{a} + \mathbf{b}$ and the angle β

$$\beta = \sphericalangle(\mathbf{a} + \mathbf{b}, \mathbf{a}) \ .$$

2. Given are two vectors **a** and **b**

$$a = 6\,\text{cm}\,; \quad \sphericalangle\,(\mathbf{a}, \mathbf{e}_1) = 36° \ ,$$
$$b = 7\,\text{cm}\,; \quad \sphericalangle\,(\mathbf{b}, \mathbf{e}_1) = 180° \ .$$

Determine sum and difference of the two vectors as well as the angles each of them encloses with the $\mathbf{e}_1$-axis
3. Find the equation of the straight line which passes through the point P_0 whose position vector is

$$\mathbf{r}_0 = x_0\mathbf{e}_1 + y_0\mathbf{e}_2 + z_0\mathbf{e}_3$$

and which is parallel to the vector

$$\mathbf{f} = a\mathbf{e}_1 + b\mathbf{e}_2 + c\mathbf{e}_3$$

Exercise 1.3.3 Prove:

1.

$$(\mathbf{a} \times \mathbf{b})^2 = a^2b^2 - (\mathbf{a} \cdot \mathbf{b})^2 \ ,$$

2.

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) ,$$

3.

$$(\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})] = [\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})]^2 .$$

Exercise 1.3.4 Let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be unit vectors along x, y, z directions, respectively.

1. For the vectors

$$\mathbf{a} = 2\mathbf{e}_1 + 4\mathbf{e}_2 + 2\mathbf{e}_3$$

   and

$$\mathbf{b} = 3\mathbf{e}_1 - 2\mathbf{e}_2 - 7\mathbf{e}_3$$

   find the components along the above unit vectors for the following expressions: $(\mathbf{a} + \mathbf{b})$, $(\mathbf{a} - \mathbf{b})$, $(-\mathbf{a})$, $6(2\mathbf{a} - 3\mathbf{b})$. Calculate the lengths of these vectors and demonstrate the validity of the triangle inequality:

$$|\mathbf{a} + \mathbf{b}| \leq a + b .$$

2. Calculate:

$$\mathbf{a} \times \mathbf{b} , \quad (\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b}) , \quad \mathbf{a} \cdot (\mathbf{a} - \mathbf{b}) .$$

3. Calculate the area of the parallelogram spanned by the vectors **a** and **b** and determine the unit vector orthogonal to this area.

Exercise 1.3.5 Prove Thales' theorem by use of the vector calculation.

Exercise 1.3.6 Prove the distributive law for the multiplication of vectors **a**, **b** by a **negative** real number α:

$$\alpha(\mathbf{a} + \mathbf{b}) = \alpha\mathbf{a} + \alpha\mathbf{b}$$

Exercise 1.3.7 Decompose the vector **b** into a parallel and a perpendicular part relatively to vector **a** (Fig. 1.56):

$$\mathbf{b} = \mathbf{b}_{\|} + \mathbf{b}_{\perp}$$

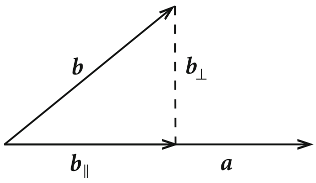


Fig. 1.56 Decomposition of vector **b** into a perpendicular and parallel component with respect to vector **a**

and show:

$$\mathbf{b}_\parallel = \frac{1}{a^2}(\mathbf{a}\cdot\mathbf{b})\,\mathbf{a}\ ,$$

$$\mathbf{b}_\perp = \frac{1}{a^2}\mathbf{a}\times(\mathbf{b}\times\mathbf{a})$$

Exercise 1.3.8 Verify the following equality

$$(\mathbf{a}-\mathbf{b})\cdot[(\mathbf{a}+\mathbf{b})\times\mathbf{c}] = 2\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$$

Exercise 1.3.9 Calculate for the three vectors

$$\mathbf{a} = (-1,\ 2,\ -3)\ ,\quad \mathbf{b} = (3,\ -1,\ 5)\ ,\quad \mathbf{c} = (-1,\ 0,\ 2)$$

the following expressions:

$$\begin{array}{lll} \mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})\ , & (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}\ , & |(\mathbf{a}\times\mathbf{b})\times\mathbf{c}|\ , \\ |\mathbf{a}\times(\mathbf{b}\times\mathbf{c})|\ , & (\mathbf{a}\times\mathbf{b})\times(\mathbf{b}\times\mathbf{c})\ , & (\mathbf{a}\times\mathbf{b})(\mathbf{b}\cdot\mathbf{c})\ . \end{array}$$

Exercise 1.3.10 Calculate:

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) + (\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{d}) + (\mathbf{c}\times\mathbf{a})\cdot(\mathbf{b}\times\mathbf{d})\ .$$

Exercise 1.3.11 **a** and **b** are two noncollinear vectors. Does the equation

$$\mathbf{a}\times\mathbf{y} = \mathbf{b}$$

have a solution for **y**? Justify!

Exercise 1.3.12 Prove the Jacobi identity (1.170):

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) + \mathbf{b}\times(\mathbf{c}\times\mathbf{a}) + \mathbf{c}\times(\mathbf{a}\times\mathbf{b}) = \mathbf{0}\ .$$

Exercise 1.3.13 $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ are three noncoplanar (not lying in the same plane) vectors. Three so-called reciprocal vectors $\mathbf{b}_1$, $\mathbf{b}_2$, $\mathbf{b}_3$ are defined by:

$$\mathbf{b}_1 = \frac{\mathbf{a}_2 \times \mathbf{a}_3}{\mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3)} ,$$

$\mathbf{b}_2$, $\mathbf{b}_3$ are given by cyclic permutation of the indexes (1, 2, 3).

1. Show for $i, j = 1, 2, 3$:

$$\mathbf{a}_i \cdot \mathbf{b}_j = \delta_{ij} .$$

2. Verify:

$$\mathbf{b}_1 \cdot (\mathbf{b}_2 \times \mathbf{b}_3) = [\mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3)]^{-1} .$$

3. Show that the $\mathbf{a}_i$ are the reciprocal vectors of the $\mathbf{b}_j$!
4. If $\mathbf{e}_i$, $i = 1, 2, 3$, are three orthonormal basis vectors. Find the corresponding reciprocal vectors!

Exercise 1.3.14 For two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}_2$ the following relations are found:

1.

$$\mathbf{a} \cdot \mathbf{b} = 4a_1b_1 - 2a_1b_2 - 2a_2b_1 + 3a_2b_2 ,$$

2.

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_2b_1 + 2a_1b_2 .$$

Are these products scalar products? Justify!

Exercise 1.3.15 Consider the ensemble V of real polynomials in one variable (degree $\leq$ 3):

$$V = \left\{p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 ; \quad a_0, \dots , a_3 \in \mathbb{R}\right\}$$

1. Show that V is a vector space over the body of real numbers.
2. Are the following elements linearly independent?

   (a)

$$p_1(x) = x^2 - 2x ; \quad p_2(x) = 7x^2 - x^3 ; \quad p_3(x) = 8x^2 + 11 ,$$

(b)

$$p_1(x) = -18x^2 + 15\ ; \quad p_2(x) = 3x^3 + 6x^2 - 5\ ; \quad p_3(x) = -x^3\ .$$

1.4 Vector-Valued Functions

By a *'vector-valued function'* one understands a function of one independent variable to which not just one single dependent variable is assigned but in fact $n > 1$ entities which together form an n-dimensional vector:

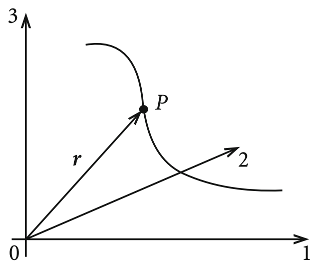
$$f : M \subset \mathbb{R}_1 \longrightarrow V \subset \mathbb{R}_n\ .$$

In this section we want to work out some important properties of such functions, which have a wide field of application in Theoretical Physics. We presume that the basic rules concerning continuity, differentiation, and integration of functions of **one** independent variable are known, e.g. from our introductory Sects. 1.1 and 1.2. We shall combine these tools with the vector algebra developed in the last chapter.

1.4.1 Parametrization of Space Curves

In physics space curves are typical examples of vector-valued functions. To start with we choose in the E_3 an arbitrary but fixed origin of coordinates $\mathcal{O}$. Then the momentary position P of a **'particle'** is determined by the position vector $\mathbf{r} = \overrightarrow{0P}$ (Fig. 1.57). By a **'particle'** we understand a physical body of mass m but with negligible extension in all directions. Later we will introduce for it the term **'mass point'**. In course of time the particle will in general change its position, i.e. $\mathbf{r}$ will change direction and magnitude. In a time-**in**dependent, complete orthonormal system (CONS) $\mathbf{e}_i$ the components of the position vector become *normal* time-

Fig. 1.57 Definition of the space curve of a particle by the temporally variable position vector



dependent functions:

$$\mathbf{r}(t) = \sum_{j=1}^{3} x_j(t)\mathbf{e}_j \equiv (x_1(t),\, x_2(t),\, x_3(t)) \ . \tag{1.202}$$

This is called the **trajectory** or the **path line** of the particle.

The set of space points the particle passes through over the time define the so-called

$$\textbf{space curve} := \ \{\mathbf{r}(t),\ t_a \le t \le t_e\} \ . \tag{1.203}$$

One calls (1.202) a **parametrization** of the space curve (1.203). The independent parameter in this case is the time t. Of course there also exist other possibilities of parametrization as we will see later in this section. Furthermore, it is clear that *different* path lines may parametrize the *same* space curve. For example this is already true when one and the same space curve is run through in opposite directions or in different time intervals.

Examples

1. **Circular motion in the xz-plane**

   Let the circle have the radius R and let its center point be at the origin of coordinates (Fig. 1.58). Then a self-evident parametrization is via the angle φ:

$$\begin{aligned} M &= \{\varphi\ ;\quad 0 \le \varphi \le 2\pi\}\ , \\ \mathbf{r}(\varphi) &= R(\cos\varphi,\, 0,\, \sin\varphi)\ . \end{aligned} \tag{1.204}$$

   Another parametrization can use, e.g., the x component x_1 :

$$\begin{aligned} M &= \{x_1\ ;\quad -R \le x_1 \le +R\}\ , \\ \mathbf{r}(x_1) &= \left(x_1,\, 0,\, \pm\sqrt{R^2 - x_1^2}\right)\ , \end{aligned}$$

   where the plus sign holds for the upper, the minus sign for the lower half-plane.

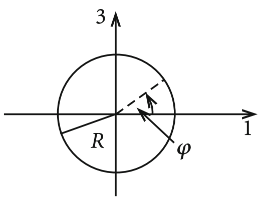


Fig. 1.58 Parametrization of a circular motion by the angle φ

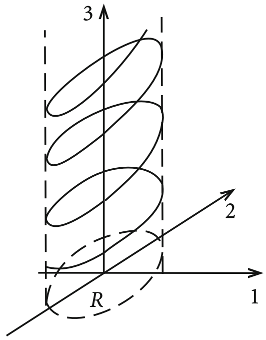


Fig. 1.59 Helical line as parametrized space curve

2. **Helical line**

   Let the independent parameter be t with

$$M = \{t\,; \quad -\infty < t < +\infty\}\ ,$$
$$\mathbf{r}(t) = (R\cos\omega t, R\sin\omega t,\ b\,t)\ . \tag{1.205}$$

R, b and ω are constants (Fig. 1.59). After one circulation $\omega\,\Delta t = 2\pi$ x and y components come back again to their initial values, while the z component has increased by the **pitch of the screw** (also called **height of ascent**) z_0:

$$z_0 = b\,\Delta t = b\frac{2\pi}{\omega}\ . \tag{1.206}$$

The **continuity of path lines** is defined analogously to that of normal functions (see Sect. 1.1.5).

Definition 1.4.1 $\mathbf{r}(t)$ is *continuous* at $t = t_0$, if for each $\varepsilon > 0$ there exists a $\delta(\varepsilon,\ t_0)$ so that for $|t - t_0| < \delta$ is always valid $|\mathbf{r}(t) - \mathbf{r}(t_0)| < \varepsilon$.

If one realizes that

$$|\mathbf{r}(t) - \mathbf{r}\,(t_0)| = \sqrt{[x_1(t) - x_1\,(t_0)]^2 + [x_2(t) - x_2\,(t_0)]^2 + [x_3(t) - x_3\,(t_0)]^2} \leq$$
$$\leq \sqrt{3}\max_{i=1,2,3}|x_i(t) - x_i\,(t_0)|\ .$$

then it becomes clear that $\mathbf{r}(t)$ is continuous if and only if **all** component functions are continuous in the ordinary sense.

1.4.2 Differentiation of Vector-Valued Functions

We consider a vector-valued function $\mathbf{a}(t)$ and look into the differential changes of the vector, i.e. the changes due to very small changes in time. Practically such a time interval, being determined by the measuring process, is of course always finite. Mathematically, however, an infinitely small time interval shall be considered. Furthermore, instead of time t any other parameter can also be used in the following formulae. The vector-valued function $\mathbf{a}(t)$ in general has at different times (parameters) t and $t + \Delta t$ different magnitudes and/or different directions. The magnitude of the difference vector

$$\Delta\mathbf{a} = \mathbf{a}(t + \Delta t) - \mathbf{a}(t)$$

will become smaller with decreasing time difference Δt, whereby its direction can change continuously in order to arrive for very small Δt in the corresponding direction of the respective tangent.

Definition 1.4.2 Derivation of a Vector-Valued Function

$$\frac{d\mathbf{a}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{a}(t + \Delta t) - \mathbf{a}(t)}{\Delta t} \; . \tag{1.207}$$

This definition clearly presumes that such a limiting vector does exist at all (Fig. 1.60). For time-derivatives sometimes one writes briefly:

$$\dot{\mathbf{a}}(t) \equiv \frac{d\mathbf{a}}{dt} \; .$$

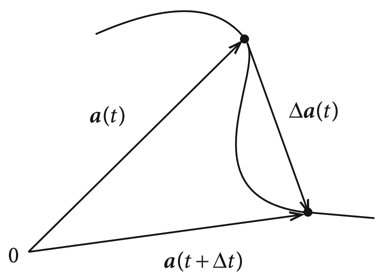


Fig. 1.60 To the definition of the derivative of a vector-valued function

We represent $\mathbf{a}(t)$ in a time-independent basis system $\{\mathbf{e}_i\}$:

$$\mathbf{a}(t) = \sum_j a_j(t)\mathbf{e}_j \ .$$

Then it holds:

$$\mathbf{a}(t + \Delta t) - \mathbf{a}(t) = \sum_j \left[a_j(t + \Delta t) - a_j(t)\right] \mathbf{e}_j \ .$$

Therewith the differentiation of a vector-valued function is obviously and completely expressed in terms of derivatives of the time-dependent component functions:

$$\dot{\mathbf{a}}(t) = \frac{d\mathbf{a}}{dt} = \sum_j \dot{a}_j(t)\mathbf{e}_j \ . \tag{1.208}$$

Correspondingly it holds also for all higher derivatives:

$$\frac{d^n}{dt^n}\mathbf{a}(t) = \sum_j \left(\frac{d^n}{dt^n}a_j(t)\right) \mathbf{e}_j \ ; \quad n = 0, 1, 2, \ \ldots \ . \tag{1.209}$$

Then it is not difficult to prove the following **rules of differentiation**

$$1)\ \ \frac{d}{dt}\left[\mathbf{a}(t) + \mathbf{b}(t)\right] = \dot{\mathbf{a}}(t) + \dot{\mathbf{b}}(t) \ , \tag{1.210}$$

$$2)\ \ \frac{d}{dt}\left[f(t)\,\mathbf{a}(t)\right] = \dot{f}(t)\,\mathbf{a}(t) + f(t)\,\dot{\mathbf{a}}(t) \ , \tag{1.211}$$

if $f(t)$ is a differentiable, scalar function,

$$3)\ \ \frac{d}{dt}\left[\mathbf{a}(t) \cdot \mathbf{b}(t)\right] = \dot{\mathbf{a}}(t) \cdot \mathbf{b}(t) + \mathbf{a}(t) \cdot \dot{\mathbf{b}}(t) \ , \tag{1.212}$$

$$4)\ \ \frac{d}{dt}\left[\mathbf{a}(t) \times \mathbf{b}(t)\right] = \dot{\mathbf{a}}(t) \times \mathbf{b}(t) + \mathbf{a}(t) \times \dot{\mathbf{b}}(t) \ . \tag{1.213}$$

In 4) we have to be very careful about the correct order of the factors.

Examples

$$\textbf{a) velocity: } \mathbf{v}(t) = \dot{\mathbf{r}}(t) \tag{1.214}$$

(always tangential to the path line) ,

$$\text{acceleration: } \mathbf{a}(t) = \dot{\mathbf{v}}(t) = \ddot{\mathbf{r}}(t) \ . \tag{1.215}$$

$$\textbf{b) unit vector:} \quad \mathbf{e}_a(t) = \frac{\mathbf{a}(t)}{|\mathbf{a}(t)|} \; .$$

$$\begin{aligned} \mathbf{e}_a^2(t) = 1 \;&\Longrightarrow\; \frac{d}{dt}\mathbf{e}_a^2(t) = 0 \overset{(1.212)}{=} 2\mathbf{e}_a(t)\cdot\dot{\mathbf{e}}_a(t) \\ &\Longrightarrow\; \frac{d}{dt}\mathbf{e}_a(t) \;\perp\; \mathbf{e}_a(t) \; . \end{aligned} \tag{1.216}$$

The derivation of a unit vector with respect to a parameter yields a vector which is always orthogonal to the original unit vector.

1.4.3 Arc Length

The **integration of vector-valued functions** can also be transferred to the corresponding integration of parameter-dependent component functions:

$$\int\limits_{t_a}^{t_e} \mathbf{a}(t)\,dt = \sum_{j=1}^{3} \mathbf{e}_j \int\limits_{t_a}^{t_e} a_j(t)\,dt \; . \tag{1.217}$$

If the basis vectors are parameter-independent they can be drawn in front of the integral. Thus in such a case one integrates the vector by integrating its components in the ordinary manner. However, it should be expressly indicated that the so defined integral of course depends on the special choice of the parameters and therefore does not at all represent a genuine curve property. During the course of this book we will meet other integrals of totally different type. However, at this stage we will make do with (1.217).

From now on, temporarily, we want to concentrate ourselves exclusively on space curves and path lines as examples of vector-valued functions. Thereby we assume for the following that the curve under consideration is *'smooth'*.

Definition 1.4.3 A space curve is denoted as *smooth*, if there exists at least one continuously differentiable parametrization $\mathbf{r} = \mathbf{r}(t)$ for which at no point we have:

$$\frac{d\mathbf{r}}{dt} = 0$$

For such smooth space curves it often appears convenient to use the so-called **arc length** s as curve parameter.

Definition 1.4.4 The **arc length** s is the length of the space curve, measured along the curved line starting from an arbitrarily chosen initial point.

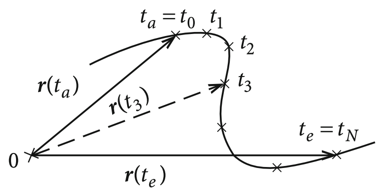


Fig. 1.61 Definition of the arc length as curve parameter

This we want to explain a bit in more detail. For this purpose and to be concrete, at first we still consider the time as the curve parameter and divide the time interval from $t_a = t_0$ to $t_e = t_N$ into N partial intervals Δt_N such that (see the marks on the space curve (Fig. 1.61)):

$$t_n = t_a + n\Delta t_N \; ; \quad n = 0, 1, 2, \ \ldots \ , N \quad \text{with } t_0 = t_a, t_N = t_e \; .$$

These time marks correspond to position vectors $\mathbf{r}(t_n)$. If we connect the time marks by straight lines then we get a **polygonal line** of the length:

$$L_N\,(t_a, t_e) = \sum_{n=0}^{N-1} |\mathbf{r}\,(t_{n+1}) - \mathbf{r}\,(t_n)| = \sum_{n=0}^{N-1} \left| \frac{\mathbf{r}\,(t_{n+1}) - \mathbf{r}\,(t_n)}{\Delta t_N} \right| \Delta t_N \; .$$

In the limit $N \to \infty$ the length L_N of the polygonal line corresponds to the arc length s between the endpoints $\mathbf{r}(t_a)$ and $\mathbf{r}(t_e)$. $N \to \infty$, however, implies that Δt_N approaches zero. Then, according to (1.207), we have, after the sum symbol, just the derivative of the position vector with respect to time:

$$\frac{\mathbf{r}\,(t_{n+1}) - \mathbf{r}\,(t_n)}{\Delta t_N} \xrightarrow[N\to\infty \Longleftrightarrow \Delta t_N \to 0]{} \left. \frac{d\mathbf{r}}{dt} \right|_{t=t_n} \; .$$

So the sum becomes an integral in *Riemannien sense*. If we now replace t_e by t then we have as arc length:

$$s(t) = \int_{t_a}^{t} \left| \frac{d\mathbf{r}(t')}{dt'} \right| dt' \; . \tag{1.218}$$

Furthermore, we have also shown that for **differential** changes of the arc length it holds:

$$\frac{ds}{dt} = \left| \frac{d\mathbf{r}(t)}{dt} \right| \; > \; 0 \tag{1.219}$$

Thus according to (1.218) we calculate the arc length $s(t)$ by use of the path line $\mathbf{r} = \mathbf{r}(t)$. The arc length is obviously a monotonically increasing function of t

which therefore can be uniquely inverted to give $t(s)$. Therewith we obtain the unambiguous parametrization of the space curve by the arc length s;

$$\mathbf{r}(t) \to \mathbf{r}\,(t(s)) = \mathbf{r}(s)\ . \tag{1.220}$$

This representation is denoted as **natural parametrization** of the space curve.

Examples

1. **Circular motion**
   In (1.204) we set $\varphi = \omega t$ (*uniform circular motion*) getting therewith as path line:

$$\mathbf{r}(t) = R(\cos\omega t,\ 0,\ \sin\omega t)$$

$$\begin{aligned}
&\Longrightarrow \frac{d\mathbf{r}}{dt} = R\omega(-\sin\omega t,\ 0,\ \cos\omega t)\\
&\Longrightarrow \left|\frac{d\mathbf{r}}{dt}\right| = R\omega\\
&\Longrightarrow s(t) = \int_0^t R\omega\, dt' = R\omega t \qquad\qquad (t_a = 0)\\
&\Longrightarrow t(s) = \frac{s}{R\omega}\ .
\end{aligned}$$

   Therewith we have the natural representation of the circular motion:

$$\mathbf{r}(s) = R\left(\cos\frac{s}{R},\ 0,\ \sin\frac{s}{R}\right)\ . \tag{1.221}$$

   After going through a full circle we must have:

$$\frac{s}{R} = 2\pi$$

   That corresponds to the arc length $s = 2\pi R$ being just the circumference.
2. **Helical line**
   We derive from (1.205):

$$\begin{aligned}
&\frac{d\mathbf{r}}{dt} = (-R\omega\sin\omega t,\ R\omega\cos\omega t,\ b)\\
\Longrightarrow &\left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{R^2\omega^2 + b^2}
\end{aligned}$$

$$\Longrightarrow s(t) = \sqrt{R^2\omega^2 + b^2}\; t$$
$$\Longrightarrow \;\; t(s) = \frac{s}{\sqrt{R^2\omega^2 + b^2}} \; .$$

Then we get the natural representation of the helical line:

$$\mathbf{r}(s) = \left(R\cos\frac{\omega s}{\sqrt{R^2\omega^2 + b^2}} \, , \; R\sin\frac{\omega s}{\sqrt{R^2\omega^2 + b^2}} \, , \; \frac{bs}{\sqrt{R^2\omega^2 + b^2}} \right) . \tag{1.222}$$

1.4.4 Moving Trihedron

In this section we introduce a new system of orthonormal basis vectors, the directions of which can be different from point to point on the space curve. Thus they are functions of the arc length, in a certain sense accompanying the mass point as it moves along the space curve. One therefore speaks of a **moving trihedron** consisting of

$\hat{\mathbf{t}}$: tangent-unit vector ,
$\hat{\mathbf{n}}$: normal-unit vector ,
$\hat{\mathbf{b}}$: binormal-unit vector .

The three unit vectors build an orthonormal right-handed trihedron. That means:

$$\hat{\mathbf{t}} = \hat{\mathbf{n}} \times \hat{\mathbf{b}} \quad \text{and cyclic} \; . \tag{1.223}$$

We know that the vector $\dot{\mathbf{r}}(t) = \frac{d}{dt}\mathbf{r}(t)$ is oriented **tangentially** to the path line. The **tangent-unit vector** is therefore defined in an obvious manner as follows:

$$\hat{\mathbf{t}} = \frac{\dfrac{d\mathbf{r}}{dt}}{\left|\dfrac{d\mathbf{r}}{dt}\right|} = \frac{\dfrac{d\mathbf{r}}{dt}}{\dfrac{ds}{dt}} \; . \tag{1.224}$$

On the right-hand side we have used Eq. (1.219). If $\mathbf{r}$ is parametrized by the arc length s, $\mathbf{r} = \mathbf{r}(s)$, then we can exploit in (1.224) the chain rule (1.87):

$$\hat{\mathbf{t}} = \frac{d\mathbf{r}(s)}{ds} = \hat{\mathbf{t}}(s) \; . \tag{1.225}$$

$\hat{\mathbf{t}}$ thus lies tangentially to the path line in direction of increasing arc length (Fig. 1.62). $\hat{\mathbf{t}}(s)$ can change its direction as function of s so that it can be considered

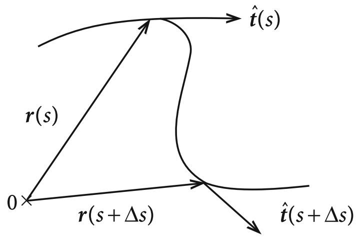


Fig. 1.62 Illustration of the tangent-unit vector

as a measure of the curvature of the path. Logically consistently one defines:

$$\begin{aligned} \kappa &= \left|\frac{d\hat{\mathbf{t}}(s)}{ds}\right| && \textbf{curvature} \;, \\ \rho &= \kappa^{-1} && \textbf{radius of curvature} \;. \end{aligned} \tag{1.226}$$

If the direction of $\hat{\mathbf{t}}(s)$ is constant for all s then the path is obviously a straight line. In this case κ is zero and $\rho = \infty$.

Since $\hat{\mathbf{t}}$ is oriented tangentially to the path line the two other unit vectors must lie within the plane perpendicular to the tangent. Because of (1.216) the vector

$$\mathbf{N} = \frac{d\hat{\mathbf{t}}}{ds}$$

will definitely be orthogonal to $\hat{\mathbf{t}}$. In addition, normalizing to unity will result in a unit vector which is called

$$\textbf{normal-unit vector:} \qquad \hat{\mathbf{n}} = \frac{\frac{d\hat{\mathbf{t}}(s)}{ds}}{\left|\frac{d\hat{\mathbf{t}}(s)}{ds}\right|} = \frac{1}{\kappa}\frac{d\hat{\mathbf{t}}(s)}{ds} = \hat{\mathbf{n}}(s) \;. \tag{1.227}$$

The plane spanned by the vectors $\hat{\mathbf{n}}$ and $\hat{\mathbf{t}}$ is referred to as **osculating plane**. For a complete characterization of the motion in space we still need a third unit vector, namely the

$$\textbf{binormal-unit vector} \qquad \hat{\mathbf{b}}(s) = \hat{\mathbf{t}}(s) \times \hat{\mathbf{n}}(s) \;. \tag{1.228}$$

$\hat{\mathbf{b}}$ stands perpendicular to the osculating plane. If the motion happens in a **fixed** plane then this plane is simultaneously also the osculating plane, and therefore is independent of s. Consequently, the direction of $\hat{\mathbf{b}}$ is certainly constant, the

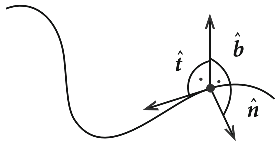


Fig. 1.63 Representation of the moving trihedron

magnitude is anyway constant, so that it must generally hold:

$$\hat{\mathbf{b}} = \text{const}\ , \quad \text{if the motion happens in a } \textbf{fixed} \text{ plane}\ .$$

If however $\hat{\mathbf{b}}$ does explicitly change with s, then it provides obviously a measure to which degree the space curve is screwing itself out of the osculating plane (Fig. 1.63). Therefore again, the derivative with respect to s will be of interest:

$$\begin{aligned} \tfrac{d\hat{\mathbf{b}}}{ds} = \tfrac{d\hat{\mathbf{t}}}{ds} \times \hat{\mathbf{n}} + \hat{\mathbf{t}} \times \tfrac{d\hat{\mathbf{n}}}{ds} = \kappa\, \hat{\mathbf{n}} \times \hat{\mathbf{n}} + \hat{\mathbf{t}} \times \tfrac{d\hat{\mathbf{n}}}{ds} \\ \Longrightarrow \tfrac{d\hat{\mathbf{b}}}{ds} = \hat{\mathbf{t}} \times \tfrac{d\hat{\mathbf{n}}}{ds}\ . \end{aligned} \tag{1.229}$$

From this we can conclude:

$$\frac{d}{ds}\hat{\mathbf{b}} \perp \hat{\mathbf{t}}\ .$$

Furthermore, since $\hat{\mathbf{b}}$ is a unit vector we have,

$$\frac{d}{ds}\hat{\mathbf{b}} \perp \hat{\mathbf{b}}\ ,$$

so that the following ansatz appears reasonable:

$$\frac{d}{ds}\hat{\mathbf{b}} = -\tau \hat{\mathbf{n}}\ . \tag{1.230}$$

The binormal is thus twisting perpendicular to $\hat{\mathbf{t}}$ into the direction of the principal normal $\hat{\mathbf{n}}$:

$$\begin{aligned} \tau: &\quad \textbf{torsion of the space curve} \\ \sigma = 1/\tau: &\quad \textbf{torsion radius}\ . \end{aligned}$$

We still do not have the change of the normal-unit vector $\hat{\mathbf{n}}$ with the arc length s:

$$\hat{\mathbf{n}} = \hat{\mathbf{b}} \times \hat{\mathbf{t}} \Longrightarrow \frac{d\hat{\mathbf{n}}}{ds} = \frac{d\hat{\mathbf{b}}}{ds} \times \hat{\mathbf{t}} + \hat{\mathbf{b}} \times \frac{d\hat{\mathbf{t}}}{ds} = -\tau\, \hat{\mathbf{n}}\, \times\, \hat{\mathbf{t}} + \kappa\, \hat{\mathbf{b}} \times \hat{\mathbf{n}} = \tau \hat{\mathbf{b}} - \kappa \hat{\mathbf{t}}\ .$$

The three relations which describe the change of the moving trihedron as function of the arc length s are known as the **Frenet's formulae**:

$$\begin{aligned}\frac{d\hat{\mathbf{t}}}{ds} &= \kappa\hat{\mathbf{n}}\,,\\ \frac{d\hat{\mathbf{b}}}{ds} &= -\tau\hat{\mathbf{n}}\,,\\ \frac{d\hat{\mathbf{n}}}{ds} &= \tau\hat{\mathbf{b}}-\kappa\hat{\mathbf{t}}\,.\end{aligned} \tag{1.231}$$

1.4.4.1 Applications

1. **Circular motion:**
   With the *natural* representation of the space curve $\mathbf{r} = \mathbf{r}(s)$ (1.221) the **tangent-unit vector** is very easily calculated:

$$\hat{\mathbf{t}} = \frac{d\mathbf{r}}{ds} = \left(-\sin\frac{s}{R},\, 0,\, \cos\frac{s}{R}\right)\,. \tag{1.232}$$

   It is obviously a vector of length 1. Differentiating once more with respect to s yields the curvature κ:

$$\begin{aligned}\frac{d\hat{\mathbf{t}}}{ds} &= \frac{1}{R}\left(-\cos\frac{s}{R},\, 0,\, -\sin\frac{s}{R}\right)\\ \Longrightarrow \kappa &= \left|\frac{d\hat{\mathbf{t}}}{ds}\right| = \frac{1}{R}\,.\end{aligned} \tag{1.233}$$

   For the **radius of curvature** we thus have found the self-evident result:

$$\rho = R\,. \tag{1.234}$$

   The **normal-unit vector** $\hat{\mathbf{n}}$ lies in the xz plane pointing to the center of the circle (Fig. 1.64) (Verify $\hat{\mathbf{n}}\cdot\hat{\mathbf{t}} = 0$!).

$$\hat{\mathbf{n}} = \rho\frac{d\hat{\mathbf{t}}}{ds} = \left(-\cos\frac{s}{R},\, 0,\, -\sin\frac{s}{R}\right) \tag{1.235}$$

   Since the motion takes place n a fixed plane we have to expect that the binormal-unit vector $\hat{\mathbf{b}}(s)$ is constant with respect to direction and

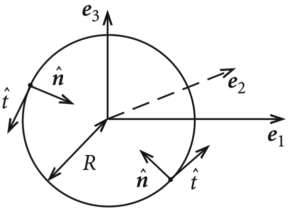


Fig. 1.64 Normal- and tangent-unit vectors at a circle

magnitude:

$$\begin{aligned}\hat{\mathbf{b}}(s) &= \mathbf{e}_1\,(t_2n_3 - t_3n_2) + \mathbf{e}_2\,(t_3n_1 - t_1n_3) + \mathbf{e}_3\,(t_1n_2 - t_2n_1) = \\ &= \mathbf{e}_1 \cdot \mathbf{0} + \mathbf{e}_2 \left(-\cos^2\frac{s}{R} - \sin^2\frac{s}{R}\right) + \mathbf{e}_3 \cdot 0\,.\end{aligned}$$

This indeed is the case:

$$\hat{\mathbf{b}}(s) = (0,\,-1,\,0)\,. \tag{1.236}$$

The unit vector points into the negative y-direction.

2. **Helical line**

According to (1.222), introducing the abbreviation $\Delta = 1/\sqrt{R^2\omega^2 + b^2}$, we have for the helical line:

$$\hat{\mathbf{t}} = \frac{d\mathbf{r}}{ds} = (-R\omega\Delta\sin(\omega s\Delta),\,R\omega\Delta\cos(\omega s\Delta),\,b\Delta)\,. \tag{1.237}$$

For the magnitude of $\hat{\mathbf{t}}$ one finds

$$|\hat{\mathbf{t}}| = \sqrt{(R^2\omega^2 + b^2)\Delta^2} = 1\,,$$

as it must be.

$$\frac{d\hat{\mathbf{t}}}{ds} = \left(-R\omega^2\Delta^2\,\cos(\omega s\Delta),\,-R\omega^2\Delta^2\,\sin(\omega s\Delta),\,0\right)\,.$$

The curvature κ then reads:

$$\kappa = \left|\frac{d\hat{\mathbf{t}}}{ds}\right| = R\omega^2\Delta^2 = \frac{R\omega^2}{R^2\omega^2 + b^2}\,. \tag{1.238}$$

The curvature of the helical line is obviously smaller than that of the circle, as it is geometrically evident since the elongation along the spiral axis of course reduces the curvature.

$$\text{radius of curvature}: \quad \rho = \frac{R^2\omega^2 + b^2}{R\omega^2} > R\,. \tag{1.239}$$

The **normal-unit vector** lies in the xy plane and points into the inside of the screw

$$\hat{\mathbf{n}} = (-\cos(\omega s\Delta),\, -\sin(\omega s\Delta),\, 0)\ . \tag{1.240}$$

The **binormal-unit vector** is now a function of the arc length s because the motion is not bounded to a fixed plane:

$$\begin{aligned} \hat{\mathbf{b}}(s) &= \mathbf{e}_1\,[+b\Delta\sin(\omega\, s\Delta)] + \mathbf{e}_2\,[-b\Delta\cos(\omega\, s\Delta)] + \\ &\quad + \mathbf{e}_3\left[R\omega\Delta\sin^2(\omega\, s\Delta) + R\omega\Delta\cos^2(\omega\, s\Delta)\right] \\ \Longrightarrow \hat{\mathbf{b}}(s) &= \Delta\,(b\sin(\omega\, s\Delta),\, -b\cos(\omega\, s\Delta), R\omega)\ . \end{aligned} \tag{1.241}$$

The torsion τ of the space curve is calculated according to (1.230) by a comparison of

$$\frac{d\hat{\mathbf{b}}}{ds} = b\,\omega\Delta^2\,(\cos(\omega\, s\Delta),\, \sin(\omega\, s\Delta),\, 0)$$

with $\hat{\mathbf{n}}$ (1.240) so that we get

$$\tau = b\,\omega\,\Delta^2\ . \tag{1.242}$$

The torsion radius

$$\sigma = \frac{1}{\tau} = \frac{R^2\omega^2 + b^2}{b\,\omega} \tag{1.243}$$

will become infinitely large for $b \to 0$ (circular motion).

3. **Velocity and acceleration of a mass point**

   According to (1.214) the velocity $\mathbf{v}$ is always tangentially oriented to the path line $\mathbf{r}(t)$:

$$\begin{aligned} \mathbf{v}(t) &= \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\frac{ds}{dt} = \frac{ds}{dt}\hat{\mathbf{t}} \\ \Longrightarrow |\mathbf{v}(\mathbf{t})| &= \frac{ds}{dt}\ . \end{aligned} \tag{1.244}$$

Differentiating once more with respect to time gives the acceleration $\mathbf{a}$:

$$\mathbf{a}(t) = \frac{d^2\mathbf{r}}{dt^2} = \dot{v}\hat{\mathbf{t}} + v\frac{d\hat{\mathbf{t}}}{dt} = \dot{v}\hat{\mathbf{t}} + v\frac{d\hat{\mathbf{t}}}{ds}\frac{ds}{dt}$$

$$\Longrightarrow \mathbf{a}(t) = \dot{v}\hat{\mathbf{t}} + \frac{v^2}{\rho}\hat{\mathbf{n}}\,. \tag{1.245}$$

The acceleration vector thus always lies in the osculating plane. One distinguishes:

$$a_t = \dot{v} \quad \textbf{(tangential acceleration)} \tag{1.246}$$

and

$$a_n = \frac{v^2}{\rho} \quad \textbf{(normal , centripetal acceleration)}\,. \tag{1.247}$$

We notice that for curved path lines ($\rho \neq \infty$) an accelerated motion occurs even when the velocity magnitude v does not change with time ($\dot{v} = 0$). An exception is only the straight line ($\rho = \infty$), only.

1.4.5 Exercises

Exercise 1.4.1 $\mathbf{e}_1'$ and $\mathbf{e}_2'$ are two orthonormal vectors which define the x' axis and the y' axis, respectively. A mass point moves along the path line:

$$\mathbf{r}(t) = \frac{1}{\sqrt{2}}\left(a_1\cos\omega t + a_2\sin\omega t\right)\,\mathbf{e}_1' + \frac{1}{\sqrt{2}}\left(-a_1\cos\omega t + a_2\sin\omega t\right)\,\mathbf{e}_2'\,,$$

a_1, a_2, ω are constant and > 0.

1. Go over from $\mathbf{e}_1'$, $\mathbf{e}_2'$ to a new basis $\mathbf{e}_1$, $\mathbf{e}_2$, i.e. to new x and y axis, and that in such a way that the representation of the space curve becomes especially simple. What is the parameter representation of the space curve in the x, y-system with ωt as parameter?
2. Which geometrical form does the space curve have?
3. Determine the angles

$$\varphi(t) = \sphericalangle\left(\mathbf{e}_1, \mathbf{r}(t)\right)\,,$$
$$\psi(t) = \sphericalangle\left(\mathbf{e}_2, \mathbf{r}(t)\right)\,.$$

4. Calculate the magnitudes of $\mathbf{r}(t)$, $\mathbf{v}(t) = \dot{\mathbf{r}}(t)$, $\mathbf{a}(t) = \ddot{\mathbf{r}}(t)$. Which relation does exist between $|\mathbf{r}(t)|$ and $|\mathbf{a}(t)|$?

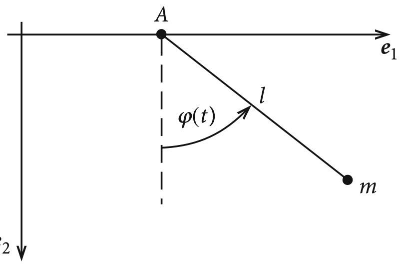


Fig. 1.65 Mass point m on a thread which is fixed on an horizontally mobile suspension A

2.

5. Calculate $\dot{r}(t) = \frac{d}{dt}|\mathbf{r}(t)|$.
6. Determine the angles:

$$\begin{aligned}
\alpha(t) &= \sphericalangle(\mathbf{r}(t), \mathbf{v}(t)) \ , \\
\beta(t) &= \sphericalangle(\mathbf{v}(t), \mathbf{a}(t)) \ , \\
\gamma(t) &= \sphericalangle(\mathbf{r}(t), \mathbf{a}(t)) \ .
\end{aligned}$$

Exercise 1.4.2

1. Determine the parameter representation of the cycloid. This curve is described by a fixed point on a circle where the latter rolls off on a straight line.
2. What is the parameter representation of a mass point on a thread which swings back and forth with a time-dependent angle $\varphi(t)$ where simultaneously the suspension A moves with constant velocity v in $\mathbf{e}_1$ direction (Fig. 1.65)?

Exercise 1.4.3 Calculate for the path line

$$\mathbf{r}(t) = e^{-\sin t}\mathbf{e}_1 + \frac{1}{\cot t}\mathbf{e}_2 + \ln\left(1 + t^2\right)\mathbf{e}_3$$

the expressions:

$$1)\,|\mathbf{r}(t)|\ ; \quad 2)\,\dot{\mathbf{r}}(t)\ ; \quad 3)\,|\dot{\mathbf{r}}(t)|\ ; \quad 4)\,\ddot{\mathbf{r}}(t)\ ; \quad 5)\,|\ddot{\mathbf{r}}(t)|$$

and that always for the time $t = 0$.

Exercise 1.4.4 Prove the following rules of differentiation for vector-valued functions $\mathbf{a}(t)$, $\mathbf{b}(t)$:

$$1)\ \frac{d}{dt}\left[\mathbf{a}(t)\cdot\mathbf{b}(t)\right] = \dot{\mathbf{a}}(t)\cdot\mathbf{b}(t) + \mathbf{a}(t)\cdot\dot{\mathbf{b}}(t)\ ,$$

$$2)\ \frac{d}{dt}\left[\mathbf{a}(t)\times\mathbf{b}(t)\right] = \dot{\mathbf{a}}(t)\times\mathbf{b}(t) + \mathbf{a}(t)\times\dot{\mathbf{b}}(t)\ ,$$

$$3)\ \mathbf{a}(t)\frac{d\mathbf{a}(t)}{dt} = \left|\mathbf{a}(t)\right|\frac{d}{dt}\left|\mathbf{a}(t)\right|\ .$$

Exercise 1.4.5 For the following path line

$$\mathbf{r}(t) = \left(3\sin\frac{t}{t_0},\ 4\frac{t}{t_0},\ 3\cos\frac{t}{t_0}\right) .$$

calculate:

1. the arc length $s(t)$ where $s(t=0)=0$,
2. the tangent-unit vector $\hat{\mathbf{t}}$,
3. the curvature κ and the radius of curvature ρ of the curve,
4. the normal-unit vector $\hat{\mathbf{n}}$,
5. the moving trihedron $(\hat{\mathbf{t}},\ \hat{\mathbf{n}},\ \hat{\mathbf{b}})$ for $t = 5\pi t_0$,
6. the torsion τ of the space curve.

Exercise 1.4.6 Show that the curvature κ of a space curve fulfills the relation

$$\kappa = \frac{1}{|\dot{\mathbf{r}}|^3}|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|$$

Exercise 1.4.7 Express in a as simple as possible manner

$$\frac{d\mathbf{r}}{ds} \cdot \left(\frac{d^2\mathbf{r}}{ds^2} \times \frac{d^3\mathbf{r}}{ds^3}\right)$$

in terms of the curvature κ and the torsion τ of the space curve.

Exercise 1.4.8 Given is the path line

$$\mathbf{r}(t) = \left(t,\ t^2,\ \left(\frac{2}{3}\right)t^3\right) .$$

The components are assumed to have coefficients of magnitude 1 in order to provide for correct dimensions.

1. Determine the arc length $s(t)$ where $s(t=0)=0$.
2. Calculate the tangent-unit vector $\hat{\mathbf{t}}$ as function of time t.
3. Express the curvature κ as a function of t.
4. Determine the moving trihedron as function of t.
5. Derive the torsion τ as function of t.

Exercise 1.4.9

1. Calculate the curvature, the torsion, and the moving trihedron of the space curve

$$\mathbf{r}(\varphi) = R(\varphi + \sin\varphi, 1 + \cos\varphi, 0) .$$

2. Determine the curvature of the planar space curve

$$\mathbf{r}(\varphi) = (\varphi, f(\varphi), 0) \ .$$

1.5 Fields

In the last section we have become acquainted with vector-valued functions as e.g. the path line of a particle. Therewith we describe the trajectory of the particle through space. However, we do not yet know what *'happens'* to the mass point on its path, which situations it encounters. For instance, the temperature might differ at different space points and therefore could influence therewith the nature of motion. The electric field intensity can be space dependent what would be of importance for the path of a charged particle. For the description of physical phenomena it is therefore very often necessary to attach to each space point $\mathbf{r}$ the value $A(\mathbf{r})$ of a certain physical quantity. This can be a scalar, a vector, a tensor ..., as e.g. the temperature, the mass density, the charge density as scalars or the gravitational force, the electric field strength, the flow velocity of a liquid as vectors and the stress tensor as a tensorial quantity. One speaks of a scalar, vectorial, tensorial field of the physical quantity A. In general the attached values will still depend on time: $A = A(\mathbf{r}, t)$. The following considerations will, however, be restricted to time independent, i.e. **static** fields. An orthonormal basis is assumed to be given.

1.5.1 Classification of the Fields

Definition 1.5.1 A **scalar field** is the ensemble of numerical values $\varphi(\mathbf{r}) = \varphi(x_1, x_2, x_3)$ of a physical quantity φ which are ascribed to each point $\mathbf{r} = (x_1, x_2, x_3)$ in a particular region of space:

$$M \subset \mathbb{R}_3 \xrightarrow{\varphi} N \subset \mathbb{R}_1 \ .$$

Then it is a **scalar-valued** function of **three** independent variables. The domain of definition M is fixed by the physical problem under study.

Graphically such fields are exhibited by two-dimensional profiles in which the areas $\varphi(\mathbf{r}) = \text{const}$ appear as so-called **contour lines**. The distance between the lines corresponds to a predetermined increase or decrease of the value of the constant (Fig. 1.66).

There are some other possibilities of field characterization. For instance, one can plot φ in dependence of **one** especially significant variable keeping thereby the other variables constant (Fig. 1.67).

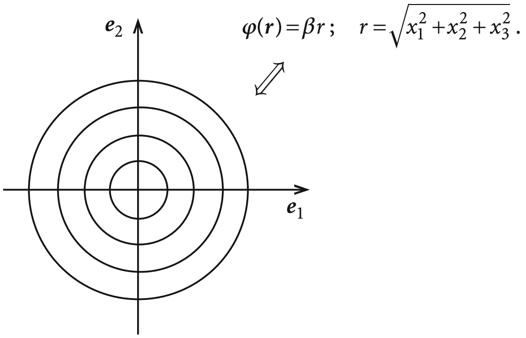


Fig. 1.66 Contour lines of the scalar field $\varphi(\mathbf{r}) = \beta\, r$

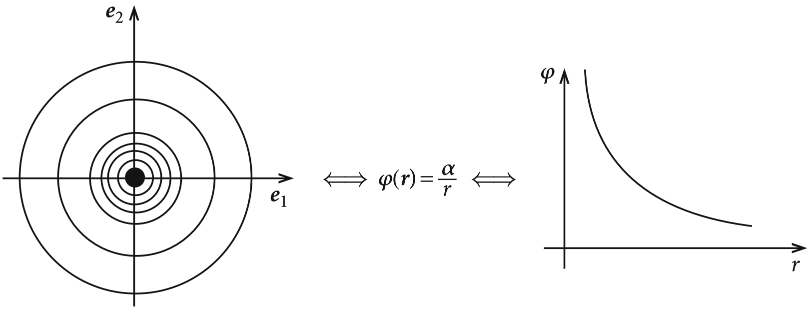


Fig. 1.67 The scalar field $\varphi(\mathbf{r}) = (\alpha/r)$ represented by its contour lines (*left*) and by its radial dependency (*right*)

Definition 1.5.2 The **vector field** is the collection of vectors, each marked by a direction and a magnitude (length, norm)

$$\mathbf{a}(\mathbf{r}) = (a_1\,(x_1, x_2, x_3)\,,\; a_2\,(x_1, x_2, x_3)\,,\; a_3\,(x_1, x_2, x_3))\;\;,$$

which are dedicated to each point $\mathbf{r} = (x_1, x_2, x_3)$ in a region of space M of interest:

$$M \subset \mathbb{R}_3 \to N \subset \mathbb{R}_3\;.$$

Hence it is about a **vector-valued** function of **three** independent variables.

Examples

$$\mathbf{a}(\mathbf{r}) = \alpha \mathbf{r}\;,$$

$$\mathbf{a}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\frac{\mathbf{r}}{r^3} \quad \text{(electrical field of a point charge } q)\;,$$

$$\mathbf{a}(\mathbf{r}) = \frac{\alpha}{\beta^2 + x_2^2 + x_3^2}\mathbf{e}_1 \; ; \quad \alpha, \beta = \text{const} \; ,$$

$$\mathbf{a}(\mathbf{r}) = \frac{1}{r}[\boldsymbol{\omega} \times \mathbf{r}] \; ; \quad \boldsymbol{\omega} = \omega_0 \mathbf{e}_3 \; ; \quad \omega_0 = \text{const} \; .$$

Graphically these vector fields can be exhibited by two-dimensional profiles (cuts) in which the areas of constant field strength $|\mathbf{a}(\mathbf{r})| = \text{const}$ appear as **contour lines** to which the field itself is locally attached as a vector arrow (Fig. 1.68).

Example

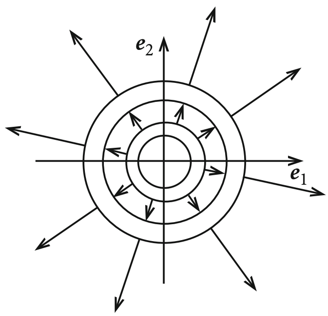
$$\mathbf{a}(\mathbf{r}) = \alpha \mathbf{r} \quad (\alpha > 0) \; .$$

The length of the vector arrow is equal to $\alpha\, r$ and the direction of the arrow perpendicular to the circles $|\mathbf{a}(\mathbf{r})|$ =const.

A further frequently used possibility of representation applies so-called **'field (force) lines'**, the local directions of which characterize the respective field direction while their **line density** is a measure of the **strength** of the field (see Fig. 1.69).

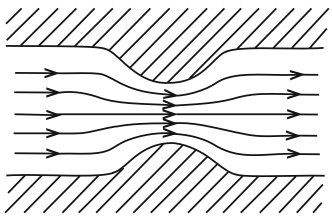
In the following we want to investigate the special properties of fields, where, however, because of the necessary conciseness of the presentation extensive and precise considerations must be left to relevant mathematics courses.

Fig. 1.68 Representation of the vector field $\alpha\mathbf{r}$



Length of the arrow: $a{\cdot}r$
Direction : radial, perpendicular of the circler $|a(r)|$ = const

Fig. 1.69 *Field line* representation of the velocity of a flowing liquid



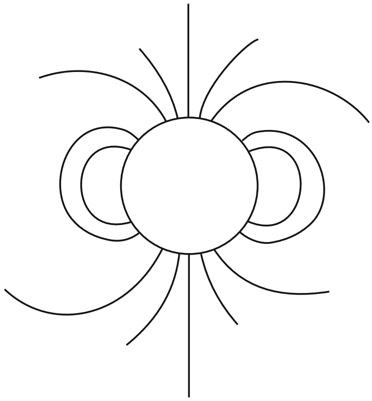


Fig. 1.70 Schematic *field line* representation of the earth magnetic field

Since the fields (Fig. 1.70) are functions of **several** independent variables terms such as continuity, derivative and integral must be handled with care.

Definition 1.5.3

1. A scalar field $\varphi(\mathbf{r})$ is called *'continuous'* at the point $\mathbf{r}_0$, if there does exist to each $\varepsilon > 0$ a $\delta(\mathbf{r}_0, \varepsilon) > 0$ so that for all $\mathbf{r}$ with $|\mathbf{r} - \mathbf{r}_0| < \delta$ it holds:

$$|\varphi(\mathbf{r}) - \varphi(\mathbf{r}_0)| < \varepsilon$$

2. The field φ is called continuous in a region of space M if it is continuous in **each** point of M.
3. A vector field $\mathbf{a}(\mathbf{r}) = (a_1(\mathbf{r}), a_2(\mathbf{r}), a_3(\mathbf{r}))$ is continuous at $\mathbf{r}_0$ if this is true in the above sense for each of the scalar component fields $a_i(\mathbf{r})$.

We have to investigate a little further when we think of the derivatives of fields.

1.5.2 Partial Derivatives

Now we are interested in how a field changes from space point to space point. Information about this will be given by the **derivative of the field with respect to position**. We comment on this *operation* at first for a scalar field. Generalizations to vector fields will then be not too difficult. We demand basically only that the criteria which we derive for scalar functions are fulfilled by **each** of the component function.

We first consider the **change of the field** φ along a way

parallel to an axis of the coordinates, then, strictly speaking, on this path the field depends only on **one** true variable since the other two are held constant. Then one can differentiate with respect to this effectively single variable in the usual manner,

$$\lim_{\Delta x_1 \to 0} \frac{\varphi\left(x_1 + \Delta x_1, x_2, x_3\right) - \varphi\left(x_1, x_2, x_3\right)}{\Delta x_1} \equiv \left(\frac{\partial \varphi}{\partial x_1}\right)_{x_2, x_3} , \tag{1.248}$$

and one speaks of a **partial derivative of** φ **with respect to** x_1.

$$\left(\text{Notations: } \left(\frac{\partial \varphi}{\partial x_1}\right)_{x_2, x_3} \Longleftrightarrow \frac{\partial \varphi}{\partial x_1} \Longleftrightarrow \partial_{x_1} \varphi \Longleftrightarrow \partial_1 \varphi \Longleftrightarrow \varphi_{x_1}\right) .$$

During the process of differentiation the other variables are strictly kept constant. The result is again a scalar field which depends on the three variables x_1, x_2, x_3. The partial derivatives with respect to the two other variables are of course defined fully analogously:

$$\lim_{\Delta x_2 \to 0} \frac{\varphi\left(x_1, x_2 + \Delta x_2, x_3\right) - \varphi\left(x_1, x_2, x_3\right)}{\Delta x_2} = \left(\frac{\partial \varphi}{\partial x_2}\right)_{x_1, x_3} = \partial_2 \varphi , \tag{1.249}$$

$$\lim_{\Delta x_3 \to 0} \frac{\varphi\left(x_1, x_2, x_3 + \Delta x_3\right) - \varphi\left(x_1, x_2, x_3\right)}{\Delta x_3} = \left(\frac{\partial \varphi}{\partial x_3}\right)_{x_1, x_2} = \partial_3 \varphi . \tag{1.250}$$

Examples

$$\begin{aligned}
&\varphi = x_1 x_2^5 + x_3 \Longrightarrow \partial_1 \varphi = x_2^5 ; \ \partial_2 \varphi = 5 x_1 x_2^4 ; \ \partial_3 \varphi = 1 , \\
&\varphi = x_3 \ln x_1 \Longrightarrow \partial_1 \varphi = \frac{x_3}{x_1} ; \ \partial_2 \varphi = 0 ; \ \partial_3 \varphi = \ln x_1 , \\
&\varphi = r = \sqrt{x_1^2 + x_2^2 + x_3^2} \Longrightarrow \partial_1 \varphi = \frac{x_1}{r} ; \ \partial_2 \varphi = \frac{x_2}{r} ; \ \partial_3 \varphi = \frac{x_3}{r} .
\end{aligned}$$

Vector fields are differentiated partially by differentiating partially each of its scalar component functions.

Examples

•

$$\mathbf{a}(\mathbf{r}) = \alpha \mathbf{r} = \alpha\left(x_1, x_2, x_3\right)$$

$$\begin{aligned}
\Longrightarrow \partial_1 \mathbf{a} &= \alpha(1,0,0) = \alpha \mathbf{e}_1 , \\
\partial_2 \mathbf{a} &= \alpha(0,1,0) = \alpha \mathbf{e}_2 , \\
\partial_3 \mathbf{a} &= \alpha(0,0,1) = \alpha \mathbf{e}_3 .
\end{aligned}$$

•

$$\mathbf{a}(\mathbf{r}) = \alpha \frac{\mathbf{r}}{r^3} \quad \text{(e.g. electrical field)}$$

$$\begin{aligned} \Longrightarrow \partial_1 a_1(\mathbf{r}) &= \partial_1 \left(\alpha \frac{x_1}{r^3}\right) = \alpha \left(\frac{1}{r^3} - \frac{3x_1}{r^4}\frac{x_1}{r}\right) = \frac{\alpha}{r^5}\left(r^2 - 3x_1^2\right) \,, \\ \partial_1 a_2(\mathbf{r}) &= \partial_1 \left(\alpha \frac{x_2}{r^3}\right) = -3\alpha \frac{x_2 x_1}{r^5} \,, \\ \partial_1 a_3(\mathbf{r}) &= \partial_1 \left(\alpha \frac{x_3}{r^3}\right) = -3\alpha \frac{x_3 x_1}{r^5} \,. \end{aligned}$$

Altogether we then get:

$$\partial_1 \mathbf{a}(\mathbf{r}) = \frac{\alpha}{r^5}\left(r^2 - 3x_1^2, \; -3x_1x_2, \; -3x_1x_3\right) \,.$$

The other two partial derivatives are recommended as exercises.

Looking at the definition (1.248) one realizes that for the partial derivative practically the same **differentiation rules** are valid as for scalar and vectorial functions of **one** variable:

$$\partial_i \left(\varphi_1 + \varphi_2\right) = \partial_i \varphi_1 + \partial_i \varphi_2 \,, \tag{1.251}$$

$$\partial_i (\mathbf{a} \cdot \mathbf{b}) = (\partial_i \mathbf{a}) \cdot \mathbf{b} + \mathbf{a} \cdot (\partial_i \mathbf{b}) \,, \tag{1.252}$$

$$\partial_i (\mathbf{a} \times \mathbf{b}) = (\partial_i \mathbf{a}) \times \mathbf{b} + \mathbf{a} \times (\partial_i \mathbf{b}) \,. \tag{1.253}$$

Since the partial derivative of a field is again a field **multiple differentiations** are recursively definable:

$$\frac{\partial^2 \varphi}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(\frac{\partial \varphi}{\partial x_i}\right) \,, \tag{1.254}$$

$$\frac{\partial^n \varphi}{\partial x_i^n} = \frac{\partial}{\partial x_i}\left(\frac{\partial^{n-1} \varphi}{\partial x_i^{n-1}}\right) = \frac{\partial}{\partial x_i}\left[\frac{\partial}{\partial x_i}\left(\frac{\partial^{n-2} \varphi}{\partial x_i^{n-2}}\right)\right] \,. \tag{1.255}$$

Even **mixed derivatives** make sense:

$$\frac{\partial^2 \varphi}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial \varphi}{\partial x_j}\right) \,. \tag{1.256}$$

In general, however, one has to respect the sequence of the derivatives. The differentiation processes are to be performed *step by step one after the other from the right to the left*. In the case, however, when the field has continuous partial derivatives at least up to second order one can prove the permutability of the differentiations:

$$\frac{\partial^2\varphi}{\partial x_i \partial x_j} = \frac{\partial^2\varphi}{\partial x_j \partial x_i} \ . \tag{1.257}$$

The explicit proof of this statement must be reserved for relevant textbooks of mathematics.

Example

$$\begin{aligned}
\varphi = x_1^5 + x_2^3 x_3 \implies & \frac{\partial\varphi}{\partial x_1} = 5x_1^4 \ ; \quad \frac{\partial^2\varphi}{\partial x_1^2} = 20x_1^3 \ ; \quad \ldots \\
& \frac{\partial\varphi}{\partial x_2} = 3x_2^2 x_3 \ ; \quad \frac{\partial\varphi}{\partial x_3} = x_2^3 \ ; \\
& \frac{\partial^2\varphi}{\partial x_1\,\partial x_2} = 0 = \frac{\partial^2\varphi}{\partial x_2\,\partial x_1} \ ; \\
& \frac{\partial^2\varphi}{\partial x_2\,\partial x_3} = 3x_2^2 = \frac{\partial^2\varphi}{\partial x_3\,\partial x_2} \quad \text{and so on.}
\end{aligned}$$

All that we have learned up to now in connection with partial derivatives could be transferred more or less directly to the already familiar differentiation rules for scalar functions of **one single** variable. The situation is somewhat different for the **chain rule** which we know in the form:

$$\frac{df\,[x(t)]}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt} \tag{1.258}$$

In case of more than one variable nothing changes as long as each of these variables depends on a **different** parameter:

$$\varphi\,[x_1(t_1),\, x_2(t_2),\, x_3(t_3)] \implies \frac{d\varphi}{dt_1} = \frac{\partial\varphi}{\partial x_1}\frac{dx_1}{dt_1} \ . \tag{1.259}$$

It becomes interesting when all the components depend on the same parameter t. That means all the variables simultaneously change as functions of t:

$$\varphi\,[\mathbf{r}(t)] = \varphi\,[x_1(t),\, x_2(t),\, x_3(t)] \ .$$

We set

$$\Delta x_i = x_i(t + \Delta t) - x_i(t)$$

and calculate therewith the difference quotient D:

$$D = \frac{\varphi\left[x_1(t+\Delta t),\, x_2(t+\Delta t),\, x_3(t+\Delta t)\right] - \varphi\left[x_1(t),\, x_2(t),\, x_3(t)\right]}{\Delta t}\ .$$

Later we shall interpret the limiting value of D due to the transition $\Delta t \to 0$ as **derivative of φ with respect to** t. For this purpose we reformulate D a little bit:

$$\begin{aligned}
D \;=\; & \frac{1}{\Delta t}\left[\varphi\left(x_1+\Delta x_1, x_2+\Delta x_2, x_3+\Delta x_3\right) - \varphi\left(x_1, x_2+\Delta x_2, x_3+\Delta x_3\right) + \right. \\
& +\varphi\left(x_1, x_2+\Delta x_2, x_3+\Delta x_3\right) - \varphi\left(x_1, x_2, x_3+\Delta x_3\right) + \\
& \left. +\varphi\left(x_1, x_2, x_3+\Delta x_3\right) - \varphi\left(x_1, x_2, x_3\right)\right] = \\
= \;& \frac{1}{\Delta x_1}\left[\varphi\left(x_1+\Delta x_1, x_2+\Delta x_2, x_3+\Delta x_3\right) - \right. \\
& \left. -\varphi\left(x_1, x_2+\Delta x_2, x_3+\Delta x_3\right)\right]\frac{\Delta x_1}{\Delta t} + \\
& +\frac{1}{\Delta x_2}\left[\varphi\left(x_1, x_2+\Delta x_2, x_3+\Delta x_3\right) - \varphi\left(x_1, x_2, x_3+\Delta x_3\right)\right]\frac{\Delta x_2}{\Delta t} + \\
& +\frac{1}{\Delta x_3}\left[\varphi\left(x_1, x_2, x_3+\Delta x_3\right) - \varphi\left(x_1, x_2, x_3\right)\right]\frac{\Delta x_3}{\Delta t}.
\end{aligned}$$

We can now conclude, because of the continuity of the functions $x_i(t)$, that $\Delta x_i \to 0$ if $\Delta t \to 0$. If we furthermore presume continuity of the first partial derivatives of φ then it obviously follows:

$$\lim_{\Delta t \to 0} D = \frac{\partial \varphi}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial \varphi}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial \varphi}{\partial x_3}\frac{dx_3}{dt}\ .$$

One denotes this limit as the **total derivative of φ with respect to** t:

$$\frac{d\varphi}{dt} = \sum_{i=1}^{3} \frac{\partial \varphi}{\partial x_i}\frac{dx_i}{dt} \tag{1.260}$$

and denotes

$$d\varphi = \sum_{i=1}^{3} \frac{\partial \varphi}{\partial x_i} dx_i \tag{1.261}$$

as the **total differential of the function** φ.

1.5.3 Gradient

With the aid of the partial derivative we have the possibility to find out how a field alters as we proceed along one of the axis of coordinates. We want to investigate now how a **scalar** field changes along an arbitrary (!) space direction $\mathbf{e}$, i.e. we are interested in the term

$$\begin{aligned} \Delta\varphi &= \varphi(\mathbf{r} + \Delta\mathbf{r}) - \varphi(\mathbf{r}) \ , \\ \Delta\mathbf{r} &= (\Delta x_1,\ \Delta x_2,\ \Delta x_3) \uparrow\uparrow \mathbf{e} \ . \end{aligned} \tag{1.262}$$

If $\Delta\mathbf{r}$ were, e.g., parallel to the 1-axis then for *sufficiently small* changes $\Delta\mathbf{r} = \Delta x_1 \mathbf{e}_1$ we would have:

$$\Delta\varphi = \frac{\partial\varphi}{\partial x_1}\Delta x_1 \qquad [\varphi(\mathbf{r} + \Delta\mathbf{r}) = \varphi\,(x_1 + \Delta x_1, x_2, x_3)] \ .$$

This presumption is not in general fulfilled (Fig. 1.71). It is,however, possible to realize it by a proper rotation of the coordinate axes. The physical field φ is of course not affected by such a redefinition of the axes directions. We execute the rotation in such a way that the *new* 1 axis coincides with the $\mathbf{e}$ direction. Then we must have:

$$\Delta\varphi = \frac{\partial\varphi}{\partial\bar{x}_1}\Delta\bar{x}_1 \ . \tag{1.263}$$

We now can express $\Delta\mathbf{r}$ in the *new* and *old* system of coordinates, respectively, as follows:

$$\Delta\mathbf{r} = \Delta\bar{x}_1\bar{\mathbf{e}}_1 = \Delta x_1\mathbf{e}_1 + \Delta x_2\mathbf{e}_2 + \Delta x_3\mathbf{e}_3 \ . \tag{1.264}$$

From this relation it follows in particular that after scalar multiplication with $\mathbf{e}_i$:

$$\Delta x_i = \Delta\bar{x}_1\,(\bar{\mathbf{e}}_1 \cdot \mathbf{e}_i) \ , \tag{1.265}$$

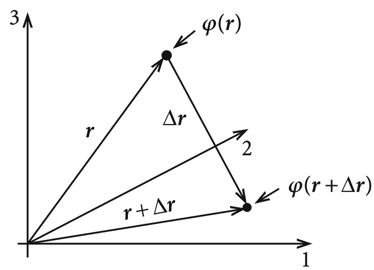


Fig. 1.71 To the introduction of the gradient

so that we can write for *sufficiently small* shifts along the $\bar{x}_1$-axis:

$$\frac{dx_i}{d\bar{x}_1} = \bar{\mathbf{e}}_1 \cdot \mathbf{e}_i \ . \tag{1.266}$$

This we exploit together with (1.265) and the chain rule in Eq. (1.263):

$$\Delta\varphi = \sum_{j=1}^{3} \frac{\partial\varphi}{\partial x_j}\frac{dx_j}{d\bar{x}_1}\Delta\bar{x}_1 = \sum_{j=1}^{3} \frac{\partial\varphi}{\partial x_j}\left(\bar{\mathbf{e}}_1 \cdot \mathbf{e}_j\right)\Delta\bar{x}_1 \ .$$

The change in the field in an arbitrary space direction is thus additively composed of the corresponding changes in the three basis directions:

$$\Delta\varphi = \sum_{j=1}^{3} \frac{\partial\varphi}{\partial x_j}\Delta x_j \ . \tag{1.267}$$

The result has the form of a scalar product between the vectors

$$(\Delta x_1,\ \Delta x_2,\ \Delta x_3) \quad \text{and} \quad \left(\frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}, \frac{\partial\varphi}{\partial x_3}\right) \ .$$

This leads to the following definition:

Definition 1.5.4 To a continuously differentiable **scalar** field $\varphi(\mathbf{r})$ a **vectorial** field is ascribed, called the **gradient field**:

$$\mathrm{grad}\varphi = \left(\frac{\partial\varphi}{\partial x_1}, \frac{\partial\varphi}{\partial x_2}, \frac{\partial\varphi}{\partial x_3}\right) \ . \tag{1.268}$$

So one denotes as **gradient of** φ the vector whose i-th component is just the partial derivative of φ with respect to x_i.

Definition 1.5.5 The vector-differential operator

$$\nabla \equiv \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) = \mathbf{e}_1\frac{\partial}{\partial x_1} + \mathbf{e}_2\frac{\partial}{\partial x_2} + \mathbf{e}_3\frac{\partial}{\partial x_3} \tag{1.269}$$

is called the **'nabla-operator'**.

The operator acts on all functions to the right of it. One can write:

$$\mathrm{grad}\varphi = \nabla\varphi \ , \tag{1.270}$$

and for the field change $\Delta\varphi$ in Eq. (1.267) now holds:

$$\Delta\varphi = \mathrm{grad}\varphi \cdot \Delta\mathbf{r} = \nabla\varphi \cdot \Delta\mathbf{r} \,. \tag{1.271}$$

For the interpretation of the gradient vector we inspect in particular a direction in which φ does **not** change:

$$0 = \mathrm{grad}\varphi \cdot \Delta\mathbf{r} \iff \mathrm{grad}\varphi \perp \Delta\mathbf{r} \,.$$

We see that the gradient vector grad $\varphi = \nabla\varphi$ is oriented perpendicular to the planes $\varphi =$ const. The magnitude $|\mathrm{grad}\varphi|$ is a measure of the degree of the change in φ if one proceeds perpendicular to the $\varphi =$ const planes.

By using the calculation rules (1.251) and (1.252) for partial differentiations one proves directly the following **rules of gradient formation**:

$$\mathrm{grad}\,(\varphi_1 + \varphi_2) = \mathrm{grad}\varphi_1 + \mathrm{grad}\varphi_2 \,, \tag{1.272}$$

$$\mathrm{grad}\,(\varphi_1\varphi_2) = \varphi_2\mathrm{grad}\varphi_1 + \varphi_1\mathrm{grad}\varphi_2 \,. \tag{1.273}$$

We want to practice, what we have derived, by some examples:

Examples

1. $\mathrm{grad}(\mathbf{a}\cdot\mathbf{r}) = \,?$ ($\mathbf{a}$: constant vector)

$$\mathbf{a}\cdot\mathbf{r} = \sum_{j=1}^{3} a_j x_j \Longrightarrow \frac{\partial(\mathbf{a}\cdot\mathbf{r})}{\partial x_i} = a_i \Longrightarrow \mathrm{grad}(\mathbf{a}\cdot\mathbf{r}) = \mathbf{a} \,. \tag{1.274}$$

2. $\mathrm{grad}\, r = \,?$ $\left(r = \sqrt{x_1^2 + x_2^2 + x_3^2}\right)$

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r} \Longrightarrow \mathrm{grad} r = \frac{\mathbf{r}}{r} = \mathbf{e}_r \tag{1.275}$$

3. $\mathrm{grad}\, 1/r^2 = \,?$

$$\frac{\partial}{\partial x_i}\frac{1}{r^2} = \left(\frac{d}{dr}\frac{1}{r^2}\right)\frac{\partial r}{\partial x_i} = -\frac{2}{r^3}\frac{x_i}{r} \Longrightarrow \mathrm{grad}\frac{1}{r^2} = -\frac{2}{r^3}\mathbf{e}_r \,. \tag{1.276}$$

4. $\mathrm{grad} f(r) = \,?$

$$\frac{\partial}{\partial x_i} f(r) = \left(\frac{df}{dr}\right)\frac{\partial r}{\partial x_i} = f'(r)\frac{x_i}{r} \Longrightarrow \mathrm{grad} f(r) = f'(r)\mathbf{e}_r \,. \tag{1.277}$$

2., 3. are special examples for $f(r)$.

1.5.4 Divergence and Curl (Rotation)

The *gradient, nabla operator* introduced in the last section acts exclusively on scalar fields φ, while the resulting gradient field grad$\varphi = \nabla\varphi$ is itself a vector. An obvious question then is whether it is possible to apply the nabla operator ∇, formally defined in (1.269) as vector-differential operator, also to vectors. The answer is yes! There are again two kinds of application, similar to the previously discussed multiplicative connection of two *ordinary* vectors, one in the sense of a scalar product, the other in the sense of a vector product.

Definition 1.5.6 Let $\mathbf{a}(\mathbf{r}) \equiv (a_1(\mathbf{r}), a_2(\mathbf{r}), a_3(\mathbf{r}))$ be a continuously differentiable vector field.

Then one calls

$$\sum_{j=1}^{3} \frac{\partial a_j}{\partial x_j} \equiv \operatorname{div} \mathbf{a}(\mathbf{r}) \equiv \nabla \cdot \mathbf{a}(\mathbf{r}) \tag{1.278}$$

the **divergence** (the *source field*) of $\mathbf{a}(\mathbf{r})$.

By this definition, to a given vector field $\mathbf{a}(\mathbf{r})$ a new scalar field div $\mathbf{a}(\mathbf{r})$ is assigned. The *illustrative* interpretation of div $\mathbf{a}(\mathbf{r})$ as a *source field* of $\mathbf{a}(\mathbf{r})$ will become understandable later by some examples from physics.

The reader should prove as an exercise the following **calculation rules**:

$$\operatorname{div}(\mathbf{a} + \mathbf{b}) = \operatorname{div}\mathbf{a} + \operatorname{div}\mathbf{b} \;, \tag{1.279}$$

$$\operatorname{div}(\gamma\, \mathbf{a}) = \gamma \operatorname{div}\mathbf{a} \;; \quad \gamma \in \mathbb{R} \;, \tag{1.280}$$

$$\operatorname{div}(\varphi\, \mathbf{a}) = \varphi \operatorname{div}\mathbf{a} + \mathbf{a} \cdot \operatorname{grad}\varphi \tag{1.281}$$

(φ: scalar field; $\mathbf{a}$: vectorial field).

Via the divergence we introduce a further important operator:

Definition 1.5.7 *Divergence of a gradient field:*

$$\operatorname{div} \operatorname{grad} \varphi = \sum_{j=1}^{3} \frac{\partial^2 \varphi}{\partial x_j^2} \equiv \Delta\varphi \;,$$

where

$$\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \tag{1.282}$$

is called the **Laplace operator**.

Examples

$$1)\ \boldsymbol{\alpha}:\ \text{constant vector} \Longrightarrow \operatorname{div}\boldsymbol{\alpha} = 0\ . \tag{1.283}$$

$$2) \operatorname{div}\mathbf{r} = \sum_{j=1}^{3} \frac{\partial x_j}{\partial x_j} = 3\ . \tag{1.284}$$

3) $\boldsymbol{\alpha}$: constant vector

$$\operatorname{div}(\mathbf{r}\times\boldsymbol{\alpha}) = \sum_{k=1}^{3} \frac{\partial}{\partial x_k}(\mathbf{r}\times\boldsymbol{\alpha})_k = \sum_{i,j,k} \frac{\partial}{\partial x_k}\left(\varepsilon_{ijk}x_i\alpha_j\right) =$$

$$= \sum_{i,j,k} \varepsilon_{ijk}\delta_{ik}\alpha_j = \sum_{i,j} \varepsilon_{iji}\alpha_j = 0\ . \tag{1.285}$$

One calls $(\mathbf{r}\times\boldsymbol{\alpha})$ a *source-free field*.

The *vectorial* application of the nabla operator on a vector field leads to the following definition:

Definition 1.5.8 Let $\mathbf{a}(\mathbf{r}) \equiv (a_1(\mathbf{r}), a_2(\mathbf{r}), a_3(\mathbf{r}))$ be a continuously differentiable vector field.

Then

$$\operatorname{curl}\mathbf{a} = \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}\right)\mathbf{e}_1 + \left(\frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1}\right)\mathbf{e}_2 + \left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right)\mathbf{e}_3$$

is denoted as **curl or rotation** (the *curl field*) of $\mathbf{a}(\mathbf{r})$. Short-hand notation:

$$\operatorname{curl}\mathbf{a} \equiv \nabla\times\mathbf{a} = \sum_{i,j,k} \varepsilon_{ijk}\left(\frac{\partial}{\partial x_i}a_j\right)\mathbf{e}_k\ . \tag{1.286}$$

By this operation the vector field $\mathbf{a}(\mathbf{r})$ is related to another vector field. The *illustrative* interpretation of curl $\mathbf{a}$ as *curl field* of $\mathbf{a}$ will later become evident in connection with certain examples.

The following properties and calculation rules are rather directly derivable from the bare definition of the curl.

1.

$$\nabla\times(\mathbf{a}+\mathbf{b}) = \nabla\times\mathbf{a} + \nabla\times\mathbf{b}\ . \tag{1.287}$$

2.

$$\nabla\times(\alpha\mathbf{a}) = \alpha\nabla\times\mathbf{a}\ ; \quad \alpha\in\mathbb{R}\ . \tag{1.288}$$

3.

$$\nabla \times (\varphi \mathbf{a}) = \varphi \nabla \times \mathbf{a} + (\nabla \varphi) \times \mathbf{a} \tag{1.289}$$

(φ : scalar field; proof as Exercise 1.5.7!)

4.

$$\nabla \times (\nabla \varphi) = 0 \tag{1.290}$$

(φ two times continuously differentiable)

The statement, which is very important for later considerations, is that the **gradient fields are always curl-free**. We demonstrate the correctness of this statement by inspecting the 1-component:

$$(\nabla \times \nabla \varphi)_1 = \frac{\partial}{\partial x_2}(\nabla \varphi)_3 - \frac{\partial}{\partial x_3}(\nabla \varphi)_2 = \frac{\partial^2 \varphi}{\partial x_2\, \partial x_3} - \frac{\partial^2 \varphi}{\partial x_3\, \partial x_2} = 0$$

(according to (1.257)). One can show the same for the other components.

5.

$$\nabla \cdot (\nabla \times \mathbf{a}) = 0 \tag{1.291}$$

($\mathbf{a}$: two times continuously differentiable)

Curl-fields are always source-free!

Proof

$$\begin{aligned}
\nabla \cdot (\nabla \times \mathbf{a}) &= \sum_{j=1}^{3} \frac{\partial}{\partial x_j}(\nabla \times \mathbf{a})_j = \sum_j \frac{\partial}{\partial x_j} \sum_{l,m} \varepsilon_{lmj} \frac{\partial a_m}{\partial x_l} = \\
&= \sum_m \sum_{l,j} \varepsilon_{lmj} \frac{\partial^2 a_m}{\partial x_j\, \partial x_l} = \\
&= \sum_m \frac{1}{2} \left(\sum_{l,j} \varepsilon_{lmj} \frac{\partial^2 a_m}{\partial x_j\, \partial x_l} + \sum_{j,l} \varepsilon_{jml} \frac{\partial^2 a_m}{\partial x_l\, \partial x_j} \right) = \\
&\overset{(1.257)}{=} \frac{1}{2} \sum_m \sum_{j,l} \underbrace{\left(\varepsilon_{lmj} + \varepsilon_{jml}\right)}_{(=0 \ \text{why?})} \frac{\partial^2 a_m}{\partial x_j\, \partial x_l} = 0\,.
\end{aligned}$$

6.

$$\nabla \times [f(r)\mathbf{r}] = 0\,. \tag{1.292}$$

$f(r)$ may be any scalar-valued function which depends only on $r = |\mathbf{r}|$. The proof of this important relation will be provided in Exercise 1.5.7.

7.

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \Delta \mathbf{a} \,. \tag{1.293}$$

This statement can be verified component by component (Proof in Exercise 1.5.7!).

1.5.5 Exercises

Exercise 1.5.1 Given are the following vector fields:

(a) $\mathbf{a}(\mathbf{r}) = \frac{1}{r}[\boldsymbol{\omega} \times \mathbf{r}]\,; \quad \boldsymbol{\omega} = \omega_0 \mathbf{e}_3\,; \quad \omega_0 = \text{const}\,,$
(b) $\mathbf{a}(\mathbf{r}) = \alpha \mathbf{r}\,; \quad \alpha < 0\,,$
(c) $\mathbf{a}(\mathbf{r}) = \alpha\,(x_1 + x_2)\,\mathbf{e}_1 + \alpha\,(x_2 - x_1)\,\mathbf{e}_2\,; \quad \alpha > 0\,,$
(d) $\mathbf{a}(\mathbf{r}) = \frac{\alpha}{x_2^2 + x_3^2 + \beta^2}\mathbf{e}_1\,; \quad \alpha, \beta > 0\,.$

1. Plot the pictures of the field lines for cuts perpendicular to the x_3-axis ($x_3 = 0$).
2. Calculate the partial derivatives of the fields!
3. Calculate $\nabla \cdot \mathbf{a}(\mathbf{r})$ and $\nabla \times \mathbf{a}(\mathbf{r})$.

Exercise 1.5.2 To a good approximation the scalar electrostatic potential of a point charge embedded in a plasma ('gas' consisting of charged particles) can be described by the following formula:

$$\varphi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0} \frac{e^{-\alpha r}}{r}$$

1. Determine the partial derivatives of φ and write down $\nabla\varphi$.
2. Calculate $\Delta\varphi$ where Δ is the Laplace operator (1.282)

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\,.$$

Exercise 1.5.3 A prolate atomic nucleus can be described as an ellipsoid of revolution (*'cigar'*):

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} = 1\,.$$

1. Determine the outwardly pointing surface normal vector $\mathbf{n}$!

2. Calculate and plot **n** at the points
   (a) $(a/\sqrt{2},\, a/\sqrt{2},\, 0,)$,
   (b) $(a/\sqrt{3},\, a/\sqrt{3},\, b/\sqrt{3})$,
   (c) $(-a/2,\, a/\sqrt{2},\, -b/2)$,
   (d) $(0,\, 0,\, b)$,
   (e) $(0,\, -a,\, 0)$.

Exercise 1.5.4

1. Given are the scalar fields

$$\varphi_1 = \cos(\boldsymbol{\alpha} \cdot \mathbf{r}) \; ; \quad \varphi_2 = e^{-\gamma r^2} \; (\boldsymbol{\alpha} = \mathbf{const}\;,\; \gamma = \text{const}) \; .$$

   Calculate the gradient fields grad φ_i and their sources

$$\nabla \cdot \nabla\varphi_i = \Delta\varphi_i \; .$$

2. Calculate the divergence of the unit vector $\mathbf{e}_r = r^{-1}\mathbf{r}$.
3. Under what conditions is the vector field $\mathbf{a}(\mathbf{r}) = f(r)\mathbf{r}$ source-free?
4. Determine the divergence of the vector field $\mathbf{a}(\mathbf{r}) = \nabla\varphi_1 \times \nabla\varphi_2$ (φ_1, φ_2: two times continuously differentiable scalar fields).
5. φ shall be a scalar field, **a** a vector field. Prove:

$$\nabla \cdot (\varphi\mathbf{a}) = \varphi\nabla \cdot \mathbf{a} + \mathbf{a} \cdot \nabla\varphi \; .$$

Exercise 1.5.5 How must the constant γ be chosen so that the vector field

$$\mathbf{a}(\mathbf{r}) \equiv \left(\gamma x_1 x_2 - x_3^3, (\gamma - 2)x_1^2, (1 - \gamma)x_1 x_3^2\right)$$

becomes *'curl-free'* ($\nabla \times \mathbf{a} = 0$)? Is it also possible to make $\mathbf{a}(\mathbf{r})$ *'source-free'* ($\nabla \cdot \mathbf{a} = 0$)?

Exercise 1.5.6

1. Show that the vector field

$$\mathbf{b}(\mathbf{r}) = \left(x_2 x_3 + 12x_1 x_2,\; x_1 x_3 - 8x_2 x_3^3 + 6x_1^2,\; x_1 x_2 - 12x_2^2 x_3^2\right)$$

   is *'curl-free'* ($\nabla \times \mathbf{b} = 0$).
2. Determine a scalar field $\varphi(\mathbf{r})$ if:

$$\nabla\varphi(\mathbf{r}) = \mathbf{b}(\mathbf{r})$$

Exercise 1.5.7

1. Show: $\nabla \times [f(r)\mathbf{r}] = 0$.
2. φ shall be a scalar field, **a** a vector field.
   Prove: $\nabla \times (\varphi \mathbf{a}) = \varphi \nabla \times \mathbf{a} + (\nabla \varphi) \times \mathbf{a}$.
3. Verify: $\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \Delta \mathbf{a}$.
   The components of **a** are two times continuously differentiable.
4. What do we find for $\nabla \times \left(\frac{1}{2}\boldsymbol{\alpha} \times \mathbf{r}\right)$ if $\boldsymbol{\alpha}$ is a constant vector?

Exercise 1.5.8

1. Prove:

$$\frac{\partial}{\partial x_i}(\mathbf{a} \times \mathbf{b}) = \left(\frac{\partial}{\partial x_i}\mathbf{a}\right) \times \mathbf{b} + \mathbf{a} \times \left(\frac{\partial}{\partial x_i}\mathbf{b}\right) \; ; \quad i = 1, 2, 3$$

   $\mathbf{a}(\mathbf{r}), \mathbf{b}(\mathbf{r})$: vector fields; $\mathbf{r} = (x_1, x_2, x_3)$.
2. Prove:

$$\nabla(\varphi_1 \varphi_2) = \varphi_1 \nabla \varphi_2 + \varphi_2 \nabla \varphi_1$$

   $\varphi_1(\mathbf{r}), \varphi_2(\mathbf{r})$: scalar fields.
3. Let $\mathbf{a}(\mathbf{r})$ and $\mathbf{b}(\mathbf{r})$ be two vector fields.
   Express

$$\nabla \cdot (\mathbf{a} \times \mathbf{b})$$

   by $\nabla \times \mathbf{a}$ and $\nabla \times \mathbf{b}$!
4. $\varphi_1(\mathbf{r})$ and $\varphi_2(\mathbf{r})$ shall be two times continuously differentiable scalar fields. Calculate the divergence of the vector field

$$\mathbf{d}(\mathbf{r}) = \nabla \varphi_1(\mathbf{r}) \times \nabla \varphi_2(\mathbf{r}) \; .$$

1.6 Matrices and Determinants

Matrices and determinants are important auxiliary means for the mathematician with the aid of which many statements and formulations can be written in an elegant, compact, and neatly arranged manner. Therefore, the prospective physicist must learn the correct handling of matrices and determinants as soon as possible. Here we want to gather the most important theorems and definitions for matrices and determinants and demonstrate their usefulness by some simple applications.

1.6.1 Matrices

Definition 1.6.1 A rectangular array of numbers ($a_{ij} \in \mathbb{R}$) of the kind

$$A \equiv \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \equiv (a_{ij})_{\substack{i=1,\dots,m \\ j=1,\dots,n}} \tag{1.294}$$

is called an $(m \times n)$**-matrix**. It consists of m **rows** ($i = 1, 2, \dots, m$) and n **columns** ($j = 1, 2, \dots, n$). If $m = n$ then one speaks of a **square matrix**.

Definition 1.6.2 Two matrices $A = (a_{ij})$, $B = (b_{ij})$ are equal (identical) if:

$$a_{ij} = b_{ij} \ , \qquad \forall\, i, j \tag{1.295}$$

Above all A and B must be of the same $(m \times n)$-type.

In the following we define and list up a few special matrices:

1. By a **zero matrix** one understands a matrix all the elements of which are zero.
2. A **symmetric matrix** is an $(n \times n)$-matrix the elements of which obey:

$$a_{ij} = a_{ji} \ , \quad \forall\, i, j \tag{1.296}$$

It is symmetric for reflection at the principal diagonal.

Example

$$A = \begin{pmatrix} 1 & 5 & -1 \\ 5 & 2 & 4 \\ -1 & 4 & 3 \end{pmatrix} .$$

3. A **diagonal matrix** has non-zero elements only on the principal diagonal:

$$d_{ij} = d_i \cdot \delta_{ij} \quad \forall_{ij} \iff \begin{pmatrix} d_1 & & & \\ & d_2 & & 0 \\ & & & \\ & 0 & \ddots & \\ & & & d_n \end{pmatrix} . \tag{1.297}$$

4. A **unit matrix** $\mathbb{1}$ is a special diagonal matrix with

$$\mathbb{1}_{ij} = \delta_{ij} \iff \begin{pmatrix} 1 & & & \\ & 1 & & 0 \\ & & \ddots & \\ & 0 & & \\ & & & 1 \end{pmatrix} . \tag{1.298}$$

5. To each given $(m \times n)$-matrix $A = (a_{ij})$ belongs a corresponding **transposed matrix** A^T resulting from an interchange of rows and columns:

$$A^T = \left(a^T_{ij} = a_{ji}\right) = \begin{pmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix} . \tag{1.299}$$

   A^T is a $(n \times m)$-matrix.
6. A **column vector** is a $(n \times 1)$-matrix.
7. A **row vector** is a $(1 \times n)$-matrix.

One can interpret the rows (columns) of a matrix as row- (column-) **vectors**. The maximal number of linearly independent row vectors (column vectors) of a given matrix is denoted as **row rank (column rank)**. Since one can show very generally that always row rank and column rank are identical one speaks only of the **'rank of a matrix'**.

Example

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 4 & 1 & 2 \end{pmatrix} .$$

The row rank is 2 since the two row vectors $(3, 0, 1)$ and $(4, 1, 2)$ are not proportional to each other and therefore are linearly independent. The column vectors $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ are also linearly independent, while this does not hold for the third vector $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ because of: $\begin{pmatrix} 3 \\ 4 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Hence the column rank is also 2.

1.6.2 Calculation Rules for Matrices

Let us first agree upon what we want to understand as the sum of two matrices:

Definition 1.6.3 If $A = (a_{ij})$, $B = (b_{ij})$ are two $(m \times n)$-matrices then the **sum** is given as the matrix $C = A + B = (c_{ij})$ with the elements:

$$c_{ij} = a_{ij} + b_{ij} \ , \quad \forall\, i,j \ . \tag{1.300}$$

C is again a $(m \times n)$-matrix.

Example

$$\begin{aligned} A &= \begin{pmatrix} 6 & 3 & 0 \\ 1 & 4 & 5 \end{pmatrix} \\ B &= \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix} \end{aligned} \implies C = A + B = \begin{pmatrix} 7 & 6 & 5 \\ 3 & 8 & 11 \end{pmatrix} \ .$$

The so defined addition is obviously commutative as well as associative.

The next step concerns the multiplication of a matrix by a real number:

Definition 1.6.4 If $A = (a_{ij})$ is a $(m \times n)$-matrix then the matrix λA $(\lambda \in \mathbb{R})$ is to understand as the $(m \times n)$-matrix:

$$\lambda A = \left(\lambda a_{ij}\right) \ . \tag{1.301}$$

Hence **each** matrix element is multiplied by λ

Example

$$3 \begin{pmatrix} 5 & -3 & 1 \\ 0 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 15 & -9 & 3 \\ 0 & 6 & -3 \end{pmatrix} \ .$$

We know from normal vectors, which represent nothing else than special matrices, namely $(n \times 1)$- and $(1 \times n)$-matrices, respectively, that they can be multiplicatively connected in form of scalar products. That is generalized correspondingly for matrices.

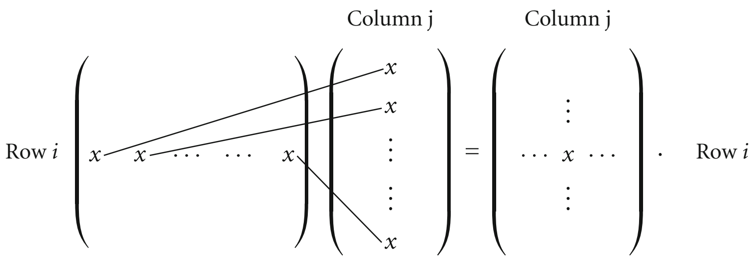
Definition 1.6.5 Let $A = (a_{ij})$ be a $(m \times n)$-matrix and $B = (b_{ij})$ a $(n \times r)$-matrix, i.e. the number of columns in A is equal to the number of rows in B. Then one understands by **product matrix**

$$C = A \cdot B = \left(c_{ij}\right)$$

an $(m \times r)$-matrix with the elements

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} \; . \tag{1.302}$$

Thus the element c_{ij} of the product matrix is just the scalar product of the i-th row vector in A with the j-th column vector in B.



It can directly be seen that this definition incorporates as special case the scalar product of two ordinary vectors. It is important to stress once more that $A \cdot B$ makes sense only if the number of columns in A is the same as the number of rows in B.

Example

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 4 & 5 & 6 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 1 & 4 \\ 5 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \Longrightarrow \quad A \cdot B = \begin{pmatrix} 15 & -2 & 5 \\ 25 & -1 & 22 \end{pmatrix} .$$

In general the matrix multiplication is not commutative:

$$A \cdot B \neq B \cdot A \qquad \text{(in general)} \; . \tag{1.303}$$

For $m \neq r$ this is immediately clear since then $B \cdot A$ is not defined. For $m = r$ $A \cdot B$ would be an $(m \times m)$-matrix and $B \cdot A$ an $(n \times n)$-matrix. Commutativity would then come into play, if at all, only for $m = r = n$, i.e. for square matrices. But even then the product is in general not commutative as is shown by the following example:

Example

$$A = \begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix} , \quad B = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$$

$$\Longrightarrow A \cdot B = \begin{pmatrix} 6 & 4 \\ 10 & 9 \end{pmatrix} ; \quad B \cdot A = \begin{pmatrix} 4 & 5 \\ 6 & 11 \end{pmatrix}$$

$$\Longrightarrow A \cdot B \neq B \cdot A \, .$$

In the next section we will get to know of a first important application of the matrix notation.

1.6.3 Transformation of Coordinates (Rotations)

Let Σ, $\overline{\Sigma}$ be two systems of coordinates specified by the orthonormal basis vectors (Fig. 1.72):

$$\mathbf{e}_1, \, \mathbf{e}_2, \, \mathbf{e}_3 \quad \text{and} \quad \bar{\mathbf{e}}_1, \, \bar{\mathbf{e}}_2, \, \bar{\mathbf{e}}_3 \ , \ \text{respectively} \, .$$

Translations are relatively uninteresting. We therefore assume that the origins of Σ and $\overline{\Sigma}$ coincide. Let us now consider an arbitrarily chosen position vector $\mathbf{r}$:

$$\begin{aligned} \mathbf{r} &= (x_1, x_2, x_3) \ \text{ in } \Sigma \qquad [\mathbf{r}\,(\Sigma)] \\ \mathbf{r} &= (\bar{x}_1, \bar{x}_2, \bar{x}_3) \ \text{ in } \overline{\Sigma} \qquad \left[\mathbf{r}\,\left(\overline{\Sigma}\right)\right] \, . \end{aligned}$$

Let us presume that the elements x_i in Σ are known while the elements $\bar{x}_j$ in $\overline{\Sigma}$ are to be determined. $\mathbf{r}$ itself is of course independent of the special choice of the system of coordinates, both with respect to direction as well as magnitude.

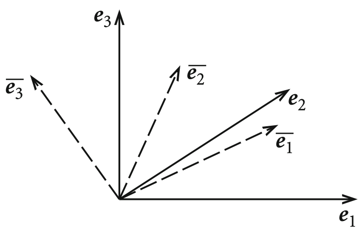


Fig. 1.72 Rotation of a system of coordinates

Therefore:

$$\sum_{j=1}^{3} x_j \mathbf{e}_j = \sum_{j=1}^{3} \bar{x}_j \bar{\mathbf{e}}_j \ . \tag{1.304}$$

The basis vectors $\bar{\mathbf{e}}_j$ can be represented in Σ:

$$\bar{\mathbf{e}}_j = \sum_k d_{jk} \mathbf{e}_k \ . \tag{1.305}$$

We determine the expansion coefficients d_{jk} by scalar multiplication of this equation by $\mathbf{e}_m$:

$$d_{jm} = \bar{\mathbf{e}}_j \cdot \mathbf{e}_m = \cos\varphi_{jm} \ . \tag{1.306}$$

φ_{jm} is the angle enclosed by the j-th axis in $\overline{\Sigma}$ and the m-th axis in Σ. The ensemble of real numbers d_{jm} defines the (3×3)-**rotation matrix** D:

$$D = \left(d_{ij} = \cos\varphi_{ij}\right) = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix} \ . \tag{1.307}$$

Some important properties of the rotation matrix are the direct consequences of the orthonormality of the basis vectors $\bar{\mathbf{e}}_j$:

$$\bar{\mathbf{e}}_i \cdot \bar{\mathbf{e}}_j = \delta_{ij} = \sum_{k,m} d_{ik} d_{jm} \left(\mathbf{e}_k \cdot \mathbf{e}_m\right) = \sum_m d_{im} d_{jm} \ .$$

This refers to the scalar product of two row vectors of the rotation matrix D. Hence the rows of the rotation matrix D are obviously **orthonormalized**:

$$\sum_m d_{im} d_{jm} = \sum_m \cos\varphi_{im} \cos\varphi_{jm} = \delta_{ij} \ . \tag{1.308}$$

To get more information about D we multiply (1.304) scalarly by the basis vector $\bar{\mathbf{e}}_i$:

$$\bar{x}_i = \sum_{j=1}^{3} x_j \left(\mathbf{e}_j \cdot \bar{\mathbf{e}}_i\right) = \sum_{j=1}^{3} \cos\varphi_{ij} x_j \, ; \quad i = 1, 2, 3 \ . \tag{1.309}$$

In matrix notation this linear system of equations reads:

$$\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{pmatrix} = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Longleftrightarrow \mathbf{r}\left(\overline{\Sigma}\right) = D \cdot \mathbf{r}(\Sigma) \; . \tag{1.310}$$

Inspecting this expression component by component we can satisfy ourselves of the correctness of this relation. Hence D obviously describes the rotation $\Sigma \to \overline{\Sigma}$.

We introduce via

$$D^{-1}D = DD^{-1} = \mathbb{1} \tag{1.311}$$

the **inverse matrix** D^{-1} belonging to D and apply this to (1.310):

$$\begin{aligned} D^{-1}\,\mathbf{r}\left(\overline{\Sigma}\right) &= D^{-1}D\,\mathbf{r}(\Sigma) = E\,\mathbf{r}(\Sigma) = \mathbf{r}(\Sigma) \\ D^{-1}\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{pmatrix} &= D^{-1}D\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = E\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \; . \end{aligned} \tag{1.312}$$

D^{-1} describes apparently the back rotation from $\overline{\Sigma}$ to Σ. We get the elements of D^{-1} by scalarly multiplying (1.304) now by $\mathbf{e}_i$:

$$x_i = \sum_{j=1}^{3} \bar{x}_j \left(\bar{\mathbf{e}}_j \cdot \mathbf{e}_i\right) = \sum_{j=1}^{3} \cos\varphi_{ji}\,\bar{x}_j \; ; \quad i = 1,2,3 \; , \tag{1.313}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} d_{11} & d_{21} & d_{31} \\ d_{12} & d_{22} & d_{32} \\ d_{13} & d_{23} & d_{33} \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{pmatrix} \Longleftrightarrow \mathbf{r}(\Sigma) = D^{-1}\,\mathbf{r}\left(\overline{\Sigma}\right) . \tag{1.314}$$

D^{-1} thus results from D by interchanging rows and columns and therefore, according to (1.299) D^{-1} is the transposed matrix of D:

$$D^{-1} = D^T = \left(\left(d^{-1}\right)_{ij} = d_{ji}\right) \; . \tag{1.315}$$

From (1.311) we then get the relations:

$$\begin{aligned} \delta_{ij} &= \sum_m d_{im}\left(d^{-1}\right)_{mj} = \sum_m d_{im}d_{jm} \; , \\ \delta_{ij} &= \sum_m \left(d^{-1}\right)_{im} d_{mj} = \sum_m d_{mi}d_{mj} \; . \end{aligned} \tag{1.316}$$

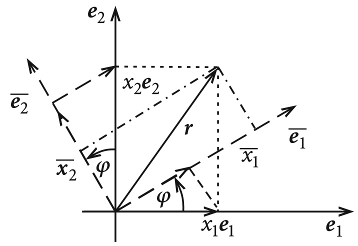


Fig. 1.73 Rotation of the axes of coordinates in the plane

The first equation is identical to (1.308) expressing the orthonormality of the rows of the rotation matrix which is already known. The second equation tells us that the **columns, too, are orthonormal**.

Examples

(1) Rotation in the plane

We start with a purely geometrical consideration (Fig. 1.73):

$$x_1\mathbf{e}_1 = x_1\cos\varphi\,\bar{\mathbf{e}}_1 - x_1\sin\varphi\,\bar{\mathbf{e}}_2\ ,$$
$$x_2\mathbf{e}_2 = x_2\cos\varphi\,\bar{\mathbf{e}}_2 + x_2\sin\varphi\,\bar{\mathbf{e}}_1\ .$$

It follows:

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 = (x_1\cos\varphi + x_2\sin\varphi)\ \bar{\mathbf{e}}_1 + (x_2\cos\varphi - x_1\sin\varphi)\ \bar{\mathbf{e}}_2 \overset{!}{=}$$
$$\overset{!}{=} \bar{x}_1\bar{\mathbf{e}}_1 + \bar{x}_2\bar{\mathbf{e}}_2\ .$$

The comparison yields:

$$\bar{x}_1 = x_1\cos\varphi + x_2\sin\varphi\ ,$$
$$\bar{x}_2 = x_2\cos\varphi - x_1\sin\varphi\ . \tag{1.317}$$

Which result would we have got by the use of the rotation matrix?

$$\cos\varphi_{11} = \bar{\mathbf{e}}_1\cdot\mathbf{e}_1 = \cos\varphi\ ; \qquad \cos\varphi_{12} = \bar{\mathbf{e}}_1\cdot\mathbf{e}_2 = \cos(\pi/2-\varphi)\ ;$$
$$\cos\varphi_{21} = \bar{\mathbf{e}}_2\cdot\mathbf{e}_1 = \cos(\pi/2+\varphi)\ ; \qquad \cos\varphi_{22} = \bar{\mathbf{e}}_2\cdot\mathbf{e}_2 = \cos\varphi\ .$$

Therewith the rotation matrix D has the following form:

$$D = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}\ . \tag{1.318}$$

The orthonormality of rows and columns is obvious. $D^{-1} = D^T$ of course corresponds to a rotation by the angle $(-\varphi)$.

$$\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} = D \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \cos\varphi + x_2 \sin\varphi \\ -x_1 \sin\varphi + x_2 \cos\varphi \end{pmatrix}$$

This result is identical to (1.317) as it should be.

(2) Multiple rotation in the plane

We execute two rotations by the angles φ_1, φ_2 in series:

$$D_i = \begin{pmatrix} \cos\varphi_i & \sin\varphi_i \\ -\sin\varphi_i & \cos\varphi_i \end{pmatrix} ; \qquad i = 1, 2 ,$$

$$\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} = D_2 \left[D_1 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right] = (D_2 \cdot D_1) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} .$$

The total rotation is mediated by the product matrix $D_2 \cdot D_1$. For this holds:

$$D_2 \cdot D_1 = \begin{pmatrix} \cos\varphi_2 \cos\varphi_1 - \sin\varphi_2 \sin\varphi_1 & \cos\varphi_2 \sin\varphi_1 + \sin\varphi_2 \cos\varphi_1 \\ -\sin\varphi_2 \cos\varphi_1 - \cos\varphi_2 \sin\varphi_1 & -\sin\varphi_2 \sin\varphi_1 + \cos\varphi_2 \cos\varphi_1 \end{pmatrix} .$$

With the aid of the addition theorems (1.60) and (1.61)

$$\cos(x + y) = \cos x \cos y - \sin x \sin y ,$$

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

we can cast $D_2 \cdot D_1$ into the form

$$D_2 \cdot D_1 = \begin{pmatrix} \cos(\varphi_1 + \varphi_2) & \sin(\varphi_1 + \varphi_2) \\ -\sin(\varphi_1 + \varphi_2) & \cos(\varphi_1 + \varphi_2) \end{pmatrix} = D_1 \cdot D_2 \tag{1.319}$$

which apparently fits our expectation.

(3) Space rotation around the 3-axis

The rotation around the 3-axis (z-axis) means that the φ_{ij} for $i, j = 1, 2$ are to be chosen as in example (1). The 3-axis remains fixed, i.e. $\bar{\mathbf{e}}_3 = \mathbf{e}_3$:

$$\varphi_{33} = 0 ; \quad \varphi_{31} = \varphi_{13} = \varphi_{23} = \varphi_{32} = \pi/2 .$$

That means for the rotation matrix:

$$D = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{1.320}$$

We have already compiled quite a number of typical properties of the rotation matrix. Let us now assume that a *'complete orthonormal basis system' (CONS)* $\{\mathbf{e}_i\}$ and an arbitrary matrix D are given. Let us find the conditions which must be fulfilled by D in order to describe a rotation. Firstly the orthonormality of rows (1.308) and columns (1.309) must be realized. That, however, is not quite sufficient, since we still have to require that the new system of coordinates, too, must represent a right system, i.e. along with

$$\mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3) = 1$$

it should also hold:

$$\bar{\mathbf{e}}_1 \cdot (\bar{\mathbf{e}}_2 \times \bar{\mathbf{e}}_3) = 1 \tag{1.321}$$

This is ensured by the use of the **'determinant'** of D which must be equal to $+1$. That leads us to a new term which is dealt with in the next section.

1.6.4 Determinants

Definition 1.6.6 If

$$A = \left(a_{ij}\right) = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$$

is an $(n \times n)$-matrix then one defines as **'determinant'** of A the following number:

$$\det A = \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} = \sum_P (\mathrm{sign} P)\, a_{1p(1)} \cdot a_{2p(2)} \cdot \ldots a_{np(n)} \ . \tag{1.322}$$

Here the sequence of numbers

$$[p(1), \dots, p(n)] \equiv P(1, 2, \dots, n)$$

represents a special **permutation** of the *natural* sequence

$$(1, 2, \ldots, n)\ .$$

The sum contains all the thinkable permutations P, including the identity. The expression in (1.322) thus consists of $n!$ summands (remember (1.52) : $n! = 1 \cdot 2 \cdot 3 \ldots \cdot n$; read: *n-factorial*). Each summand obviously contains exactly **one** element from each row and **one** element from each column of the matrix A.

$$\operatorname{sign} P : \textit{sign of the permutation } P\ .$$

Each permutation can be realized successively by pairwise permutation of neighboring elements (*transposition*). The sign of the permutation is positive if the number of transpositions necessary to reach the respective permuted sequence of numbers is even. Otherwise it is negative.

Example

$$\begin{aligned} P(123) = (231) & \\ & \text{realizable by } \textbf{two} \text{ transpositions:} \\ & (123) \to (213) \to (231) \\ & \Longrightarrow \operatorname{sign} P = +1\ . \end{aligned}$$

The general definition (1.322) looks rather complicated. Let us therefore inspect how one can calculate $\det A$ explicitly.

$$n = 1: \quad \det A = |a_{11}| = a_{11}\ . \tag{1.323}$$

$$\begin{aligned} n = 2: \quad \det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} &= \operatorname{sign}(12) a_{11} a_{22} + \operatorname{sign}(21) a_{12} a_{21} = \\ &= a_{11} a_{22} - a_{12} a_{21}\ . \end{aligned} \tag{1.324}$$

Scheme (Rule of Thumb)

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}\ .$$

The connecting lines symbolize the products of the various summands, solid line with positive sign, broken line with negative sign.

$$n = 3:$$

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} .$$

There appear $3! = 6$ summands:

P	sign P
123	+1
132	−1
213	−1
231	+1
312	+1
321	−1

This means:

$$\det A = a_{11}\,(a_{22}\,a_{33} - a_{23}\,a_{32}) - a_{12}\,(a_{21}\,a_{33} - a_{23}\,a_{31}) + \\ + a_{13}\,(a_{21}\,a_{32} - a_{22}\,a_{31}) \;.$$

With (1.324) this expression can also be written in the following form:

$$\det A = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} . \tag{1.325}$$

This is called the *determinant expansion with respect to the first row* (see (1.327)).

Scheme (Sarrus-Rule)

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \iff \begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \\ (-) & (-) & (-) & (+) & (+) & (+) \end{array} . \tag{1.326}$$

It goes without saying that for $n \geq 4$ the representation becomes very soon very much more complicated. The majority of applications in Theoretical Physics, however, fortunately manages with $n \leq 3$. If not, the so-called **expansion theorem** helps, which we quote here without proof:

Theorem 1.6.1 *Expansion with respect to a row*

$$\det A = a_{i1}U_{i1} + a_{i2}U_{i2} + \ldots + a_{in}U_{in} = \sum_{j=1}^{n} a_{ij}U_{ij} \tag{1.327}$$

$U_{ij} = (-1)^{i+j}A_{ij}$: ***algebraic complement to*** a_{ij},
A_{ij}: ***subdeterminant*** = *determinant of the* $((n-1)\times(n-1))$*-matrix originating from A by eliminating the i-th row and the j-th column.*

The calculation of the $(n\times n)$-determinant is replaced by the expansion rule to that of $((n-1)\times(n-1))$-determinants. To the latter one can apply again the expansion theorem thereby reducing the dimensions of the remaining determinants further on. After $(n-2)$-fold expansion (1.324) comes into operation. The practical evaluation appears to be the simpler the more zeros are in the row of expansion. In this respect, in order to enhance the number of zeros in the row, one or more of the following calculation rules for equivalent rearrangings of the determinant may be helpful.

1.6.5 Calculation Rules for Determinants

Some of the important properties of the determinant can be read off rather directly from the definition (1.322):

1. Multiplication of a row or a column by a real number α

$$\begin{vmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ \alpha a_{i1} & \ldots & \alpha a_{in} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{vmatrix} = \alpha \begin{vmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \ldots & a_{in} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{vmatrix} . \tag{1.328}$$

After (1.322) the proof is immediately clear since each of the $n!$ summands in $\det A$ contains exactly **one** element from, respectively, each row and each column of A. In particular it holds:

$$\det(\alpha A) = \alpha^n \det A . \tag{1.329}$$

2. Likewise directly from the definition (1.322) it follows for the addition with respect to a row or a column, respectively:

$$\begin{vmatrix} a_{11}+b_{11} & \dots & a_{1n}+b_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} =$$

$$\begin{vmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} + \begin{vmatrix} b_{11} & \dots & b_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix} \tag{1.330}$$

3. The permutation of two neighboring rows (columns) changes the sign of the determinant. For a proof one should remember that thereby sign P reverses since the number of transpositions necessary for P alters by ± 1.
4. The matrix A may possess two identical rows (columns). By a sufficient number of permutations we can bring these two rows (columns) into neighboring positions ($A \rightarrow A'$). The value of $\det A$ can thereby have changed at most by its sign:

$$\det A = \pm \det A' \ .$$

Now we interchange still once more in A' the two identical rows (columns) where the matrix A' does not change, however, the determinant does:

$$\det A' = -\det A' \ .$$

That means the determinant must vanish:

$$\det A' = 0 = \det A \ .$$

5.

$$\det A = \det A^T \tag{1.331}$$

The proof is recommended as Exercise 1.6.4. It exploits again directly the definition (1.322). The statement (1.331) has the important consequence that one can expand a determinant obviously not only with respect to a row but also with respect to a column. Namely, along with (1.327) we also have:

$$\det A = \det A^T = \sum_{j=1}^{n} a_{ij}^T U_{ij}^T = \sum_{j=1}^{n} a_{ji} U_{ji} \; . \tag{1.332}$$

6. If one adds to a certain row (column) the elements of another row (column) multiplied by any real number α then the determinant remains unaffected:

$$\begin{vmatrix} \vdots & & \vdots \\ a_{i1} + \alpha a_{j1} & \dots & a_{in} + \alpha a_{jn} \\ \vdots & & \vdots \\ a_{j1} & \dots & a_{jn} \\ \vdots & & \vdots \end{vmatrix} = \tag{1.333}$$

$$= \begin{vmatrix} \vdots & & \vdots \\ a_{i1} & \dots & a_{in} \\ \vdots & & \vdots \\ a_{j1} & \dots & a_{jn} \\ \vdots & & \vdots \end{vmatrix} + \alpha \underbrace{\begin{vmatrix} \vdots & & \vdots \\ a_{j1} & \dots & a_{jn} \\ \vdots & & \vdots \\ a_{j1} & \dots & a_{jn} \\ \vdots & & \vdots \end{vmatrix}}_{=0} .$$

7. Multiplication theorem (without proof!):

$$\det(A \cdot B) = \det A \cdot \det B \; . \tag{1.334}$$

8. For a matrix with triangle shape

$$TR = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ 0 & & & a_{nn} \end{pmatrix}$$

one easily finds by expansion with respect to the first column:

$$\det TR = a_{11} \cdot a_{22} \cdot \dots \cdot a_{nn} \; .$$

In particular it follows then for the diagonal matrix D from (1.297):

$$\det D = d_1 \cdot d_2 \cdot \dots \cdot d_n \; .$$

That means for the unit matrix $\mathbb{1}$ (1.298):

$$\det \mathbb{1} = 1 \tag{1.335}$$

9. Multiplying the elements of a row (column) of a determinant with the algebraic complement U_{ij} of **another** row (column) and summing these products yields zero:

$$\sum_{k=1}^{n} a_{ik} U_{jk} = 0 \qquad \text{(rows)}\ ,$$
$$\sum_{k=1}^{n} a_{ki} U_{kj} = 0 \qquad \text{(columns)}\ . \tag{1.336}$$

Proof Let B be an $(n \times n)$-matrix, which except for the j-th row is identical to A. In the j-th row of B there appears once more the i-th row of A. Because of point 4. then:

$$\det B = 0\ .$$

One expands B according to (1.327) with respect to the j-th row:

$$0 = \det B = \sum_k b_{jk} U_{jk} = \sum_k a_{ik} U_{jk}\ ; \quad \text{q. e. d.}$$

1.6.6 *Special Applications*

1.6.6.1 Inverse Matrix

Definition 1.6.7 $A = (a_{ij})$ is a given $(n \times n)$-matrix. Then one denotes as its **inverse matrix**

$$A^{-1} = \left(\left(a^{-1}\right)_{ij}\right)$$

just the $(n \times n)$-matrix, for which holds:

$$A^{-1}A = A\,A^{-1} = \mathbb{1}\ . \tag{1.337}$$

Theorem 1.6.2 A^{-1} *exists only when* $\det A \neq 0$. *The elements are then found by:*

$$\left(a^{-1}\right)_{ij} = \frac{U_{ji}}{\det A}\ . \tag{1.338}$$

(Note the order of the indexes!)

Proof Let $\widehat{A} = (\alpha_{ij} = U_{ji})$ be an $(n \times n)$-matrix. With the expansion theorems (1.327) and (1.332) we find:

$$\det A = \sum_j a_{ij} U_{ij} = \sum_j a_{ij} \alpha_{ji} = \left(A \cdot \widehat{A}\right)_{ii} ,$$

$$\det A = \sum_i a_{ij} U_{ij} = \sum_i \alpha_{ji} a_{ij} = \left(\widehat{A} \cdot A\right)_{jj} .$$

The diagonal elements of the product matrices $A \cdot \widehat{A}$ and $\widehat{A} \cdot A$ are thus all identical to $\det A$. What about the non-diagonal elements? With (1.336) one finds:

$$\left(A \cdot \widehat{A}\right)_{ij} = \sum_k a_{ik} \alpha_{kj} = \sum_k a_{ik} U_{jk} = 0 \quad \text{for } i \neq j .$$

It follows that $A \cdot \widehat{A}$ and $\widehat{A} \cdot A$ are diagonal matrices with

$$A \cdot \widehat{A} = \widehat{A} \cdot A = \det A \cdot \mathbb{1} .$$

With $\det A \neq 0$ and by comparison with (1.337) the theorem is proved:

$$\frac{\widehat{A}}{\det A} = A^{-1} \Longleftrightarrow \frac{U_{ji}}{\det A} = \left(a^{-1}\right)_{ij} .$$

1.6.6.2 Vector Product

The vector product can be written as determinant in a very memorable form. According to (1.196) holds:

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= \sum_{i,j,k} \varepsilon_{ijk} a_i b_j \mathbf{e}_k = \\
&= \mathbf{e}_1 (a_2 b_3 - a_3 b_2) + \mathbf{e}_2 (a_3 b_1 - a_1 b_3) + \mathbf{e}_3 (a_1 b_2 - a_2 b_1) = \\
&= \mathbf{e}_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{e}_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{e}_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} .
\end{aligned}$$

This can be written as (3×3)-determinant:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} . \tag{1.339}$$

1.6.6.3 Curl (Rotation)

Also this vector differential operator can be expressed as determinant:

$$\nabla \times \mathbf{a} =\equiv \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ a_1 & a_2 & a_3 \end{vmatrix} . \tag{1.340}$$

1.6.6.4 Scalar Triple Product

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \equiv \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} . \tag{1.341}$$

One recognizes the correctness of this representation by (1.339) or by a direct evaluation. A cyclic permutation of the vectors in the scalar triple product in any case means two interchanges in the determinant each involving two rows, so that the value of the determinant remains unchanged.

In particular for orthonormalized basis vectors $\mathbf{e}_i$ we get:

$$\mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3) \equiv \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 . \tag{1.342}$$

1.6.6.5 Rotation Matrix

We remember the question which came up in connection with (1.321). Under what conditions an arbitrary matrix D based in a given CONS $\{\mathbf{e}_i\}$ is a rotation matrix? At first it must satisfy the orthonormality relations (1.308) and (1.316):

$$\sum_m d_{im} d_{jm} = \delta_{ij} ,$$

$$\sum_m d_{mi} d_{mj} = \delta_{ij} .$$

What is more, the *new* basis system $\{\bar{\mathbf{e}}_j\}$ originating from the original system $\{\mathbf{e}_i\}$ by rotation shall again be a right-handed trihedron, i.e. (1.342) must also be valid for the $\bar{\mathbf{e}}_j$. That is not yet guaranteed by the conditions (1.308) and (1.316). For instance, if we replace in the i-th row of D the d_{ij} by $(-d_{ij})$, the orthonormality relations will still be valid. On the other hand, however, according to (1.305) $\{\bar{\mathbf{e}}_i\}$ transfers into $(-\bar{\mathbf{e}}_i)$. Thus the right-handed trihedron becomes a left-handed one. However, we notice with (1.305):

$$\bar{\mathbf{e}}_1 \cdot (\bar{\mathbf{e}}_2 \times \bar{\mathbf{e}}_3) = \sum_{m,n,p} d_{1m} d_{2n} d_{3p}\, \mathbf{e}_m \cdot \left(\mathbf{e}_n \times \mathbf{e}_p\right) =$$
$$= \sum_{m,n,p} \varepsilon_{mnp} d_{1m} d_{2n} d_{3p} = \det D\ . \tag{1.343}$$

That means that besides the orthonormality of rows and columns a rotation matrix D still must fulfill:

$$\det D = 1 \tag{1.344}$$

1.6.6.6 Linear Systems of Equations

As the fourth very important field of application of determinants we finally discuss solutions and solvability conditions for linear systems of equations. We ask ourselves under which conditions a system of n equations with n unknowns $x_1, \ldots, x_n$ of the following type

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & = & b_1 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \ldots & + & a_{nn}x_n & = & b_n \end{array} \tag{1.345}$$

possesses a uniquely determined solution. Let us assume that the coefficients a_{ij} are all real. They build up the so-called **matrix of coefficients** A:

$$A \equiv \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} . \tag{1.346}$$

If only one of the b_i in (1.345) turns out to be unequal zero, one speaks of an **inhomogeneous** system of equations. If all $b_i = 0$, then it is a **homogeneous** system of equations.

Now we multiply each of the n equations in (1.345) by the corresponding algebraic complement U_{ik}, where k is kept fixed while i is the respective row index:

$$\begin{array}{ccccccccc} \left[a_{11}x_1\right. & + & a_{12}x_2 & + & \ldots & + & \left.a_{1n}x_n\right] & U_{1k} = & b_1 U_{1k} \\ \vdots & & \vdots & & & & \vdots & \vdots & \vdots \\ \left[a_{n1}x_1\right. & + & a_{n2}x_2 & + & \ldots & + & \left.a_{nn}x_n\right] & U_{nk} = & b_n U_{nk}\ . \end{array}$$

We then add all the equations together:

$$\sum_{j=1}^{n}\left(\sum_{i=1}^{n} a_{ij}U_{ik}\right) x_j = \sum_{j=1}^{n} b_j U_{jk} \ .$$

Because of (1.336) the expression in the bracket vanishes for $j \neq k$ so that we are left with:

$$\sum_{i=1}^{n} a_{ik}U_{ik}x_k = \sum_{j=1}^{n} b_j U_{jk}$$

On the left-hand side we recognize $detA$, expanded according to (1.332) with respect to the k-th column:

$$\det A \cdot x_k = \sum_{j=1}^{n} b_j U_{jk} \ . \tag{1.347}$$

We define a new matrix A_k as the matrix identical to A except for the fact that the k-th column is replaced by a column vector which is built up by the inhomogeneities of the linear system of equations (1.345):

$$\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

But then the right-hand side of (1.347) is just $\det A_k$, expanded according to the k-th column:

$$x_k \det A = \det A_k \ . \tag{1.348}$$

Therewith follows the

Cramer's rule.

The linear inhomogeneous system of equations (1.345) possesses a unique solution only when

$$\det A \neq 0 \ .$$

The solution is then given by:

$$x_k = \frac{\det A_k}{\det A} \qquad k = 1, 2, \ldots, n \,. \tag{1.349}$$

Let us illustrate the procedure by the following

Example

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \,, \\ 3x_1 + 2x_2 + x_3 &= 4 \,, \\ 5x_1 - 3x_2 + x_3 &= 0 \end{aligned}$$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 5 & -3 & 1 \end{pmatrix} \Longrightarrow \det A = -12 \,,$$

$$A_1 = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 2 & 1 \\ 0 & -3 & 1 \end{pmatrix} \Longrightarrow \det A_1 = -6 \,,$$

$$A_2 = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \\ 5 & 0 & 1 \end{pmatrix} \Longrightarrow \det A_2 = -12 \,,$$

$$A_3 = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 2 & 4 \\ 5 & -3 & 0 \end{pmatrix} \Longrightarrow \det A_3 = -6 \,.$$

According to Cramer's rule the system of equations is uniquely solvable since $\det A \neq 0$ and the solution is:

$$x_1 = \frac{1}{2} \,; \quad x_2 = 1 \,; \quad x_3 = \frac{1}{2} \,.$$

We now consider the special case of **homogeneous** systems of equations, i.e. we assume that all b_i in (1.345) are zero. But then it must also hold that $\det A_k \equiv 0$, so that according to (1.348) what remains is to solve:

$$x_k \det A = 0 \tag{1.350}$$

If $\det A \neq 0$, then the homogeneous system of equations has only the trivial *zero solution*, which of course always exists:

$$x_1 = x_2 = \ldots = x_n = 0 \,. \tag{1.351}$$

Hence, non-trivial solutions of a homogeneous system of equations can be expected only if

$$\det A = 0 \tag{1.352}$$

That means, however, that not all rows (columns) can be linearly independent. For the rank of the matrix one therefore has to conclude:

$$\operatorname{rank} A = m < n \; . \tag{1.353}$$

Let us presume that the first m equations in (1.345) are the linearly independent ones. (If that is not the case we can arbitrarily interchange the order of the equations!) Then we can write for these equations:

$$\begin{array}{ccccccc} a_{11}x_1 & + \ldots + & a_{1m}x_m & = -\,(& a_{1m+1}x_{m+1} & + \ldots + & a_{1n}x_n) \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + \ldots + & a_{mm}x_m & = -\,(& a_{mm+1}x_{m+1} & + \ldots + & a_{mn}x_n) \; . \end{array} \tag{1.354}$$

For the $(m \times m)$ matrix of coefficients A',

$$A' = \begin{pmatrix} a_{11} & \ldots & a_{1m} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mm} \end{pmatrix} , \tag{1.355}$$

one can now assume

$$\det A' \neq 0$$

so that Cramer's rule (1.349) becomes applicable. The matrix A_k then has, as k-th column vector, the following expression:

$$\begin{pmatrix} -\sum\limits_{j=m+1}^{n} a_{1j}x_j \\ \vdots \\ -\sum\limits_{j=m+1}^{n} a_{mj}x_j \end{pmatrix} \tag{1.356}$$

The solution thus will still depend on the arbitrarily choosable parameters $x_{m+1}, \ldots, x_n$.

Example

$$\begin{aligned} x_1 &\quad +4x_2 &\quad -x_3 &\quad = 0\,, \\ 2x_1 &\quad -3x_2 &\quad +x_3 &\quad = 0\,, \\ 4x_1 &\quad +16x_2 &\quad -4x_3 &\quad = 0\,; \end{aligned} \qquad A = \begin{pmatrix} 1 & 4 & -1 \\ 2 & -3 & 1 \\ 4 & 16 & -4 \end{pmatrix}.$$

It is obviously

$$\det A = 0\,.$$

The first two rows are linearly independent:

$$\begin{aligned} x_1 + 4x_2 &= x_3 \\ 2x_1 - 3x_2 &= -x_3 \end{aligned} \quad \Longrightarrow \quad \det A' = -11\,.$$

With

$$\det A_1 = \begin{vmatrix} x_3 & 4 \\ -x_3 & -3 \end{vmatrix} = x_3\,,$$

$$\det A_2 = \begin{vmatrix} 1 & x_3 \\ 2 & -x_3 \end{vmatrix} = -3x_3$$

follows

$$x_1 = -\frac{x_3}{11}\,; \quad x_2 = \frac{3}{11}x_3\,,$$

where x_3 remains arbitrary.

1.6.7 Exercises

Exercise 1.6.1 Construct for the matrices

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 0 & 4 \\ 0 & 0 & 5 \end{pmatrix}\,, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

the product matrices $A \cdot B$, $B \cdot A$.

Exercise 1.6.2

$$A \equiv (a_{ij}) : (m \times n) - \text{matrix}$$
$$B \equiv (b_{ij}) : (n \times r) - \text{matrix}$$

1. Show that for the transposed matrices holds:

$$(A \cdot B)^T = B^T A^T$$

2. Let $m = n$. Then A^{-1} is the inverse matrix of A if it fulfills the relation

$$A^{-1} \cdot A = A \cdot A^{-1} = \mathbb{1}$$

   Prove the validity of

$$(A^{-1})^T = (A^T)^{-1}$$

3. Let $m = n = r$. Verify the relation

$$(A \cdot B)^{-1} = B^{-1} A^{-1}$$

Exercise 1.6.3 Calculate the following determinants:

$$1) \begin{vmatrix} 4 & 3 & 2 \\ 1 & 0 & -1 \\ 5 & 2 & 2 \end{vmatrix} , \qquad 2) \begin{vmatrix} 1 & 6 & 8 & 7 \\ -2 & 3 & 11 & 5 \\ 5 & 0 & 6 & 7 \\ -1 & 9 & 19 & 12 \end{vmatrix} ,$$

$$3) \begin{vmatrix} 4 & 3 & 0 & 1 \\ 6 & 7 & 8 & -1 \\ 0 & 1 & 0 & 7 \\ 3 & -4 & 0 & 6 \end{vmatrix} .$$

Exercise 1.6.4

1. Let A^T be the transposed matrix of the $(n \times n)$-matrix A. Prove:

$$\det A^T = \det A \ .$$

2. Let B be an antisymmetric $(n \times n)$-matrix

$$B = (b_{ij}) \qquad \text{with} \qquad b_{ij} = -b_{ji} \ .$$

Demonstrate that

$$\det B = 0 ,$$

must be if n is an odd integer.

Exercise 1.6.5 The matrix A is given by

$$A = \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix} .$$

Show that

$$\det A = \left(a^2 + b^2 + c^2 + d^2\right)^2 .$$

Hint: Multiply first A by its transposed matrix A^T.

Exercise 1.6.6 Inspect the following systems of equations with respect to solvability and, if solvable, find the solution

$$\begin{array}{rrrrrrrl}
1) & 2x_1 & + & x_2 & + & 5x_3 & = & -21 , \\
 & x_1 & + & 5x_2 & + & 2x_3 & = & 19 , \\
 & 5x_1 & + & 2x_2 & + & x_3 & = & 2 . \\
2) & x_1 & - & x_2 & + & 3x_3 & = & 4 , \\
 & 9x_1 & + & 3x_2 & - & 12x_3 & = & -3 , \\
 & 3x_1 & + & x_2 & - & 4x_3 & = & -1 . \\
3) & x_1 & + & x_2 & - & x_3 & = & 0 , \\
 & -x_1 & + & 3x_2 & + & x_3 & = & 0 , \\
 & x_2 & + & x_3 & & & = & 0 . \\
4) & 2x_1 & - & 3x_2 & + & x_3 & = & 0 , \\
 & 4x_1 & + & 4x_2 & - & x_3 & = & 0 , \\
 & x_1 & - & \frac{3}{2}x_2 & + & \frac{1}{2}x_3 & = & 0 .
\end{array}$$

Exercise 1.6.7 Given is the matrix A:

$$A = \begin{pmatrix} -\frac{1}{2}\sqrt{2} & 0 & -\frac{1}{2}\sqrt{2} \\ 0 & 1 & 0 \\ \frac{1}{2}\sqrt{2} & 0 & -\frac{1}{2}\sqrt{2} \end{pmatrix} .$$

1. Does A describe a rotation? If yes what kind of rotation?
2. How do the vectors

$$\mathbf{a} = (0, -2, 1) \ , \quad \mathbf{b} = (3, 5, -4)$$

   change after the rotation? Calculate the scalar product $\mathbf{a} \cdot \mathbf{b}$ before and after the rotation.

Exercise 1.6.8

1. Determine for the matrices

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix} \ ; \quad B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & \frac{1}{2} & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

   the product matrices AB and BA!
2. Calculate the determinants of A and B as well as those of AB and BA!
3. Are A and B rotation matrices? Give reasons for your answer!
4. Determine the inverse matrix A^{-1}!

Exercise 1.6.9 Prove the following statements!

1. During a rotation the length of a vector is unchanged.
2. For the elements d_{ij} of the rotation matrix the relations

$$d_{ij} = U_{ij} \ , \quad i,j = 1, 2, 3 \ ,$$

   are valid, where U_{ij} is the algebraic complement to d_{ij}.

Exercise 1.6.10 D_1 and D_2 are two rotation matrices. Show that rows and columns of the product matrix $D = D_1 \cdot D_2$ are orthonormal.

1.7 Coordinate Systems

1.7.1 Transformation of Variables, Jacobian Determinant

For our considerations so far we have presupposed, directly or at least indirectly, a Cartesian system of coordinates. However, in subsequent applications we shall use, as a rule, those coordinate systems which best fit the underlying problem with respect to its inherent symmetry. That will then not necessarily be the Cartesian coordinates. Therefore we consider in the following the principles for the transition from one set of coordinates to another one.

Let us inspect first, as an introductory example, **plane polar coordinates** by which the position of a point P in the plane can *almost always* be defined as

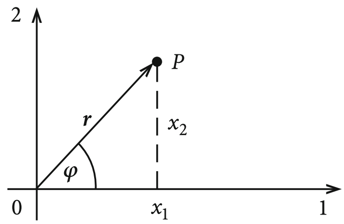


Fig. 1.74 To the definition of plane polar coordinates

conveniently as by Cartesian coordinates x_1, x_2. In Fig. 1.74 r is the distance between P and the origin of coordinates $\mathcal{O}$ and φ is the angle between the straight line $\overline{OP}$ and the 1-axis.

The mapping

$$(r, \varphi) \Longrightarrow (x_1, x_2)$$

is described by the *transformation formulae*

$$\begin{aligned} x_1 &= r\cos\varphi = x_1(r, \varphi) \ , \\ x_2 &= r\sin\varphi = x_2(r, \varphi) \end{aligned} \tag{1.357}$$

One speaks of a two-dimensional **point transformation** which maps the (r, φ)-plane *point by point* onto the (x_1,x_2)-plane. We must reasonably require from the *new* coordinates that they catch each point of the plane. This is here obviously the case. However, it should also be guaranteed that each point $P \cong (x_1, x_2)$ is uniquely ascribed to a definite (r, φ) pair. But here difficulties appear with $(x_1 = 0, \ x_2 = 0)$ since **all** pairs $(0, \varphi)$ are mapped on $(0, 0)$. The mapping (1.357) is for $r = 0$ not uniquely reversible, but otherwise for $r \neq 0$:

$$\begin{aligned} r &= \sqrt{x_1^2 + x_2^2} \ , \\ \varphi &= \arctan\frac{x_2}{x_1} \ . \end{aligned} \tag{1.358}$$

The trigonometric function arc tangent has to be restricted to the branch which delivers the values $0 \leq \varphi < 2\pi$. Hence the transformation (1.357) is **almost always** reversible.

Let us now consider a **general transformation of variables** in a d-dimensional space:

$$x_i = x_i\,(y_1, \ldots, y_d) \ ; \quad i = 1, \ldots, d \ . \tag{1.359}$$

As in the introductory example we require:

1. **Each** point of the space under consideration must be specifiable by the **generalized** coordinates y_i.
2. The transformation must be **'almost always locally reversible'**.

That means:

(a) **'Locally reversible'**: To any arbitrary point P there exists a neighborhood $U(P)$ in which the mapping is absolutely unique, i.e. to each set of d elements $(x_1, \dots, x_d)$ there belongs **exactly one** set of d elements $(y_1, \dots, y_d)$.
(b) **'Almost always'**: The condition of local reversibility is allowed to be violated at most in regions of lower dimensionality $d' < d$. The transformation between Cartesian coordinates and plane polar coordinates is, as we have seen, *almost always locally reversible* except for the one-dimensional manifold $\{r = 0;\ 0 \leq \varphi \leq 2\pi\}$.

How can we check the local reversibility? P may be an arbitrarily chosen but fixed point of the d dimensional space with the coordinates

$$(x_1, \dots, x_d) \quad \text{and} \quad (y_1, \dots, y_d) \ , \text{ respectively} .$$

A differentially small neighborhood of P will then be covered by:

$$(y_1 + dy_1, \dots, y_d + dy_d) \ .$$

For the corresponding coordinates x_i one thus has to assume :

$$dx_i = x_i\,(y_1 + dy_1, \dots, y_d + dy_d) - x_i\,(y_1, \dots, y_d)\,; \quad i = 1, \dots, d \ .$$

Since the coordinates of P have to remain fixed, the requirement of a one-to-one mapping means that the differential changes dy_i are in one-to-one relation to the differential changes dx_i. For the latter according to (1.261) we have:

$$dx_i = \sum_{j=1}^{d} \frac{\partial x_i}{\partial y_j}\, dy_j \Big|_P \ ; \quad i = 1, \dots, d \ . \tag{1.360}$$

With the so-called **Jacobian matrix**,

$$F^{(xy)} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_d} \\ \vdots & & \vdots \\ \frac{\partial x_d}{\partial y_1} & \cdots & \frac{\partial x_d}{\partial y_d} \end{pmatrix} ; \quad F_{ij}^{(xy)} = \frac{\partial x_i}{\partial y_j} \ , \tag{1.361}$$

which of course depends on the coordinates of the point under consideration P, we can write (1.360) also in matrix form:

$$\begin{pmatrix} dx_1 \\ \vdots \\ dx_d \end{pmatrix} = F_P^{(xy)} \begin{pmatrix} dy_1 \\ \vdots \\ dy_d \end{pmatrix} . \tag{1.362}$$

An inversion is possible exactly then when the inverse $\left(F_P^{(xy)}\right)^{-1}$ does exist. According to (1.338), however, that means that the so-called **'Jacobian determinant'** must be **unequal zero**:

$$\det F^{(xy)} = \frac{\partial(x_1, \ldots, x_d)}{\partial(y_1, \ldots, y_d)} = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_d} \\ \vdots & & \vdots \\ \frac{\partial x_d}{\partial y_1} & \cdots & \frac{\partial x_d}{\partial y_d} \end{vmatrix} \tag{1.363}$$

Let us formulate this issue as follows:

Theorem 1.7.1 *The transformation of variables*

$$x_i = x_i\,(y_1, \ldots, y_d)\,; \quad i = 1, 2, \ldots, d$$

with continuously partially differentiable functions x_i is in the proximity of a point P bijective, i.e. uniquely solvable, if and only if:

$$\left.\frac{\partial(x_1, \ldots, x_d)}{\partial(y_1, \ldots, y_d)}\right|_P \neq 0 \tag{1.364}$$

As an example we consider plane polar coordinates $d = 2$:

$$\frac{\partial x_1}{\partial r} = \cos\varphi\ , \quad \frac{\partial x_1}{\partial \varphi} = -r\sin\varphi\ ,$$
$$\frac{\partial x_2}{\partial r} = \sin\varphi\ , \quad \frac{\partial x_2}{\partial \varphi} = r\cos\varphi$$

$$\Longrightarrow \frac{\partial(x_1, x_2)}{\partial(r, \varphi)} = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r\ .$$

We see that the mapping is everywhere locally reversible except for $r = 0$.

The following statement is important and also is easily provable:

Theorem 1.7.2 *Let*

$$\begin{matrix} x_i = x_i\,(y_1, \ldots, y_d) \\ y_i = y_i\,(z_1, \ldots, z_d) \end{matrix} \; ; \qquad i = 1, \ldots, d$$

be two continuously partially differentiable transformations. Then it holds for the double transformation:

$$x_i = x_i\,[y_1\,(z_1, \ldots, z_d)\,, \ldots, y_d\,(z_1, \ldots, z_d)]\ ,$$

$$\frac{\partial\,(x_1, \ldots, x_d)}{\partial\,(z_1, \ldots, z_d)} = \frac{\partial\,(x_1, \ldots, x_d)}{\partial\,(y_1, \ldots, y_d)} \cdot \frac{\partial\,(y_1, \ldots, y_d)}{\partial\,(z_1, \ldots, z_d)}\ . \tag{1.365}$$

Proof The proof uses the **chain rule** (1.87):

$$\frac{\partial x_i}{\partial z_j} = \sum_{k=1}^{d} \frac{\partial x_i}{\partial y_k}\frac{\partial y_k}{\partial z_j} \Longleftrightarrow F^{(x,z)} = F^{(x,y)} \cdot F^{(y,z)}\ .$$

Applying the multiplication theorem (1.334) immediately leads to the above statement:

$$\det F^{(x,z)} = \det F^{(x,y)} \det F^{(y,z)}\ .$$

In particular it follows from this theorem for the special case $z_i = x_i$:

$$\frac{\partial(y_1, \ldots, y_d)}{\partial(x_1, \ldots, x_d)} = \frac{1}{\frac{\partial(x_1,\ldots,x_d)}{\partial(y_1,\ldots,y_d)}}\ . \tag{1.366}$$

That means: If $\frac{\partial(x_1,\ldots,x_d)}{\partial(y_1,\ldots,y_d)} \neq 0$, then it must also be true that $\frac{\partial(y_1,\ldots,y_d)}{\partial(x_1,\ldots,x_d)} \neq 0$. This, in turn, corresponds to the almost self-evident conclusion that

$$x_i = x_i\,(y_1, \ldots, y_d)\ ; \quad i = 1, 2, \ldots, d$$

together with

$$y_j = y_j\,(x_1, \ldots, x_d)\ ; \quad j = 1, 2, \ldots, d$$

represents an unambiguously reversible transformation.

For the cases $d = 2$ and $d = 3$, which we are of course most interested in, the Jacobian determinant has a rather illustrative geometrical meaning. For $d = 2$, it indicates how a surface element will be changed by the transformation and for

$d = 3$ it characterizes the change of a volume element. Let us inspect the situation for $d = 3$ in a bit more detail. For this purpose we first introduce a new term, namely the **'coordinate line'**.

Definition 1.7.1 If in all formulae of the transformation

$$\mathbf{x} = \mathbf{x}(y_1, \ldots, y_d)$$

one keeps $(d-1)$ of the d coordinates y_i constant, i.e. $y_i =$ const for $i \neq j$, then it results a space curve parametrized by y_j which is called the y_j**-coordinate line**.

Examples $(d = 2)$

(a) *Cartesian coordinates:*
The coordinate lines build a *rectangular, rectilinear* grid (Fig. 1.75).

(b) *Plane polar coordinates:*
The lines $\varphi =$ const are again straight lines, the lines $r =$ const, however, are circles (Fig. 1.76). One therefore speaks of **'curvilinear coordinates'**. Nevertheless, one recognizes that the network of coordinate lines is locally still rectangular (**curvilinear-orthogonal**).

We now consider an infinitesimally small volume element dV in the three-dimensional space which is restricted by such curvilinear coordinate lines. For sufficiently small edges one can approximate the small volume by a parallelepiped

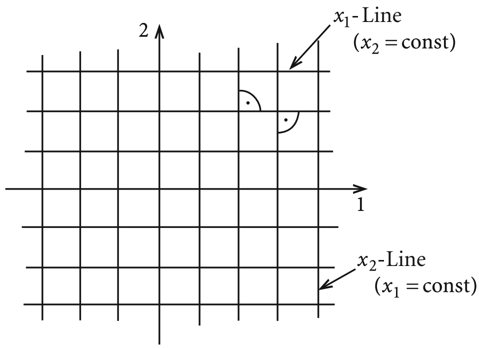


Fig. 1.75 Coordinate lines in case of Cartesian coordinates

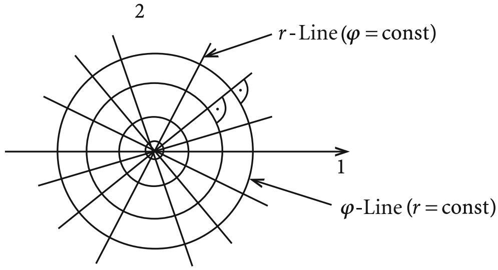


Fig. 1.76 Coordinate lines in case of plane polar coordinates

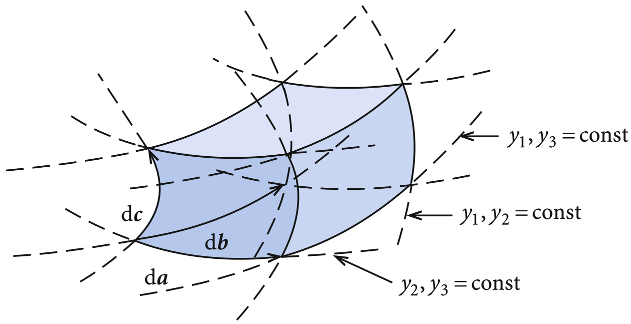


Fig. 1.77 Coordinate lines in the case of arbitrary curvilinear coordinates

bounded by the vectors:

$$d\mathbf{a} \equiv \left(\frac{\partial x_1}{\partial y_1}dy_1, \frac{\partial x_2}{\partial y_1}dy_1, \frac{\partial x_3}{\partial y_1}dy_1\right) \equiv \frac{\partial \mathbf{r}}{\partial y_1}dy_1 \ ,$$

$$d\mathbf{b} \equiv \left(\frac{\partial x_1}{\partial y_2}dy_2, \frac{\partial x_2}{\partial y_2}dy_2, \frac{\partial x_3}{\partial y_2}dy_2\right) \equiv \frac{\partial \mathbf{r}}{\partial y_2}dy_2 \ ,$$

$$d\mathbf{c} \equiv \left(\frac{\partial x_1}{\partial y_3}dy_3, \frac{\partial x_2}{\partial y_3}dy_3, \frac{\partial x_3}{\partial y_3}dy_3\right) \equiv \frac{\partial \mathbf{r}}{\partial y_3}dy_3 \ .$$

The volume dV of the parallelepiped is then given by the scalar triple product built by $d\mathbf{a}$, $d\mathbf{b}$, $d\mathbf{c}$ (Fig. 1.77). For this holds:

$$dV = \begin{vmatrix} \frac{\partial x_1}{\partial y_1}dy_1 & \frac{\partial x_2}{\partial y_1}dy_1 & \frac{\partial x_3}{\partial y_1}dy_1 \\ \frac{\partial x_1}{\partial y_2}dy_2 & \frac{\partial x_2}{\partial y_2}dy_2 & \frac{\partial x_3}{\partial y_2}y_2 \\ \frac{\partial x_1}{\partial y_3}dy_3 & \frac{\partial x_2}{\partial y_3}dy_3 & \frac{\partial x_3}{\partial y_3}dy_3 \end{vmatrix} =$$

$$\overset{(1.328)}{=} dy_1 dy_2 dy_3 \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_2}{\partial y_1} & \frac{\partial x_3}{\partial y_1} \\ \frac{\partial x_1}{\partial y_2} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_3}{\partial y_2} \\ \frac{\partial x_1}{\partial y_3} & \frac{\partial x_2}{\partial y_3} & \frac{\partial x_3}{\partial y_3} \end{vmatrix} \overset{(1.331)}{=} dy_1 dy_2 dy_3 \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \frac{\partial x_1}{\partial y_3} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \frac{\partial x_2}{\partial y_3} \\ \frac{\partial x_3}{\partial y_1} & \frac{\partial x_3}{\partial y_2} & \frac{\partial x_3}{\partial y_3} \end{vmatrix} =$$

$$= \frac{\partial(x_1, x_2, x_3)}{\partial(y_1, y_2, y_3)} dy_1\, dy_2\, dy_3 = dx_1\, dx_2\, dx_3 \ .$$

Hence the Jacobian determinant describes indeed how the representation of the volume element will change as a consequence of the variable transformation. The relation (1.367) is of course not only valid for $d = 3$ but holds with an analogous generalization for all dimensions d. This turns out to be especially important for the change of variables in multiple integrals.

1.7.2 Curvilinear Coordinates

We want to investigate with which basis vectors the curvilinear coordinates are to be described. We first start with the already familiar Cartesian coordinates,

$$x_1, x_2, x_3 \ ,$$

defined by the CONS:

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \ ; \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \ ; \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \ . \tag{1.367}$$

For the position vector $\mathbf{r}$ we then have,

$$\mathbf{r} = \sum_{j=1}^{3} x_j \mathbf{e}_j \ ,$$

and for its differential:

$$d\mathbf{r} = \sum_{j=1}^{3} dx_j \mathbf{e}_j = \sum_{j=1}^{3} \frac{\partial \mathbf{r}}{\partial x_j} dx_j \ .$$

That means

$$\mathbf{e}_j = \frac{\partial \mathbf{r}}{\partial x_j} \ , \tag{1.368}$$

which obviously agrees with (1.367). $\mathbf{e}_j$ is the tangent-unit vector to the x_j coordinate line.

This we now generalize to arbitrary curvilinear coordinates y_1, y_2, y_3: The **basis vectors** are defined in such a way that they are oriented **tangentially to the coordinate lines**. The vector $\partial\mathbf{r}/\partial y_i$ lies obviously tangentially to the y_i coordinate

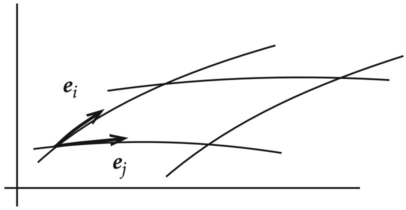


Fig. 1.78 Basis vectors for curvilinear coordinates

line which, however, in general is not normalized to one. With

$$b_{y_i} = \left| \frac{\partial \mathbf{r}}{\partial y_i} \right| \tag{1.369}$$

one then obtains as unit vector:

$$\mathbf{e}_{y_i} = b_{y_i}^{-1} \frac{\partial \mathbf{r}}{\partial y_i} \ . \tag{1.370}$$

In contrast to the Cartesian basis vectors (1.368) the above unit vectors in general do not form a space-fixed orthonormal trihedron but rather will be position dependent, namely the so-called **'local trihedron'** (Fig. 1.78).

Example: Plane Polar Coordinates

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial \varphi} &= (-r \sin \varphi,\ r \cos \varphi) \ , \\
b_\varphi &= \left| \frac{\partial \mathbf{r}}{\partial \varphi} \right| = r \ , \\
\frac{\partial \mathbf{r}}{\partial r} &= (\cos \varphi,\ \sin \varphi) \ , \\
b_r &= \left| \frac{\partial \mathbf{r}}{\partial r} \right| = 1 \ .
\end{aligned}$$

This yields as basis vectors:

$$\mathbf{e}_\varphi = (-\sin \varphi, \cos \varphi) \ ; \qquad \mathbf{e}_r = (\cos \varphi, \sin \varphi) \ . \tag{1.371}$$

These basis vectors are evidently orthonormal. One speaks of

curvilinear-orthogonal

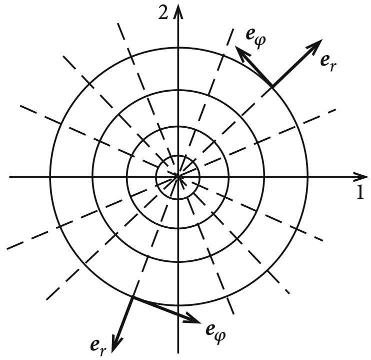


Fig. 1.79 Basis vectors for plane polar coordinates

basis vectors if

$$\mathbf{e}_{y_i} \cdot \mathbf{e}_{y_j} = \delta_{ij} \tag{1.372}$$

is fulfilled (Fig. 1.79).

For the differential of the position vector **r** one finds with curvilinear coordinates:

$$d\mathbf{r} = \sum_{j=1}^{3} \frac{\partial \mathbf{r}}{\partial y_j} dy_j = \sum_{j=1}^{3} b_{y_j} dy_j \mathbf{e}_{y_j} \ . \tag{1.373}$$

Example: Plane Polar Coordinates

$$d\mathbf{r} = dr\,\mathbf{e}_r + r\,d\varphi\,\mathbf{e}_\varphi \ . \tag{1.374}$$

To conclude we still want to rewrite the **vector-differential operators**, introduced in Sect. 1.5.3, for curvilinear coordinates:

(a) Gradient

For the y_i component of the gradient of a scalar, sufficiently often differentiable field φ holds:

$$\begin{aligned}\nabla_{y_i}\varphi &= \mathbf{e}_{y_i} \cdot \nabla\varphi = b_{y_i}^{-1} \frac{\partial \mathbf{r}}{\partial y_i} \cdot \nabla\varphi = \\ &= b_{y_i}^{-1} \left(\frac{\partial x_1}{\partial y_i} \frac{\partial \varphi}{\partial x_1} + \frac{\partial x_2}{\partial y_i} \frac{\partial \varphi}{\partial x_2} + \frac{\partial x_3}{\partial y_i} \frac{\partial \varphi}{\partial x_3} \right) \ .\end{aligned}$$

With the chain rule (1.260) we get:

$$\nabla_{y_i}\varphi = b_{y_i}^{-1}\frac{\partial \varphi}{\partial y_i}\,. \tag{1.375}$$

The **nabla-operator** introduced in (1.269) has here the more general shape:

$$\nabla = \left(b_{y_1}^{-1}\frac{\partial}{\partial y_1},\, b_{y_2}^{-1}\frac{\partial}{\partial y_2},\, b_{y_3}^{-1}\frac{\partial}{\partial y_3}\right) = \sum_{j=1}^{3}\mathbf{e}_{y_j} b_{y_j}^{-1}\frac{\partial}{\partial y_j}\,. \tag{1.376}$$

(b) Divergence

Let

$$\mathbf{a} = \sum_{i=1}^{3} a_{y_i}\mathbf{e}_{y_i}$$

be a sufficiently often partially differentiable vector field. Then we have:

$$\nabla\cdot\mathbf{a} = \frac{1}{b_{y_1}b_{y_2}b_{y_3}}\left[\frac{\partial}{\partial y_1}\left(b_{y_2}b_{y_3}a_{y_1}\right) + \frac{\partial}{\partial y_2}\left(b_{y_3}b_{y_1}a_{y_2}\right) + \frac{\partial}{\partial y_3}\left(b_{y_1}b_{y_2}a_{y_3}\right)\right]\,. \tag{1.377}$$

Proof In the first step with (1.376) we have:

$$\begin{aligned}\nabla\cdot\mathbf{a} &= \sum_{i,j}\left(\mathbf{e}_{y_i}b_{y_i}^{-1}\frac{\partial}{\partial y_i}\right)\cdot\left(a_{y_j}\mathbf{e}_{y_j}\right) = \\ &= \sum_i \frac{1}{b_{y_i}}\frac{\partial a_{y_i}}{\partial y_i} + \sum_{i,j}\frac{a_{y_j}}{b_{y_i}}\mathbf{e}_{y_i}\cdot\frac{\partial \mathbf{e}_{y_j}}{\partial y_i}\,. \end{aligned}\tag{1.378}$$

We exploit

$$\frac{\partial^2\mathbf{r}}{\partial y_i\partial y_j} = \frac{\partial^2\mathbf{r}}{\partial y_j\partial y_i}$$

and deduce with (1.370):

$$\begin{aligned}\frac{\partial}{\partial y_i}\left(b_{y_j}\mathbf{e}_{y_j}\right) &= \frac{\partial}{\partial y_j}\left(b_{y_i}\mathbf{e}_{y_i}\right)\\ \Longleftrightarrow b_{y_j}\frac{\partial}{\partial y_i}\mathbf{e}_{y_j} + \frac{\partial b_{y_j}}{\partial y_i}\mathbf{e}_{y_j} &= b_{y_i}\frac{\partial \mathbf{e}_{y_i}}{\partial y_j} + \frac{\partial b_{y_i}}{\partial y_j}\mathbf{e}_{y_i}\,.\end{aligned}$$

We multiply this expression scalarly by $\mathbf{e}_{y_i}$:

$$b_{y_j}\mathbf{e}_{y_i} \cdot \frac{\partial}{\partial y_i}\mathbf{e}_{y_j} + \delta_{ij}\frac{\partial b_{y_j}}{\partial y_i} = b_{y_i}\mathbf{e}_{y_i} \cdot \frac{\partial \mathbf{e}_{y_i}}{\partial y_j} + \frac{\partial b_{y_i}}{\partial y_j},$$

$$\mathbf{e}_{y_i} \cdot \frac{\partial \mathbf{e}_{y_i}}{\partial y_j} = \frac{1}{2}\frac{\partial}{\partial y_j}\left(\mathbf{e}_{y_i}^2\right) = 0 \,.$$

so that we have:

$$b_{y_j}\mathbf{e}_{y_i} \cdot \frac{\partial}{\partial y_i}\mathbf{e}_{y_j} = \frac{\partial b_{y_i}}{\partial y_j} - \delta_{ij}\frac{\partial b_{y_j}}{\partial y_i} = \begin{cases} 0 & \text{for } i = j\,, \\ \dfrac{\partial b_{y_i}}{\partial y_j} & \text{for } i \neq j\,. \end{cases}$$

This result we now use in (1.378):

$$\nabla \cdot \mathbf{a} = \sum_i b_{y_i}^{-1}\frac{\partial a_{y_i}}{\partial y_i} + \sum_{i,j}^{i\neq j}\frac{a_{y_j}}{b_{y_i}b_{y_j}}\frac{\partial b_{y_i}}{\partial y_j} = \sum_i b_{y_i}^{-1}\left(\frac{\partial a_{y_i}}{\partial y_i} + \sum_j^{\neq i}\frac{a_{y_i}}{b_{y_j}}\frac{\partial b_{y_j}}{\partial y_i}\right) =$$

$$= \frac{1}{b_{y_1}b_{y_2}b_{y_3}}\left[\frac{\partial}{\partial y_1}\left(a_{y_1}b_{y_2}b_{y_3}\right) + \ldots\right] \,; \quad \text{q. e. d.}$$

In the second step, in the double sum we have interchanged the indexes i and j.

(c) Curl (Rotation)

Analogously to the derivation of the divergence one gets the expression for the rotation:

$$\nabla \times \mathbf{a} = \frac{1}{b_{y_1}b_{y_2}b_{y_3}}\begin{vmatrix} b_{y_1}\mathbf{e}_{y_1} & b_{y_2}\mathbf{e}_{y_2} & b_{y_3}\mathbf{e}_{y_3} \\ \dfrac{\partial}{\partial y_1} & \dfrac{\partial}{\partial y_2} & \dfrac{\partial}{\partial y_3} \\ b_{y_1}a_{y_1} & b_{y_2}a_{y_2} & b_{y_3}a_{y_3} \end{vmatrix} \,. \tag{1.379}$$

1.7.3 Cylindrical Coordinates

Cylindrical coordinates (ρ, φ, z) correspond to the planar polar coordinates (ρ, φ) which are for the three-dimensional space supplemented by an additional vertical coordinate z. They are conveniently used for problems which exhibit a rotation symmetry with respect to a fixed axis. The latter is then declared as x_3 axis (Fig. 1.80).

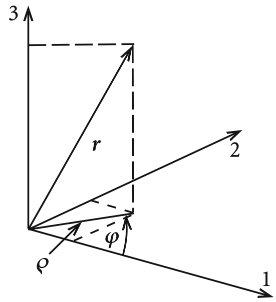


Fig. 1.80 Cylindrical coordinates

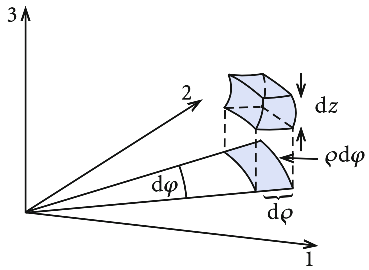


Fig. 1.81 Volume element in cylindrical coordinates

1.7.3.1 Transformation Formulae

$$
\begin{aligned}
x_1 &= \rho\cos\varphi \ , \\
x_2 &= \rho\sin\varphi \ , \\
x_3 &= \quad z \ .
\end{aligned}
\tag{1.380}
$$

1.7.3.2 Jacobian Determinant

$$
\frac{\partial\,(x_1, x_2, x_3)}{\partial(\rho, \varphi, z)} = \begin{vmatrix} \cos\varphi & -\rho\sin\varphi & 0 \\ \sin\varphi & \rho\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \rho \ . \tag{1.381}
$$

Thus the mapping is uniquely reversible except for $\rho = 0$.

The volume element, which is the volume increase due to infinitesimal changes of the coordinates, can be read off from Fig. 1.81:

$$
dV = \rho\, d\rho\, d\varphi\, dz \ . \tag{1.382}
$$

This follows already from the general relation (1.367):

$$
dV = \frac{\partial\,(x_1, x_2, x_3)}{\partial(\rho, \varphi, z)} d\rho\, d\varphi\, dz \ . \tag{1.383}
$$

1.7.3.3 Coordinate Lines [$\cong \mathbf{r} = \mathbf{r}(y_i : y_j = \text{const for } j \neq i)$]

ρ line: radial ray in the x_1, x_2 plane starting from the z-axis.
φ line: circle in the x_1, x_2 plane with its center on the z-axis.
z line: straight line parallel to the x_3 axis.

We derive the unit vectors:

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial \rho} &= (\cos\varphi,\ \sin\varphi,\ 0) \Longrightarrow b_\rho = 1\ , \\
\frac{\partial \mathbf{r}}{\partial \varphi} &= (-\rho\sin\varphi,\ \rho\cos\varphi,\ 0) \Longrightarrow b_\varphi = \rho\ , \\
\frac{\partial \mathbf{r}}{\partial z} &= (0,\ 0,\ 1) \Longrightarrow b_z = 1\ .
\end{aligned} \tag{1.384}$$

Therewith the **unit vectors in cylindrical coordinates** are given by:

$$\begin{aligned}
\mathbf{e}_\rho &= (\cos\varphi,\ \sin\varphi,\ 0)\ , \\
\mathbf{e}_\varphi &= (-\sin\varphi,\ \cos\varphi,\ 0)\ , \\
\mathbf{e}_z &= (0,\ 0,\ 1)\ .
\end{aligned} \tag{1.385}$$

These are curvilinear-orthogonal and are oriented tangentially at the respective coordinate line. For the differential of the position vector according to (1.373) we have with cylindrical coordinates:

$$d\mathbf{r} = d\rho\, \mathbf{e}_\rho + \rho\, d\varphi\, \mathbf{e}_\varphi + dz\, \mathbf{e}_z\ . \tag{1.386}$$

1.7.3.4 Gradient

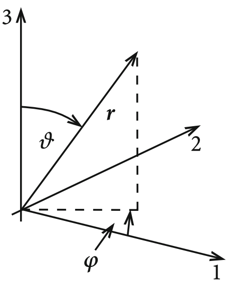
It follows immediately from (1.376):

$$\nabla \equiv \left(\frac{\partial}{\partial \rho}, \frac{1}{\rho}\frac{\partial}{\partial \varphi}, \frac{\partial}{\partial z}\right) = \mathbf{e}_\rho \frac{\partial}{\partial \rho} + \mathbf{e}_\varphi \frac{1}{\rho}\frac{\partial}{\partial \varphi} + \mathbf{e}_z \frac{\partial}{\partial z}\ . \tag{1.387}$$

Divergence and curl can be read off with (1.384) directly from (1.377) and (1.379).

1.7.4 Spherical Coordinates

Spherical (polar) coordinates are especially suited to problems with radial symmetry.



r: Length of the position vector,
ϑ: $\sphericalangle(\boldsymbol{r},\, x_3\text{-axis})$ with $0 \le \vartheta \le \pi$ (*polar angle*),
φ: $\sphericalangle$ (projection of r on x_1, x_2-plane, x_1-axis) with $0 \le \varphi \le 2\pi$ (azimuth)

Fig. 1.82 To the definition of spherical coordinates

1.7.4.1 Transformation Formulae

$$
\begin{aligned}
x_1 &= r \sin\vartheta \cos\varphi \ , \\
x_2 &= r \sin\vartheta \sin\varphi \ , \\
x_3 &= r \cos\vartheta \ .
\end{aligned}
\tag{1.388}
$$

r is the length of the position vector (Fig. 1.82);
$\vartheta = \sphericalangle(\mathbf{r}, x_3 \text{ axis})$ with $0 \le \vartheta \le \pi$ (*'polar angle'*);
$\varphi = \sphericalangle$(projection of $\mathbf{r}$ onto the x_1, x_2 plane, x_1 axis) with $0 \le \varphi \le 2\pi$ (*'azimuthal angle'*)

1.7.4.2 Jacobian Determinant

$$
\begin{aligned}
\frac{\partial\,(x_1, x_2, x_3)}{\partial(r, \vartheta, \varphi)} &= \begin{vmatrix} \sin\vartheta\cos\varphi & r\cos\vartheta\cos\varphi & -r\sin\vartheta\sin\varphi \\ \sin\vartheta\sin\varphi & r\cos\vartheta\sin\varphi & r\sin\vartheta\cos\varphi \\ \cos\vartheta & -r\sin\vartheta & 0 \end{vmatrix} = \\
&= r^2\cos^2\vartheta\sin\vartheta\cos^2\varphi + r^2\sin^3\vartheta\sin^2\varphi + \\
&\quad + r^2\sin\vartheta\cos^2\vartheta\sin^2\varphi + r^2\sin^3\vartheta\cos^2\varphi = \\
&= r^2\sin\vartheta \ .
\end{aligned}
\tag{1.389}
$$

So the mapping is uniquely reversible except for $r = 0$ and/or $\vartheta = 0, \pi$.

1.7.4.3 Volume Element

$$dV = \frac{\partial(x_1, x_2, x_3)}{\partial(r, \vartheta, \varphi)} \, dr \, d\vartheta \, d\varphi = r^2 \sin\vartheta \, dr \, d\vartheta \, d\varphi \; . \tag{1.390}$$

One should try to visualize this result geometrically!

As an example of use we calculate the volume of a sphere with radius R. For this purpose we have to sum up all volume elements dV within the sphere in *Riemannian sense*.

$$V = \int\limits_{\text{sphere}} dV = \int\limits_0^R \int\limits_0^\pi \int\limits_0^{2\pi} r^2 \, dr \, \sin\vartheta \, d\vartheta \, d\varphi = \varphi \,\Big|_0^{2\pi} \cdot (-\cos\vartheta) \,\Big|_0^\pi \cdot \frac{r^3}{3}\Big|_0^R = \frac{4\pi}{3} R^3 \; .$$

Compare this with the much more cumbersome calculation in Sect. 1.2.5 to appreciate the usefulness of curvilinear (here spherical) coordinates!

1.7.4.4 Coordinate Lines

r line: radial ray starting from the origin of coordinates.
φ line: circle, parallel to the x_1, x_2 plane with its center on the x_3-axis.
ϑ line: marginated by a semicircle with the origin of coordinates as its center and by the x_3 axis.

For the unit vectors we need:

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial r} &= (\sin\vartheta \cos\varphi, \, \sin\vartheta \sin\varphi, \, \cos\vartheta) \Longrightarrow b_r = 1 \; , \\
\frac{\partial \mathbf{r}}{\partial \vartheta} &= r(\cos\vartheta \cos\varphi, \, \cos\vartheta \sin\varphi, \, -\sin\vartheta) \Longrightarrow b_\vartheta = r \; , \\
\frac{\partial \mathbf{r}}{\partial \varphi} &= r(-\sin\vartheta \sin\varphi, \, \sin\vartheta \cos\varphi, \, 0) \Longrightarrow b_\varphi = r \sin\vartheta \; .
\end{aligned} \tag{1.391}$$

This yields the **curvilinear unit vectors** (Fig. 1.83):

$$\begin{aligned}
\mathbf{e}_r &= (\sin\vartheta \cos\varphi, \, \sin\vartheta \sin\varphi, \, \cos\vartheta) \; , \\
\mathbf{e}_\vartheta &= (\cos\vartheta \cos\varphi, \, \cos\vartheta \sin\varphi, \, -\sin\vartheta) \; , \\
\mathbf{e}_\varphi &= (-\sin\varphi, \, \cos\varphi, \, 0) \; .
\end{aligned} \tag{1.392}$$

By construction these basis vectors are oriented tangentially to the coordinate lines. Obviously they are curvilinear-orthogonal. For the differential $d\mathbf{r}$ of the

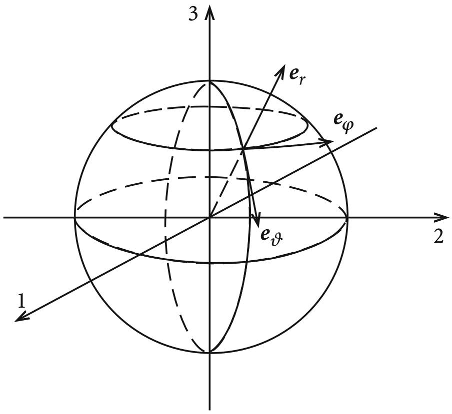


Fig. 1.83 Basis vectors for spherical coordinates

position vector we find according to (1.373) and (1.385):

$$d\mathbf{r} = dr\,\mathbf{e}_r + r\,d\vartheta\,\mathbf{e}_\vartheta + r\sin\vartheta\,d\varphi\,\mathbf{e}_\varphi\ . \tag{1.393}$$

1.7.4.5 Nabla Operator (Gradient)

$$\nabla \equiv \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \vartheta}, \frac{1}{r\sin\vartheta}\frac{\partial}{\partial \varphi}\right) \equiv \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\vartheta\frac{1}{r}\frac{\partial}{\partial \vartheta} + \mathbf{e}_\varphi\frac{1}{r\sin\vartheta}\frac{\partial}{\partial \varphi}. \tag{1.394}$$

Divergence and curl can be directly found with (1.391) from (1.377) and (1.379), respectively.

1.7.5 Exercises

Exercise 1.7.1

1. Verify for the variable transformation

$$x_i = x_i\,(y_1, y_2)\ ;\quad i = 1, 2$$

the following symmetries:

$$\frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} = \frac{\partial(x_2, x_1)}{\partial(y_2, y_1)} = -\frac{\partial(x_1, x_2)}{\partial(y_2, y_1)}\ .$$

2. Calculate the Jacobian determinants

$$\frac{\partial(x_1, x_2)}{\partial(x_1, x_2)} \quad \text{and} \quad \frac{\partial(x_1, y_2)}{\partial(y_1, y_2)} \; .$$

Exercise 1.7.2 Derive for

$$x = x(y, z) \; ,$$
$$y = y(x, z) \; ,$$
$$z = z(x, y)$$

the following relations:

$$\left(\frac{\partial x}{\partial y}\right)_z = \left[\left(\frac{\partial y}{\partial x}\right)_z\right]^{-1} \quad \text{and} \quad \left(\frac{\partial x}{\partial y}\right)_z \left(\frac{\partial y}{\partial z}\right)_x \left(\frac{\partial z}{\partial x}\right)_y = -1 \; .$$

Exercise 1.7.3 x_1, x_2, x_3 are Cartesian coordinates. Parabolic cylindrical coordinates (u, v, z) satisfy the transformation formulae:

$$x_1 = \frac{1}{2}\left(u^2 - v^2\right) \; ,$$
$$x_2 = u\,v \; ,$$
$$x_3 = z \; .$$

1. Calculate the Jacobian determinant

$$\frac{\partial(x_1, x_2, x_3)}{\partial(u, v, z)} \; .$$

2. How does the volume element $dV = dx_1 dx_2 dx_3$ transform itself?
3. Determine the unit vectors

$$\mathbf{e}_u, \; \mathbf{e}_v, \; \mathbf{e}_z \; !$$

   Illustrate the coordinate lines!
4. Derive the differential $d\mathbf{r}$ of the position vector and the nabla-operator ∇ in parabolic cylindrical coordinates.

Exercise 1.7.4 If a point P has the Cartesian coordinates $(3, 3)$. What are its plane polar coordinates?

Exercise 1.7.5 Represent the points $P_i = (x_i, y_i, z_i)$:

$$P_1 = (1, 0, 1)\ ; \quad P_2 = (0, 1, -1)\ ; \quad P_3 = (0, -3, 0)$$

by

1. spherical coordinates (r, ϑ, φ),
2. cylindrical coordinates (ρ, φ, z)!

Exercise 1.7.6 How does the equation of the circle with radius R look like in Cartesian coordinates and in plane polar coordinates, respectively.

Exercise 1.7.7 Formulate the vector field

$$\mathbf{a} = x_3\mathbf{e}_1 + 2x_1\mathbf{e}_2 + x_2\mathbf{e}_3$$

in cylindrical coordinates and in spherical coordinates!

Exercise 1.7.8

1. Calculate the area of circle by use of
   (a) Cartesian coordinates (x, y) (Fig. 1.84),
   (b) plane polar coordinates (ρ, φ) (Fig. 1.85).
2. Calculate the volume of a sphere with radius R!
3. Calculate the cylinder segment plotted in Fig. 1.86. Thereby R_1 is the inner radius, R_2 the outer radius and z_0 the height of the cylinder!

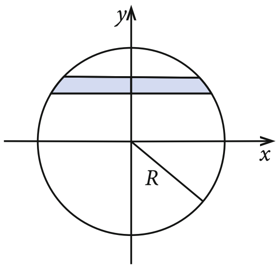


Fig. 1.84 Calculation of the area of a circle by use of Cartesian coordinates

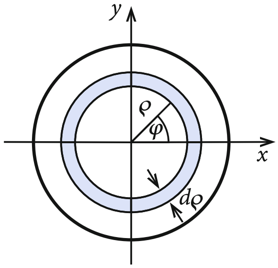


Fig. 1.85 Calculation of the area of a circle by use of planar polar coordinates

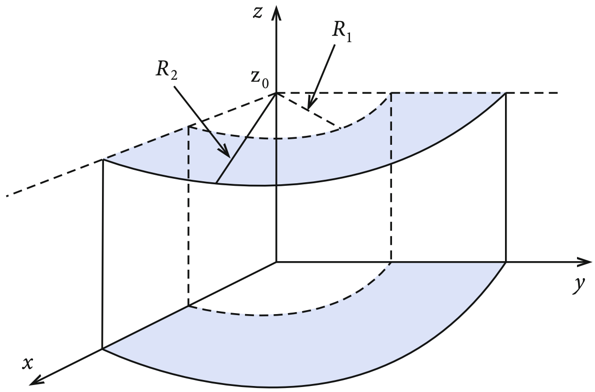


Fig. 1.86 To the calculation of the volume of a cylinder segment

1.8 Self-Examination Questions

To Section 1.1

1. Denominate the most important types of numbers!
2. What does one understand by a convergent, divergent sequence of numbers?
3. Which calculation rules hold for convergent sequences of numbers?
4. How are harmonic and geometric series defined?
5. What does one understand by the domain of definition D and the co-domain W of a function $f(x)$?
6. When is $f(x)$ continuous at x_0?
7. When does $f(x)$ have a unique inverse function f^{-1}?
8. Give the series expansions of the cosine and the sine function!
9. To which function the logarithm to the basis a is the inverse function?
10. When is $f(x)$ differentiable at x_0?
11. How is the quotient $f(x)/g(x)$ $(g(x) \neq 0)$ to be differentiated?
12. What does the chain rule tell us?
13. Under which conditions does l'Hospital's rule become useful?
14. When does $f(x)$ exhibit a maximum, a minimum, an inflection point at x_0?

To Section 1.2

1. What is the relationship between differentiation and integration?
2. What do we understand by antiderivative (primitive function) of $f(x)$?
3. Formulate the antiderivative of $\sin x$!
4. What is the illustrative meaning of the (definite) integral of the function $f(x)$?
5. How does the value of a definite integral change when we interchange the lower and the upper limits?
6. What does the mean value theorem of integral calculus imply?
7. Formulate the fundamental theorem of calculus!
8. Explain the technique of integration by parts!

9. When is the substitution of the variable useful?
10. What is to be taken care of in multiple integrals with non-constant limits of integration?

To Section 1.3

1. Which parameters are needed to define a vector?
2. Which vector does not have a well-defined direction?
3. Which *multiplicative* connections do exist for vectors?
4. Formulate Schwarz's inequality! Try to outline the proof!
5. What is a linear vector space? When is it called unitary?
6. What is the illustrative meaning of the magnitude of a vector product? How can we fix its direction?
7. What is the difference between a polar and an axial vector?
8. What is a pseudoscalar?
9. Formulate the sine (cosine) rule!
10. Which geometrical meaning can be given to the scalar triple product?
11. How can we treat the double vector product?
12. How is the basis of a linear vector space defined?
13. What do we interpret as directional cosine?
14. Give the component representation of the scalar product between two vectors!
15. Find the component representations of the vector product, the double vector product, and the scalar triple product!

To Section 1.4

1. What is a space curve? How is the path line (trajectory) of a mass point defined?
2. How does one *parametrize* a space curve?
3. What is a vector-valued function?
4. Parametrize the planar circular motion and helical line!
5. Define the continuity of space curves!
6. How is the derivative of a vector-valued function defined?
7. What is the arc length of a space curve?
8. What is called the *natural parametrization* of a space curve?
9. Which are the unit vectors of the *moving trihedron*?
10. Explain the terms curvature, radius of curvature, osculating plane, torsion, and torsion radius!
11. Formulate Frenet's formulae!
12. Which space curve has the smaller curvature: the circle or the helical line, if both have the same radius in the *xy* plane?
13. Which torsion radius has the circular motion?
14. Which direction does the normal-unit vector of the helical line have?
15. What do you understand by tangent acceleration and centripetal acceleration of a mass point?

To Section 1.5

1. What is a scalar field and what is a vector field? Give examples!
2. Interpret the term *contour line*! What is a *field line*?
3. Define the continuity of fields!
4. What do we understand by the partial derivative of a scalar field with respect to a space coordinate?
5. Give the *total derivative* of a scalar field with respect to a space coordinate!
6. What is a gradient field? Which direction does the gradient vector have?
7. Define the divergence and the curl of a vector field!
8. How is the Laplace operator defined?
9. When is a vector field source-free and when is it curl-free?
10. What can generally be said about the curl (rotation) of a gradient field, what about the divergence of a curl-field?

To Section 1.6

1. What is a matrix?
2. What do we understand in particular by a zero matrix, a diagonal matrix, a unit matrix, a symmetric matrix, a transposed matrix?
3. How is the rank of a matrix defined?
4. Explain the sum of two matrices, the multiplication of a matrix with a real number, and the product of two matrices!
5. Is the matrix multiplication commutative?
6. How is the rotation matrix defined?
7. Show that the columns and the rows of the rotation matrix are orthonormal!
8. How is the transposed rotation matrix related to the inverse rotation matrix?
9. How does the rotation matrix look like for the special case of a rotation by the angle φ within the plane?
10. Which conditions are to be fulfilled by a rotation matrix?
11. How is the determinant of a square matrix defined?
12. When does the Sarrus-rule help?
13. What is meant by the algebraic complement to a certain matrix element?
14. How can we expand a determinant with respect to a row?
15. Justify why it is allowed to add to a row (column) of a determinant another row (column) of the same determinant multiplied by a real number α without changing the value of the determinant.
16. When does the inverse to a given matrix exist? How can we calculate the elements of the inverse matrix?
17. Write down the vector product of two vectors, the rotation of a vector, and the scalar triple product of three non-complanar vectors in terms of determinants!
18. Under which condition is a linear inhomogeneous system of equations uniquely solvable? What does the Cramer's rule tell us?
19. When does a homogeneous system of equations possess non-trivial solutions?

To Section 1.7

1. Which general conditions must be fulfilled by a correct transformation of variables?
2. What do we understand by a Jacobian determinant?
3. What is a coordinate line?
4. When do we speak of a curvilinear-orthogonal coordinate?
5. How does one calculate the volume element $dV = dx_1\, dx_2\, dx_3$ after transformation of variables $(x_1, x_2, x_3) \to (y_1, y_2, y_3)$ in the new variables y_1, y_2, y_3?
6. How are the basis vectors of curvilinear systems of coordinates oriented relatively to the coordinate lines? How can we determine such basis vectors?
7. How does the nabla operator generally look like by the use of curvilinear coordinates?
8. What are the transformation formulae between Cartesian and cylindrical (spherical) coordinates?
9. Formulate the volume element dV in cylindrical (spherical) coordinates!
10. Characterize the coordinate lines for cylindrical and spherical coordinates!

Chapter 2
Mechanics of the Free Mass Point

The concept of the *mass point* is basic to the theory of mechanics. As we have already defined earlier (Sect. 1.4.1) we understand by the term **'mass point'** a physical body of mass m but with negligible extension in all directions. One has to note that the concept of mass point does not necessarily presume *small bodies*. The term mass point is rather used for problems for which it is sufficient to observe only the behavior of one salient point of the macroscopic body, e.g. the center of gravity, without considering the movement of all the other points of the body. So it is to treat even the whole earth as a mass point if one is interested only in the path of the earth around the sun. That is obviously no longer allowed when we want to understand the origin of earth tides. We denote a mass point as **free** when it can react to the applied forces **without** being bound by any restraining condition.

2.1 Kinematics

Kinematics compiles the mathematical and physical terms and theorems necessary to describe the movement of a mass point, at first without explicitly questioning for the cause of this motion. The necessary preparations for this purpose have been made in the introductory Chap. 1. Hence we can restrict ourselves here to a concise recapitulation.

W. Nolting, *Theoretical Physics 1*, DOI 10.1007/978-3-319-40108-9_2

2.1.1 Velocity and Acceleration

The motion of a mass point is characterized by:

$$\begin{aligned} \text{position vector :} \quad & \mathbf{r}(t) \ , \\ \text{velocity vector :} \quad & \mathbf{v}(t) = \dot{\mathbf{r}}(t) \ , \\ \text{acceleration vector :} \quad & \mathbf{a}(t) = \ddot{\mathbf{r}}(t) \ . \end{aligned}$$

Higher time derivatives do not interest us in mechanics; very often they even fail to exist because in many realistic cases the acceleration is not a continuous function of time.

The **typical task** for mechanics consists of the calculation of the path line (trajectory) $\mathbf{r}(t)$ on the basis of a given acceleration $\mathbf{a}(t) = \ddot{\mathbf{r}}(t)$. For this purpose one has obviously to integrate $\mathbf{a}(t)$ twice with respect to time. After each integration an integration constant appears which remains undetermined unless we have two **initial conditions** at our disposal to fix these constants. In this connection let us assume that we know the velocity and the position of the mass point (particle) at a certain time t_0, i.e.

$$\mathbf{a}(t) \ \text{ for all } t \ , \ \ \mathbf{v}\left(t_0\right) \ , \ \text{ and } \mathbf{r}\left(t_0\right)$$

are given. Then the velocity of the particle is determined by

$$\mathbf{v}(t) = \mathbf{v}\left(t_0\right) + \int\limits_{t_0}^{t} dt' \mathbf{a}\left(t'\right) \tag{2.1}$$

and the position vector by:

$$\mathbf{r}(t) = \mathbf{r}\left(t_0\right) + \mathbf{v}\left(t_0\right)\left(t - t_0\right) + \int\limits_{t_0}^{t} \left[\int\limits_{t_0}^{t'} dt'' \mathbf{a}\left(t''\right) \right] dt' \ . \tag{2.2}$$

Before we inspect these relations using simple examples we want to formulate the characteristic parameters of a mass point $\mathbf{r}(t)$, $\mathbf{v}(t)$, $\mathbf{a}(t)$ in different systems of coordinates.

(a) Cartesian Coordinates

The trajectory is described by the three time-dependent component functions $x_1(t), x_2(t), x_3(t)$:

$$\mathbf{r}(t) = \left(x_1(t), x_2(t), x_3(t)\right) = \sum_{j=1}^{3} x_j(t) \mathbf{e}_j \ . \tag{2.3}$$

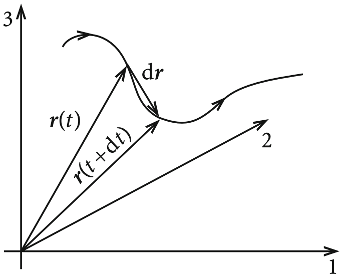


Fig. 2.1 Trajectory of a mass point in Cartesian coordinates

The basis vectors are time-independent and fixed in space (Fig. 2.1). The velocity

$$\mathbf{v}(t) = \sum_{j=1}^{3} \dot{x}_j(t)\mathbf{e}_j \tag{2.4}$$

is a vector which is oriented tangentially to the trajectory. It provides information about the distance covered by the mass point in the time interval dt. For a comparison with the experiment, however, one has to bear in mind that a measurement always happens in a *finite* time interval so that the mathematical limit in (2.4) is in a sense a fiction which can only be *'guessed'* by performing finer and finer experiments.

The temporal change of the velocity is called **acceleration**:

$$\mathbf{a}(t) = \sum_{j=1}^{3} \ddot{x}_j(t)\mathbf{e}_j \tag{2.5}$$

(b) Natural Coordinates

The **'moving trihedron'**, discussed in Sect. 1.4.4, represents a coordinate system directly attached to the space curve. We found in (1.244) and (1.245):

$$\mathbf{v}(t) = v\hat{\mathbf{t}}\ ; \quad v = \frac{ds}{dt}\ ; \tag{2.6}$$

$$\mathbf{a}(t) = \dot{v}\hat{\mathbf{t}} + \frac{v^2}{\rho}\hat{\mathbf{n}}\ . \tag{2.7}$$

$\hat{\mathbf{t}}$ is the tangent-unit vector, s the arc length, ρ the radius of curvatures, and $\hat{\mathbf{n}}$ the normal-unit vector. $\hat{\mathbf{t}}$ lies tangentially on the path line and $\hat{\mathbf{n}}$ describes the change of the $\hat{\mathbf{t}}$ direction with s (1.227). The vector of acceleration always lies in the so-called **osculating plane** spanned by the vectors $\hat{\mathbf{n}}$ and $\hat{\mathbf{t}}$ (Fig. 2.2) and is decomposed into

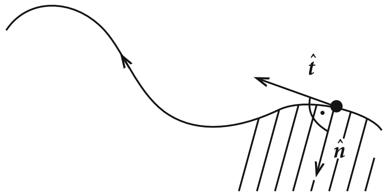


Fig. 2.2 Definition of the osculating plane spanned by the tangent- and the normal-unit vector

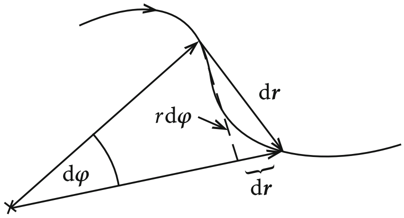


Fig. 2.3 Representation of the mass point velocity in plane polar coordinates

two parts which are due to, respectively, the change of magnitude and the change of direction of the velocity.

(c) Plane Polar Coordinates

These coordinates, which we considered several times in Sect. 1.7 in diverse examples, are of course applicable only for motions in a fixed plane. The basis vectors $\mathbf{e}_\varphi, \mathbf{e}_r$ are given in Eq. (1.371). For the position vector holds:

$$\mathbf{r}(t) = r(t)\mathbf{e}_r \; . \tag{2.8}$$

For the differential $d\mathbf{r}$ we find with (1.374):

$$d\mathbf{r} = dr\,\mathbf{e}_r + r\,d\varphi\,\mathbf{e}_\varphi \; .$$

The velocity follows immediately from above (Fig. 2.3):

$$\mathbf{v}(t) = \dot{r}\,\mathbf{e}_r + r\dot{\varphi}\,\mathbf{e}_\varphi \; . \tag{2.9}$$

Furthermore, one can also directly differentiate (2.8):

$$\mathbf{v}(t) = \dot{r}\,\mathbf{e}_r + r\,\dot{\mathbf{e}}_r \; . \tag{2.10}$$

The comparison of these two expressions for the velocity yields the time-derivative of the unit vector $\mathbf{e}_r$:

$$\dot{\mathbf{e}}_r = \dot{\varphi}\,\mathbf{e}_\varphi \; . \tag{2.11}$$

According to (1.216) the time-derivative of the unit vector $\mathbf{e}_\varphi$ must be orthogonal to $\mathbf{e}_\varphi$ and therefore parallel or antiparallel to $\mathbf{e}_r$:

$$\dot{\mathbf{e}}_\varphi = \alpha\, \mathbf{e}_r \ .$$

Because of $\mathbf{e}_r \cdot \mathbf{e}_\varphi = 0$ it is $\dot{\mathbf{e}}_r \cdot \mathbf{e}_\varphi = -\mathbf{e}_r \cdot \dot{\mathbf{e}}_\varphi$ and therewith

$$\alpha = \mathbf{e}_r \cdot \dot{\mathbf{e}}_\varphi = -\dot{\mathbf{e}}_r \cdot \mathbf{e}_\varphi = -\dot{\varphi}\, \mathbf{e}_\varphi \cdot \mathbf{e}_\varphi = -\dot{\varphi} \ .$$

Hence we have:

$$\dot{\mathbf{e}}_\varphi = -\dot{\varphi}\, \mathbf{e}_r \ . \tag{2.12}$$

The differentiation with respect to time in (2.9) thus results in the following expression for the **acceleration**:

$$\begin{aligned} \mathbf{a}(t) &= a_r \mathbf{e}_r + a_\varphi \mathbf{e}_\varphi \ , \\ a_r &= \ddot{r} - r\,\dot{\varphi}^2 \ , \\ a_\varphi &= r\ddot{\varphi} + 2\dot{r}\dot{\varphi} \ . \end{aligned} \tag{2.13}$$

(d) Cylindrical Coordinates

These were broadly discussed in Sect. 1.7.3. For the **position vector** we have here:

$$\mathbf{r}(t) = \rho\, \mathbf{e}_\rho + z\, \mathbf{e}_z \ . \tag{2.14}$$

The differential appeared already in (1.386):

$$d\mathbf{r} = d\rho\, \mathbf{e}_\rho + \rho\, d\varphi\, \mathbf{e}_\varphi + dz\, \mathbf{e}_z \ . \tag{2.15}$$

This yields the **velocity**:

$$\mathbf{v}(t) = \dot{\rho}\, \mathbf{e}_\rho + \rho\, \dot{\varphi}\, \mathbf{e}_\varphi + \dot{z}\, \mathbf{e}_z \ . \tag{2.16}$$

$\mathbf{e}_z$ is constant with respect to direction as well as magnitude, i.e. $\dot{\mathbf{e}}_z = \mathbf{0}$. Both the other unit vectors, however, can change as function of time. The differentiation of (2.14) yields:

$$\dot{\mathbf{r}}(t) = \dot{\rho}\, \mathbf{e}_\rho + \rho\, \dot{\mathbf{e}}_\rho + \dot{z}\, \mathbf{e}_z \ .$$

The comparison with (2.16) leads to

$$\dot{\mathbf{e}}_\rho = \dot{\varphi}\, \mathbf{e}_\varphi \ . \tag{2.17}$$

$\dot{\mathbf{e}}_\varphi$ is perpendicular to $\mathbf{e}_\varphi$:

$$\dot{\mathbf{e}}_\varphi = \alpha \mathbf{e}_\rho + \beta \mathbf{e}_z \ .$$

Because of

$$\begin{aligned}
\mathbf{e}_\varphi \cdot \mathbf{e}_\rho = 0 &\Longrightarrow \dot{\mathbf{e}}_\varphi \cdot \mathbf{e}_\rho = -\mathbf{e}_\varphi \cdot \dot{\mathbf{e}}_\rho \ , \\
\mathbf{e}_\varphi \cdot \mathbf{e}_z = 0 &\Longrightarrow \dot{\mathbf{e}}_\varphi \cdot \mathbf{e}_z = -\dot{\mathbf{e}}_z \cdot \mathbf{e}_\varphi = 0
\end{aligned}$$

the first consequence is $\beta = 0$ and furthermore:

$$\alpha = \mathbf{e}_\rho \cdot \dot{\mathbf{e}}_\varphi = -\dot{\mathbf{e}}_\rho \cdot \mathbf{e}_\varphi = -\dot{\varphi} \ .$$

This shows the change of the basis vector $\mathbf{e}_\varphi$ as function of time:

$$\dot{\mathbf{e}}_\varphi = -\dot{\varphi}\, \mathbf{e}_\rho \ . \tag{2.18}$$

Hereafter it does not pose any difficulty to fix the **acceleration** in cylindrical coordinates by time differentiation of (2.16):

$$\begin{aligned}
\mathbf{a}(t) &= a_\rho\, \mathbf{e}_\rho + a_\varphi\, \mathbf{e}_\varphi + a_z\, \mathbf{e}_z \ , \\
a_\rho &= \ddot{\rho} - \rho\, \dot{\varphi}^2 \ , \\
a_\varphi &= \rho \ddot{\varphi} + 2\dot{\rho}\, \dot{\varphi} \ , \\
a_z &= \ddot{z} \ .
\end{aligned} \tag{2.19}$$

(e) Spherical Coordinates

These coordinates have been introduced in Sect. 1.7.4. The **position vector** is written as:

$$\mathbf{r}(t) = r\, \mathbf{e}_r \ . \tag{2.20}$$

With the differential derived in (1.393)

$$d\mathbf{r} = dr\, \mathbf{e}_r + r d\vartheta\, \mathbf{e}_\vartheta + r \sin\vartheta\, d\varphi\, \mathbf{e}_\varphi$$

it follows immediately for the **velocity**:

$$\mathbf{v}(t) = \dot{r}\, \mathbf{e}_r + r\, \dot{\vartheta}\, \mathbf{e}_\vartheta + r \sin\vartheta\, \dot{\varphi}\, \mathbf{e}_\varphi \ . \tag{2.21}$$

The calculation of the acceleration turns out to be rather lengthy. First we differentiate (2.20) with respect to time,

$$\dot{\mathbf{r}}(t) = \dot{r}\,\mathbf{e}_r + r\,\dot{\mathbf{e}}_r\ ,$$

and compare this with (2.21):

$$\dot{\mathbf{e}}_r = \dot{\vartheta}\,\mathbf{e}_\vartheta + \sin\vartheta\,\dot{\varphi}\,\mathbf{e}_\varphi\ . \tag{2.22}$$

We still need the time-derivatives of the two other basis vectors. Since both are unit vectors $\dot{\mathbf{e}}_\vartheta$ and $\dot{\mathbf{e}}_\varphi$ are orthogonal to $\mathbf{e}_\vartheta$ and $\mathbf{e}_\varphi$, respectively:

$$\begin{aligned}\dot{\mathbf{e}}_\vartheta &= \alpha\,\mathbf{e}_\varphi + \beta\mathbf{e}_r\ ,\\ \dot{\mathbf{e}}_\varphi &= \gamma\,\mathbf{e}_\vartheta + \delta\mathbf{e}_r\ .\end{aligned}$$

Furthermore it holds:

$$\begin{aligned}0 = \mathbf{e}_\vartheta\cdot\mathbf{e}_r = \mathbf{e}_\vartheta\cdot\mathbf{e}_\varphi = \mathbf{e}_\varphi\cdot\mathbf{e}_r\\ \Longrightarrow \dot{\mathbf{e}}_\vartheta\cdot\mathbf{e}_r = -\mathbf{e}_\vartheta\cdot\dot{\mathbf{e}}_r\ .\end{aligned}$$

It follows:

$$\begin{aligned}\beta &= \dot{\mathbf{e}}_\vartheta\cdot\mathbf{e}_r = -\mathbf{e}_\vartheta\cdot\dot{\mathbf{e}}_r = -\dot{\vartheta}\ ,\\ \alpha &= \dot{\mathbf{e}}_\vartheta\cdot\mathbf{e}_\varphi = -\mathbf{e}_\vartheta\cdot\dot{\mathbf{e}}_\varphi = -\gamma\ ,\\ \delta &= \dot{\mathbf{e}}_\varphi\cdot\mathbf{e}_r = -\mathbf{e}_\varphi\cdot\dot{\mathbf{e}}_r = -\sin\vartheta\,\dot{\varphi}\ .\end{aligned}$$

We have now the intermediate result:

$$\begin{aligned}\dot{\mathbf{e}}_\vartheta &= \alpha\mathbf{e}_\varphi - \dot{\vartheta}\,\mathbf{e}_r\ ,\\ \dot{\mathbf{e}}_\varphi &= -\alpha\,\mathbf{e}_\vartheta - \sin\vartheta\,\dot{\varphi}\,\mathbf{e}_r\ .\end{aligned}$$

Obviously we still need a further conditional equation. $\mathbf{e}_\varphi$ has in Cartesian coordinates a vanishing x_3 component (1.392). That holds of course also for $\dot{\mathbf{e}}_\varphi$. Thus we can conclude with (1.392):

$$0 = -\alpha(-\sin\vartheta) - \sin\vartheta\,\dot{\varphi}\cos\vartheta \Longrightarrow \alpha = \dot{\varphi}\cos\vartheta\ .$$

That leads to:

$$\dot{\mathbf{e}}_\vartheta = \dot{\varphi}\cos\vartheta\,\mathbf{e}_\varphi - \dot{\vartheta}\,\mathbf{e}_r\ , \tag{2.23}$$

$$\dot{\mathbf{e}}_\varphi = -\dot{\varphi}\cos\vartheta\,\mathbf{e}_\vartheta - \sin\vartheta\,\dot{\varphi}\,\mathbf{e}_r\ . \tag{2.24}$$

By a further differentiation in (2.21) we eventually arrive at the **acceleration in spherical coordinates**:

$$\begin{aligned}\mathbf{a}(t) &= a_r \mathbf{e}_r + a_\vartheta \mathbf{e}_\vartheta + a_\varphi \mathbf{e}_\varphi \ , \\ a_r &= \ddot{r} - r\dot{\vartheta}^2 - r\sin^2\vartheta\,\dot{\varphi}^2 \ , \\ a_\vartheta &= r\ddot{\vartheta} + 2\dot{r}\dot{\vartheta} - r\sin\vartheta\cos\vartheta\,\dot{\varphi}^2 \ , \\ a_\varphi &= r\sin\vartheta\,\ddot{\varphi} + 2\sin\vartheta\,\dot{r}\,\dot{\varphi} + 2r\cos\vartheta\,\dot{\vartheta}\,\dot{\varphi} \ .\end{aligned} \tag{2.25}$$

2.1.2 Simple Examples

(a) Mass Point on a Straight Line

We can describe the motion without referring to any special system of coordinates. If $\mathbf{c}$ is a vector in the direction of the motion and $\mathbf{b}$ a vector perpendicular to it then we can write for the position vector of the mass point (Fig. 2.4):

$$\mathbf{r}(t) = \mathbf{b} + \alpha(t)\mathbf{c} \ . \tag{2.26}$$

From this, the respective time derivatives give us the velocity and acceleration:

$$\mathbf{v}(t) = \dot{\alpha}(t)\,\mathbf{c} \ ; \quad \mathbf{a}(t) = \ddot{\alpha}(t)\,\mathbf{c} \ . \tag{2.27}$$

(b) Uniform Straight-Line Motion

Therewith it is meant the most simple form of motion, namely the one without any acceleration:

$$\mathbf{a}(t) = 0 \ ; \quad \mathbf{v}(t) = \mathbf{v}_0 \quad \text{for all } \ t \ .$$

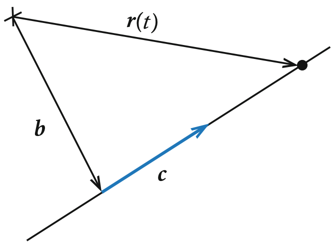


Fig. 2.4 Rectilinear motion of a mass point

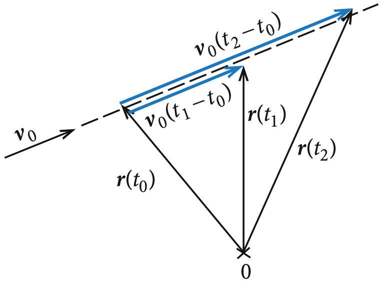


Fig. 2.5 Acceleration-free motion of a mass point

The third summand in (2.2) then disappears:

$$\mathbf{r}(t) = \mathbf{r}(t_0) + \mathbf{v}_0 (t - t_0) \ . \tag{2.28}$$

This formally agrees with (2.26). The motion is thus carried out rectilinearly in the direction of the constant velocity vector $\mathbf{v}_0$. It is called *'uniform'* since the same distances are covered in equal time intervals (Fig. 2.5).

(c) Uniformly Accelerated Motion

Now we assume a constant acceleration

$$\mathbf{a}(t) = \mathbf{a}_0 \tag{2.29}$$

That means in (2.2):

$$\begin{aligned} \int\limits_{t_0}^{t} \left[\int\limits_{t_0}^{t'} dt'' \mathbf{a}\left(t''\right) \right] dt' &= \int\limits_{t_0}^{t} \left[\mathbf{a}_0 \left(t' - t_0\right)\right] dt' = \\ &= \mathbf{a}_0 \left(\frac{t^2}{2} - \frac{t_0^2}{2} \right) - \mathbf{a}_0 t_0 (t - t_0) = \\ &= \frac{1}{2} \mathbf{a}_0 (t - t_0)^2 \ . \end{aligned}$$

We therewith get as path line:

$$\mathbf{r}(t) = \mathbf{r}(t_0) + \mathbf{v}(t_0)(t - t_0) + \frac{1}{2} \mathbf{a}_0 (t - t_0)^2 \ . \tag{2.30}$$

The velocity of the mass point increases linearly with time (Fig. 2.6):

$$\mathbf{v}(t) = \mathbf{v}(t_0) + \mathbf{a}_0 (t - t_0) \ . \tag{2.31}$$

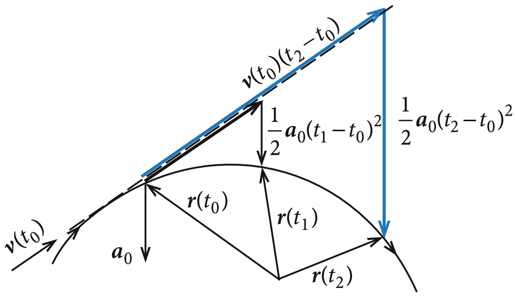


Fig. 2.6 Typical course of a uniformly accelerated motion

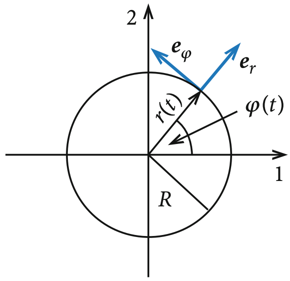


Fig. 2.7 Circular motion of a mass point

The trajectory results from a superposition of a uniform straight-line motion in the direction of the initial velocity $\mathbf{v}(t_0)$ and a straight-line accelerated motion in the direction of $\mathbf{a}_0$.

(d) Circular Motion

This we have already extensively investigated in Sect. 1.4.4 in connection with the introduction of natural coordinates. Other suitable coordinates are the plane polar coordinates. Since the radius of the circle is constant it follows from (2.8), (2.9) and (2.13):

$$\mathbf{r}(t) = R\,\mathbf{e}_r\ , \quad \mathbf{v}(t) = R\,\dot{\varphi}\,\mathbf{e}_\varphi\ , \tag{2.32}$$

$$\mathbf{a}(t) = a_r\,\mathbf{e}_r + a_\varphi\,\mathbf{e}_\varphi\ , \quad a_r = -R\,\dot{\varphi}^2\ , \quad a_\varphi = R\,\ddot{\varphi}\ . \tag{2.33}$$

$\dot{\varphi}(t)$ denotes the change in angle per time unit (Fig. 2.7). One therefore defines:

$$\omega = \dot{\varphi} \qquad\qquad \textbf{angular velocity}\ . \tag{2.34}$$

Therewith it also holds:

$$v = R\,\omega \qquad \text{(velocity magnitude)}\,, \tag{2.35}$$

$$a_r = -\,R\,\omega^2 \qquad \text{(centripetal acceleration)}\,, \tag{2.36}$$

$$a_\varphi = R\dot{\omega} \qquad \text{(tangential acceleration)} \tag{2.37}$$

(compare with (1.246) and (1.247)). A special case is:

$$\omega = \text{const} \Longleftrightarrow \textbf{uniform circular motion}\,. \tag{2.38}$$

Sometimes it appears reasonable to assign to the angular velocity an **axial** vector in the direction of the axis of rotation. In the present case that is the 3-axis:

$$\boldsymbol{\omega} = \omega\,\mathbf{e}_3\,. \tag{2.39}$$

Hence the magnitude of this vector is ω. So we can write:

$$\mathbf{v}(t) = \boldsymbol{\omega} \times \mathbf{r}(t) = \omega\,R\,\mathbf{e}_\varphi\,. \tag{2.40}$$

2.1.3 Exercises

Exercise 2.1.1 A mass point moves on a circular path with constant velocity $v = 50$ cm/s. Thereby the velocity vector $\mathbf{v}$ changes its direction in 2 s by 60°.

1. Calculate the velocity change $|\Delta\mathbf{v}|$ in the time interval of 2 s.
2. What is the magnitude of centripetal acceleration of the uniform circular motion?

Exercise 2.1.2

1. A body rotates around an axis through the origin of coordinates with the angular velocity

$$\boldsymbol{\omega} = (-1,\ 2,\ 1)\,.$$

   What is the velocity of the point P of the body with the position vector

$$\mathbf{r}_P = (2,\ 0,\ 1)\ ?$$

2. How would its velocity change when the rotation axis is shifted parallel in such a way that the origin (on the axis) now comes to $\mathbf{a} = (1,\ 1,\ 1)$?

Exercise 2.1.3 Consider the equation of motion

$$\ddot{\mathbf{r}} = -\mathbf{g}$$

of a particle in the earth's field near the earth surface. The x_3-axis of a Cartesian system of coordinates is upwardly oriented, i.e. $\mathbf{g} = (0,\ 0,\ g)$.

1. Find the solution of the equation of motion for the case that the particle starts at the time $t = 0$ from the origin with the initial velocity

$$\mathbf{v}_0 = (v_{0_1}\ ,\ v_{0_2}\ ,\ v_{0_3})$$

2. Show that the motion is restricted to a fixed plane. What is the direction of the surface normal of the orbital plane?
3. Now choose the direction of the initial velocity as $1'$-axis of a new system of coordinates with the same origin given by the unit vector $\mathbf{e}'_1$. Find another unit vector $\mathbf{e}'_2$ orthogonal to $\mathbf{e}'_1$ which together with $\mathbf{e}'_1$ spans the orbital plane and defines the $2'$-axis.
4. Choose $\mathbf{e}'_3$ so that $\mathbf{e}'_1$, $\mathbf{e}'_2$, $\mathbf{e}'_3$ represent an orthonormal right-handed system.

Exercise 2.1.4 For the movement of a crashing earth satellite, which experiences the gravitational as well as a frictional force, the following space dependent acceleration is found:

$$\mathbf{a} = -\frac{\gamma}{r^2}\mathbf{e}_r - \alpha(r)\dot{\mathbf{r}}\ ; \quad \alpha > 0$$

r : distance from the earth's center.

1. Which conditional equations are fulfilled by the components a_r, a_ϑ, a_φ of the acceleration in spherical coordinates?
2. How should $\alpha(r)$ and β be chosen so that

$$\begin{aligned} r(t) &= r_0(1-\beta t)^{2/3} \\ \vartheta(t) &= -\vartheta_0 \ln(1-\beta t)^{2/3}\ ; \quad \vartheta_0 > 0 \\ \varphi(t) &\equiv \text{const} \end{aligned}$$

   solve the conditional equations. Calculate the trajectory $r = r(\vartheta)$.
3. Calculate the magnitude $|\mathbf{v}|$ of the velocity!

2.2 Fundamental Laws of Dynamics

Up to now we have restricted ourselves to **describe** the motion of a mass point without investigating the primary **cause** of the motion. From now on, the latter will be the focus of our considerations. The goal is to develop procedures by which one can derive the explicit movement of the mass point from a known driving cause.

We start with a few very general remarks concerning the challenges and possibilities of every physical theory; here, however, with the special perspective on *Classical Mechanics*. Like any physical theory mechanics also is based on

definitions and theorems

The definitions are reasonably separated into **basis definitions** and **following definitions**:

$$\begin{tikzcd} & \textit{definitions} \arrow[dl] \arrow[dr] & \\ \textit{basis definitions} & & \textit{following definitions} \end{tikzcd}$$

By basis definitions we mean concepts like position, time, mass, ..., which are no further commented on in the course of the theory. Following definitions are entities *derived* from the basis definitions such as velocity, acceleration, momentum, ... Analogously we have also to decompose the *theorems*:

$$\begin{tikzcd} & \textit{theorems} \arrow[dl] \arrow[dr] & \\ \textit{axioms, principles} & & \textit{conclusions} \end{tikzcd}$$

Axioms are a matter of basic empirical facts which are mathematically not provable and will not be further justified within the theory. In the framework of Classical Mechanics these are **'Newton's axioms of motion'**. By **conclusions** we understand the actual results of the physical theory. By use of the concept of the *'mathematical proof'* they emerge out of the basis definitions and axioms which together are called the **postulates** of the theory.

The *'ultimate judge'* of any physical theory is the experiment. The value of a theory is measured by the degree of agreement of its conclusions with the manifestations of nature. It is known today that Classical Mechanics is not able to correctly describe all movements and manifestations of the inanimate nature. In particular in atomic and subatomic regions modifications have become necessary. But one can regard Classical Mechanics as a self-consistent limiting case of a higher all-embracing theory, if it is finally found.

2.2.1 Newton's Laws of Motion

When formulating the fundamental laws of dynamics we find ourselves in a harsh dilemma. We have to introduce **two new terms**, namely

force and mass

The physical term **'force'** can be defined only indirectly via its effect. If we want to **change the state of motion or the shape of a body** by exertion of our muscles, e.g., it needs an effort, which must be the bigger the greater the temporal change in velocity (acceleration) should be or the stronger the deformation has to result. This effort is called **'force'**. As an immediate sensation it can not be defined more precisely. The direction along which we let our muscles work fixes the direction of the velocity change and the direction of the deformation, respectively. That has the important implication:

force is a vectorial physical quantity.

As a matter of fact, we observe all over in our environment changes in the state of motion of bodies, and that, too, without being influenced by our muscles. We interpret the cause also as **force** which in the same manner as our muscles act on the bodies. The investigation of the nature of such forces constitutes a central task of physics.

We are left with the simple statement

force = cause of movement

In this form the statement is certainly not generally valid and can quickly be disproved by several counterexamples. A disk gliding on a frozen surface moves with almost constant velocity, and without any application of force. A body which in principle is at rest appears to move if one observes it from a moving train, i.e. the state of motion does depend on the system of coordinates chosen. In order to investigate this issue we start with the definition

Force-Free Body
A body which does not experience any external influence.
This definition contains a rather risky, albeit plausible extrapolation of our daily experience. A completely isolated body does not exist!

Axiom 2.1 (Lex Prima, Galilei's Law of Inertia) *There are systems of coordinates in which a force-free body (mass point) persists in the state of rest or in the state of uniform straight-line motion. Such systems shall be called **'inertial systems'**.*

Newton's original formulation is a bit less restrictive:
Each body persists in the state of rest or in the state of uniform straight-line motion if it does not experience any forces to change its state.
Next we ask ourselves how the bodies behave in such special inertial systems under the influence of forces. Here again we have to make use of our daily experience. We observe that to produce the same acceleration of different bodies with identical volumes different exertions are necessary. It is easier to move a block of wood than a block of iron. The effect of the force is obviously also dependent on a material property of the body which is to be moved. This property opposes, as we observe, the change of motion a certain **resistance of inertia** which does **not** depend on the actual strength of the influencing force.

Postulate
Every body (every particle) possesses a **scalar property** given by a positive real number which we call

inertial mass m_{in}

Definition 2.2.1 The product of inertial mass and velocity is denoted as

$$\textbf{(linear) momentum} : \qquad \mathbf{p} = m_{in}\,\mathbf{v}\ . \tag{2.41}$$

Therewith we now formulate:

Axiom 2.2 (Lex Secunda, Law of Motion) *The rate of change in the momentum is proportional to the impact of the driving force and takes place in the direction of the force:*

$$\mathbf{F} = \dot{\mathbf{p}} = \frac{d}{dt}\,(m_{in}\,\mathbf{v})\ . \tag{2.42}$$

It is important to stress that this axiom is exclusively formulated for the inertial systems defined by Axiom 2.1. Let us add some **auxiliary remarks**:

1. If the mass **does not** depend on time, then, but really only then, we have:

$$\mathbf{F} = m_{in}\ddot{\mathbf{r}} = m_{in}\,\mathbf{a}\ . \tag{2.43}$$

   This relation can be regarded as **basic dynamical equation** of Classical Mechanics. Like the most physical laws it also has the form of a differential equation from which one eventually arrives at the path of the particle $\mathbf{r}(t)$ by continued integration provided the force is known. The dynamical equation will therefore be at the center of the following considerations.
2. In the definition (2.41) of the momentum the mass m_{in} is considered as constant. In **relativistic mechanics** the latter remains true only if we understand by mass the **rest mass** m_0. In the definition of the momentum we then have to interpret m_{in} as

$$m_{in} = \frac{m_0}{\sqrt{1 - v^2/c^2}} \tag{2.44}$$

   v is here the particle velocity and c the velocity of light in vacuum. The latter represents an absolute upper bound for v. However, in almost all the cases which we are interested in here it is $v \ll c$ and therefore $m_{in} \approx m_0$.
3. Temporal changes of mass do appear of course not only in the relativistic mechanics. Examples are

   the rocket, the car with internal-combustion engine, ...

4. In Newton's original formulation only a proportionality between $\mathbf{F}$ and $\dot{\mathbf{p}}$ is postulated. But since up to now we are not able to concretely define force or mass, nothing can hinder us to choose the equality sign.
5. The law of motion (2.43) anyway allows us already to define the ratio of force and mass:

$$\frac{\mathbf{F}}{m_{in}} = \mathbf{a} \ .$$

The acceleration on the right-hand side is measurable as well as well-defined. One should notice, however, that Eq. (2.43) actually does not define either force or mass.

As yet we have discussed only the action of a force on a mass point (body), but not the retroaction of the mass point on the source of force. That is the subject of

Axiom 2.3 (Lex Tertia, Law of Reaction, 'actio=reactio')

$$\mathbf{F}_{12}: \quad \textit{Force of body } 2 \textit{ on body } 1 \ ,$$
$$\mathbf{F}_{21}: \quad \textit{Force of body } 1 \textit{ on body } 2 \ .$$

Action and retroaction are equal:

$$\mathbf{F}_{12} = -\mathbf{F}_{21} \ . \tag{2.45}$$

Example Support pressure of a sphere shown in Fig. 2.8.

This third axiom now provides the way to indeed define the inertial mass. If we combine (2.43) with (2.45) so it holds for two mass points which execute forces on each other if all other influences are *switched off*:

$$m_{in,1}\,\mathbf{a}_1 = -m_{in,2}\,\mathbf{a}_2 \ . \tag{2.46}$$

In this equation the forces are completely eliminated so that the mass ratio is fixed by the measurement of accelerations. Let us consider a practical **realization**:
On two mass points we let two forces equal in magnitude and opposite in direction act. This can be realized by cutting a compressed spring connecting the two mass points (Fig. 2.9). One observes that the ratio of, respectively, the velocities v_1, v_2

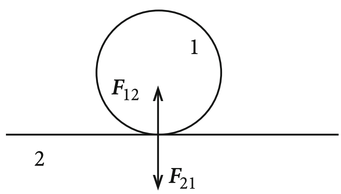


Fig. 2.8 Support pressure of a sphere as example for 'actio=reactio'

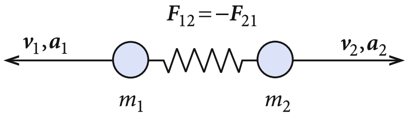


Fig. 2.9 Thought experiment for fixing the inertial mass

and accelerations a_1, a_2 is independent of the acting force $|\mathbf{F}_{12}|$. This shows that the *mass* is indeed a material property and is independent of the strength of the acting forces. We can now introduce a **mass standard** having therewith uniquely defined the measurement of the mass. We can add the mass, more precisely the *inertial mass*, to the basis definitions, whilst the definition of the force then represents according to (2.42) a following definition.

SI: International System of Units

$$[m_{in}] = 1\,\text{kg}\ ,$$
$$[\mathbf{F}] = 1\,\text{N}(= 1\,\text{Newton}) = 1\,\text{kg}\,\text{m}\,\text{s}^{-2}\ .$$

The last axiom that is still to be considered is almost a matter of course after we have identified beforehand the force as a vectorial entity:

Axiom 2.4 (Corollarium, Superposition Principle) *If several forces* $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ *act on a mass point then these add up to a* **resultant** *like normal vectors :*

$$\mathbf{F} = \sum_{i=1}^{n} \mathbf{F}_i\ . \tag{2.47}$$

2.2.2 *Forces*

At the beginning of this chapter we recognized as the elementary task of each physical theory, in particular the Classical Mechanics, to derive *conclusions* from preformulated postulates (basis definitions, axioms). The axioms and the fundamental definition of mass are now available. The law of motion (2.42) and (2.43), respectively, have become the **principal dynamical equation of Classical Mechanics**. This equation is to be solved. As a rule, mathematically that means, for a given force $\mathbf{F}$, one has to solve a **differential equation of second order**.

More precise than the term force in this connection is, strictly speaking, the concept of the

'Force Field'

$$\mathbf{F} = \mathbf{F}\,(\mathbf{r},\ \dot{\mathbf{r}},\ t)\ .$$

To each space point a force that acts on the mass point is assigned, which in general can even be time dependent and, additionally, may depend on the particle velocity. Dependence on acceleration $\ddot{\mathbf{r}}$, however, will not appear.

All the matter is built by elementary constituents (molecules, atoms, nucleons, electrons, …). Therefore, in the last analysis each force can be traced back to the interactions between these elementary constituents. To do this in all detail, however, is beyond the framework of Classical Mechanics which only asks for the **consequences** and not for the **elementary causes** of the forces. Normally one restricts oneself to mathematically as simple as possible and empirically reasoned

model representations

Some frequently used examples are listed in the following:

(a) Weight, Gravitational Force

Each body is **'heavy'**. 1 m^3 of iron is *'heavier'* than 1 cm^3 of iron. By this everyday experience a new material quantity is documented which is denoted as

gravitational (heavy) mass m_h .

It manifests itself in the **'gravitational force'**

$$\mathbf{F}_g = m_h \, \mathbf{g} \,, \tag{2.48}$$

which acts on a stationary (motionless) mass point in the **gravitational field** of the earth. $\mathbf{g}$ close to the earth's surface is a nearly constant vector always pointing **downwards** in direction to the earth's center. If we define this direction as the negative x_3 direction of a Cartesian coordinate system then we write

$$\mathbf{g} = -(0,\, 0,\, g)\,; \quad g = 9.81\,\mathrm{m\,s^{-2}} \quad \textbf{'gravity acceleration'.} \tag{2.49}$$

The gravitational mass m_h, which in homogeneous materials turns out to be proportional to the volume, can be determined via the gravitational force (2.48), e.g., by use of a spring balance (Fig. 2.10). The deflection Δx of the spring caused by m_h can be normalized which fixes the unit of the heavy mass. As mass standard a platinum-iridium brick is stored in a special laboratory near Paris. Thereafter 1 kilogram (1 kg) corresponds exactly to the mass of 1 dm^3 of water at a temperature of 4 °C.

As **weight** of a body one denotes the force $\mathbf{F}_g$ from (2.48) which acts on the body on the earth's surface. Here the mass of 1 kg experiences the gravitational force of 9.81 N.

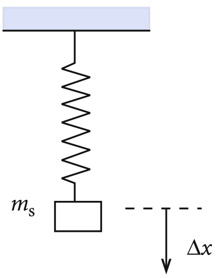


Fig. 2.10 Thought experiment for the description of the gravitational (heavy) mass

The *inertial mass* m_{in} has been introduced as *resistance of inertia* with which a body opposes a change of its state of motion. Because of the different experimental situations the identity

$$m_h = m_{in} = m \tag{2.50}$$

is therefore **not at all** a matter of course. However, it can be experimentally shown that for **all** bodies the ratio m_h/m_{in} is constant so that in any case at least $m_h \propto m_{in}$ holds. To demonstrate this one measures the acceleration of a body with the gravitational mass m_h during its free fall in the earth's gravitational field. One finds that

$$a = \frac{m_h}{m_{in}} g \tag{2.51}$$

is independent of the respective substance so that it necessarily follows that

$$m_h \propto m_{in} \tag{2.52}$$

Einstein's Equivalence Principle
The measuring methods for m_h and m_{in} are in principle equivalent. Therefore Eq. (2.50) is valid.
This principle represents the basis of the **'general theory of relativity'**. For the following we thus drop the indexes *in* and *h*.

(b) Central Forces

Forces of the type

$$\mathbf{F}(\mathbf{r}) = f(\mathbf{r}, \dot{\mathbf{r}}, t)\ \mathbf{r} = (f(\mathbf{r}, \dot{\mathbf{r}}, t)\ r)\ \mathbf{e}_r \tag{2.53}$$

appear very often in nature. The force acts radially from a center at $\mathbf{r} = \mathbf{0}$ outwardly ($f > 0$) or inwardly towards the center ($f < 0$).

Examples

(1) **Isotropic harmonic oscillator**

$$f(r) = \text{const} < 0\,. \tag{2.54}$$

(2) **Gravitational force**,
executed by a mass M, located at the origin, on a particle with mass m at the point $\mathbf{r}$:

$$f(r) = -\gamma\frac{mM}{r^3}\,. \tag{2.55}$$

(3) **Coulomb force**,
executed by a charge q_1, located at the origin, on another charge q_2 at $\mathbf{r}$:

$$f(r) = \frac{q_1 q_2}{4\pi\,\varepsilon_0 r^3}\,. \tag{2.56}$$

In the end, practically all classical interactions can be traced back to either (2.55) or (2.56). The constants γ, ε_0, q_i will be explained later.

(c) Lorentz Force

It is the force experienced by a particle with the charge q in an electromagnetic field:

$$\mathbf{F} = q\,[\mathbf{E}(\mathbf{r},\,t) + (\mathbf{v}\times\mathbf{B}(\mathbf{r},\,t))] \tag{2.57}$$

($\mathbf{B}$: magnetic induction; $\mathbf{E}$: electric field strength). The special aspect of this force is its dependence on the particle velocity $\mathbf{v}$. The same is the case also for the

(d) Frictional Force

$$\mathbf{F} = -\alpha(v)\cdot\mathbf{v}\,. \tag{2.58}$$

In many respects this represents a very complicated type of force for which, strictly speaking, up to now there does not exist a closed satisfying theory. It merely appears confirmed that to a good approximation the dependency on $(-\mathbf{v})$ holds. The most frequently used entries for the coefficient α are:

$$\alpha(v) = \alpha = \text{const} \quad \text{(Stokes-friction)}\,, \tag{2.59}$$

$$\alpha(v) = \alpha \cdot v \quad \text{(Newton-friction)}\,. \tag{2.60}$$

2.2.3 Inertial Systems, Galilean Transformation

Newton's axioms deal with the motion of physical bodies. But motion is a relative term; the motion of a body can be defined only relative to a system of coordinates. However, regarding the choice of such systems of coordinates there are hardly any limits. Coordinate systems which are solely rigidly shifted or rigidly inclined to each other are completely equivalent with respect to the dynamics of the mass point. The components of the trajectory $\mathbf{r}(t)$ will of course change from system to system, but not the geometrical shape of the path or the temporal process of the particle motion.

If the different frames of reference are moving relatively to each other then of course the situation is different. A mass point which in a certain frame moves straight-line uniformly will experience an acceleration in another frame which is rotating relative to the first. Hence, Newton's axioms make sense only if they are referred to a definite system of coordinates or, at least, to a definite class of systems.

The genuine coordinate systems of Classical Mechanics are the **'inertial systems'**, introduced by Axiom 2.1, in which a force-free mass point moves on a straight line with

$$\mathbf{v} = \text{const}$$

We want to investigate these systems, which are obviously somehow highlighted, in a little more detail. For this purpose we study the forces which act on a mass point in two different systems of coordinates moving relative to each other. For simplicity we choose two Cartesian systems. In both systems the observer sits at the origin of coordinates.

1. Statement:
Not all systems of coordinates are inertial systems.

This statement is more or less trivial. In a system, which rotates relative to an inertial system, a force-free mass point executes an accelerated motion.

2. Statement:
There exists at least one inertial system, for instance that in which the fixed stars are at rest.

In a certain sense, here Newton's fiction of the **'absolute space'** is hidden away. This idea is lost in the theory of Special Relativity. However, there is no need to postulate here the existence of the absolute space. The second statement refers only to the indisputable fact that there indeed exist systems in which Newton's mechanics is valid.

We determine the totality of all inertial systems by finding out which transformations of coordinates transfer one inertial system into another one.

Let Σ and $\overline{\Sigma}$ be two different coordinate systems where we assume $\Sigma = \overline{\Sigma}$ at $t = 0$. Let Σ be an inertial system. We know that $\overline{\Sigma}$ is also an inertial system only if

$$m\ddot{\mathbf{r}} = 0 \qquad \textit{results in:} \qquad m\ddot{\bar{\mathbf{r}}} = 0$$

A time-dependent rotation of $\overline{\Sigma}$ relative to Σ is therewith excluded from the very beginning because it always automatically generates an acceleration connected with the change of the direction of velocity. A constant inclination (time-independent rotation) is of course thinkable since it does not lead to any acceleration. But that is not interesting here. Hence, we can restrict our considerations to systems moving relatively with parallel (Cartesian) axes.

The mass point m is described at time t by the position vector (Fig. 2.11)

$$\mathbf{r}(t) = \mathbf{r}_0(t) + \bar{\mathbf{r}}(t)$$

The transformation is completely characterized by $\mathbf{r}_0 = \mathbf{r}_0(t)$. For the acceleration of the mass point holds:

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_0 + \ddot{\bar{\mathbf{r}}} \, .$$

$\overline{\Sigma}$ is obviously also an inertial system exactly then when the transformation fulfills the condition:

$$\ddot{\mathbf{r}}_0 = 0 \Longleftrightarrow \mathbf{r}_0(t) = \mathbf{v}_0 t \tag{2.61}$$

This equation defines a so-called **'Galilean transformation'** which transforms one inertial system into another inertial system.

$$\mathbf{r} = \mathbf{v}_0 t + \bar{\mathbf{r}} \, ; \quad t = \bar{t} \, . \tag{2.62}$$

Notice that we have not transformed time as well. This implies the assumption of an **absolute time**, a view no longer maintainable in Special Relativity. There the

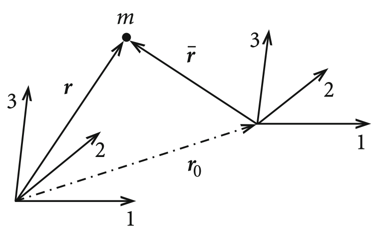


Fig. 2.11 Position vector of the mass point m in two reference systems that move relative to each other

Galilean transformation is to be replaced by the **Lorentz-transformation** which affects also the time variable.

3. Statement:
There are infinitely many inertial systems moving relatively to each other with constant velocities.

In such systems it holds:

$$\bar{\mathbf{F}} = \mathbf{F} \iff m\ddot{\bar{\mathbf{r}}} = m\ddot{\mathbf{r}} \ , \tag{2.63}$$

so that not only the first but also the second Newton's axiom remains unaffected by the transformation. One should bear in mind, however, that in case of a space and velocity dependent force $\mathbf{F} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t)$ the vectors $\mathbf{r}$ and $\dot{\mathbf{r}}$ have to be transformed properly.

2.2.4 Rotating Reference Systems, Pseudo Forces (Fictitious Forces)

In this section we want to discuss an example for non-inertial systems. We consider two coordinate systems $\Sigma, \overline{\Sigma}$, the origins of which, for simplicity, shall coincide. Let Σ be an inertial system while $\overline{\Sigma}$ rotates relative to Σ with constant angular velocity ω around the x_3 axis. The application of cylindrical coordinates (Sect. 1.7.3) is certainly convenient in this case.

$$\Sigma: \ (\rho, \varphi, z) \ ; \quad \overline{\Sigma}: \ (\overline{\rho} \ , \overline{\varphi}, \overline{z}) \ .$$

The following relations of the coordinates are obvious:

$$\rho = \overline{\rho} \ ; \quad \varphi = \overline{\varphi} + \omega t \ ; \quad z = \overline{z} \ . \tag{2.64}$$

According to (2.19) in Σ we have for the force components:

$$F_\rho = m\left(\ddot{\rho} - \rho\dot{\varphi}^2\right) \ ; \quad F_\varphi = m\left(\rho\ddot{\varphi} + 2\dot{\rho}\,\dot{\varphi}\right) \ ; \quad F_z = m\ddot{z} \ . \tag{2.65}$$

We now want to convert these force components into the rotating reference system $\overline{\Sigma}$ (Fig. 2.12). From (2.64) follows:

$$\begin{aligned}
\dot{\rho} &= \dot{\overline{\rho}} \ ; \quad \dot{\varphi} = \dot{\overline{\varphi}} + \omega \ ; \quad \dot{z} = \dot{\overline{z}} \ , \\
\ddot{\rho} &= \ddot{\overline{\rho}} \ ; \quad \ddot{\varphi} = \ddot{\overline{\varphi}} \ ; \quad \ddot{z} = \ddot{\overline{z}} \ .
\end{aligned}$$

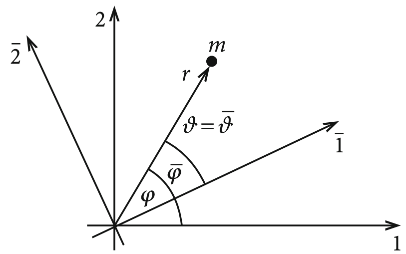


Fig. 2.12 Two reference frames with a common origin rotating relative to each other

By insertion into (2.65) we obtain the force equations in $\overline{\Sigma}$:

$$m\left(\ddot{\overline{\rho}} - \overline{\rho}\,\dot{\overline{\varphi}}^2\right) = F_\rho + 2m\overline{\rho}\,\omega\dot{\overline{\varphi}} + m\omega^2\overline{\rho} = \overline{F}_\rho \tag{2.66}$$

$$m\left(\overline{\rho}\,\ddot{\overline{\varphi}} + 2\dot{\overline{\rho}}\,\dot{\overline{\varphi}}\right) = F_\varphi - 2m\,\omega\dot{\overline{\rho}} \qquad = \overline{F}_\varphi \tag{2.67}$$

$$m\ddot{\overline{z}} \qquad = \overline{F}_z\;. \tag{2.68}$$

If $\overline{\Sigma}$ were an inertial system, then we would have had: $F_\rho = \overline{F}_\rho$, $F_\varphi = \overline{F}_\varphi$, $F_z = \overline{F}_z$. However, since $\overline{\Sigma}$ is **not** an inertial system there appear additional forces which one calls

'pseudo forces'

even though they exhibit rather real consequences. They are called *pseudo* because they appear only in non-inertial systems and because they appear there *'to bring the Newton mechanics into order'*. They take care that a force-free mass point experiences in the non-inertial system $\overline{\Sigma}$ such a *pseudo force* so that, observed from the inertial system Σ, its motion appears uniformly rectilinear.

2.2.5 Arbitrarily Accelerated Reference Systems

We consider two coordinate systems which are arbitrarily accelerated relative to each other:

$$\Sigma:\ (x_1, x_2, x_3)\ ;\quad (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)\ ,$$

$$\overline{\Sigma}:\ (\bar{x}_1, \bar{x}_2, \bar{x}_3)\ ;\quad (\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2\ , \bar{\mathbf{e}}_3)\ .$$

Let Σ be an inertial system. The full relative movement can be thought to be composed of a motion of the origin of $\overline{\Sigma}$ and a rotation of the axes of $\overline{\Sigma}$ around its own origin, both relative to Σ (Fig. 2.13).

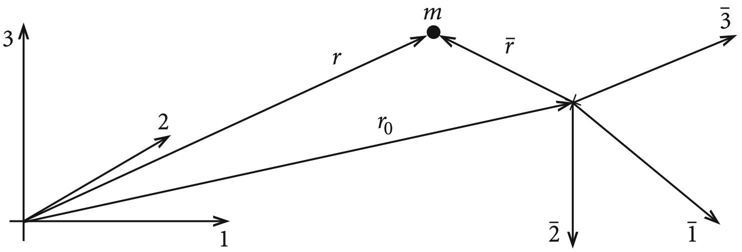


Fig. 2.13 Position vector of a mass point in two relative to each other arbitrarily accelerated reference systems

It holds for the position vector of the mass point m:

$$\mathbf{r} = \mathbf{r}_0 + \bar{\mathbf{r}} = \mathbf{r}_0 + \sum_{j=1}^{3} \bar{x}_j \bar{\mathbf{e}}_j \ . \tag{2.69}$$

This we use to calculate the velocities in both systems by time differentiation:

$$\overline{\Sigma}: \quad \dot{\bar{\mathbf{r}}} = \sum_{j=1}^{3} \dot{\bar{x}}_j \, \bar{\mathbf{e}}_j \ . \tag{2.70}$$

For a co-rotating observer the axes in $\overline{\Sigma}$ of course do not change, but surely for the observer in Σ:

$$\Sigma: \quad \dot{\mathbf{r}} = \dot{\mathbf{r}}_0 + \sum_{j=1}^{3} \left(\dot{\bar{x}}_j \, \bar{\mathbf{e}}_j + \bar{x}_j \, \dot{\bar{\mathbf{e}}}_j \right) \ . \tag{2.71}$$

It is easy to interpret the three terms on the right-hand side:
$\dot{\mathbf{r}}_0$: Relative velocity of the origins of coordinates.
$\sum_j \dot{\bar{x}}_j \, \bar{\mathbf{e}}_j$: Velocity of the mass point in $\overline{\Sigma}$ (2.70).
$\sum_j \bar{x}_j \, \dot{\bar{\mathbf{e}}}_j$: Velocity of a **rigidly** with $\overline{\Sigma}$ co-rotating point seen from Σ. For such a point the directions of the axes change, but not the components $\bar{x}_j$.
We reformulate this last term with the aid of the angular velocity $\boldsymbol{\omega}$ which describes the rotation of $\overline{\Sigma}$ around its own origin. $\boldsymbol{\omega}$ has the direction of the momentary axis of rotation. The velocity of the rigidly co-rotating point is perpendicular to $\bar{\mathbf{r}}$ and also perpendicular to $\boldsymbol{\omega}$ (Fig. 2.14). For the magnitude holds:

$$\begin{aligned} \delta\bar{r} &= |\bar{\mathbf{r}}| \sin\alpha \, \omega \, dt \\ &= \left| \left(\bar{\mathbf{r}} \times \boldsymbol{\omega} \right) \right| dt \ . \end{aligned}$$

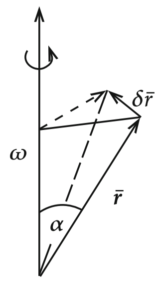


Fig. 2.14 Temporal change of the position vector of a mass point which is rigidly co-rotating with a certain reference system, observed from a space-fixed system

Altogether we thus have:

$$\frac{\delta \bar{\mathbf{r}}}{dt} = \sum_{j=1}^{3} \bar{x}_j \dot{\bar{\mathbf{e}}}_j = \boldsymbol{\omega} \times \bar{\mathbf{r}} \,. \tag{2.72}$$

This we insert into Eq. (2.71):

$$\dot{\mathbf{r}} = \dot{\mathbf{r}}_0 + \dot{\bar{\mathbf{r}}} + \boldsymbol{\omega} \times \bar{\mathbf{r}} \,. \tag{2.73}$$

With (2.69) this can also be read as follows:

$$\frac{d}{dt}\left(\mathbf{r} - \mathbf{r}_0\right) = \frac{d}{dt}\bar{\mathbf{r}} = \dot{\bar{\mathbf{r}}} + \boldsymbol{\omega} \times \bar{\mathbf{r}} \,. \tag{2.74}$$

time derivative done in Σ | time derivative done in $\overline{\Sigma}$

This equation provides very generally a prescription how one has to differentiate with respect to time a vector in the inertial system Σ, which is represented in a rotating reference system $\overline{\Sigma}$.

$$\frac{d}{dt} = \frac{\bar{d}}{dt} + \boldsymbol{\omega} \,\times \,. \tag{2.75}$$

derivative in Σ | derivative in $\overline{\Sigma}$, which only concerns the components | influence of the rotation

We directly apply this prescription once more to (2.73):

$$\begin{aligned}\frac{d}{dt}\left(\dot{\mathbf{r}} - \dot{\mathbf{r}}_0\right) &= \frac{d}{dt}\left(\dot{\bar{\mathbf{r}}} + \boldsymbol{\omega} \times \bar{\mathbf{r}}\right) = \frac{d}{dt}\dot{\bar{\mathbf{r}}} + \frac{d}{dt}(\boldsymbol{\omega} \times \bar{\mathbf{r}}) = \\ &= \ddot{\bar{\mathbf{r}}} + \left(\boldsymbol{\omega} \times \dot{\bar{\mathbf{r}}}\right) + \left(\left(\dot{\boldsymbol{\omega}} \times \bar{\mathbf{r}}\right) + \left(\boldsymbol{\omega} \times \dot{\bar{\mathbf{r}}}\right)\right) + \left(\boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \bar{\mathbf{r}}\right)\right) = \\ &= \ddot{\bar{\mathbf{r}}} + \left(\boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \bar{\mathbf{r}}\right)\right) + 2\left(\boldsymbol{\omega} \times \dot{\bar{\mathbf{r}}}\right) + \left(\dot{\boldsymbol{\omega}} \times \bar{\mathbf{r}}\right) . \end{aligned} \tag{2.76}$$

That eventually gives the **equation of motion in the non-inertial systems $\overline{\Sigma}$**:

$$m\ddot{\bar{\mathbf{r}}} = \mathbf{F} - m\ddot{\mathbf{r}}_0 - m\,\boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \bar{\mathbf{r}}\right) - m\left(\dot{\boldsymbol{\omega}} \times \bar{\mathbf{r}}\right) - 2m\left(\boldsymbol{\omega} \times \dot{\bar{\mathbf{r}}}\right) , \tag{2.77}$$

$$\bar{\mathbf{F}}_c = -2m\left(\boldsymbol{\omega} \times \dot{\bar{\mathbf{r}}}\right) : \quad \textbf{Coriolis force} , \tag{2.78}$$

$$\bar{\mathbf{F}}_z = -m\left(\boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \bar{\mathbf{r}}\right)\right) : \quad \textbf{centrifugal force} . \tag{2.79}$$

The implication of these rather involved **pseudo forces**, which show up in addition to $\mathbf{F}$ on the right-hand side of the equation of motion (2.77), is again nothing other that they fix the motion of a force-free mass point in the non-inertial system $\overline{\Sigma}$ in such a (complicated) manner that this motion appears rectilinearly for an observer in the inertial system Σ. In the final analysis they rely on the *inertia* of the particle and are therefore sometimes also called:

'inertia forces'

2.2.6 Exercises

Exercise 2.2.1 Let Σ and $\overline{\Sigma}$ be two Cartesian systems of coordinates moving relative to each other with parallel axes. The position of a particle at an arbitrary time t is described in Σ by

$$\mathbf{r}(t) = \left(6\alpha_1 t^2 - 4\alpha_2 t\right)\,\mathbf{e}_1 - 3\alpha_3 t^3 \mathbf{e}_2 + 3\alpha_4 \mathbf{e}_3$$

and in $\overline{\Sigma}$ by

$$\bar{\mathbf{r}}(t) = \left(6\alpha_1 t^2 + 3\alpha_2 t\right)\,\mathbf{e}_1 - \left(3\alpha_3 t^3 - 11\alpha_5\right)\,\mathbf{e}_2 + 4\alpha_6 t \mathbf{e}_3$$

1. What is the velocity of $\overline{\Sigma}$ relative to Σ?
2. Which acceleration does the particle experience in, respectively, Σ and $\overline{\Sigma}$?
3. If Σ is an inertial system, is then $\overline{\Sigma}$, too, an inertial system?

Exercise 2.2.2 In an inertial system the time t' is measured with a somewhat *'inaccurate'* clock. The *'true'* time in the inertial system is t. However, it is found that:

$$t' = t + \alpha(t) \ .$$

With the *'inaccurate'* clock it is (misleadingly) observed for the force-free, one-dimensional movement of a mass point m an acceleration to be of the form:

$$a' = \frac{F'}{m} = \frac{d^2x}{dt'^2} \neq 0$$

Calculate the correspondingly acting force F'!

Exercise 2.2.3 Although equations of motion in inertial systems are simpler, one describes movements on the earth normally in the reference system co-rotating with the earth (laboratory coordinate (lab) system). This system, strictly speaking, is no longer an inertial system because of the earth's rotation.

On the earth's surface let there be a Cartesian coordinate system $\overline{\Sigma}$ fixed at a certain point whose geographical latitude angle is φ:

$\bar{x}_3$ axis: vertically upwards
$\bar{x}_2$ axis: northward
$\bar{x}_1$ axis: eastward.

The angular velocity of the earth amounts to

$$|\boldsymbol{\omega}| = \frac{2\pi}{24} h^{-1} = 7.27 \cdot 10^{-5}\, s^{-1} \ .$$

1. How does the equation of motion of a mass point appear in this coordinate system close to the earth's surface? Neglect terms of order ω^2!
2. Calculate the acceleration $\ddot{\mathbf{r}}_0$ of the origin of $\overline{\Sigma}$ relative to a reference system Σ fixed and at rest in the earth's center.
3. How big is the *true* earth's acceleration $\hat{\mathbf{g}}$ **measured** in $\overline{\Sigma}$? How does the earth's surface adjust itself?
4. How does the Coriolis force depend on the geographical latitude?
5. Locate the coordinate system $\overline{\Sigma}$ in such a way that the $\bar{x}_3$ axis stands perpendicular to the **real** earth's surface. Which equations of motion are then to be solved for a mass point near the earth's surface? The Coriolis force can be taken to a good approximation from 4. since $\mathbf{g}$ and $\hat{\mathbf{g}}$ enclose only a very small angle.
6. A body is dropped from rest in a free fall from the height H. Solve the equations of motion in 5. under the assumption that $\dot{\bar{x}}_1$ and $\dot{\bar{x}}_2$ remain small during the time of the fall. Determine the eastward-deviation as a consequence of the earth's rotation!

2.3 Simple Problems of Dynamics

The basic program of Classical Mechanics consists of the calculation of the path of motion of a physical system with the aid of Newton's laws of motion (2.42) and (2.43), respectively. For this purpose the force **F** must be known. The solution of the fundamental task generally takes place in three steps:

1. Setting up the equation of motion,
2. Solution of the differential equation by use of purely mathematical methods,
3. Physical interpretation of the solution.

Until otherwise stated, let us consider in the following treatments the mass m as time-independent material constant so that we can apply the law of motion in the form (2.43).

The simplest problem is of course given by the **force-free motion**, the result of which must agree with Axiom 2.1. The equation of motion has the form:

$$\mathbf{F} = m\ddot{\mathbf{r}} \equiv \mathbf{0} . \tag{2.80}$$

Strictly speaking, this equation must be solved separately for **each** component. It actually represents therefore a short-hand notation for a set of three equation of the type,

$$\begin{aligned} m\ddot{x}_1 &= 0 , \\ m\ddot{x}_2 &= 0 , \\ m\ddot{x}_3 &= 0 , \end{aligned} \tag{2.81}$$

where each of them is a so-called **differential equation of second order**. In simple cases, such as the present one, it is, however, reasonable to discuss directly the more compact representation (2.80), the solution of which is immediately found:

$$\mathbf{r}(t) = \mathbf{v}_0 t + \mathbf{r}_0 . \tag{2.82}$$

It describes either a motionless mass point ($\mathbf{v}_0 = \mathbf{0}$) or a mass point moving with constant velocity $\mathbf{v}_0$. The mass m does not influence the solution. What is the meaning of the two constant vectors $\mathbf{v}_0$ and $\mathbf{r}_0$? That is already indicated by the chosen notation:

$$\begin{aligned} \mathbf{v}_0 &= \dot{\mathbf{r}}(t=0) : \quad \text{velocity at time } t = 0 , \\ \mathbf{r}_0 &= \mathbf{r}(t=0) : \quad \text{particle position at time } t = 0 . \end{aligned}$$

The motion of the mass-point is completely fixed if the initial position $\mathbf{r}_0$ and the initial velocity $\mathbf{v}_0$ are given. Since these are vectors that means the specification of six **initial conditions**, two per each of the three equations in (2.81).

2.3.1 *Motion in the Homogeneous Gravitational Field*

According to the above-given program we have to at first formulate the equation of motion. Using (2.48) together with (2.43) and exploiting the equality of inertial and heavy mass we can write:

$$\ddot{\mathbf{r}} = \mathbf{g} \; ; \qquad \mathbf{g} = (0, 0, -g) \; . \tag{2.83}$$

The mass is eliminated; in the gravitational field all bodies are therefrom equally accelerated. It results in a

uniformly accelerated motion

as we have discussed it already in Sect. 2.1.2. We can directly take the former results (2.30) and (2.31):

$$\mathbf{v}(t) = \mathbf{v}(t_0) + \mathbf{g} \cdot (t - t_0) \;, \tag{2.84}$$

$$\mathbf{r}(t) = \mathbf{r}(t_0) + \mathbf{v}(t_0)(t - t_0) + \frac{1}{2}\mathbf{g} \cdot (t - t_0)^2 \;. \tag{2.85}$$

This is the purely mathematical result which we want to interpret physically a little bit more:

To begin with, we recognize that the actual geometrical shape of the path line depends strongly on the initial conditions $\mathbf{r}(t_0)$, $\mathbf{v}(t_0)$. That we demonstrate with two **special cases**:

(a) Free Fall from the Height *h*

The initial conditions in this case are ($t_0 = 0$):

$$\begin{aligned} \mathbf{r}(t = 0) &= (0, 0, h) \;, \\ \mathbf{v}(t = 0) &= \mathbf{0} \;. \end{aligned} \tag{2.86}$$

Then we have the solution:

$$x_1(t) = x_2(t) = 0 \; ; \quad \dot{x}_1(t) = \dot{x}_2(t) = 0 \; .$$

Hence, it turns out to be a one-dimensional motion (Fig. 2.15):

$$x_3(t) = h - \frac{1}{2} g\, t^2 \; ; \quad \dot{x}_3(t) = -g\, t \; . \tag{2.87}$$

As **'fall time'** t_F one denotes the time the body needs to arrive at the earth's surface ($x_3 = 0$).

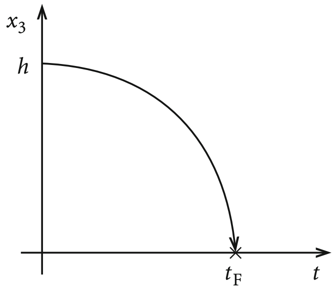


Fig. 2.15 Time-dependence of the distance between a mass m and the earth's surface during the free fall in the gravitational field

$$\begin{aligned} x_3\,(t_F) &\overset{!}{=} 0 = h - \tfrac{1}{2} g\, t_F^2 \\ &\Longrightarrow t_F = \sqrt{2h/g}\,. \end{aligned} \tag{2.88}$$

For the final velocity at the impingement then holds:

$$v_F = |\dot{x}_3\,(t_F)| = \sqrt{2hg}\,. \tag{2.89}$$

(b) Vertical Throw Upwards

This corresponds to the initial conditions ($t_0 = 0$):

$$\begin{aligned} \mathbf{r}(t = 0) &= 0\,, \\ \mathbf{v}(t = 0) &= (0, 0, v_0)\;\;. \end{aligned} \tag{2.90}$$

Inserting into (2.84) and (2.85) we first get:

$$x_1(t) = x_2(t) = 0\,; \quad \dot{x}_1(t) = \dot{x}_2(t) = 0\,.$$

Thus it is again a one-dimensional motion:

$$x_3(t) = v_0 t - \frac{1}{2} g\, t^2\,; \quad \dot{x}_3(t) = v_0 - g\, t\,. \tag{2.91}$$

We add a brief interpretation of the result (Fig. 2.16): The velocity of the thrown body decreases at first with increasing time becoming zero as soon as the maximum height is reached. That is the case after the time t_H:

$$\begin{aligned} \dot{x}_3\,(t_H) &\overset{!}{=} 0 = v_0 - g\, t_H \\ &\Longrightarrow t_H = \frac{v_0}{g}\,. \end{aligned} \tag{2.92}$$

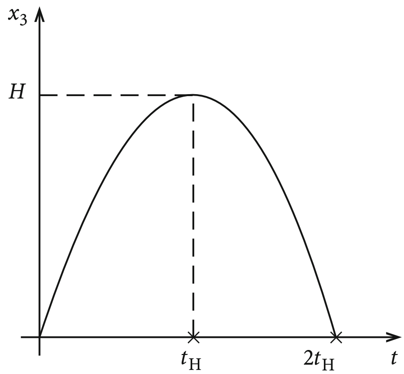


Fig. 2.16 Time-dependent devolution of the distance of a mass m from the earth's bottom during the vertical throw upwards in the gravitational field

Subsequently the direction of the motion reverses and $\dot{x}_3(t)$ becomes negative. For the maximal flight altitude holds:

$$H = x_3\,(t_H) = \frac{v_0^2}{2g}\;. \tag{2.93}$$

After the time $2t_H$ the projectile reaches the earth's surface again with the velocity $-v_0$ at the impingement.

For arbitrary initial conditions we have to evaluate the general result (2.84), (2.85) in the same manner as demonstrated for the above two special cases. The general procedure was already performed in Sect. 2.1.2 (see Fig. 2.6, **'trajectory parabola'**). One can show (Exercise 2.1.3) that thereby the motion happens always in a fixed plane spanned $\mathbf{v}(t = t_0)$ and $\mathbf{g}$.

Up to now we could refer to previous calculations and results. If we now turn to somewhat more sophisticated problems of motion then we have to *integrate explicitly* a linear differential equation of second order. For this reason we want to first deal with the general theory of linear differential equations in the form of a short mathematical insertion.

2.3.2 Linear Differential Equations

We refer to

$$x^{(n)}(t) = \frac{d^n}{dt^n}x(t) \tag{2.94}$$

as the n-th derivative of the function $x(t)$. A relation which contains one or more derivatives of a given function, where the n-th derivative appears as the highest,

$$f\left(x^{(n)}, x^{(n-1)}, \ldots, \dot{x}, x, t\right) = 0\;, \tag{2.95}$$

is called a **differential equation of n-th order**. The goal is to derive the **solution function** $x(t)$ from such a relation. The basic dynamical equation of Classical Mechanics (2.43) written in Cartesian coordinates, e.g., has just this shape:

$$m\ddot{x}_i - F_i\left(\dot{x}_1, \dot{x}_2, \dot{x}_3, x_1, x_2, x_3, t\right) = 0\ , \quad i = 1, 2, 3\ . \tag{2.96}$$

This is a coupled system of three differential equations of second order for the three functions $x_1(t), x_2(t), x_3(t)$.

Let us first focus, however, on a general relation of the type (2.95). The central statement is formulated in the following

Theorem 2.3.1 *The **general** solution of a differential equation of n-th order (2.95) is an ensemble of solutions*

$$x = x\left(t\,|\gamma_1, \gamma_2, \ldots, \gamma_n\right)\ ,$$

*which depends on n **independent** parameters $\gamma_1, \gamma_2, \ldots, \gamma_n$. Every set of γ_i's which are fixed in advance then leads to a **special (particular)** solution.*

One should compare, e.g., the solution (2.85) for the motion in the homogeneous gravitational field with this theorem. It represents the solution of a differential equation of second order. For each component solution $x_i(t)$ there appear two independent parameters $x_i(t_0)$ and $v_i(t_0)$. Equation (2.85) therefore turns out to be the **general solution**. **Special solutions** we found in the examples (a) and (b) by fixing the *initial values* in (2.86) and (2.90), respectively.

It is important that the reverse of the above theorem is also valid.

Theorem 2.3.2 *If the solution of a differential equation of n-th order (2.95) does depend on n **independent** parameters then it is the **general** solution.*

It is usual but not at all necessary to identify the parameters $\gamma_1, \ldots, \gamma_n$ with the *initial values* $x(t_0), \dot{x}(t_0), \ldots, x^{(n-1)}(t_0)$.

The special case important for us is the

<u>linear</u> differential equation.

So one denotes a relation of the type (2.95) in which the derivatives $x^{(j)}$ appear solely linearly,

$$\sum_{j=0}^{n} \alpha_j(t)\, x^{(j)}(t) = \beta(t)\ , \tag{2.97}$$

where the differential equation with $\beta(t) \equiv 0$ is called *homogeneous* and with $\beta(t) \not\equiv 0$ *inhomogeneous*.

We consider first the **homogeneous, linear differential equations**. For these the

superposition principle

holds. This confirms that when $x_1(t)$ and $x_2(t)$ solve the differential equation then $c_1 x_1(t) + c_2 x_2(t)$ with arbitrary coefficients c_1, c_2 also solves it. Because of the linearity of the differential equation the proof is obvious.

Furthermore, as for normal vectors, one can define a **linear independency** of solutions:

m solution functions $x_1(t), \ldots, x_m(t)$ are called **linearly independent** if

$$\sum_{j=1}^{m} \alpha_j x_j(t) = 0 \tag{2.98}$$

is an identity **only** for $\alpha_1 = \alpha_2 = \ldots = \alpha_m = 0$.

If m is the maximal number of linearly independent solution functions then one can write the **general** solution $x(t|\gamma_1, \ldots, \gamma_n)$ for **any** fixed choice of the parameters γ_i in the form:

$$x\,(t\,|\gamma_1, \ldots, \gamma_n) = \sum_{j=1}^{m} \alpha_j x_j(t)\ . \tag{2.99}$$

If that were not possible then $x(t|..)$ itself would be a linearly independent solution and therefore m is not the maximal number. Furthermore, the right-hand side in principle depends on m independent parameters α_j. That means that m must **not be smaller** than n because otherwise $x(t|..)$ would not be the general solution. However, it is also true that m must **not be greater** than n because otherwise $x(t|..)$ would depend on more than n independent parameters. Consequently, we must have $m = n$. We conclude:

The general solution of the homogeneous, linear differential equation of n-th order can be written as linear combination of n linearly independent (special) solution functions.

In a certain sense, this fact can be used as **recipe** for the solution of a homogeneous, linear differential equation of n-th order. One tries to find by *'guessing and trying'* n linearly independent solutions. Then we can be sure that every linear combination of them represents the general solution.

Let us now inspect the **inhomogeneous differential equation of n-th order**. We presume to have found with $x(t|\gamma_1, \ldots, \gamma_n)$ the general solution of the associated *homogeneous* equation and, additionally, with $x_0(t)$ a special solution of the *inhomogeneous* equation. Then it becomes immediately clear, because of the linearity of the differential equation, that

$$\bar{x}\,(t|\gamma_1, \ldots, \gamma_n) = x\,(t|\gamma_1, \ldots, \gamma_n) + x_0(t) \tag{2.100}$$

is in the first place certainly a solution of the inhomogeneous equation. But what's more, it is already the *general* solution since it depends already on n independent

parameters. Out of this fact we derive a practicable **recipe** for solving linear, inhomogeneous differential equations:
Look for the general solution of the associated homogeneous differential equation and try to find any special solution of the inhomogeneous equation. According to (2.100) the sum is then already the required general solution of the inhomogeneous differential equation.
We shall apply this recipe over and over in the following.

2.3.3 Motion with Friction in the Homogeneous Gravitational Field

Every moving macroscopic body becomes to a certain degree decelerated by interaction with its environment. Thus during the motion **frictional forces** appear which are opposed to the motion. Although little is known till today about the causes of friction it is clear that it must be a **macroscopic phenomenon**. The equations of motion of atomic and nuclear physics do **not** contain friction terms.

(a) Friction in Gases and Liquids

In viscous media the ansatz (2.58) can be considered as a good approximation:

$$\mathbf{F}_\mathrm{R} = -\alpha(v)\mathbf{v} \ . \tag{2.101}$$

where $\alpha(v)$ has to be determined empirically. The versions given in (2.59) and (2.60) are special types:

(1) Newton's law of friction

$$\mathbf{F}_\mathrm{R} = -\alpha\, v\, \mathbf{v} \ . \tag{2.102}$$

For the usefulness of this ansatz the velocity of the moving body mast exceed a certain limiting value which depends on the respective *rubbing* material (fast projectiles, movement in viscous liquids, …).

(2) Stokes's law of friction
If the relative velocities in viscous media are smaller than the mentioned limiting velocity then it appears to be better to apply the ansatz:

$$\mathbf{F}_\mathrm{R} = -\alpha\, \mathbf{v} \ . \tag{2.103}$$

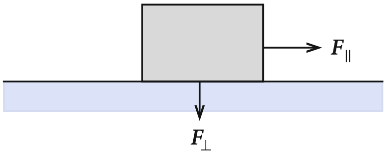


Fig. 2.17 Simple arrangement for the illustration of friction between solids

(b) Friction Between Solids

A solid body presses with the force $\mathbf{F}_\perp$ on a substratum. For forward motion only the tangential component $\mathbf{F}_\|$ of the external force plays a role (Fig. 2.17).

(1) Sliding friction
One observes that the force of friction is to a large extent independent of the supporting surface and also of the relative velocity:

$$\mathbf{F}_\mathrm{R} = -\mu_\mathrm{g} F_\perp \frac{\mathbf{v}}{v} , \quad \text{if } v > 0 . \tag{2.104}$$

One speaks of **Coulomb friction**. μ_g is the **sliding friction coefficient**.

(2) Static friction
For the case $v = 0$ static friction occurs which compensates the parallel component $\mathbf{F}_\|$ of the external force:

$$\mathbf{F}_\mathrm{R} = -\mathbf{F}_\| \ (v = 0) . \tag{2.105}$$

Of course that holds only as long as the pulling force does not exceed a certain upper bound which is fixed by the **static friction coefficient** μ_H:

$$F_\| < \mu_\mathrm{H} F_\perp . \tag{2.106}$$

Experiments show that in general $0 < \mu_\mathrm{g} < \mu_\mathrm{H}$ is valid.
After these preliminary remarks we now want to discuss the movement of a body, e.g. a parachute, in the earth's gravitational field, which is under the influence of friction. As a reasonable model we assume Stokes's friction. The equation of motion then reads:

$$m\,\ddot{\mathbf{r}} = -m\,\mathbf{g} - \alpha\,\dot{\mathbf{r}} \qquad \mathbf{g} = (0, 0, g) . \tag{2.107}$$

This is an inhomogeneous differential equation of second order,

$$m\ddot{\mathbf{r}} + \alpha\dot{\mathbf{r}} = -m\,\mathbf{g} ,$$

with the inhomogeneity $(-m\,\mathbf{g})$. In order to find the general solution of this equation we at first seek the general solution of the associated homogeneous equation:

$$m\,\ddot{\mathbf{r}} + \alpha\,\dot{\mathbf{r}} = \mathbf{0}\ . \tag{2.108}$$

Strictly speaking, we have to solve this equation for each component separately:

$$m\,\ddot{x}_i + \alpha\dot{x}_i = 0\ ; \quad i = 1, 2, 3\ . \tag{2.109}$$

For such differential equations with **constant** coefficients the following ansatz is typical and mostly successful:

$$x_i = e^{\gamma t}\ .$$

Insertion yields:

$$e^{\gamma t}\left(m\gamma^2 + \alpha\gamma\right) = 0 \Longleftrightarrow m\gamma^2 + \alpha\gamma = 0\ .$$

This equation has the solutions:

$$\gamma_1 = 0\ ; \quad \gamma_2 = -\frac{\alpha}{m}\ .$$

That corresponds to the two linearly independent solutions of (2.109):

$$x_i^{(1)}(t) = 1\ ; \quad x_i^{(2)}(t) = e^{-(\alpha/m)t}\ .$$

As explained in Sect. 2.3.2 the linear combination of these two functions represents the general solution:

$$x_i^{(0)}(t) = a_i^{(1)} + a_i^{(2)} e^{-(\alpha/m)t}\ . \tag{2.110}$$

This result corresponds to the motion under the **sole** influence of the friction (Fig. 2.18). For the general solution of the inhomogeneous equations we need

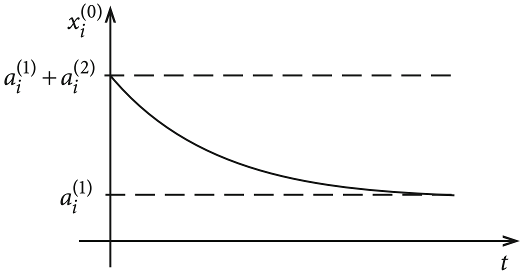


Fig. 2.18 Schematic representation of the time dependence of the three Cartesian components $(i = x, y, z)$ of the position vector of a mass point under the sole action of friction (a_i: initial conditions)

to investigate only the x_3-component since $\mathbf{g} = (0, 0, g)$. For the two other components the respective inhomogeneity vanishes so that Eq. (2.110) is already the complete solution:

$$m\ddot{x}_3 + \alpha\dot{x}_3 = -mg\ . \tag{2.111}$$

We look after a **special** solution in order to combine it then with the general solution of the homogeneous equation. We can arrive at this by the following consideration. The gravitational force will enhance the velocity of the mass point until the frictional force, increasing simultaneously with the velocity, will balance the gravitation:

$$\alpha\dot{x}_3^{(E)} = -mg \Longleftrightarrow \dot{x}_3^{(E)} = -\frac{m}{\alpha}g\ . \tag{2.112}$$

As soon as the mass point has reached this velocity, according to (2.107) a force-free motion sets in. The same motion, however, occurs when we release the mass point directly with the initial velocity $\dot{x}_3^{(E)}$. It then performs a uniform straight-line motion with the constant velocity $\dot{x}_3^{(E)}$. Therewith we have already found a special solution of the inhomogeneous equation (2.111):

$$x_3(t) = -\frac{m}{\alpha}g\,t\ . \tag{2.113}$$

That helps us to formulate the general solution for the x_3 component:

$$x_3(t) = a_3^{(1)} + a_3^{(2)}e^{-(\alpha/m)t} - \frac{m}{\alpha}g\,t\ . \tag{2.114}$$

For the two other components (2.110) is already the final solution:

$$\begin{aligned} x_2(t) &= a_2^{(1)} + a_2^{(2)}e^{-(\alpha/m)t}\ , \\ x_1(t) &= a_1^{(1)} + a_1^{(2)}e^{-(\alpha/m)t}\ . \end{aligned} \tag{2.115}$$

Each component solution contains two independent parameters. For the velocities it holds:

$$\begin{aligned} v_1(t) &= -a_1^{(2)}\frac{\alpha}{m}e^{-(\alpha/m)t}\ , \\ v_2(t) &= -a_2^{(2)}\frac{\alpha}{m}e^{-(\alpha/m)t}\ , \end{aligned} \tag{2.116}$$

$$\begin{aligned} v_3(t) &= -\left(a_3^{(2)}\frac{\alpha}{m}e^{-(\alpha/m)t} + \frac{m}{\alpha}g\right) \\ &\underset{t\to\infty}{\Longrightarrow} \quad -\frac{m}{\alpha}g\ . \end{aligned} \tag{2.117}$$

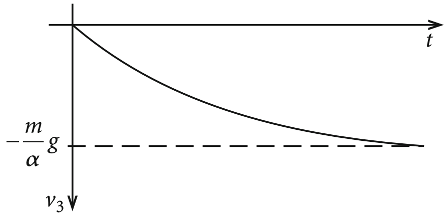


Fig. 2.19 Time dependence of the velocity of a mass m during its vertical fall with friction α in the earth's gravitational field

If we choose as initial conditions those of the vertical fall,

$$\mathbf{r}(t=0) = (0, 0, H)\ , \quad \mathbf{v}(t=0) = (0, 0, 0)\ ,$$

we first get:

$$x_1(t) = x_2(t) \equiv 0\ . \tag{2.118}$$

So it turns out to be a linear motion.

$$H = a_3^{(1)} + a_3^{(2)}\ ,$$

$$0 = a_3^{(2)}\frac{\alpha}{m} + \frac{m}{\alpha}g \Longrightarrow a_3^{(2)} = -\frac{m^2}{\alpha^2}g,\ a_3^{(1)} = H + \frac{m^2}{\alpha^2}g\ .$$

Hence, we get the concrete result:

$$v_3(t) = \frac{m}{\alpha}g\left(e^{-(\alpha/m)t} - 1\right)\ , \tag{2.119}$$

$$x_3(t) = H + \frac{m}{\alpha}g\left[\frac{m}{\alpha}\left(1 - e^{-(\alpha/m)t}\right) - t\right]\ . \tag{2.120}$$

For $t \to \infty$ the velocity $v_3(t)$ approaches the limiting value $\dot{x}_3^{(E)}$ (Fig. 2.19).

2.3.4 Simple Pendulum

As an additional simple problem of dynamics we will now discuss the simple (thread) pendulum (Fig. 2.20), which sometimes is also called **mathematical pendulum** because it represents a somewhat mathematical abstraction. One considers the motion of a mass point which is fixed by a **massless** thread. The latter has a constant length l so that the mass point performs a planar motion on a circular arc with radius l. The gravitational force acts on the mass point:

$$\mathbf{F} = m_s\,\mathbf{g}\ ; \quad \mathbf{g} = (g,\ 0,\ 0)\ . \tag{2.121}$$

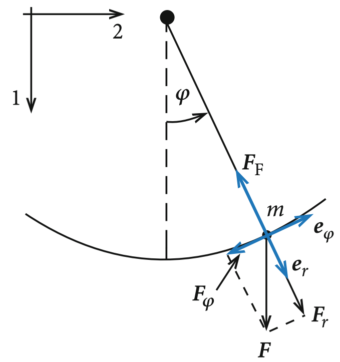


Fig. 2.20 Forces and coordinates concerning the simple (thread) pendulum

The simple pendulum is excellently suited to demonstrate the equivalence of the inertial and the gravitational mass. To show this, we will first distinguish again between these two masses.

The application of plane polar coordinates is the natural choice:

$$\begin{aligned} \mathbf{F} &= F_r \mathbf{e}_r + F_\varphi \mathbf{e}_\varphi \ , \\ F_r &= m_h \, g \cos\varphi \ , \\ F_\varphi &= -m_h \, g \sin\varphi \ . \end{aligned} \tag{2.122}$$

The **equation of motion** written in detail by use of (2.13) reads as:

$$m_{in} \left[\left(\ddot{r} - r\dot{\varphi}^2 \right) \mathbf{e}_r + \left(r\ddot{\varphi} + 2\dot{r}\dot{\varphi} \right) \mathbf{e}_\varphi \right] = \left(F_r + F_\mathrm{F} \right) \mathbf{e}_r + F_\varphi \mathbf{e}_\varphi \ .$$

F_F is called the

'thread tension'

It is about a so-called **'constraining force'** which realizes certain **'constraints'**. Here the constraint is the constant distance of the mass point from the center of rotation:

$$r = l = \text{const} \ ; \quad \dot{r} = \ddot{r} = 0 \ .$$

F_F thus prevents the free fall of the mass point and takes care for a static problem in the radial direction:

$$F_\mathrm{F} = -m_h \, g \cos\varphi - m_{in} \, l \, \dot{\varphi}^2 \ . \tag{2.123}$$

So only the movement in $\mathbf{e}_\varphi$-direction is of interest:

$$m_{in} \, l\ddot{\varphi} \ = -m_h \, g \sin\varphi \Longrightarrow \ddot{\varphi} + \frac{g}{l} \frac{m_h}{m_{in}} \sin\varphi = 0 \ . \tag{2.124}$$

The non-linear function of φ which appears together with $\ddot{\varphi}$ makes the solution a bit elaborate. The calculation leads to the so-called elliptic integrals of the first kind.

To simplify the task we restrict ourselves here to **small** deflections of the pendulum so that we can assume:

$$\sin\varphi \approx \varphi$$

The equation of motion then takes the form of an **oscillation equation**:

$$\ddot{\varphi} + \frac{g}{l}\frac{m_h}{m_{in}}\varphi = 0 \,. \tag{2.125}$$

That is again a homogeneous differential equation of second order. $\varphi(t)$ must be a function which after twofold differentiating, except for the sign, reproduces itself. Therefore

$$\varphi_1(t) = \sin\omega t \quad \text{and} \quad \varphi_2(t) = \cos\omega t$$

are two linearly independent solutions provided one chooses:

$$\omega^2 = \frac{g}{l}\frac{m_h}{m_{in}}$$

The general solution is then:

$$\varphi(t) = A\sin\omega t + B\cos\omega t \,. \tag{2.126}$$

A and B can be fixed by initial conditions:

$$A = \frac{1}{\omega}\dot{\varphi}(t=0)\,, \quad B = \varphi(t=0)\,.$$

Experimentally the **angular frequency** ω turns out to be independent of the mass of the oscillating particle. That in turn can be explained only if $m_h \propto m_{in}$. We therefore assume as in (2.50) $m_h = m_{in}$:

$$\textbf{angular frequency} \quad \omega = \sqrt{\frac{g}{l}}\,. \tag{2.127}$$

As **(oscillation) period** τ one denotes the time needed for a full oscillation, i.e. the time after which the mass point arrives again at its starting point:

$$\omega\tau = 2\pi \iff \tau = 2\pi\sqrt{\frac{l}{g}}\,. \tag{2.128}$$

This result enables a rather accurate determination of the gravitational acceleration g.

By **frequency** one means the number of full oscillations per second:

$$\nu = \frac{1}{\tau} = \frac{1}{2\pi}\sqrt{\frac{g}{l}} = \frac{\omega}{2\pi}\ . \tag{2.129}$$

The solution (2.126) corresponds to a superposition of two oscillations with the same frequency but with different amplitudes A and B. The amplitude is thereby the maximal deflection out of the equilibrium position. Instead of (2.126) we can also write:

$$\varphi(t) = \sqrt{A^2 + B^2}\left(\frac{A}{\sqrt{A^2 + B^2}}\sin\omega t + \frac{B}{\sqrt{A^2 + B^2}}\cos\omega t\right)\ .$$

If we now define

$$A_0 = \sqrt{A^2 + B^2}\ ; \quad \cos\alpha = \frac{A}{\sqrt{A^2 + B^2}}\ ; \quad \sin\alpha = \frac{B}{\sqrt{A^2 + B^2}}$$

and exploit the addition theorem (1.60)

$$\sin(x + y) = \sin x\cos y + \cos x\sin y$$

then we arrive at an alternative representation of the solution $\varphi(t)$:

$$\varphi(t) = A_0\sin(\omega t + \alpha)\ . \tag{2.130}$$

The superposition of the two oscillations in (2.126) results again in an oscillation of exactly the same frequency but with a **phase shift** α (Fig. 2.21).

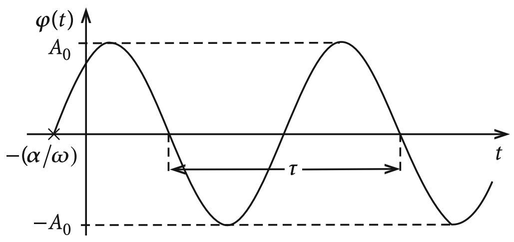


Fig. 2.21 Time dependence of the angle deflection for the thread pendulum

2.3.5 Complex Numbers

For the solution of the oscillation equation (2.125) we looked for a function which essentially reproduces itself after twofold differentiation. That happens indeed to the trigonometric functions sine and cosine. But the exponential function, which for a variety of reasons is *tractable mathematically easier*, also possesses a similar property. However, the ansatz $e^{\alpha t}$ would have led to the conditional equation

$$e^{\alpha t}\left(\alpha^2 + \frac{g}{l}\right) = 0 \, ; \quad e^{\alpha t} \neq 0 \ ,$$

an equation being not solvable for real α. The equation becomes, however, solvable if one allows for **complex numbers** which we did not yet introduce so far.

By application of complex numbers and functions many issues in Theoretical Physics turn out to be mathematically essentially simpler. it is needless to say that all measurable quantities, which we call **'observables'**, are in any case real so that we must be able uniquely to relate real and complex representations. That will be treated in this section.

(a) Imaginary Numbers

The new number type of the imaginary numbers is characterized by the fact that their squares are always **negative** real numbers.

Definition 2.3.1 'Unit of imaginary numbers'

$$i^2 = -1 \Longleftrightarrow i = \sqrt{-1} \, . \tag{2.131}$$

Each **imaginary number** can be written as

$$i \cdot y$$

with real y.

Examples

(1) $\sqrt{-4} = \sqrt{-1} \cdot \sqrt{4} = \pm 2i$,
(2) $i^3 = i \cdot i^2 = -i$,
(3) $\alpha^2 + \frac{g}{l} = 0 \Longrightarrow \alpha_{1,2} = \pm i\sqrt{\frac{g}{l}}$.

(b) Complex Numbers

Definition 2.3.2 The **complex number** z is the sum of a real and an imaginary number:

$$z = x + iy\ , \tag{2.132}$$

where x is the **real part** and y the **imaginary part** of z.
One calls

$$z^\star = x - iy \tag{2.133}$$

the **conjugated complex number** to z.

A complex number is equal to zero only if **both** real **and** imaginary part vanish. The purely real and purely imaginary numbers are special complex numbers with vanishing imaginary and real part, respectively.

(c) Calculation Rules

For setting up calculation rules we allow us to be guided by the corresponding rules of the real numbers since these can be considered as special complex numbers.

One **adds (subtracts)** two complex numbers

$$z_1 = x_1 + iy_1\ ; \quad z_2 = x_2 + iy_2\ ,$$

by adding (subtracting) separately the real and the imaginary parts:

$$z = z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)\ . \tag{2.134}$$

The **product** is given by a formal expansion taking into consideration (2.131):

$$z = z_1 z_2 = (x_1 x_2 - y_1 y_2) + i\,(x_1 y_2 + y_1 x_2)\ . \tag{2.135}$$

Obviously the product is equal to zero only if one of the two factors vanishes. In the same manner one can introduce the **quotient (ratio)** of two complex numbers,

$$z = \frac{z_1}{z_2} = \frac{z_1 z_2^\star}{z_2 z_2^*} = \frac{1}{x_2^2 + y_2^2}\left[(x_1 x_2 + y_1 y_2) + i\,(y_1 x_2 - x_1 y_2)\right], \tag{2.136}$$

where $z_2 \neq 0$ has to be stipulated.

(d) Complex Plane

One can interpret real and imaginary part of a complex number as the two components of a two-dimensional vector:

$$z = x + iy = (x, y) \ . \tag{2.137}$$

The real part then corresponds to the projection on the **real axis**, the imaginary part to that on the **imaginary axis**. **Basis vectors** of the so-called **complex plane** are then:

$$1 = (1, 0); \quad i = (0, 1) \ . \tag{2.138}$$

Like the normal two-dimensional vectors one can represent the complex numbers, too, by plane polar coordinates (**'polar representation'**) (Fig. 2.22):

$$\begin{aligned} x &= r\cos\varphi \ , \\ y &= r\sin\varphi \end{aligned} \quad \Longrightarrow \quad \begin{aligned} z &= r(\cos\varphi + i\sin\varphi) \ , \\ z^* &= r(\cos\varphi - i\sin\varphi) \ . \end{aligned} \tag{2.139}$$

Thus z^* follows from z by reflection on the real axis. One defines:

Magnitude (Absolute Value) of z

$$|z| = r = \sqrt{x^2 + y^2} \ . \tag{2.140}$$

Argument of z

$$\varphi = \arg(z) = \arctan\frac{y}{x} \ . \tag{2.141}$$

For each value of $y/x = \tan\varphi$ in between $-\infty$ and $+\infty$ there exist **two** φ-values between 0 and 2π (see Fig. 2.23). One has to take just that φ-value with which the transformation formulae (2.139) can be exactly fulfilled.

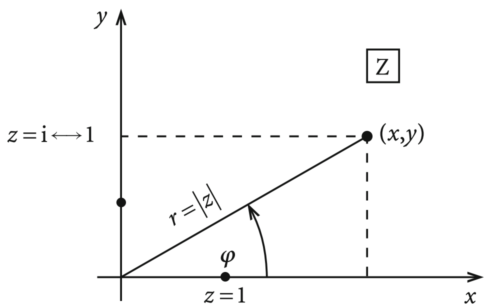


Fig. 2.22 Polar representation of a complex number

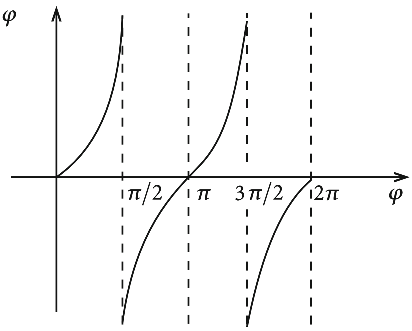


Fig. 2.23 General behavior of the trigonometric function $\tan\varphi$

For the magnitude holds

$$|z| = \sqrt{z \cdot z^*}\,, \tag{2.142}$$

as can easily be verified:

$$z \cdot z^* = (x + iy)(x - iy) = x^2 + y^2 + i(yx - xy) = x^2 + y^2 = |z|^2\,.$$

(e) Exponential form of a Complex Number

For the exponential function e^x the series expansion (1.64) holds:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}\,. \tag{2.143}$$

Corresponding expansions one knows also for the trigonometric functions sine (1.51) and cosine (1.58):

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \frac{1}{i} \sum_{n=0}^{\infty} \frac{(ix)^{2n+1}}{(2n+1)!}\,, \tag{2.144}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(ix)^{2n}}{(2n)!}\,. \tag{2.145}$$

From this one reads off the very important

Euler's Formula

$$e^{i\varphi} = \cos\varphi + i \sin\varphi \tag{2.146}$$

Therewith and according to (2.139) the complex number can now be represented also as follows:

$$z = |z|e^{i\varphi} \; . \tag{2.147}$$

Since the cosine is an even and the sine an uneven function of φ it holds:

$$e^{-i\varphi} = \cos\varphi - i\sin\varphi \tag{2.148}$$

That means for the conjugated complex number:

$$z^* = |z|e^{-i\varphi} \; . \tag{2.149}$$

The inversion formulae are also useful:

$$\cos\varphi = \frac{1}{2}\left(e^{i\varphi} + e^{-i\varphi}\right) \; ; \quad \sin\varphi = \frac{1}{2i}\left(e^{i\varphi} - e^{-i\varphi}\right) \; . \tag{2.150}$$

Notice the **important** fact that every complex number, considered as a function of φ, is periodic with the period 2π:

$$|z|e^{i\varphi} = |z|e^{i(\varphi + 2n\pi)} \; , \quad n = \pm 1, \pm 2, \ldots \tag{2.151}$$

(f) Further Calculation Rules

Multiplication: [cf. (2.135)]

$$\begin{aligned} z &= z_1 \cdot z_2 = |z_1| \cdot |z_2| \, e^{i(\varphi_1 + \varphi_2)} \\ \Longrightarrow |z| &= |z_1| \cdot |z_2| \; ; \quad \arg(z) = \varphi_1 + \varphi_2 \; . \end{aligned} \tag{2.152}$$

Division: [cf. (2.136)]

$$\begin{aligned} z &= \frac{z_1}{z_2} = \frac{|z_1|}{|z_2|} \, e^{i(\varphi_1 - \varphi_2)} \\ \Longrightarrow |z| &= \frac{|z_1|}{|z_2|} \; ; \quad \arg(z) = \varphi_1 - \varphi_2 \; . \end{aligned} \tag{2.153}$$

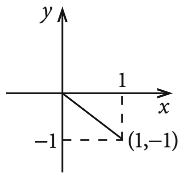


Fig. 2.24 The number $z = 1 - i$ in the complex plane

Raising to a Power: [cf. (1.1)]

$$\begin{aligned} z &= z_1^n = |z_1|^n \, e^{in\varphi_1} \\ \Longrightarrow |z| &= |z_1|^n \; ; \quad \arg(z) = n\varphi_1 \; . \end{aligned} \tag{2.154}$$

Extracting a Root: [cf. (1.7)]

$$\begin{aligned} z &= \sqrt[n]{z_1} = \sqrt[n]{|z_1|} \, e^{i\varphi/n} \\ \Longrightarrow |z| &= \sqrt[n]{|z_1|} \; ; \quad \arg(z) = \varphi/n \; . \end{aligned} \tag{2.155}$$

Examples

1. $\ln(-5) = \ln(5 \cdot e^{i\pi}) = \ln 5 + \ln e^{i\pi} = \ln 5 + i\pi$.
2. $z = 1 - i$ (see Fig. 2.24)

$$\begin{aligned} |z| &= \sqrt{2} \; , \\ \arg(z) &= \arctan\left(\frac{-1}{+1}\right) = 7\pi/4 \; , \\ z &= \sqrt{2} \, e^{i(7\pi/4)} \; . \end{aligned}$$

3. $\sqrt{1-i} = 2^{1/4} \, e^{i(7\pi/8)}$.
4. $\ln(1 + 3i) = \ln\left(\sqrt{10} \, e^{i \arctan 3}\right) = \frac{1}{2} \ln 10 + i \arctan 3$.
5. $\frac{1}{i} = -i$.
6. $|e^{i\varphi}| = \left(\cos^2\varphi + \sin^2\varphi\right)^{1/2} = \sqrt{1} = 1$.
   The complex numbers $e^{i\varphi}$ thus lie in the complex plane on the unit circle around the origin of coordinates.

2.3.6 Linear Harmonic Oscillator

The harmonic oscillator belongs to the most important and to the most intensively discussed model systems of Theoretical Physics. The range of its application range

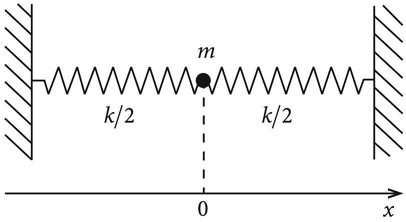


Fig. 2.25 Elastic spring as a possible realization of a free harmonic oscillator

far exceeds thee canopy of Classical Mechanics. We will be dealing again and again with this model in Electrodynamics and in particular in Quantum Theory. The relevance of this model lies above all in the fact that it belongs to the very few mathematically strictly tractable systems by which many of the fundamental principles of Theoretical Physics can be illustrated. One understands by the harmonic oscillator a self-oscillating system that obeys a characteristic equation of motion of the same type as that for the simple pendulum (2.125).

In order to discuss the basic phenomena we first have in mind an elastic spring to which a mass point m is attached. For small deflections the mass point experiences a backwards directed force being proportional to the displacement $|x|$. According to the sketched arrangement in Fig. 2.25 the gravitational force will be ignored. The movement happens one-dimensionally along the spring axis. Then Hooke's law holds:

$$F = -k\,x\,. \tag{2.156}$$

k is the **spring constant**. As equation of motion er have the following linear homogeneous differential equation:

$$m\ddot{x} + kx = 0\,. \tag{2.157}$$

In it is the same differential equation as that for the simple pendulum (2.125). From reasons which become clear later the entity

$$\omega_0 = \sqrt{\frac{k}{m}} \tag{2.158}$$

is called **eigen frequency** of the harmonic oscillator. When a physical system is described by an equation of motion of the type given in (2.157) then we always speak of a **linear harmonic oscillator**.

An interesting **non-mechanical** realization of the harmonic oscillator is represented by the electrical oscillator circuit consisting of a coil with the self-inductance L and a capacitor with the capacity C (Fig. 2.26). The electrical current I then fulfills

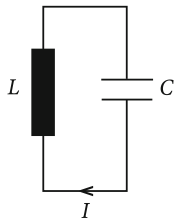


Fig. 2.26 Electrical oscillator circuit as a possible realization of the free harmonic oscillator

the differential equation

$$L\ddot{I} + \frac{1}{C} I = 0 \; ; \quad \omega_0^2 = \frac{1}{LC} \; . \tag{2.159}$$

We already solved the differential equation (2.157) in Sect. 2.3.4 (see (2.126) and (2.130)):

$$x(t) = A \sin \omega_0 t + B \cos \omega_0 t \; ; \quad x(t) = A_0 \sin(\omega_0 t + \alpha) \; . \tag{2.160}$$

It is a characteristic of the **harmonic behavior** of the oscillator that the frequency ω_0 is independent of the amplitude of the oscillation. Hence ω_0 must be considered as a pure system property.

After having introduced in the last section the complex numbers we want to solve the equation

$$\ddot{x} + \omega_0^2 x = 0$$

once more, but now with the ansatz $e^{\alpha t}$. One finds by insertion:

$$e^{\alpha t} \left(\alpha^2 + \omega_0^2 \right) = 0 \Longleftrightarrow \alpha^2 = -\omega_0^2 \; .$$

This yields two imaginary values for α

$$\alpha_\pm = \pm i \omega_0$$

and therewith the following two linearly independent solutions,

$$x_\pm(t) = e^{\pm i \omega_0 t} \; ,$$

from which we get the **general** solution:

$$x(t) = A_+ e^{i \omega_0 t} + A_- e^{-i \omega_0 t} \; . \tag{2.161}$$

When interpreting this type of solution one has to be a bit cautious. $x(t)$ must of course be a real quantity. The functions $e^{\pm i \omega_0 t}$ are, however, complex. The coefficients $A_\pm$ therefore have to fulfill certain conditions. First it follows with (2.146):

$$x(t) = (A_+ + A_-)\cos\omega_0 t + i\,(A_+ - A_-)\sin\omega_0 t\ . \tag{2.162}$$

If the quantities $A_\pm$ were real, then we would have necessarily to require that $A_+ = A_-$. However, the consequence would then be that $x(t)$ contains only **one** independent parameter and thus could not be the general solution. Consequently we have to assume that A_+ and A_- are complex. At first glance this, however, would mean that there were **four** independent parameters. But this is really not the case because the requirement of a real $x(t)$ leads to:

$$x(t) = x^*(t) \Longleftrightarrow A_+ e^{i\omega_0 t} + A_- e^{-i\omega_0 t} = A_+^*\, e^{-i\omega_0 t} + A_-^*\, e^{i\omega_0 t}\ .$$

Because of the linear independency of $e^{i\omega_0 t}$ and $e^{-i\omega_0 t}$ this equation can only be fulfilled if $A_+ = A_-^*$ and $A_- = A_+^*$ hold. So A_+ and A_- are conjugate-complex quantities,

$$A_+ = A_-^* = a + ib\ ,$$

so that indeed we are left with only **two** independent parameters a and b. Inserted into (2.162) it follows:

$$x(t) = 2a\cos\omega_0 t - 2b\sin\omega_0 t\ .$$

The two types of solution (2.161) and (2.160) are therefore absolutely equivalent.

As general solution of a homogeneous differential equation of second order both (2.160) and (2.161) still contain two free parameters which must be fixed by initial conditions. We discuss two different situations:

(a) At time $t = 0$ let the oscillator be displaced by $x = x_0$ and then released. That corresponds to the initial conditions:

$$x(t=0) = x_0\ ; \quad \dot{x}(t=0) = 0\ . \tag{2.163}$$

This we insert into (2.160):

$$x_0 = B\ ; \quad 0 = \omega_0 A \Longrightarrow A = 0\ .$$

The **special** solution then reads:

$$x(t) = x_0\cos\omega_0 t\ . \tag{2.164}$$

The initial displacement becomes the amplitude of the oscillation.

(b) Let the oscillator be kicked off from its equilibrium position with the initial velocity v_0:

$$x(0) = 0\ ; \quad \dot{x}(0) = v_0\ . \tag{2.165}$$

We use again (2.160):

$$B = 0\ ; \quad v_0 = A\omega_0\ .$$

In this case we obtain a further **special** solution:

$$x(t) = \frac{v_0}{\omega_0} \sin \omega_0 t\ . \tag{2.166}$$

2.3.7 Free Damped Linear Oscillator

Each real oscillator eventually comes to stop because of the unavoidable frictional forces. We therefore want to now include them into our considerations where, however, we will restrict ourselves to the simplest case of the Stokes's friction. Then the extended equation of motion reads:

$$m\ddot{x} = -kx - \alpha\dot{x}\ . \tag{2.167}$$

One can realize this situation by a 'tongue', dipping into a liquid and being fixed to the mass m (Fig. 2.27). While the frictional term in Eq. (2.167) in general represents a certain approximation, there exists an exact non-mechanical realization of the damped harmonic oscillator by the electrical oscillator circuit. The sum of the partial voltages in the circuit sketched in Fig. 2.28 must be zero. The electrical current therefore obeys the following differential equation:

$$L\ddot{I} + R\dot{I} + \frac{1}{C}I = 0\ . \tag{2.168}$$

The ohmic resistance R simulates the frictional term.

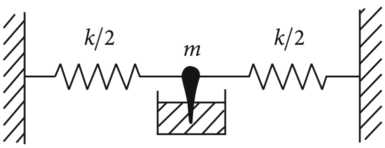


Fig. 2.27 Elastic spring as a possible realization of a free harmonic oscillator with friction

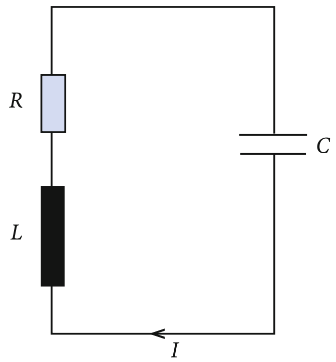


Fig. 2.28 Electrical oscillator circuit as a possible realization of a free harmonic oscillator with friction

After division by m we get from (2.167) the following homogeneous differential equation of second order:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \; ; \quad \beta = \frac{\alpha}{2m} \; . \tag{2.169}$$

As ansatz an exponential function appears again plausible:

$$x(t) = e^{\lambda t} \; .$$

It is exactly then a solution if λ fulfills the following relation:

$$\lambda^2 + 2\beta\lambda + \omega_0^2 = 0 \; .$$

Therefrom one finds:

$$\lambda_{1,2} = -\beta \pm \sqrt{\beta^2 - \omega_0^2} \; . \tag{2.170}$$

If the root is not equal to zero then we have found two linearly independent solutions. The general solution therefore reads:

$$x(t) = a_1 \, e^{\lambda_1 t} + a_2 \, e^{\lambda_2 t} \; . \tag{2.171}$$

When discussing the solutions we have to distinguish three cases.

(a) Weak Damping (Oscillatory Case)

It refers to the situation:

$$\beta < \omega_0 \; .$$

Then the root in Eq. (2.170) is purely imaginary:

$$\omega = \sqrt{\omega_0^2 - \beta^2} \Longleftrightarrow \lambda_{1,2} = -\beta \pm i\omega \ . \tag{2.172}$$

The general solution (2.171) is therewith written as:

$$x(t) = e^{-\beta t}\left(a_1\, e^{i\omega t} + a_2\, e^{-i\omega t}\right) \ . \tag{2.173}$$

A comparison with (2.161), the solution for the free oscillation, shows that it comes out as an oscillation with smaller frequency ($\omega < \omega_0$) and with an exponentially decaying amplitude as function of time.

With the initial conditions

$$x_0 = x(t = 0) \ , \quad v_0 = \dot{x}(t = 0)$$

we can bring (2.173) into yet another form :

$$x_0 = a_1 + a_2 \ , \quad v_0 = -\beta\,(a_1 + a_2) + i\omega\,(a_1 - a_2) = -\beta x_0 + i\omega\,(a_1 - a_2) \ .$$

That means:

$$x(t) = e^{-\beta t}\left(x_0 \cos\omega t + \frac{v_0 + \beta x_0}{\omega} \sin\omega t\right) \ . \tag{2.174}$$

We find a third representation herefrom by the following definitions:

$$A = \frac{1}{\omega}\sqrt{x_0^2\,\omega^2 + (v_0 + \beta x_0)^2} \ , \tag{2.175}$$

$$\left.\begin{aligned} \sin\varphi &= \tfrac{x_0}{A} \\ \cos\varphi &= \tfrac{v_0+\beta x_0}{\omega A} \end{aligned}\right\} \quad \Longrightarrow \ \varphi = \arctan\left(\frac{\omega x_0}{v_0 + \beta x_0}\right) \ . \tag{2.176}$$

Therewith we get:

$$x(t) = Ae^{-\beta t}\sin(\omega t + \varphi) \ . \tag{2.177}$$

Now A and the phase shift φ are the free parameters of the general solution. The **amplitude** of the oscillation

$$A\, e^{-\beta t}$$

is **exponentially damped**. Thus in a strict sense one cannot speak of a purely periodic motion since the initial situation is not periodically reproduced. Terms like frequency and time period are therefore no longer uniquely defined (Fig. 2.29).

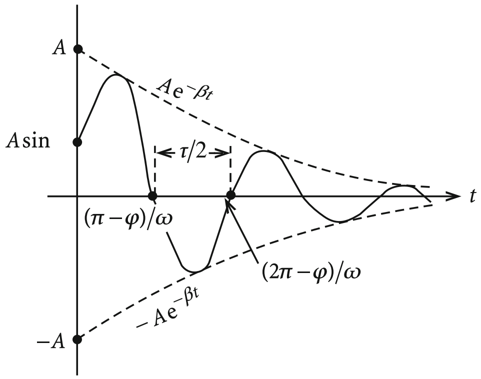


Fig. 2.29 Time dependence of the amplitude of a weakly damped linear harmonic oscillator

Merely the zero crossings are still periodic with the time separation $\tau/2$ where

$$\tau = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\omega_0^2 - \beta^2}} \,. \tag{2.178}$$

Sometimes one calls

$$Ae^{-\beta t}$$

the **envelope** of the damped oscillation.

(b) Critical Damping (Aperiodic Limiting Case)

This is the limiting case

$$\alpha^2 = 4\,k\,m\,; \qquad \beta^2 = \omega_0^2 \Longleftrightarrow \omega = 0\,. \tag{2.179}$$

Now the root in (2.170) disappears so that one gets with the ansatz $x(t) = e^{\lambda t}$ because of

$$\lambda_{1,2} = -\beta$$

only **one** special solution. From this one can not yet construct the general solution. We still need a second special solution. For that the following trick helps. In the solution (2.174) we perform the limiting process $\omega \to 0$ whereby we exploit the fact that according to (2.144) and (2.145) it must hold:

$$\cos\omega t \underset{\omega\to 0}{\longrightarrow} 1\,; \quad \sin\omega t \underset{\omega\to 0}{\longrightarrow} \omega t\,.$$

Therewith follows:

$$x(t) = e^{-\beta t} \left[x_0 + (v_0 + \beta\, x_0)\, t \right] \ . \tag{2.180}$$

This solution contains the two independent parameters x_0 and v_0. It fulfills with (2.179) the homogeneous differential equation (2.169) and is therefore the general solution.

It is interesting that one can also find the result (2.180) a little bit more systematically. The ansatz $x(t) = e^{\lambda t}$ leads to only **one** special solution. We therefore tentatively extend it:

$$x(t) = \varphi(t)\, e^{-\beta t} \ . \tag{2.181}$$

Then for the derivatives needed in (2.169) we have:

$$\begin{aligned} \dot{x}(t) &= (\dot{\varphi} - \beta\varphi)\ e^{-\beta t} \ , \\ \ddot{x}(t) &= \left(\ddot{\varphi} - 2\beta\dot{\varphi} + \beta^2\varphi\right) e^{-\beta t} \ . \end{aligned}$$

Insertion into (2.169) with $\omega_0^2 = \beta^2$ leads to $\ddot{\varphi} \equiv 0$ and therewith to

$$\varphi(t) = a_1 + a_2 t$$

and finally to

$$x(t) = (a_1 + a_2 t)\ e^{-\beta t} \ . \tag{2.182}$$

That is identical to (2.180).

The actual behavior of the solution curve very strongly depends on the initial conditions (Fig. 2.30). There does not appear any oscillation, only one zero crossing

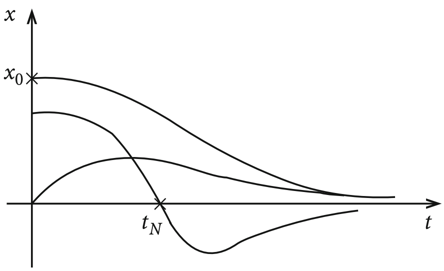


Fig. 2.30 Possible (schematic) time dependencies of the harmonic oscillator in the aperiodic limiting case (critical damping)

is still possible, and that if the initial conditions are chosen just so that

$$t = t_N = -\frac{x_0}{v_0 + \beta x_0} \tag{2.183}$$

can be realized with $t_N > 0$.

(c) Strong Damping (Creeping Case)

We now assume:

$$\beta > \omega_0 \ .$$

According to (2.170) there are now two **negative-real** solutions:

$$\lambda_{1,2} = -\beta \pm \gamma \ ; \quad 0 < \gamma = +\sqrt{\beta^2 - \omega_0^2} < \beta \ .$$

Hence, the general solution in this case reads:

$$x(t) = e^{-\beta t}\left(a_1\, e^{\gamma t} + a_2\, e^{-\gamma t}\right) \ . \tag{2.184}$$

Because $\gamma > 0$ the second summand will be quickly damped. The system is not capable of oscillation. It displays at most still one zero crossing. a_1 and a_2 are again fixed by initial conditions:

$$\begin{aligned} a_1 &= \frac{1}{2}\left(x_0 + \frac{v_0 + \beta x_0}{\gamma}\right) \ , \\ a_2 &= \frac{1}{2}\left(x_0 - \frac{v_0 + \beta x_0}{\gamma}\right) \ . \end{aligned} \tag{2.185}$$

A zero crossing happens if

$$\frac{a_1}{a_2} = -e^{-2\gamma t} \iff t = -\frac{1}{2\gamma}\ln\left(-\frac{a_1}{a_2}\right)$$

can be fulfilled. That means that

$$\frac{a_1}{a_2} < 0 \quad \text{and} \quad \left|\frac{a_1}{a_2}\right| < 1$$

must be realized by the initial conditions. One can demonstrate that in the aperiodic limiting case the system is damped quicker than in the genuine creeping case.

2.3.8 Damped Linear Oscillator Under the Influence of an External Force

Because of the unavoidable friction every oscillating process is exponentially damped unless an additional external force acts. We will now include the latter in our considerations. The equation of motion (2.169) is then to be replaced by

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{1}{m}F(t) \; . \tag{2.186}$$

We choose the same denotations as in the last section and restrict ourselves to the important special case of a periodic force:

$$F(t) = f \cos \overline{\omega} t \; . \tag{2.187}$$

One can realize the periodic force by, for instance, a wheel spinning with constant angular velocity and being connected via a drive rod to the oscillating body (Fig. 2.31).

Here again we have an exact non-mechanical realization (Fig. 2.32) by the electrical oscillator circuit if one applies to it a periodic alternating voltage $U_0 \sin \overline{\omega} t$:

$$L\ddot{I} + R\dot{I} + \frac{1}{C}I = U_0 \overline{\omega} \cos \overline{\omega} t \; . \tag{2.188}$$

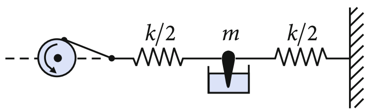


Fig. 2.31 Mechanical realization of the damped harmonic oscillator under the influence of a periodic external force

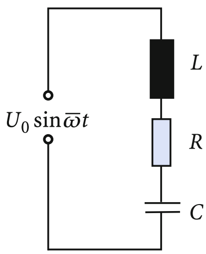


Fig. 2.32 Electrical realization of the damped harmonic oscillator under the influence of a periodic external force

The eigen frequency of the oscillator circuit is obviously:

$$\omega_0^2 = \frac{1}{LC} \,,$$

while the damping constant is given by:

$$\beta = \frac{R}{2L}$$

We look for the general solution of the inhomogeneous differential equation of second order (2.186). We know already the general solution of the associated homogeneous equation from the last section. Therefore we first try to find a special solution of the inhomogeneous differential equation. The easiest way to do this is probably if we first rewrite the differential equation (2.186) by use of complex quantities:

$$\ddot{z} + 2\beta\dot{z} + \omega_0^2 z = \frac{f}{m} e^{i\overline{\omega}t} \,. \tag{2.189}$$

Naturally, physical forces are always real. However, to calculate with the exponential function is especially comfortable. That is the reason why one uses such complex ansatz-functions. One therewith comes to a complex solution from which one eventually takes the real part as the physically relevant result. This works because of the linearity of the differential equation which prevents the real and imaginary parts from mixing.

After a certain **settling time** the oscillator will essentially follow the driving force $F(t)$. A self-evident solution ansatz therefore should be

$$z(t) = A\, e^{i\overline{\omega}t}$$

Insertion into (2.189) yields in this case a conditional equation for the *amplitude* A:

$$\left[A\left(-\overline{\omega}^2 + 2i\beta\,\overline{\omega} + \omega_0^2\right) - \frac{f}{m}\right] e^{i\overline{\omega}t} = 0 \,.$$

Thus for A must hold:

$$A = -\frac{f}{m} \frac{1}{\left(\overline{\omega}^2 - \omega_0^2\right) - 2i\beta\,\overline{\omega}} = |A|\, e^{i\varphi} \,. \tag{2.190}$$

A is of course complex:

$$A = -\frac{f}{m} \frac{\left(\overline{\omega}^2 - \omega_0^2\right) + 2i\beta\,\overline{\omega}}{\left(\overline{\omega}^2 - \omega_0^2\right)^2 + 4\beta^2\overline{\omega}^2}$$

whose magnitude is

$$|A| = \frac{f/m}{\sqrt{\left(\overline{\omega}^2 - \omega_0^2\right)^2 + 4\beta^2\overline{\omega}^2}} \,. \tag{2.191}$$

Real and imaginary parts can then be written as follows:

$$\begin{aligned} \mathrm{Re}A &= -\frac{m}{f}|A|^2\left(\overline{\omega}^2 - \omega_0^2\right) \,, \\ \mathrm{Im}A &= -2\frac{m}{f}\beta|A|^2\overline{\omega} \,. \end{aligned} \tag{2.192}$$

For $\overline{\varphi} = \arg(A)$ it therefore holds:

$$\tan\overline{\varphi} = \frac{\mathrm{Im}A}{\mathrm{Re}A} = \frac{2\beta\,\overline{\omega}}{\overline{\omega}^2 - \omega_0^2} \,. \tag{2.193}$$

Since for positive $\overline{\omega}$ the numerator ImA is always less than zero, $\overline{\varphi}$ will always lie in between $-\pi$ and 0.

We have now found a special solution for (2.189), namely:

$$z(t) = |A|\, e^{i(\overline{\omega}t + \overline{\varphi})} \,.$$

Only the real part is physically relevant which represents a special solution of (2.186):

$$x_0(t) = |A|\cos\left(\overline{\omega}t + \overline{\varphi}\right) \,. \tag{2.194}$$

Therewith the problem is in principle solved because we know the general solution of the associated homogeneous equation:

$$x_{\mathrm{inh}}(t) = x_{\mathrm{hom}}(t) + x_0(t) \,. \tag{2.195}$$

Independently of which of the three cases discussed in the last section (oscillatory case, aperiodic limiting case, creeping case) does appear, the homogeneous solution exhibits in any case an exponentially damped motion which after a sufficiently long time ($t > 1/\beta$) will hardly carry any significant weight. It plays a role only during the so-called **'settling process'**. One can use it to fulfill the given preconditions. After a certain time the mass point m oscillates with the frequency $\overline{\omega}$ of the driving force. The motion then becomes independent of the initial conditions. Therefore, we can concentrate the following discussions on the special solution $x_0(t)$.

The amplitude $|A|$ of the enforced oscillation is proportional to the amplitude f of the driving force and otherwise is essentially dependent on system properties such as $(m,\ \omega_0,\ \beta)$ as well as the frequency $\overline{\omega}$. Furthermore, $|A|$ is a symmetric function of $\overline{\omega}$. The limiting cases

$$|A|_{\overline{\omega}=0} = \frac{f}{m\,\omega_0^2} = \frac{f}{k}\,,$$
$$|A|_{\overline{\omega}\to\infty} \sim \frac{1}{\overline{\omega}^2} \longrightarrow 0 \tag{2.196}$$

one can read off directly from (2.191).

If one sets the derivative of $|A|$ with respect to $\overline{\omega}$ equal to zero one finds a conditional equation for the extreme values of $|A|$:

$$\overline{\omega}_1 = 0\,; \quad \overline{\omega}_\pm = \pm\sqrt{\omega_0^2 - 2\beta^2}\,. \tag{2.197}$$

The values $\overline{\omega}_\pm$ have a certain formal similarity to the eigen frequency ω of the damped harmonic oscillator (2.172) being, however, because of the factor 2 in front of β, **not** identical to it. The $\overline{\omega}_\pm$ are of course frequencies for A extreme values only as long as they are real, i.e. for $2\beta^2 < \omega_0^2$. In case $\overline{\omega}_\pm$ are real, then one finds at $\overline{\omega}_1$ a minimum and at $\overline{\omega}_\pm$ maxima. If however the $\overline{\omega}_\pm$ turn out to be imaginary numbers, then $|A|$ has a single maximum at $\overline{\omega}_1 = 0$ (Fig. 2.33).

The appearance of a pronounced maximum of the amplitude is called

'resonance'

The **resonance frequency** $\sqrt{\omega_0^2 - 2\beta^2}$ shifts with increasing friction to lower values. In the special case of the undamped oscillator it coincides with the eigen

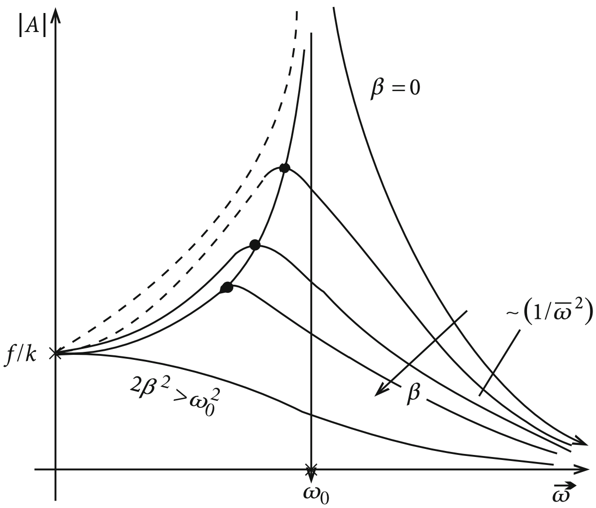


Fig. 2.33 Resonance behavior of the amplitude of the harmonic oscillator under the influence of a periodic external force for different damping strengths β

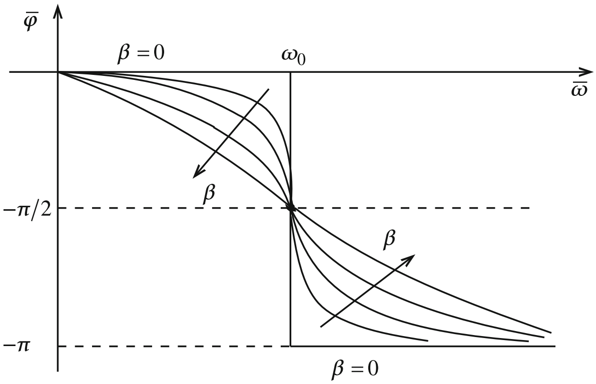


Fig. 2.34 Phase shift between the oscillation amplitude of the harmonic oscillator and the driving force as function of the frequency of the periodic external force

frequency ω_0 of the oscillator. The amplitude then becomes infinitely large and one speaks of a **resonance catastrophy**. For real systems, however, one has to take into consideration that near the resonance the amplitude can become so big that the preconditions of the harmonic oscillator are no longer fulfilled. We think, as an example, of the assumed *small* deflections of the simple thread pendulum.

Let us finally still consider the phase shift $\overline{\varphi}$ of the oscillation amplitude $|A|$ relatively to the driving force for which we have already found out in (2.192) and (2.193) that always

$$-\pi \leq \overline{\varphi} \leq 0$$

holds. The amplitude thus drags behind the force (Fig. 2.34). The displacement maximum is reached only **after** the force reaches maximum. For $\overline{\omega} = \omega_0$ the phase shift $\overline{\varphi}$ independently of β is always equal to $-\pi/2$. For the undamped oscillator $\overline{\varphi}$ jumps at $\overline{\omega} = \omega_0$ discontinuously from 0 to $-\pi$. With $\beta \neq 0$ the phase shift $\overline{\varphi}$ becomes a continuous function of $\overline{\omega}$.

2.3.9 Arbitrary One-Dimensional Space-Dependent Force

As a last simple problem of dynamics we want to discuss the case of an in principle arbitrary but one-dimensional and only space-dependent force:

$$F = F(x) \,. \tag{2.198}$$

In such a case a general procedure for solving the equation of motion

$$m\ddot{x} = F(x) \tag{2.199}$$

can be developed that ultimately reduces the problem to so-called *'quadratures'*, i.e. to the explicit evaluation of well-defined integrals. This method leads at first to purely mathematically defined auxiliary quantities (e.g. constants of integration), which, however, later will acquire fundamental physical meanings, such as energy, potential, work, power, ...

We multiply (2.199) with $\dot{x}$:

$$m\ddot{x}\,\dot{x} = F(x)\dot{x} \; .$$

This can then obviously also be written in the following form:

$$\frac{d}{dt}\left(\frac{m}{2}\dot{x}^2\right) = -\frac{d}{dt}V(x) \; , \tag{2.200}$$

if one understands by $V(x)$ the following indefinite integral:

$$V(x) = -\int^{x} F(x')dx' \; . \tag{2.201}$$

$V(x)$ is in a certain sense the antiderivative of the force $F(x)$ being therefore a known quantity except for an additive constant. The minus sign is simply a convention without any deeper physical meaning.

By the integration process the Eq. (2.200) provides a new constant which we want to denote by E:

$$\frac{m}{2}\dot{x}^2 = E - V(x) \; . \tag{2.202}$$

This equation can further be rewritten by use of the so-called *separation of variables*:

$$dt = \frac{dx}{\sqrt{\frac{2}{m}\left(E - V(x)\right)}} \; , \quad t - t_0 = \int_{x_0}^{x} \frac{dx'}{\sqrt{\frac{2}{m}\left(E - V(x')\right)}} \; . \tag{2.203}$$

Therewith the problem is in principle solved. After the evaluation of the integral we obtain

$$t = t(x)$$

and after reversal

$$x = x(t) \ .$$

The two independent parameters of this solution are now t_0 and E. Notice that x_0 is not an additional free parameter.

The expressions (2.201) and (2.203) contain some terms with a deep physical meaning. Such a double role, namely on the one hand being simply an auxiliary quantity in connection with the integration of equations of motion and on the other hand manifesting fundamental physical statements, is rather typical for many terms in physics.

(1) Work

Let us consider at first the integrand in (2.201). There is no need to further convince that the motion of a body in a force field requires an **'effort'**. One says one has to **carry out work**. A measure for that is the product of force and the covered distance. One thus defines

$$dW = -F\,dx \tag{2.204}$$

as (infinitesimal) **work** which must be done to shift the mass point by a distance dx in the force field F. For a finite piece of path it holds then:

$$W_{21} = -\int\limits_{x_1}^{x_2} F(x)dx \ . \tag{2.205}$$

If a mass point is moved **against** an acting force it experiences work from outside. This we count as positive. In case of a motion in field direction then the mass point itself executes work which we define as negative.

Examples

(a) Harmonic oscillator (spring): $F = -kx$

$$\Longrightarrow \ W_{21} = \frac{k}{2}\left(x_2^2 - x_1^2\right) \ , \tag{2.206}$$

(b) Gravitational field: $F = -mg\,\mathbf{e}_x$

$$\Longrightarrow \ W_{21} = mg\,(x_2 - x_1) \ . \tag{2.207}$$

(2) Potential, Potential Energy

If it is possible to find for a force F an antiderivative as in (2.201) then one calls it a **conservative force** and

$$V(x): \quad \textbf{potential} \ \text{ of the force } \ F .$$

In the simple special case $F = F(x)$ considered here such an antiderivative can always be found. That does not hold for velocity- and/or time-dependent force fields. In the next section we will derive the general criteria for the existence of a potential.

At this stage we have to point out a certain definition muddle in literature concerning the terms **potential** and **potential energy**. Under a *potential* one understands in the framework of Classical Mechanics the *potential energy* per mass unit. The discrimination does not appear very profound; so we will not retrace it here. But one should bear in mind that in some textbooks the two terms do not exactly mean the same.

Obviously it holds:

$$W_{21} = V(x_2) - V(x_1) \ . \tag{2.208}$$

If a mass point possesses the **potential energy** V then it is *'potentially'* able to carry out work.

Examples

(a) Harmonic oscillator (spring):

$$V(x) = k \int^{x} x' dx' = \frac{k}{2} x^2 + c \ , \tag{2.209}$$

(b) Gravitational field:

$$V(x) = mg \int^{x} dx' = mg\, x + c \ . \tag{2.210}$$

Potentials are defined only up to an additive constants. Only potential differences are unique and therefore physically meaningful.

(3) Kinetic Energy

In the Eqs. (2.200) and (2.202) there appears a quantity which is unequal zero only for moving masses ($\dot{x} \neq 0$). One calls it the kinetic energy:

$$T = \frac{m}{2} \dot{x}^2 \ . \tag{2.211}$$

One reads off from (2.200) and (2.208) that the change ΔT of the kinetic energy corresponds to the work done on the body by the external force:

$$\Delta T = -\Delta W \ . \tag{2.212}$$

Hence T has the dimension of work.

(4) Total Energy

The integration constant E represents the sum of kinetic and potential energies:

$$E = T + V = \frac{m}{2}\dot{x}^2 + V(x) \ . \tag{2.213}$$

For conservative forces such as $F(x)$ assumed here according to (2.200) the **energy conservation law** holds:

$$\frac{dE}{dt} = 0 \Longleftrightarrow E = \text{const} \ . \tag{2.214}$$

Like V of course E is also fixed only up to an additive constant.

(5) Classical Particle Paths

Our very general considerations already permit us to draw far reaching conclusions about possible particle paths. Since T is non-negative it follows from (2.213):

$$\textit{classically allowed region of motion}: \ E \geq V(x) \ , \tag{2.215}$$

$$\textit{classically forbidden region of motion}: \ E < V(x) \ , \tag{2.216}$$

$$\textit{classical turning points}: \ E = V(x) \ . \tag{2.217}$$

The supplement *classical* is important since the above statement has to be commented on when dealing with the all-embracing quantum theory.

Examples

(a) Harmonic oscillator:
Because of (2.215) it is to be expected that an oscillatory motion takes place between the two turning points $\pm x_0$. The distance between $E = E_0$ and $V(x)$ is a measure for the velocity of the mass point (Fig. 2.35). At the turning points the velocity of the particle is zero. The direction of motion reverses.

(b) General state dependence of potential:
For $x \leq x_1$ no movement is possible, and so is the case between x_2 and x_3, also. Between x_1 and x_2 an oscillatory behavior takes place, whilst a particle coming

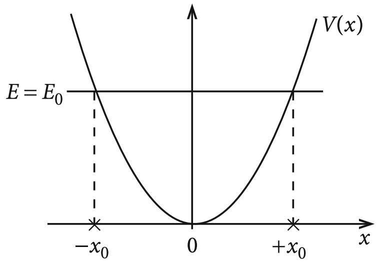


Fig. 2.35 Potential course for the linear harmonic oscillator with turning points

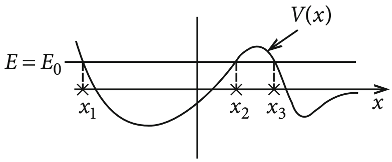


Fig. 2.36 Regions of motion of a particle of energy E_0 with a preset state dependence of potential

from $+\infty$ is reflected at x_3 (Fig. 2.36). Possible **equilibrium positions** of the particle are those points where no forces act. Obviously these are the extremal values of the potential V:

$$F = 0 = -\frac{dV}{dx} \Longleftrightarrow V \text{ extremal} .$$

In case of a maximum the particle is in an **unstable equilibrium**. The smallest position change lets it fall down the **potential wall**. In case of a minimum the particle finds itself in a **stable equilibrium**.

Finally we add a remark about the **dimension**, which is the same for T, W, V and E:

$$[E] = kg\, m^2\, s^{-2} = \text{Joule} . \tag{2.218}$$

2.3.10 Exercises

Exercise 2.3.1

1. Given the linear homogeneous differential equation of third order:

$$\sum_{j=0}^{3} \alpha_j(t) x^{(j)}(t) = 0 ,$$

demonstrate that the three solution functions

$$x_1(t) , \quad x_2(t) , \quad x_3(t)$$

are exactly then linearly independent when the so-called **Wronski-determinant**

$$W(x_1, x_2, x_3; t) \equiv \begin{vmatrix} x_1(t) & x_2(t) & x_3(t) \\ \dot{x}_1(t) & \dot{x}_2(t) & \dot{x}_3(t) \\ \ddot{x}_1(t) & \ddot{x}_2(t) & \ddot{x}_3(t) \end{vmatrix}$$

does **not** vanish.

2. Given is the linear homogeneous differential equation

$$\dddot{x}(t) - \frac{6}{t^2}\dot{x}(t) + \frac{12}{t^3}x(t) = 0 \, .$$

Examine by insertion whether or not

$$x_1(t) = \frac{1}{t^2} \, ; \quad x_2(t) = t^2 \, ; \quad x_3(t) = t^3$$

are special solutions. Are they linearly independent? What is the general solution?

Exercise 2.3.2 In the earth's gravitational field two stones are vertically thrown upwards with identical initial velocities v_0 but in a temporal separation of t_0.

1. Formulate and integrate the equations of motion!
2. After what time do the two stones meet each other?
3. How big are then their velocities at the time of their meeting?

Exercise 2.3.3 Two masses m_1 and m_2 ($m_1 < m_2$) are connected by a thread of length L. The earth's gravitational field acts along x direction (see Fig. 2.37).

1. What are the equations of motion for m_1 and m_2?
2. Calculate the accelerations of both the masses as functions of m_1 and m_2.
3. How large is the thread tension?

Exercise 2.3.4 Two masses m_1 and m_2 ($m_2 > m_1$) can move without any friction in the earth's gravitational field on two planes being inclined relatively to the horizontal by angles α and β. They are connected by a thread of constant length L, both therefore perform in principle one-dimensional motions (see Fig. 2.38).

1. Formulate the equations of motion of the two masses m_1 and m_2.
2. Express the accelerations in terms of m_1, m_2, α, β and g.
3. Calculate the thread tension S.
4. Under what conditions are the masses at rest or in uniform straight-line motion?

Exercise 2.3.5 A rope of mass m and length l is sliding over an edge under the influence of the earth's gravitational field in x direction (see Fig. 2.39). The friction between the rope and the supporting surface is neglected.

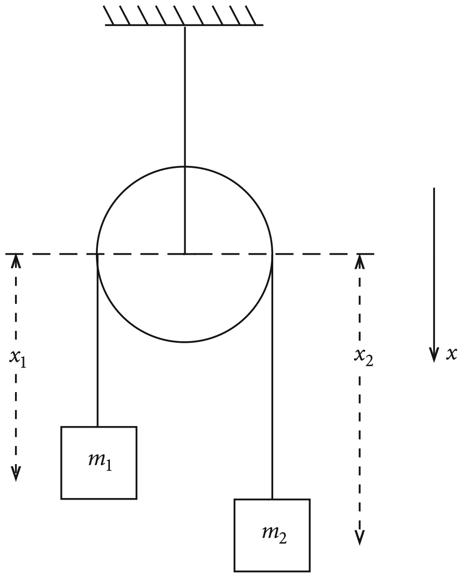


Fig. 2.37 Atwood's (free-fall) machine

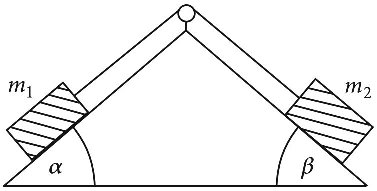


Fig. 2.38 Two masses connected with each other on a wedge in the earth's gravitational field

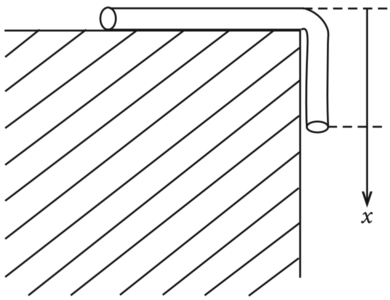


Fig. 2.39 A rope moving over an edge under the influence of the earth' gravitational field

1. What is the equation of motion?
2. Find the solution for the case that at the time $t = 0$ the rope is released where the piece x_0 of the rope is already suspended?
3. How big is the velocity when the end of the rope is just slipping over the edge?

Exercise 2.3.6 An inclined plane with the inclination angle α is balanced on a scales. On the inclined plane there is, somehow fixed, a mass m. The scales exhibits the weight.

1. The fixing is removed and the mass slides frictionlessly down the inclined plane. Will the display of the scales change?
2. How does the contact force change?

Exercise 2.3.7 Discuss the vertical throw of a mass m in the earth's gravitational field $\left(\mathbf{F} = -\gamma \frac{mM}{r^3}\mathbf{r}\right)$.

1. The initial velocity at the time of throwing of the mass from the earth's surface is v_0. Find the velocity v of the mass as function of its distance z from the earth's center.
2. How large must the minimum value of v_0 be for the mass to leave the gravity region?

Exercise 2.3.8 Test by the following arithmetic problems your capability to work with complex numbers.

1. Calculate

$$(-i)^3,\ i^{15}\ ,\ \ \sqrt{4(-25)}\ ,\ \ \ln(1+i)\ ,\ e^{i(\pi/3)}\ ,\ \ e^{i(\pi/2)}\ .$$

2. Calculate the product $z = z_1 z_2$:

$$\text{a) } z_1 = 1 + i\ ; \quad z_2 = 1 - i\ ,$$
$$\text{b) } z_1 = 3 - 2i\ ; \quad z_2 = 5 + 4i\ .$$

3. Mark the points z_i and z_i^* in the complex plane :

$$z_1 = -1 - i\ , \quad z_2 = -3 + 1/2i\ , \quad z_3 = 3 + 2i\ , \quad z_4 = 3/2i\ .$$

4. Find the polar representation of the following complex numbers:

$$z_1 = i - 1\ , \quad z_2 = -(1 + i)\ , \quad z_3 = e^{3+2i}\ , \quad z_4 = \frac{1}{2}\sqrt{3} + \frac{i}{2}\ , \quad z_5 = -i\ .$$

5. Determine real and imaginary parts of the following complex numbers:

$$z_1 = e^{1/2 + i\pi}\ , \quad z_2 = e^{-1 - i(3/2\pi)}\ ; \quad z_3 = e^{3-i}\ .$$

6. $z(t)$ is a linear function of time:

$$\text{a) } z(t) = -t + i\,2\pi t\ ,$$
$$\text{b) } z(t) = 2t - i\,3/2t\ .$$

How does the real part of $e^{z(t)}$ read and what is its period?

Exercise 2.3.9 Prove by exploiting certain properties of the complex numbers the following addition theorems of the trigonometric functions:

$$1)\ \sin(\alpha+\beta) = \sin\alpha\,\cos\beta + \sin\beta\,\cos\alpha$$
$$2)\ \cos(\alpha+\beta) = \cos\alpha\,\cos\beta - \sin\alpha\,\sin\beta$$

Exercise 2.3.10 Determine the general solution of the following inhomogeneous differential equations:

$$1)\ 7\ddot{x} - 4\dot{x} - 3x = 6\ ,$$
$$2)\ \ddot{z} - 10\dot{z} + 9z = 9t\ .$$

Exercise 2.3.11 Try to guess (‘*targetedly*’) for each of the following inhomogeneous differential equations a special solution:

$$1)\ \ddot{y} + \dot{y} + y = 2t + 3\ ,$$
$$2)\ 4\ddot{y} + 2\dot{y} + 3y = -2t + 5\ .$$

Exercise 2.3.12 Solve the differential equation:

$$\ddot{z} + 4z = 0$$

with the boundary conditions:

$$1)\ z(0) = 0\ ; \quad z(\pi/4) = 1\ ,$$
$$2)\ z(\pi/2) = -1\ ; \quad \dot{z}(\pi/2) = 1\ .$$

Exercise 2.3.13 A body of mass m is moving in the earth’s gravitational field under the influence of Newton-friction.

1. What is its equation of motion? You may restrict yourself to the vertical component.
2. At which initial velocity would a uniform straight-line motion occur?
3. Determine the time dependence of the velocity in the case where the body starts to fall at $t = 0$ with the velocity $v(t = 0) = 0$.
4. Calculate the drop distance as function of the time if the body is released at $t = 0$ in the height H. Discuss also the limiting case $\alpha \to 0$.

Exercise 2.3.14 A body of mass m is subjected to the gravitational force and Stokes-friction.

1. Write down its equation of motion. Which type of differential equation one has to deal with?
2. Determine the **general** solution of the differential equation!

3. The body is shot at time $t = 0$ from the ground under the angle of inclination $\gamma = 45°$ relative to the earth's surface with the velocity $\sqrt{2}\,v_0$. What are the initial conditions?
4. Calculate the path line $\mathbf{r} = \mathbf{r}(t)$ with the initial conditions of 3..
5. Calculate the maximum flight altitude attained by the body. After what time is this altitude reached?

Exercise 2.3.15 We discuss the general solution

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t \quad (A,\ B \text{ known})$$

of the linear harmonic oscillator

$$\ddot{x} + \omega_0^2 x = 0 .$$

1. At which time t_1 does the oscillator achieve its maximum displacement $x_{\max}$? How big is $x_{\max}$? What is the value of the acceleration at time t_1?
2. At what time t_2 does the oscillator reach its maximum velocity $\dot{x}_{\max}$? What is the value of $\dot{x}_{\max}$? How large is the displacement at t_2? Which simple relation exists between $x_{\max}$ and $\dot{x}_{\max}$?
3. At what time t_3 does the oscillator experience the maximum acceleration $\ddot{x}_{\max}$? How large is it? What are the values of displacement and velocity at time t_3?

Exercise 2.3.16 A linear harmonic oscillator ($\omega_0^2 = \frac{k}{m}$; k: spring constant, m: mass) is subjected to Stokes-friction ($F_R = -\alpha\dot{x}$) and gets in its rest position ($x = 0, \dot{x} = 0$) an external impulse at time $t = 0$:

$$F(t) = \begin{cases} \dfrac{mv_0}{t_0} & \text{for } 0 \le t \le t_0 \\ 0 & \text{otherwise} \end{cases}$$

1. Determine the deflection $x(t)$ for $0 \le t \le t_0$!
2. The force $F(t)$ performs a finite jump at $t = t_0$. Find out which boundary conditions follow from that for $t > t_0$. Fix therewith $x(t)$ for $t > t_0$.
3. Discuss the extremely short impulse of force: $t_0 \to 0$!
4. What are the results for the long-lasting impulse of force $t_0 \gg m/\alpha$?

Exercise 2.3.17 A ball-shaped waterdrop (radius R, volume V, mass m) falls vertically downwards in the earth's gravitational field. Thereby on the drop acts a friction force

$$\mathbf{F}_R = -\widehat{\alpha}R^2 \cdot \mathbf{v} \quad (\widehat{\alpha} > 0)$$

The fall starts at $t = 0$ ($\mathbf{v}(0) = 0$). In the air the volume of the drop increases because of condensation of water vapour from the atmosphere, and it is proportional to its surface:

$$\frac{\underset{\cdot}{\mathrm{V}}}{\underset{\cdot}{\mathrm{t}}} = \gamma \cdot 4\pi \cdot R^2(t) \quad (R(t=0) = R_0) \; .$$

The density ρ of the water thereby remains constant so that the mass of the drop increases. Formulate the equation of motion and try to integrate it. It is recommended to use R instead of the time t as independent variable.

Exercise 2.3.18 A particle of mass m and charge q moves under the influence of a temporally and spatially constant magnetic induction $\mathbf{B}$. It experiences the Lorentz force $\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$ where $\mathbf{v}$ is the particle velocity.

1. What is its equation of motion?
2. Show that $|\dot{r}|$ is constant.
3. Show that the angle between $\dot{\mathbf{r}}$ and $\mathbf{B}$ is constant.
4. Find out by a first integration a relation between $\mathbf{r}$ and $\dot{\mathbf{r}}$. Use the initial conditions $\mathbf{r}(t=0) = \mathbf{r}_0$ and $\dot{\mathbf{r}}(t=0) = \mathbf{v}_0$.
5. What can be said about the field-parallel component $\dot{\mathbf{r}}_{\|}$ and the component perpendicular to the field $\dot{\mathbf{r}}_{\perp}$ of the velocity $\dot{\mathbf{r}}$?
6. If $\varphi(t)$ is the angle between $\dot{\mathbf{r}}_{\perp}$ and $\mathbf{e}_1$-axis, show that

$$\varphi(t) = -\omega t + \alpha \; ; \quad \omega = \frac{qB}{m} \; ; \quad \alpha = \text{const} \; .$$

7. The choice of the directions of $\mathbf{e}_1$ and $\mathbf{e}_2$ is still available. Choose

$$\mathbf{e}_2 \uparrow\uparrow \mathbf{v}_{0\perp} = \big(\mathbf{e}_3 \times \big(\mathbf{v}_0 \times \mathbf{e}_3\big)\big) \quad \text{(see Exercise 1.3.7)} \; .$$

   Verify that then we must have

$$\mathbf{e}_1 \uparrow\uparrow (\mathbf{v}_0 \times \mathbf{e}_3) \quad \text{and } \alpha = \pi/2$$

   Give therewith the full solution for $\dot{r}(t)$.
8. Find by a further integration $\mathbf{r}(t)$.
9. Under what conditions does the particle move on a circular path perpendicular to the field $\mathbf{B}$? Express the radius R by the magnitude of the initial velocity $\mathbf{v}_0$.
10. Which geometrical form does the general solution have?

Exercise 2.3.19 A mass point moves linearly under the influence of the force

$$F = F(x) = -kx - \gamma x^3 \; .$$

Here γ is a very small correction. Calculate the period of the (weakly) anharmonic oscillation. For an approximate solution use:

$$1)\ \sqrt{1+x} \approx 1 + \frac{1}{2}x$$

$$2)\ \sqrt{1+x} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2\ .$$

2.4 Fundamental Concepts and Theorems

In this section we want to investigate in more detail some of the fundamental concepts and terms of Classical Mechanics such as

work, power, energy, angular momentum, torque (moment), ...

which, to some extent, we have already introduced in the last chapter for the special case $\mathbf{F} = F(x)\mathbf{e}_x$. For these terms, under certain conditions, **conservation laws** become valid which can provide important information about the particle's movement pattern and in addition, more technically, can substantially simplify the integration of the equations of motion.

2.4.1 Work, Power, and Energy

We start with the term **'work'** which has to be generalized for *arbitrary* force fields,

$$\mathbf{F} = \mathbf{F}\left(\mathbf{r}, \dot{\mathbf{r}}\ , t\right)$$

in analogy and compared to (2.204). To produce an infinitesimal displacement $d\mathbf{r}$ the work

$$\delta W = -\mathbf{F} \cdot d\mathbf{r} \tag{2.219}$$

has to be invested. The sign convention is the same as explained after (2.205). The symbol δ is chosen consciously since this expression does not necessarily represent a total differential as we will see in the following. Here it merely denotes an infinitesimally small quantity.

For finite pathways (Fig. 2.40) it holds:

$$W_{21} = -\int_{P_1}^{P_2} \mathbf{F}\left(\mathbf{r}, \dot{\mathbf{r}}\ , t\right) \cdot d\mathbf{r}\ . \tag{2.220}$$

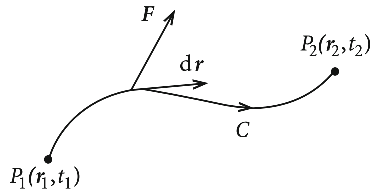


Fig. 2.40 To the illustration of the line integral of the work

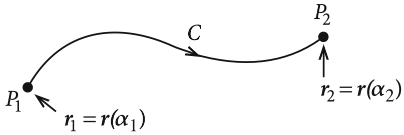


Fig. 2.41 For the evaluation of the line integral of the work by parametrization of the corresponding space curve

This quantity normally depends on:

1) force field $\mathbf{F}$,
2) endpoints P_1, P_2,
3) path C,
4) temporal course of movement.

If $\mathbf{F} = \mathbf{F}(\mathbf{r})$ then of course point 4) becomes meaningless, i.e. W_{21} depends only on the shape of the path and no longer on the temporal course of motion of the mass point along the trajectory. The integration in (2.220) represents a so-called **curvilinear (line) integral**. One evaluates such line integrals by tracing them back, in some way, to normal Riemann-integrals. That can be done with the parametrization of the space curve C introduced in Sect. 1.4.1 (Fig. 2.41). The parameter α can but need not necessarily be the time t:

$$C: \ \mathbf{r} = \mathbf{r}(\alpha) \ ; \quad \alpha_1 \le \alpha \le \alpha_2 \ ;$$

$$d\mathbf{r} = \frac{d\mathbf{r}(\alpha)}{d\alpha} d\alpha \ .$$

Therewith Eq. (2.220) can also be written as follows:

$$W_{21} = -\int_{\alpha_1}^{\alpha_2} \mathbf{F}\left(\mathbf{r}, \dot{\mathbf{r}}\ , t\right) \cdot \frac{d\mathbf{r}(\alpha)}{d\alpha} d\alpha \ . \tag{2.221}$$

The shape of the path C manifests itself in the term $\frac{d}{d\alpha}\mathbf{r}(\alpha)$. In order to become familiar with such curvilinear integrals let us insert an exercising example.

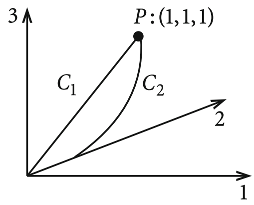


Fig. 2.42 For the demonstration of the possible path dependence of the work integral for a given force field

Example We consider the vector field

$$\mathbf{F} = \left(2x_1^2 - 3x_2,\ 4x_2x_3,\ 3x_1^2x_3\right) \tag{2.222}$$

and calculate the work along two different paths C_1 and C_2 (Fig. 2.42):

$$C_1 : \text{straight line: } \mathbf{r}(\alpha) = (\alpha, \alpha, \alpha) \ ; \quad 0 \le \alpha \le 1 \ ,$$
$$C_2 : \mathbf{r}(\alpha) = \left(\alpha, \alpha^2, \alpha^3\right) \ ; \quad 0 \le \alpha \le 1 \ .$$

At first we need:

$$\frac{d\mathbf{r}}{d\alpha} = \begin{cases} (1, 1, 1) & : \ C_1 \\ (1, 2\alpha, 3\alpha^2) & : \ C_2 \ , \end{cases}$$

$$\Longrightarrow \mathbf{F} = \begin{cases} \left(2\alpha^2 - 3\alpha, 4\alpha^2, 3\alpha^3\right) & : \ C_1 \\ \left(2\alpha^2 - 3\alpha^2, 4\alpha^5, 3\alpha^5\right) & : \ C_2 \ , \end{cases}$$

$$\Longrightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{d\alpha} = \begin{cases} 3\alpha^3 + 6\alpha^2 - 3\alpha & : \ C_1 \\ 9\alpha^7 + 8\alpha^6 - \alpha^2 & : \ C_2 \ . \end{cases}$$

Now we are able to calculate the works carried out on the two different paths:

$$W_{C_1} = -\int_0^1 \left(3\alpha^3 + 6\alpha^2 - 3\alpha\right) d\alpha = -5/4 \ ,$$

$$W_{C_2} = -\int_0^1 \left(9\alpha^7 + 8\alpha^6 - \alpha^2\right) d\alpha = -325/168 \ .$$

This example demonstrates that the work performed can be dependent on the chosen path. This path dependence is a very important point to which we shall come back more elaborately in the next section.

With (2.221) we now define as the next fundamental term, namely, the **power** P, as *'rate of work done'*:

$$P = \frac{\delta W}{dt} = -\frac{d}{dt}\int_{t_0}^{t} \mathbf{F}\left(\mathbf{r}\left(t'\right), \dot{\mathbf{r}}\left(t'\right), t'\right) \cdot \dot{\mathbf{r}}\left(t'\right)\, dt'$$

$$\Longrightarrow P = -\mathbf{F}\left(\mathbf{r}(t), \dot{\mathbf{r}}(t), t\right) \cdot \dot{\mathbf{r}}(t)\ . \tag{2.223}$$

The **dimension** comes out according to (2.218):

$$[P] = \text{Joule/s} = \text{Watt}\ . \tag{2.224}$$

The power P naturally depends, for **all** types of force fields, on the time schedule of the particle motion. We encounter P when we multiply Newton's equation of motion scalarly with the velocity:

$$m\,\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} = \mathbf{F} \cdot \dot{\mathbf{r}}\ \ .$$

The left-hand side we recognize to be the time derivative of the

Kinetic Energy

$$T = \frac{m}{2}\dot{\mathbf{r}}^2\ , \tag{2.225}$$

which we have already introduced in (2.211) for the one-dimensional motion:

$$\frac{d}{dt}T = \frac{d}{dt}\frac{m}{2}\dot{\mathbf{r}}^2 = \mathbf{F} \cdot \dot{\mathbf{r}} = -P\ . \tag{2.226}$$

The comparison with (2.223) then yields after integration from t_1 to $t_2 > t_1$:

$$W_{21} = T_1 - T_2 = \frac{m}{2}\left[\dot{\mathbf{r}}^2\left(t_1\right) - \dot{\mathbf{r}}^2\left(t_2\right)\right]\ . \tag{2.227}$$

The work done on a mass point along its path produces a change in its state of motion.

In case of a one-dimensional motion it was always possible to interpret in Eq. (2.200), which is the analogue to (2.226), the right-hand side as time derivative of a pure space function. For three-dimensional motion and **arbitrary** forces that does **not necessarily** remain valid. Forces for which this nevertheless holds are special cases which are called **'conservative'**:

$$\frac{d}{dt}V(\mathbf{r}) = -\mathbf{F} \cdot \dot{\mathbf{r}}\ . \tag{2.228}$$

One then refers to $V(\mathbf{r})$ as the **'potential of the force F'** or as the **'potential energy'**. We shall investigate in the next section how one can find out whether a given force is conservative or not. The friction is an example for a non-conservative force.

We decompose the forces acting on the mass point into conservative and non-conservative parts, where the latter are also denoted as **'dissipative'**:

$$\mathbf{F} = \mathbf{F}_{\text{cons}} + \mathbf{F}_{\text{diss}} \ .$$

$\mathbf{F}_{\text{cons}}$ has a potential $V(\mathbf{r})$. That we insert into (2.226):

$$\frac{d}{dt}\left[T + V(\mathbf{r})\right] = \mathbf{F}_{\text{diss}} \cdot \dot{\mathbf{r}} \ . \tag{2.229}$$

One defines again as

Energy of the Mass Point

$$E = \frac{m}{2}\dot{\mathbf{r}}^2 + V(\mathbf{r}) \ . \tag{2.230}$$

Equation (2.229) then represents the **energy theorem**:
The time rate of change of energy is equal to the power of the dissipative forces.
If all the forces are conservative, then we have the

energy conservation law

$$\frac{m}{2}\dot{\mathbf{r}}^2 + V(\mathbf{r}) = E = \text{const} \ . \tag{2.231}$$

One has to bear in mind that 'energy' here always means **mechanical** energy. Dissipative forces transform this energy into other types of energy such as, for instance, heat. The sum of **all** energy contributions of course remains always constant.

2.4.2 Potential

Let us now investigate under what conditions a force is conservative. For this purpose we explicitly perform the time differentiation in (2.228):

$$\begin{aligned} \frac{d}{dt}V\left(x_1, x_2, x_3\right) &= \frac{\partial V}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial V}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial V}{\partial x_3}\frac{dx_3}{dt} = \\ &= \dot{\mathbf{r}} \cdot \nabla V \ . \end{aligned}$$

Therewith we get the result that if **F** is conservative:

$$\dot{\mathbf{r}} \cdot \nabla V = -\mathbf{F} \cdot \dot{\mathbf{r}} \ . \tag{2.232}$$

From this we conclude that a force is **conservative** if it can be written as the gradient of a scalar potential. That means in particular that **F** must not depend either on $\dot{\mathbf{r}}$ or on t:

$$\mathbf{F} = \mathbf{F}(\mathbf{r}) = -\nabla V(\mathbf{r}) \ . \tag{2.233}$$

The minus sign is a convention. We presume that the potential V possesses continuous partial derivatives up to at least second order. Then, according to (1.257), the second partial derivatives of V are interchangeable:

$$\frac{\partial^2 V}{\partial x_i \partial x_j} = \frac{\partial^2 V}{\partial x_j \partial x_i} \ ; \qquad i,j = 1,2,3 \ .$$

With (2.233) this leads to:

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i} \ ; \qquad i,j = 1,2,3 \ .$$

By comparing this equation with (1.286) one recognizes that a **conservative force F** has to fulfill (cf. (1.290)):

$$\nabla \times \mathbf{F} = 0 \tag{2.234}$$

One can show that this condition is not only necessary but also sufficient:
A force **F** *has a potential if and only if* $\nabla \times \mathbf{F}$ *vanishes!*
Further we can formulate a third, now an integral criterion. With (1.261) it holds for the total differential of the scalar function V:

$$dV = \nabla V \cdot d\mathbf{r} \ .$$

If we denote by $\oint$ the line integral over a closed path then it follows:

$$\oint \nabla V \cdot d\mathbf{r} = \oint dV = V_{\text{final}} - V_{\text{initial}} = 0 \ .$$

This, however, means with (2.233):

$$\oint \mathbf{F} \cdot d\mathbf{r} = 0 \Longleftrightarrow \mathbf{F} \text{ conservative} \ . \tag{2.235}$$

A conservative force does not carry out any work on a closed path!

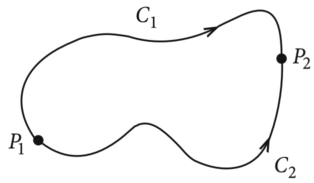


Fig. 2.43 Decomposition of a closed path

Now we can construct a closed path also by combination of two different paths C_1 and C_2 which are connected at two points P_1, P_2 (Fig. 2.43):

$$0 = \int\limits_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int\limits_{-C_2} \mathbf{F} \cdot d\mathbf{r} = \int\limits_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int\limits_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

$$\Longrightarrow \int\limits_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int\limits_{C_2} \mathbf{F} \cdot d\mathbf{r} \,. \tag{2.236}$$

A force field **F** *is conservative if and only if the work necessary to move the mass point between two space points turns out to be path-independent!*
For this reason, the force field (2.222), previously considered as an example, is **not** conservative!

If one wants to calculate the potential of a force one should first check whether or not $\nabla \times \mathbf{F} = 0$ is fulfilled. If yes, then one can exploit the path-independence in order to determine the potential at the point P via a *'calculationally convenient'* path:

$$V(P) = \int\limits_{P_0}^{P} dV = -\int\limits_{P_0}^{P} \mathbf{F} \cdot d\mathbf{r} \tag{2.237}$$

It is fixed only up to an additive constant. One therefore sets the potential of an arbitrarily chosen reference point P_0 equal to zero. Very often it appears convenient to choose the infinitely distant point. The potential $V(P)$ then corresponds to the work that is needed to bring the mass point from the reference point P_0 to P.

Examples

(a) **Linear harmonic oscillator**
As discussed elaborately in Sect. 2.3.9 the forces $F = F(x)$ of one-dimensional motion always possess a potential. The oscillator potential we have already given in (2.209):

$$V(x) = \frac{k}{2}x^2 + c \,.$$

Here one normally agrees to put $V(x = 0) = 0$ and consequently $c = 0$.

(b) **Linear harmonic oscillator with friction**
For the total force it holds according to (2.167):

$$F = -kx - \alpha\dot{x} \ .$$

Because of the $\dot{x}$-dependency it can **not** be conservative. The energy theorem (2.229) reads in this case:

$$\frac{d}{dt}\left(\frac{m}{2}\dot{x}^2 + \frac{k}{2}x^2\right) = -\alpha\dot{x}^2 \ . \tag{2.238}$$

The energy steadily decreases because of the friction.

(c) **Spatially isotropic harmonic oscillator**
This oscillator is defined by the force

$$\mathbf{F}(\mathbf{r}) = -k\mathbf{r} \ . \tag{2.239}$$

One easily checks (see (1.292)) that

$$\nabla \times \mathbf{F} = 0$$

The force thus possesses a potential which we calculate using (2.237):

$$V(\mathbf{r}) = -\int_0^{\mathbf{r}} \mathbf{F}\cdot d\mathbf{r}' = k \int_0^{\mathbf{r}=(x,y,z)} \left(x'dx' + y'dy' + z'dz'\right) =$$
$$= k\int_0^x x'dx' + k\int_0^y y'dy' + k\int_0^z z'dz' = \frac{k}{2}\left(x^2+y^2+z^2\right) \ .$$

This yields:

$$V(\mathbf{r}) = \frac{k}{2}\mathbf{r}^2 \ . \tag{2.240}$$

2.4.3 Angular Momentum and Torque (Moment)

If we multiply the basic dynamical equation (2.43) vectorially by $\mathbf{r}$,

$$m\,(\mathbf{r}\times\ddot{\mathbf{r}}) = (\mathbf{r}\times\mathbf{F}) \ , \tag{2.241}$$

then there appears on the left-hand side the time-derivative of an important physical quantity:

$$\mathbf{L} = m\,(\mathbf{r} \times \dot{\mathbf{r}}) = (\mathbf{r} \times \mathbf{p}) \qquad \textbf{angular momentum}\ . \tag{2.242}$$

Since both position $\mathbf{r}$ and momentum $\mathbf{p}$ are polar vectors the resulting $\mathbf{L}$ must be an axial vector oriented perpendicularly to the plane spanned by $\mathbf{r}$ and $\mathbf{p}$. With the further definition,

$$\mathbf{M} = (\mathbf{r} \times \mathbf{F}) \qquad \textbf{torque (moment)}\ , \tag{2.243}$$

it follows from (2.241):

$$\frac{d}{dt}\mathbf{L} = \mathbf{M}\ . \tag{2.244}$$

This equation represents the **angular-momentum law**:
The time rate of the change of angular momentum is equal to the applied torque.
If the torque is identical to zero then this theorem becomes the

Law of Conservation of Angular-Momentum

$$\mathbf{M} = 0 \Longleftrightarrow \frac{d}{dt}\mathbf{L} = 0\ ; \quad \mathbf{L} = \text{const}\ . \tag{2.245}$$

There are two possibilities for getting $\mathbf{M} = \mathbf{0}$:

$$\begin{array}{ll} 1)\ \mathbf{F} \equiv \mathbf{0} & \text{(trivial case)}\ , \\ 2)\ \mathbf{F} \uparrow\uparrow \mathbf{r} & \text{(central field)}\ . \end{array} \tag{2.246}$$

Case (1) is identical to the uniform straight-line motion of the mass point:

$$\dot{\mathbf{r}} = \mathbf{v} = \text{const}\ .$$

At first glance it appears astonishing that a uniform straight-line movement possesses any, even if constant, angular momentum. In Fig. 2.44 $\mathbf{L}$ is perpendicular to the plane of the paper with the magnitude $m\,v\,d$. Only if the reference point

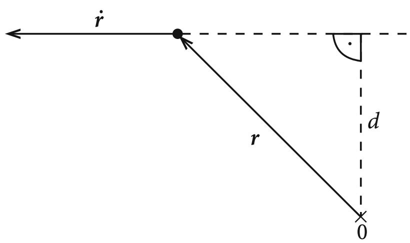


Fig. 2.44 To the angular momentum of a uniform straight-line movement

(origin of coordinates) lies on the straight line then **L** indeed disappears. That gives evidence that the angular momentum is not at all a genuine particle property, but rather depends on the choice of the reference point.

A shift of the origin of coordinates by the constant vector **a**,

$$\mathbf{r}' = \mathbf{r} + \mathbf{a}\ ; \quad \dot{\mathbf{r}}' = \dot{\mathbf{r}} \Longrightarrow \mathbf{p}' = \mathbf{p}\ ,$$

means for the angular momentum:

$$\mathbf{L}' = \left(\mathbf{r}' \times \mathbf{p}'\right) = (\mathbf{r} \times \mathbf{p}) + (\mathbf{a} \times \mathbf{p}) = \mathbf{L} + (\mathbf{a} \times \mathbf{p})\ . \tag{2.247}$$

If $\mathbf{L}$ is constant then $\mathbf{L}'$ is also constant only if simultaneously the conservation of momentum also holds $\mathbf{p} =$ const. Furthermore, it does **not necessarily** follow from $\mathbf{L} = \mathbf{0}$ that also $\mathbf{L}' = \mathbf{0}$. In general that is indeed not the case.

The second possibility for $\mathbf{M} = \mathbf{0}$ in (2.246) shall be discussed in a separate section.

2.4.4 Central Forces

A force-type of the profile

$$\mathbf{F} = f\,(\mathbf{r}, \dot{\mathbf{r}}\ , t)\ \mathbf{e}_r \tag{2.248}$$

is called a **'central force'**. Thus the force is directed along the radial rays which start from the center (origin) of the force (Fig. 2.45). For such forces the angular momentum **L**, if referred to the force center, is constant, according to (2.245).

Central forces in the general form (2.248) are not necessarily conservative. It rather holds:

$$\textit{Central force}\ \ \mathbf{F}\ \ \textit{conservative}\quad \Longleftrightarrow \quad \mathbf{F} = f(r)\,\mathbf{e}_r\ . \tag{2.249}$$

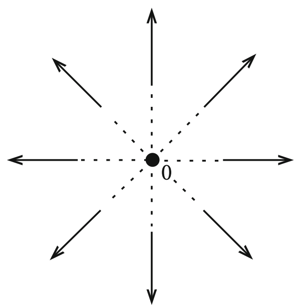


Fig. 2.45 Impact lines of a central force

It is clear that $\mathbf{F}$ must not depend on $\dot{\mathbf{r}}$ and t to be conservative. For a proof of (2.249) we therefore can restrict ourselves to forces $\mathbf{F}$ of the form:

$$\mathbf{F} = f(\mathbf{r})\,\mathbf{e}_r \ .$$

According to (2.234) the force $\mathbf{F}$ is conservative if and only if the curl of $\mathbf{F}$ vanishes. That we inspect with (1.289):

$$\nabla \times \mathbf{F} = \frac{f(\mathbf{r})}{r} \nabla \times \mathbf{r} + \left[\left(\nabla \frac{f(\mathbf{r})}{r} \right) \times \mathbf{r} \right] .$$

After (1.292) we can exploit $\nabla \times \mathbf{r} = 0$, so it remains to require:

$$0 \stackrel{!}{=} \left[\nabla \left(\frac{f(\mathbf{r})}{r} \right) \times \mathbf{r} \right] .$$

Hence the two vectors in the square bracket have to be parallel. In view of (1.271) and the subsequent discussion one realizes that the gradient vector is orthogonal to the planes $f(\mathbf{r})/r = \text{const}$. Hence these planes must simultaneously be orthogonal to $\mathbf{r}$. That means, however, that $f(\mathbf{r})/r$ has to be constant on the surface of a sphere. This is possible only if $f(\mathbf{r}) = f(r)$. That proves (2.249)!

We can add a further statement:

A conservative force $\mathbf{F}$ *is a central force if and only if it holds:*

$$V(\mathbf{r}) = V(r) \tag{2.250}$$

Proof

(a) We assume that $\mathbf{F}$ is conservative and $V(\mathbf{r}) = V(r)$ then it follows:

$$\mathbf{F} = -\nabla V(r) \stackrel{(1.277)}{=} -\frac{dV}{dr}\mathbf{e}_r \ .$$

Thus $\mathbf{F}$ is a central force of type (2.249).

(b) We assume that $\mathbf{F}$ is a conservative central force then it follows:

$$\mathbf{F} = -\nabla V = f(r)\mathbf{e}_r \Longleftrightarrow \frac{\partial V}{\partial x_i} = -\frac{f(r)}{r} x_i = \\ = -f(r)\frac{\partial r}{\partial x_i} \ .$$

We choose $\hat{f}(r)$ so, that $f(r) = \frac{d\hat{f}(r)}{dr}$. Then the last relation reads:

$$\frac{\partial}{\partial x_i} V = -\frac{\partial}{\partial x_i} \hat{f}(r) \qquad \forall i \ .$$

Therewith V can depend only on r.

If conservation of angular momentum is valid as in the case of central forces, then quite far-reaching statements about the type of motion of the mass point can already be formulated. From the definition of **L** it follows after scalar multiplication with **r**:

$$\mathbf{r} \cdot (m\ (\mathbf{r} \times \dot{\mathbf{r}})) = 0 = \mathbf{r} \cdot \mathbf{L}\ .$$

If **L** is a constant vector then this equation represents a plane which is perpendicular to **L** and contains the origin:
In the case the angular-momentum is conserved the mass point moves on a plane which is perpendicular to the angular momentum and contains the origin of coordinates!
From the constancy of $|\mathbf{L}|$ follows a further important statement. If the position vector sweeps an area dS in time dt then dS is just half of the parallelogram spanned by $\mathbf{r}(t)$ and $\mathbf{r}(t + dt)$ (see Fig. 2.46):

$$dS = \frac{1}{2}\,|(\mathbf{r}(t) \times \mathbf{r}(t + dt))| = \frac{1}{2}\,|(\mathbf{r}(t) \times (\mathbf{r}(t)\ + \dot{\mathbf{r}}(t)dt)| =$$
$$= \frac{1}{2}dt\,|(\mathbf{r}(t) \times \dot{\mathbf{r}}(t))|\ .$$

That means:

$$\frac{dS}{dt} = \frac{1}{2m}|\mathbf{L}|\ . \tag{2.251}$$

Therewith we have found the **'area conservation principle'**:
In the case of angular-momentum conservation the position (radius) vector of the mass point sweeps equal areas in equal times.

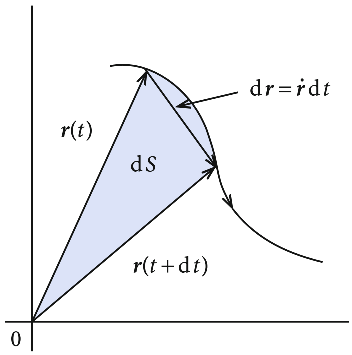


Fig. 2.46 Sketch for the derivation of the area conservation principle

2.4.5 Integration of the Equations of Motion

If the **law of conservation of angular-momentum**

$$\mathbf{L} = m\,(\mathbf{r} \times \dot{\mathbf{r}}) = \text{const}$$

or the **energy conservation law**

$$E = \frac{m}{2}\,\dot{\mathbf{r}}^2 + V(\mathbf{r}) = \text{const}$$

are valid then one speaks of

first integrals of motion

The original equations of motion are always differential equations of second order, the conservation laws, on the other hand, are only of first order. Furthermore, on the basis of the conservation laws a **general procedure** for the complete solution of the equations of motion can be developed.

We have shown that the angular-momentum conservation law holds if and only if the acting force is a central force:

$$\mathbf{F} = f\,(\mathbf{r}, \dot{\mathbf{r}}\,, t)\,\mathbf{r}\ .$$

(The trivial case $\mathbf{F} \equiv 0$ shall be excluded!)

If simultaneously the energy conservation law holds then definitely a potential must exist. Hence, the central force is conservative and must be of the form:

$$\mathbf{F} = f(r)\mathbf{r}.$$

Moreover we know that in such a case the potential can depend only on the magnitude of $\mathbf{r}$:

$$V = V(r)\ .$$

Therewith we will further evaluate the conservation laws. Because of the constancy of the angular momentum the motion will happen in a fixed plane. Let this be the xy plane. For the description we choose spherical coordinates (r, ϑ, φ) where we can directly exploit

$$\vartheta = \frac{\pi}{2} \Longrightarrow \dot{\vartheta} = 0$$

In (2.21) we derived:

$$\mathbf{r} = r\,\mathbf{e}_r \ ,$$
$$\dot{\mathbf{r}} = \dot{r}\,\mathbf{e}_r + r\dot{\vartheta}\,\mathbf{e}_\vartheta + r\sin\vartheta\,\dot{\varphi}\,\mathbf{e}_\varphi \ .$$

That means here:

$$\dot{\mathbf{r}} = \dot{r}\,\mathbf{e}_r + r\dot{\varphi}\,\mathbf{e}_\varphi \ , \tag{2.252}$$

and for the **angular momentum** we get:

$$\mathbf{L} = -m\,r^2\dot{\varphi}\,\mathbf{e}_\vartheta = m\,r^2\dot{\varphi}\,\mathbf{e}_z \ .$$

Because of

$$\dot{\mathbf{r}}^2 = \dot{\mathbf{r}}\cdot\dot{\mathbf{r}} = \dot{r}^2 + r^2\dot{\varphi}^2$$

the **energy theorem** reads:

$$E = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) + V(r) \ . \tag{2.253}$$

Using the angular-momentum law we can now eliminate $\dot{\varphi}$ from the energy theorem:

$$E = \frac{m}{2}\dot{r}^2 + \frac{L^2}{2mr^2} + V(r) \ . \tag{2.254}$$

When we introduce the **effective potential**

$$V_{\mathrm{eff}}(r) = \frac{L^2}{2mr^2} + V(r) \tag{2.255}$$

the energy theorem has mathematically the same structure as that for the one-dimensional movement discussed in Sect. 2.3.9. We therefore can proceed for the integration in the same manner. Analogously to (2.203) we now get:

$$t - t_0 = \int\limits_{r_0}^{r} \frac{dr'}{\sqrt{\frac{2}{m}\left[E - V_{\mathrm{eff}}(r')\right]}} \ . \tag{2.256}$$

By inversion one comes to:

$$r = r(t) \ .$$

For the complete solution $\mathbf{r}(t) = r(t)\big(\cos\varphi(t), \sin\varphi(t), 0\big)$ we still need $\varphi = \varphi(t)$. At first we can exploit the angular-momentum law in order to derive $\varphi = \varphi(r)$:

$$d\varphi = \frac{L}{mr^2}dt = \frac{L}{mr^2}\frac{dr}{\dot{r}} = \frac{L}{mr^2}\frac{dr}{\sqrt{\frac{2}{m}\left[E - V_{\text{eff}}(r)\right]}} \,.$$

That can formally be integrated:

$$\varphi - \varphi_0 = \int\limits_{\bar{r}_0}^{r} \frac{L\,dr'}{r'^2\sqrt{2m\left[E - V_{\text{eff}}(r')\right]}} \,. \tag{2.257}$$

By inversion we obtain herefrom the path $r = r(\varphi)$ and after insertion of $r = r(t)$ also $\varphi = \varphi(t)$.

The shape of the trajectory and its the temporal evolution depend on two essential integration constants L and E. The other constants r_0, $\bar{r}_0$, φ_0, t_0 can be fixed for the sake of convenience by a suitable choice of the system of coordinates and the time zero!

The discussion in Sect. 2.3.9 about classically forbidden and allowed regions of the respective motion can be transferred literally by simply replacing $V(x)$ by $V_{\text{eff}}(r)$.

Example: Attractive Coulomb Potential

$$V_{\text{eff}}(r) = -\frac{q_1 q_2}{4\pi\,\varepsilon_0 r} + \frac{L^2}{2m\,r^2} \,. \tag{2.258}$$

For $E = E_0 < 0$ we have a bonded oscillatory motion. For $E = E_1 > 0$ the particle can in principle run up to infinity *('scattering states')* without returning (Fig. 2.47).

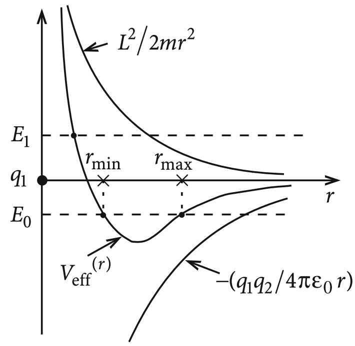


Fig. 2.47 The effective potential belonging to the Coulomb potential

2.4.6 Exercises

Exercise 2.4.1

1. Investigate whether or not the following force field is conservative:

$$\mathbf{F}(\mathbf{r}) = \left(\alpha_1 y^2 z^3 - 6\alpha_2 x z^2\right)\,\mathbf{e}_x + 2\alpha_1 x y\, z^3 \mathbf{e}_y + \left(3\alpha_1 x y^2 z^2 - 6\alpha_2 x^2 z\right)\,\mathbf{e}_z$$

2. A mass point is moved in the force field **F** from the origin $\mathcal{O}$ to the space point $P(x_0, y_0, z_0)$ along the following path (Fig. 2.48),

$$0 \xrightarrow{C_1} P_1 \xrightarrow{C_2} P_2 \xrightarrow{C_3} P\ ,$$

   i.e. piecewise along the coordinate axes.
   Give a parametrization of the path and calculate therewith the work executed on the body when it is shifted from $\mathcal{O}$ to P.
3. Does **F** have a potential? If yes, find it?

Exercise 2.4.2 Calculate the work which has to be performed against the field

$$\mathbf{F}(\mathbf{r}) = \alpha \cdot \mathbf{r} \qquad (\alpha = \text{const})$$

when going from the point P_1 to the point P_2. Evaluate the relevant line integrals along the paths in, respectively, Figs. 2.49, 2.50, and 2.51:

Determine the potential of the given force and verify therewith the above results.

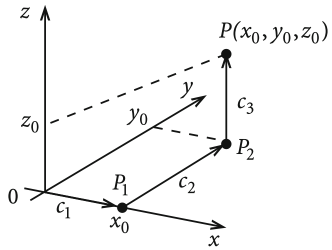


Fig. 2.48 Sketch of the path, mentioned in Exercise 2.4.1, between the points $(0, 0, 0)$ and (x_0, y_0, z_0)

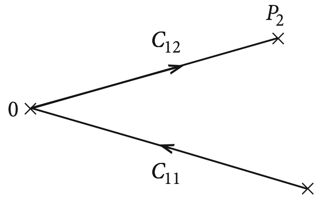


Fig. 2.49 Special path for the calculation of the work that must be carried out against the field $\mathbf{F} = \alpha \cdot \mathbf{r}$ on the way from P_1 to P_2

C1

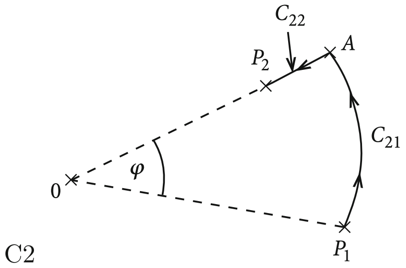


Fig. 2.50 Special path for the calculation of the work that must be carried out against the field $\mathbf{F} = \alpha \cdot \mathbf{r}$ on the way from P_1 to P_2

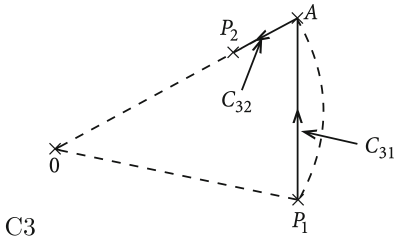


Fig. 2.51 Special path for the calculation of the work that must be carried out against the field $\mathbf{F} = \alpha \cdot \mathbf{r}$ on the way from P_1 to P_2

Exercise 2.4.3 Given is the force field

$$\mathbf{F}(x, y, z) = f\left(\frac{3}{\alpha^2}x^2 + \frac{2}{\alpha}y,\ -\frac{9}{\alpha^2}yz,\ \frac{8}{\alpha^3}xz^2\right) \quad \alpha = \text{const}\ .$$

1. Is the force conservative?
2. Which work must be done in order to move the mass point m in the field $\mathbf{F}$ on a straight line from $(0, 0, 0)$ to (α, α, α)?
3. Calculate the work for the case where as the path the polygonal line $(0, 0, 0) \to (\alpha, 0, 0) \to (\alpha, \alpha, 0) \to (\alpha, \alpha, \alpha)$ is chosen.
4. Which work must be brought up on the parabolic arc $(y = x^2, z = y^2)$ from $(0, 0, 0)$ to (α, α, α)?
5. The mass point moves on a circle with radius α within the xy-plane around the origin of coordinates. What work has to be performed for a motion over a full circle?

Exercise 2.4.4 A force

$$\mathbf{F}(\mathbf{r}) = \mathbf{F}(x, y, z) = \alpha\left(xy, -z^2, 0\right)\ .$$

acts on a mass point m.

1. Is $\mathbf{F}$ conservative? Justify your answer!
2. What work is necessary to move the mass point linearly

$$\text{from } P_1 = (0, 0, 0) \text{ to } P_2 = (1, 2, 3) \text{ ?}$$

3. Calculate the work necessary for moving the mass point on the *'curved'* path $(x = y^2, z = 3 \cdot \sqrt{y})$ from P_1 to P_2!

Exercise 2.4.5 Given is the force field

$$\mathbf{F}(\mathbf{r}) = (\mathbf{a} \times \mathbf{r}) \qquad (\mathbf{a} = \text{const}) \ .$$

Calculate the work using the same line integrals as in Exercise 2.4.2. Does a potential exist?

Exercise 2.4.6 A mass point moves in the force field

$$\mathbf{F}(\mathbf{r}) = (ay,\ ax,\ b) \ ,$$

where a, b are positive constants.

1. Show that it is a conservative force.
2. Calculate the needed work to move the mass point along a straight line from $P_0 : (0, 0, 0)$ to $P : (x, y, z)$.
3. What is the potential of the force $\mathbf{F}$?
4. How does the needed work change when the mass point is moved parallel to the coordinate axes from P_0 to P:

$$(0, 0, 0) \to (x, 0, 0) \to (x, y, 0) \to (x, y, z) \text{ ?}$$

Exercise 2.4.7 Given are the potentials:

1. $V(\mathbf{r}) = \frac{1}{2} k(x^2 + y^2 + z^2)$,
2. $V(\mathbf{r}) = \frac{m}{2} \left[(\boldsymbol{\omega} \cdot \mathbf{r})^2 - \omega^2 \mathbf{r}^2 \right]$
   ($\boldsymbol{\omega}$: constant vector).

Calculate the force $\mathbf{F} = \mathbf{F}(\mathbf{r})$ which is generated by the potentials. Are they central forces? What is the physical meaning of the given potentials?

Exercise 2.4.8 A particle of mass m moves under the influence of a conservative central force on a circular path with radius R through the origin of coordinates as plotted in Fig. 2.52.

1. Determine $r = r(\varphi)$.
2. Formulate the energy conservation law in terms of r and $dr/d\varphi$.
3. Which force acts on the mass point?

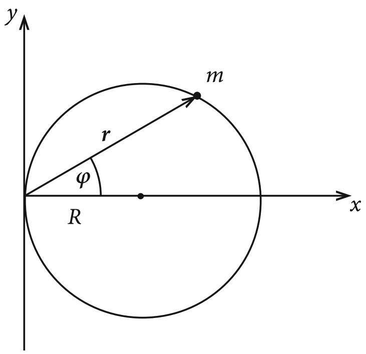


Fig. 2.52 Motion of a mass point m on a circular path

Exercise 2.4.9 A particle with the mass $m = 3g$ moves in a homogeneous time-dependent force field

$$\mathbf{F} = \left(45t^2,\ 6t - 3,\ -18t\right) \cdot 10^{-5} N$$

(t: time measured in seconds) with the initial conditions:

$$\mathbf{r}(t = 0) = (0, 0, 0)\,\text{cm}\ ,$$
$$\dot{\mathbf{r}}(t = 0) = (0, 0, 6)\,\text{cm}\,\text{s}^{-1}\ .$$

1. Calculate the velocity of the particle after 1 s.
2. Which kinetic energy does the particle have after 1 s?
3. What is the work W_{10} done by the field when moving the particle from $\mathbf{r}(t = 0)$ to $\mathbf{r}(t = 1)$?

Exercise 2.4.10 We discuss once more, just as in Exercise 2.3.15, the general solution of the linear harmonic oscillator; however now starting from the energy conservation law:

1. Why is this law valid?
2. Use the energy conservation law for calculating $x(t)$. Thereby the independent parameters shall be the total energy E and the time t_1 at which the oscillator reaches its maximum deflection $x_{\max}$.
3. Now choose the solution so that E and t_2 become the independent parameters, where t_2 is the time at which the oscillator assumes its maximum velocity.

Exercise 2.4.11 A mass point m moves frictionless in a potential $V(x)$ that has a minimum at $x = 0$ (see Fig. 2.53).

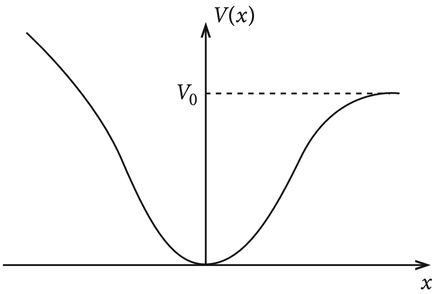


Fig. 2.53 Motion of the mass point m in a one-dimensional potential $V(x)$

1. How can one generally solve the equation of motion

$$m\ddot{x} = -\frac{dV}{dx}$$

   by a quadrature (integral)?
2. For not too high energies E, $(E < V_0)$, the mass point performs in general an anharmonic oscillation. How are the oscillation amplitudes (= turning points) for given energy E determined?

$$x_{\mathrm{min}} = -a\ , \quad x_{\mathrm{max}} = b$$

   How can we fix the oscillation period with the result from 1.? Which symmetry properties must $V(x)$ have in order to ensure $a = b$ for all energies E?
3. Calculate with the result from 2. the period of the **harmonic** oscillation.
4. Under what conditions for $V(x)$ and E will the oscillation period τ become infinite with a finite oscillation width b?
5. Let us assume an equation of motion of the form

$$m\ddot{x} = -m\left(\omega_0^2 x + \varepsilon x^3\right) = -\frac{dV}{dx}\ ,$$

   where for the energy E of the mass point holds:

$$\varepsilon E \ll m\omega_0^4$$

   Calculate approximately the oscillation period τ in dependence of the oscillation amplitude a up to linear terms in ε!
6. For the potential $V(x)$ from 5. solve the equation of motion of the mass point up to linear terms in ε by use of the quadrature from 1.. Choose as initial condition $x(0) = 0$. Calculate within the same approximation out of $E = V(-a)$ the initial velocity $\dot{x}(0)$.

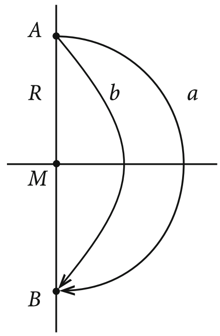


Fig. 2.54 Different ways of the mass point m in the gravitational potential

Exercise 2.4.12 A mass point m moves in the gravitational field

$$V(\mathbf{r}) = -\frac{\alpha}{r} \; ; \; \alpha = \gamma m M$$

($m \ll M$; γ: gravitational constant)

1. Calculate the time which the mass point needs to come from point A to B (see Fig. 2.54) if it travels
   (a) on a circle (time t_a)
   (b) on a parabola (time t_b)
2. Calculate for both cases the kinetic energy at A and the corresponding velocity. How much time would the mass point need if it traveled with the same velocity (uniformly in a straight-line) **directly** from A to B?
3. On the mass point in addition to the gravitational attraction also acts a small friction force

$$\mathbf{F}_R = -m\hat{\alpha}\dot{\mathbf{r}} \quad \left(|\mathbf{F}_R| \ll \left|\frac{\alpha}{r^2}\right|\right)$$

   (1) Write down the equation of motion!
   (2) Starting from a circular path (radius R, angular velocity ω_0) it shall be shown that for small times the ansatz

$$r = R(1 + c_1 t) \; ; \; \dot{\varphi} = \omega_0(1 + c_2 t)$$

   fulfills the equation of motion to first approximation. Calculate c_1 and c_2 and estimate for which times the approach is useful.
   (3) In the framework of the approximation (2) how do the radius, angular velocity, path velocity, kinetic, potential, and total energy change with time?
   (4) How much of the (mechanical) energy is destroyed by friction? Where does it come from?

Exercise 2.4.13 A mass point moves on an ellipse in the xy plane

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and runs through it three times in 2 s.

1. What is the trajectory

$$\mathbf{r}(t) = (x(t), y(t), z(t)) \ ,$$

   if $x(t) = a \cos \omega t$ is given?
2. Which force does act on the mass point?
3. Calculate the angular momentum of the mass point. Why should this be constant with respect to direction as well as magnitude?
4. Calculate the area ΔS which the position vector sweeps in 1 s.

2.5 Planetary Motion

The potential

$$V(r) = -\frac{\alpha}{r} \tag{2.259}$$

is the most important example that leads to a central force field. It has significant applications in celestial mechanics and for the semiclassical atom model. We want to investigate its properties in connection with the special example of the planetary motion around the sun.

Starting point for the solution of the equation of motion in a conservative central field is the validity of the energy and the angular-momentum conservation law which manifests itself in the equation (m: planet mass):

$$E = \frac{m}{2}\dot{r}^2 + \frac{L^2}{2m\,r^2} + V(r) \tag{2.260}$$

Here $V(r)$ is the **gravitational potential**:

$$V(r) = -\gamma \frac{mM}{r} \tag{2.261}$$

(M: mass of the sun; γ: Newton's gravitational constant $(6.67 \cdot 10^{-11} \mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-3})$).

For the explicit solution of the problem, however, we will not choose the general procedure described in the last section, but prefer a more direct integration. For this purpose we introduce a new variable

$$s = \frac{1}{r}$$

and try at first to determine s as function of φ:

$$\frac{ds}{d\varphi} = \frac{d}{dt}\left(\frac{1}{r}\right)\frac{dt}{d\varphi} = -\frac{\dot{r}}{r^2}\frac{1}{\dot{\varphi}} = -\frac{\dot{r}}{r^2}\frac{mr^2}{L}$$

$$\Longrightarrow \dot{r} = -\frac{L}{m}\frac{ds}{d\varphi} \; . \tag{2.262}$$

With $V(1/s) = \overline{V}(s) = -\gamma\, m M\, s$ (2.260) reads:

$$\frac{L^2}{2m}\left[\left(\frac{ds}{d\varphi}\right)^2 + s^2\right] + \overline{V}(s) = E \; . \tag{2.263}$$

We differentiate this equation once more with respect to φ:

$$\frac{L^2}{2m}\left[2\frac{ds}{d\varphi}\frac{d^2 s}{d\varphi^2} + 2s\frac{ds}{d\varphi}\right] + \frac{d\overline{V}}{ds}\frac{ds}{d\varphi} = 0 \; .$$

It follows:

$$\frac{d^2 s}{d\varphi^2} + s = -\frac{m}{L^2}\frac{d\overline{V}}{ds} = \gamma\, m^2 \frac{M}{L^2} \; . \tag{2.264}$$

This is an inhomogeneous differential equation of second order. The **general solution** of the associated **homogeneous equation** reads:

$$s_0(\varphi) = \alpha \sin\varphi + \beta\cos\varphi \; .$$

It is not difficult to identify the following function as a special solution of the full inhomogeneous equation:

$$s_1(\varphi) \equiv \gamma\, m^2 \frac{M}{L^2} \; .$$

That leads immediately to the **general solution of the inhomogeneous differential equation of second order**:

$$s(\varphi) = \alpha \sin\varphi + \beta\cos\varphi + \gamma\, m^2 \frac{M}{L^2} \; . \tag{2.265}$$

The two independent parameters α and β are fixed by the initial conditions. So we demand that the point closest to the sun (s maximal) is found at $\varphi = 0$:

$$\begin{aligned}
\left.\frac{ds}{d\varphi}\right|_{\varphi=0} &\stackrel{!}{=} 0 = \left.(\alpha\cos\varphi - \beta\sin\varphi)\right|_{\varphi=0} = \alpha \ , \\
\left.\frac{d^2s}{d\varphi^2}\right|_{\varphi=0} &= \left.(-\alpha\sin\varphi - \beta\cos\varphi)\right|_{\varphi=0} = -\beta \stackrel{!}{\leq} 0 \\
&\Longrightarrow \beta \geq 0 \ .
\end{aligned}$$

That leads to the trajectory:

$$s = \frac{1}{r} = \beta\cos\varphi + \gamma\, m^2\frac{M}{L^2} \ . \tag{2.266}$$

We introduce the following constants:

$$k = \frac{L^2}{\gamma\, M\, m^2} \ ; \quad \beta = \frac{\varepsilon}{k} \geq 0 \ . \tag{2.267}$$

Therewith we have found:

$$\frac{1}{r} = \frac{1}{k}(1 + \varepsilon\cos\varphi) \ . \tag{2.268}$$

This is the equation of a **conic section** in polar coordinates. Therewith the geometrical shapes of the planetary paths are determined. The parameter ε, which is related to the positive integration constant β, can assume arbitrary positive values:

$$\begin{aligned}
\varepsilon < 1 : &\quad \text{ellipse} \ , \\
\varepsilon = 1 : &\quad \text{parabola} \ , \\
\varepsilon > 1 : &\quad \text{hyperbola} \ .
\end{aligned}$$

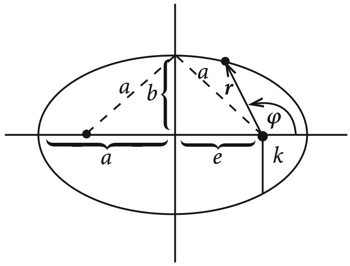
Finally we want to consider how the crucial integration constants L and E will influence the shape of the path:

(a) Ellipse

The ellipse is the geometric locus of all those points for which the sum of their distances to two different **'focal points'** is constant $= 2a$ (Fig. 2.55). Therewith evidently a represents the semi-major axis. Furthermore we have according to Pythagoras's theorem:

$$a^2 = e^2 + b^2 \implies b^2 = a^2 - e^2 \ .$$

Fig. 2.55 Characteristic parameters of the ellipse



With (2.268) we have for the *'point closest to the sun'*:

$$r_0 = r(\varphi = 0) = a - e = \frac{k}{1+\varepsilon}$$

and for the point *'point farthest from the sun'*:

$$r_1 = r(\varphi = \pi) = a + e = \frac{k}{1-\varepsilon}\ .$$

Combining the last two equations leads to the **numerical eccentricity**:

$$\varepsilon = \frac{e}{a}\ .$$

This we insert into r_0,

$$r_0 = a - e = \frac{ka}{a+e}\ ,$$

finding therewith

$$\frac{b^2}{a} = k = \frac{L^2}{\gamma M m^2}\ . \tag{2.269}$$

Thus the angular momentum L influences both the semi-axes!

We recognize the influence of the energy when we inspect (2.268) for the point closest to the sun:

$$\dot{r}(\varphi = 0) \overset{(2.262)}{=} -\frac{L}{m}\frac{ds}{d\varphi}\bigg|_{\varphi=0} = 0\ .$$

This yields for the total energy E:

$$E = \frac{L^2}{2m\, r_0^2} - \gamma \frac{m\, M}{r_0} = \gamma\, m\, M \left(\frac{k}{2r_0^2} - \frac{1}{r_0} \right) =$$

$$= \gamma\, m\, M \frac{a^2 - e^2 - 2a(a-e)}{2a(a-e)^2}$$

$$\Longrightarrow E = -\frac{\gamma\, m\, M}{2a} \Longrightarrow a = -\frac{\gamma\, m\, M}{2E} \; . \tag{2.270}$$

We see that the energy E determines uniquely the semi-major axis a of the ellipse. Since it is a bounded motion we have $E < 0$. For the semi-minor axis it then immediately results from (2.269):

$$b = \frac{L}{\sqrt{-2\, m\, E}} \; . \tag{2.271}$$

(b) Hyperbola

The path of hyperbola is often marked by the

'impact parameter' d**,**

which is the distance by which the particle would pass the scattering center if there were no deviation, and by the angle ϑ by which it is actually deflected when flying around the center (see Fig. 2.56). How are these quantities linked with L and E? The directions of the asymptotes ($r \to \infty$) according to (2.268) are given by

$$\cos \varphi_\infty = -\frac{1}{\varepsilon}$$

Obviously it holds:

$$\pi - \vartheta = 2\,(\pi - \varphi_\infty) \Longrightarrow \vartheta/2 = \varphi_\infty - \pi/2 \; .$$

Hence it follows:

$$\sin \vartheta/2 = \sin\,(\varphi_\infty - \pi/2) = -\cos \varphi_\infty = 1/\varepsilon \; .$$

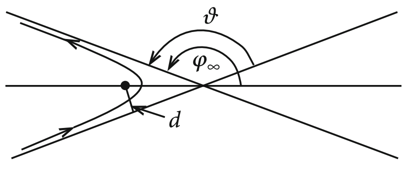


Fig. 2.56 Characteristic parameters of the hyperbola

Let $\dot{\mathbf{r}}_\infty$ be the velocity of the particle at the point at infinity. From the energy conservation law follows then:

$$E = \frac{m}{2}\dot{\mathbf{r}}_\infty^2 > 0 \tag{2.272}$$

and from the angular-momentum conservation law:

$$L = m\,|(\mathbf{r}\times\dot{\mathbf{r}})| = m\,|(\mathbf{r}_\infty\times\dot{\mathbf{r}}_\infty)| = m\,d\,|\dot{\mathbf{r}}_\infty|\ .$$

That yields the relation:

$$L^2 = 2\,m\,E\,d^2\ . \tag{2.273}$$

Just as for the ellipse it also holds here for the point closest to the sun $\dot{r}_0 = 0$ and $r_0 = k/(1+\epsilon)$ and thereby for the energy:

$$\begin{aligned}E &= \frac{L^2}{2\,m\,r_0^2} - \gamma\,m\,M\frac{1}{r_0} = \gamma\,M\,m\left(\frac{k}{2r_0^2} - \frac{1}{r_0}\right) = \\ &= \gamma\,M\,m\left[\frac{(1+\varepsilon)^2}{2k} - \frac{(1+\varepsilon)}{k}\right] = \gamma\,M\,m\frac{(\varepsilon+1)(\varepsilon-1)}{2k}\ .\end{aligned}$$

It follows:

$$\varepsilon^2 - 1 = \frac{2k\,E}{\gamma\,M\,m} = \frac{2L^2E}{\gamma^2M^2m^3} = \frac{4E^2d^2}{\gamma^2M^2m^2} = \frac{1}{\sin^2\vartheta/2} - 1 = \cot^2\frac{\vartheta}{2}\ .$$

For the hyperbolic path we have therewith found the following relations for the impact parameter d and the deflection angle ϑ:

$$d = \frac{L}{\sqrt{2m\,E}}\ ; \quad \tan\frac{\vartheta}{2} = \frac{\gamma\,M\,m}{2\,d\,E}\ . \tag{2.274}$$

We notice that the energy E and the angular momentum L uniquely establish d and ϑ. These relations play an important role also in atomic physics, since the deflection of charged particles due to the positively charged nucleus takes place by the same type of potential α/r.

(c) Curve Sketching

We close this section with a descriptive discussion of the types of motion in the gravitational potential (see Fig. 2.57).

The mass point can reach only those regions for which holds:

$$V_{\text{eff}}(r) = \frac{L^2}{2\,m\,r^2} - \gamma\frac{m\,M}{r} \le E \tag{2.275}$$

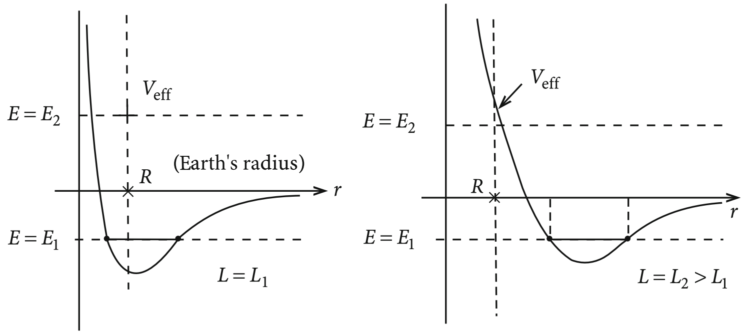


Fig. 2.57 The effective potential belonging to the gravitational potential for two different values of the angular momentum L

The angular momentum contributes a repulsive term to the effective potential which dominates for small r.

For negative energies ($E = E_1$) there is always only a finite region of allowed values for the magnitude of the position vector. A satellite, as an example, will thus always stay within the attraction region of the earth if $E < 0$. On the other hand, it should not penetrate the earth. Therefore, the region $r \leq R$ must be excluded by a sufficiently large angular momentum. The required minimum angular momentum determines the minimum velocity tangential to the earth's surface. This consideration leads to (see Exercise 2.5.4):

First Cosmic Velocity

$$v_1 = \sqrt{gR} = 7.9\,\mathrm{km\,s^{-1}}\ . \tag{2.276}$$

In order to leave the attraction region of the earth the satellite needs at least the energy $E = 0$. On the earth's surface it has the potential $-\gamma\frac{mM}{R}$ where the gravitational force amounts to $m\,g = \gamma\frac{mM}{R^2}$. From that follows

$$0 = \frac{m}{2}\,v_2^2 - m\,g\,R\ .$$

The satellite therefore needs as minimum initial velocity the so-called

Second Cosmic Velocity

$$v_2 = \sqrt{2g\,R} = 11.2\,\mathrm{km}\,s^{-1}\ . \tag{2.277}$$

(d) Kepler's Laws Let us finally still recall Kepler's laws, which we have derived in this section in a rather general manner:

1. *The planets move along ellipses with the sun at one of the focal points.*
2. *The radius vector from the sun to the planet sweeps equal areas in equal times ('areal velocity is constant').*
3. *The ratio of the squares of the orbital periods of two planets is the same as that of the cubes of the respective semi-major axes of the ellipses.*

The first and second law follow directly from the energy and angular-momentum conservation laws. The second law is nothing else but the area conservation principle (2.251). The validity of the third law is not yet clear up to now. This is, however, easily done when we inspect the total area of the ellipse and apply suitably the area conservation principle (2.251):

$$s = \pi\, a\, b = \tau \frac{ds}{dt} = \tau \frac{L}{2m} \qquad (\tau : \text{orbital period})$$

$$\Longrightarrow \frac{\tau^2}{a^3} = \frac{\pi^2 b^2 4m^2}{L^2 a} = \frac{4m^2\pi^2 k}{L^2} = \frac{4\pi^2}{\gamma M} = \text{const}\,. \tag{2.278}$$

2.5.1 Exercises

Exercise 2.5.1 The ellipse is the geometric locus of all those points $M = (x, y)$ for which the sum of their distances to two given fixed points $F_1 = (e, 0)$ and $F_2 = (-e, 0)$ (F_1, F_2: focal points) is constant $= 2a$ (Fig. 2.58).

1. Express b by a and e.
2. Determine the equation of the ellipse in Cartesian coordinates.
3. Determine the equation of the ellipse in polar coordinates, i.e. find $r = r(\varphi)$. For that use the quantities $k = b^2/a$ and $\varepsilon = e/a < 1$ (ε: eccentricity).
4. Determine the parameter form of the ellipse: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}$.

Exercise 2.5.2 A particle of mass m possesses in a force field the potential

$$V(\mathbf{r}) = \frac{\alpha}{r^2}\,.$$

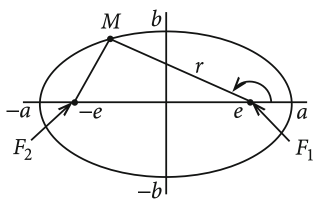


Fig. 2.58 Characteristic parameters of the ellipse

1. What can generally be said about force, energy, and angular momentum?
2. Time zero and origin of coordinates are chosen just so that for $\alpha > 0$ (repulsive potential) it holds:

$$r_{\min} = r(t = 0)\ , \quad \varphi(r_{\min}) = 0$$

   Calculate $r_{\min}$ as function of L and E.
3. Determine the function $r = r(t)$ and the path $r = r(\varphi)$ for $E > 0$ and $\alpha > 0$. What is the path for the special case $\alpha = 0$?
4. Under which conditions will an attractive potential ($\alpha < 0$) lead to a *bound motion*? Determine for this case $r_{\max}$.
5. Calculate with the initial condition $r(t = 0) = r_{\max}$ the time t_0, after which the particle will land in the center $r = 0$.
6. Calculate the trajectory $r = r(\varphi)$ with $\varphi(r_{\max}) = 0$.

Exercise 2.5.3 The vector

$$\mathbf{A} = (\dot{\mathbf{r}} \times \mathbf{L}) + V(r)\,\mathbf{r} \quad (\mathbf{L}:\ \text{angular momentum})$$

is denoted as *'Lenz vector'* belonging to the central potential $V(r)$.

1. Demonstrate that for the potential

$$V(r) = -\frac{\alpha}{r} \quad (\alpha > 0\ ,\ \text{Kepler, Coulomb})$$

   the Lenz vector is a conserved quantity.
2. Calculate the magnitude of $\mathbf{A}$.
3. Bring the path equation by use of the Lenz vector into the form

$$\frac{1}{r} = \frac{1 + \varepsilon\cos\varphi}{k} \quad (\varphi = \sphericalangle(\mathbf{A}\ ,\mathbf{r}))$$

   and express the parameters k and ε by the mass m, the constant α, the total energy E and the angular momentum L. Hint: Investigate the scalar product $\mathbf{A} \cdot \mathbf{r}$.

Exercise 2.5.4

1. How does the effective potential $V_{\text{eff}}(r)$ for the path of an earth satellite of mass m read? Assume that the earth satellite moves on a circular path. Determine the radius R_0 as a function of the rotational frequency of the satellite. Find the radius in case of a geostationary orbit?
2. The satellite moves in the attractive region of the earth. Determine the *first cosmic velocity* (2.276) as the minimum velocity tangential to the earth's surface which is necessary for the satellite in order not to drop back to earth!
3. Which minimum initial velocity (*second cosmic velocity*, (2.277)) must the satellite get to be able to leave the attraction region of the earth?

Exercise 2.5.5 For a conservative central force field

$$\mathbf{F}(\mathbf{r}) = f(r)\,\mathbf{e}_r$$

the path line $r = r(\varphi)$ is given. Using this path line the function $f(r)$ can be deduced.

1. Verify the relation

$$f(r) = \frac{L^2}{mr^4}\left(\frac{d^2r}{d\varphi^2} - \frac{2}{r}\left(\frac{dr}{d\varphi}\right)^2 - r\right) .$$

   L is the magnitude of the angular momentum.
2. The path line may be an ellipse with the force center in one of its focal points. Show that it must hold:

$$f(r) \propto -\frac{1}{r^2}$$

3. The path line is given by

$$r = r_0\, e^{-\varphi}$$

   Find $f(r)$?

Exercise 2.5.6

1. Show that the force

$$\mathbf{F}(\mathbf{r}) = -\frac{\alpha}{r^n}\mathbf{e}_r \;\; ; \;\; \alpha > 0 \; ; \; n > 1$$

   is conservative!
2. Verify that the angular momentum $\mathbf{L}$ of a particle of mass m in the force field $\mathbf{F}$ from 1. is a conserved quantity. What does follow from that for the particle motion?
3. For the field $\mathbf{F}$ from 1. the energy theorem is brought into the form

$$E = \frac{m}{2}\,\dot{r}^2 + V_{\text{eff}}(r) .$$

   How does the effective potential $V_{\text{eff}}(r)$ look like?
4. Determine under which conditions concerning n the particle can move in a stable **circular orbit** ($r \equiv r_0$). Calculate the radius r_0!

2.6 Self-Examination Questions

To Section 2.1

1. What does the term *mass point* mean?
2. What is provided by kinematics?
3. What do we understand by path line, position vector, velocity, and acceleration of a mass point?
4. Formulate the components of the velocity of a mass point in, respectively, cylindrical and spherical coordinates!
5. Sketch the position and the velocity vector of a mass point for the uniform straight-line and the uniformly accelerated motion!
6. What is characteristic for a uniform circular motion?

To Section 2.2

1. List Newton's axioms.
2. What do we understand as *inertial* and *gravitational (heavy)* mass? What is the connection between them?
3. Which law is called the *basic dynamical equation* of Classical Mechanics?
4. What is a central force?
5. What is an inertial system?
6. What are the rules of a Galilean transformation?
7. Define the term *pseudo force*.
8. Interpret the meaning of the Coriolis force and the centrifugal force!

To Section 2.3

1. What is the equation of a force-free motion?
2. Which kind of movement does the mass point perform in the homogeneous gravitational field?
3. How does the final velocity of a body of mass m, which is dropped with the initial velocity zero from the height h in the earth's gravitational field, depend on the height h and the mass m at the impingement on the earth's surface?
4. Which equation is called a linear differential equation of n-th order?
5. Describe the general procedure for solving linear inhomogeneous differential equations.
6. What are the *most popular* ansaetze for frictional forces?
7. How does the equation of motion of a material body read when the motion takes place in the earth's gravitational field under the influence of Stokes's friction? Which type of differential equation does one get? Find a special solution!
8. What do we understand by a *simple (mathematical, thread) pendulum*?
9. What is thread tension? What kind of force is it?
10. Formulate the oscillation equation of the simple pendulum!
11. With respect to the example of the thread pendulum interpret the terms oscillation period, frequency, and angular frequency!

12. How can we use the thread pendulum to demonstrate the equivalence of inertial and heavy mass?
13. How is the unit i of imaginary numbers defined?
14. What is meant by the polar representation of a complex number?
15. What do we understand by Euler's formula?
16. Define the harmonic oscillator and name some possibilities of realization. What is called the eigen-frequency of an oscillator?
17. Formulate the equation of motion of the free damped linear harmonic oscillator (Stokes's friction). Distinguish the oscillatory case, the aperiodic limiting case, and the creeping case!
18. Plot qualitatively for the aperiodic limiting case the solution $x(t)$ of the damped harmonic oscillator. How many zero crossings are possible?
19. In which case is the harmonic oscillator more strongly damped, in the aperiodic limiting case or in the creeping case?
20. Formulate the equation of motion of the linear damped harmonic oscillator under the influence of a time-dependent external force $F(t)$. Find a mechanical and a non-mechanical realization!
21. Describe the term *resonance*. How is the *resonance frequency* influenced by friction? Plot qualitatively the behavior of the oscillation amplitude as function of the frequency of a periodic external force, and that for different values of the damping!
22. What is meant by phase shift? How does it depend in the case of a damped oscillator on the frequency of the driving periodic force?
23. Derive by integration of the basic dynamical equation for one-dimensional motion the fundamental terms of work, potential and kinetic energy, as well as the total energy. Define therewith the classically allowed and the classically forbidden regions of motion!
24. Starting from a given potential discuss qualitatively the one-dimensional motion of a mass point.

To Section 2.4

1. Which work has to be done to move a mass point in the field $F = F(r, \dot{r}, t)$ by the distance dr? Discuss in particular the choice of the sign!
2. Which factors do determine the work to be done for shifts of the mass point via finite distances?
3. How is the power P defined? What is the dimension of power?
4. When are forces *conservative*? Name a few criteria!
5. What do we understand by the *potential* of a force?
6. Formulate the energy theorem!
7. How does the potential of the spatially isotropic harmonic oscillator look like?
8. Define angular momentum and torque!
9. Investigate with the example of a uniform straight-line motion whether or not the angular momentum is a pure particle property! How does L depend on the choice of the reference point?
10. What is the angular-momentum law?

11. Under which conditions is a central force conservative?
12. When does conservation of angular-momentum hold?
13. What is stated by the area conservation principle?
14. How can we construct by use of angular-momentum conservation law and energy conservation law a general procedure for the solution of the equation of motion?

To Section 2.5

1. Which type of potential determines the planetary motion?
2. What are the types of geometrical shapes of planetary paths?
3. If the planetary path is an ellipse how do angular momentum L and total energy E determine the two semi axes?
4. What are the indicators for the path of hyperbola?
5. What is meant by the impact parameter d?
6. How do the impact parameter d and the deflection angle ϑ depend on L and E?
7. Define the first and second cosmic velocities!
8. State and interpret Kepler's laws!

Chapter 3
Mechanics of Many-Particle Systems

The most real physical systems are composed of many single particles which are influencing each other, i.e. interacting with each other. Think of the atoms of a solid, of a several-atom molecule, of the planetary system of the sun, ... Normally it is inexpedient or even impossible to consider separately the equation of motion for each of the mass points of the many-particle system. One therefore gathers the particles into a

mass-point system

and tries to derive statements about the **total** system as a whole. Let N be the total number of mass points in this system which we number from $i = 1$ to $i = N$:

$$
\begin{aligned}
m_i &: \text{ mass of the } i\text{-th particle },\\
\mathbf{r}_i &: \text{ position vector of the } i\text{-th particle },\\
\mathbf{F}_i &: \text{ total force acting on particle } i ,\\
\mathbf{F}_i^{(\mathrm{ex})} &: \textbf{ external} \text{ force acting on particle } i ,\\
\mathbf{F}_{ij} &: \text{ force executed by the particle } j \text{ on the particle } i \text{ (}\textbf{internal} \text{ force)} .
\end{aligned}
$$

One distinguishes between **internal** and **external** forces. By *'internal'* forces one understands those forces which are executed from the particles of a mass-point system on each other. *'External'* forces have their origin outside the system, as e.g. the gravitational force. We denote the mass-point system as **closed** if no external forces are present. In mechanics, in fact practically in all branches of physics, one takes into consideration as internal forces only **two-particle forces**, which exclusively depend on the positions and possibly also on the velocities of **two** particles.

W. Nolting, *Theoretical Physics 1*, DOI 10.1007/978-3-319-40108-9_3

For each single particle, of course, Newton's equation of motion holds:

$$m_i\ddot{\mathbf{r}}_i = \mathbf{F}_i = \mathbf{F}_i^{(\mathrm{ex})} + \sum_j \mathbf{F}_{ij} \; . \tag{3.1}$$

Obviously the third of Newton's axioms is important for the treatment of internal forces:

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji} \; ; \quad \mathbf{F}_{ii} = 0 \; . \tag{3.2}$$

We discuss at first some **conservation laws** which indicate under which conditions certain mechanical quantities are time invariant.

3.1 Conservation Laws

3.1.1 Principle of Conservation of Linear Momentum (Center of Mass Theorem)

We add up the equations of motion (3.1) for all N particles. Because of (3.2) the contributions of the internal forces fall out of the sum:

$$\sum_{i,j} \mathbf{F}_{ij} = \frac{1}{2} \sum_{i,j} \left(\mathbf{F}_{ij} + \mathbf{F}_{ji}\right) = 0 \; . \tag{3.3}$$

So we are left with:

$$\sum_{i=1}^{N} m_i\ddot{\mathbf{r}}_i = \sum_{i=1}^{N} \mathbf{F}_i^{(\mathrm{ex})} \; .$$

This equation is given a compact form by the following definitions:

Definition 3.1.1

$$M = \sum_i m_i \; : \; \text{total mass} \; , \tag{3.4}$$

$$\mathbf{R} = \frac{1}{M} \sum_i m_i \, \mathbf{r}_i \; : \; \text{center of mass} \; , \tag{3.5}$$

$$\mathbf{P} = \sum_i m_i\dot{\mathbf{r}}_i = M\,\dot{\mathbf{R}} \; : \; \text{total linear momentum} \; , \tag{3.6}$$

$$\mathbf{F}^{(\mathrm{ex})} = \sum_i \mathbf{F}_i^{(\mathrm{ex})} \; : \; \text{total external force} \; . \tag{3.7}$$

Therewith follows:

$$M\ddot{\mathbf{R}} = \sum_i \mathbf{F}_i^{(\mathrm{ex})} = \mathbf{F}^{(\mathrm{ex})} \ . \tag{3.8}$$

That is the

center of mass theorem:

The center of mass of a mass-point system moves as if the total mass is concentrated at this point and all external forces are acting on it.

The internal forces have no influence on the movement of the center of mass. The center of mass theorem provides retroactively the justification for the introduction of the mass-point concept. As long as one is not interested in particular details of the single particle movements one can indeed replace the motion of the total body consisting of N individual particles by that of a single mass point, namely, the center of gravity.

The center of mass theorem corresponds to the **principle of linear momentum** of the N particle system:

$$\dot{\mathbf{P}} = \mathbf{F}^{(\mathrm{ex})} \ . \tag{3.9}$$

Momentum Conservation Law

$$\mathbf{F}^{(\mathrm{ex})} \equiv 0 \iff \mathbf{P} = \mathrm{const} \ . \tag{3.10}$$

In the case of vanishing total external force the total linear momentum remains constant both with respect to direction and magnitude.

Examples

(1) **Exploding grenade:**
The motion of the center of mass remains uninfluenced by the explosion.

(2) **Rocket:**
The expulsion of the exhaust gas is compensated by the forward motion of the rocket.

3.1.2 Conservation of Angular Momentum

We define as **total angular momentum** of the N particle system:

$$\mathbf{L} = \sum_{i=1}^{N} \mathbf{L}_i = \sum_{i=1}^{N} m_i \left(\mathbf{r}_i \times \dot{\mathbf{r}}_i\right) \ . \tag{3.11}$$

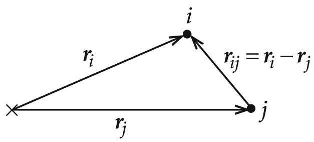


Fig. 3.1 The distance vector $\mathbf{r}_{ij}$ for the determination of the contribution of the internal forces to the total angular momentum

For its time-dependence holds:

$$\dot{\mathbf{L}} = \sum_i m_i \left[(\dot{\mathbf{r}}_i \times \dot{\mathbf{r}}_i) + (\mathbf{r}_i \times \ddot{\mathbf{r}}_i)\right] = \sum_i m_i \left(\mathbf{r}_i \times \ddot{\mathbf{r}}_i\right) = \sum_i \left(\mathbf{r}_i \times \mathbf{F}_i\right) =$$

$$= \sum_i \left(\mathbf{r}_i \times \mathbf{F}_i^{(\mathrm{ex})}\right) + \sum_{i,j} \left(\mathbf{r}_i \times \mathbf{F}_{ij}\right) \ .$$

Again we are able to demonstrate that the contribution of the internal forces drops out (Fig. 3.1):

$$\sum_{i,j} \left(\mathbf{r}_i \times \mathbf{F}_{ij}\right) = \frac{1}{2} \sum_{i,j} \left[\left(\mathbf{r}_i \times \mathbf{F}_{ij}\right) + \left(\mathbf{r}_j \times \mathbf{F}_{ji}\right)\right] = \frac{1}{2} \sum_{i,j} \left(\mathbf{r}_{ij} \times \mathbf{F}_{ij}\right) = 0 \ ,$$

since, as a general rule, the two-body forces have the property:

$$\mathbf{F}_{ij} \propto \mathbf{r}_{ij} \ .$$

With

$$\mathbf{M}_i^{(\mathrm{ex})} = \left(\mathbf{r}_i \times \mathbf{F}_i^{(\mathrm{ex})}\right) \qquad \textbf{external torque} \tag{3.12}$$

we therewith have derived the **angular-momentum law**:

$$\frac{d}{dt} \mathbf{L} = \sum_{i=1}^{N} \left(\mathbf{r}_i \times \mathbf{F}_i^{(\mathrm{ex})}\right) = \sum_{i=1}^{N} \mathbf{M}_i^{(\mathrm{ex})} = \mathbf{M}^{(\mathrm{ex})} \ . \tag{3.13}$$

The time rate of change of the total angular momentum is equal to the sum of the **external** torques. The internal forces have no influence.

In a **closed** system the **angular-momentum conservation law** also holds:

$$\mathbf{M}^{(\mathrm{ex})} = 0 \Longleftrightarrow \mathbf{L} = \text{const} \ . \tag{3.14}$$

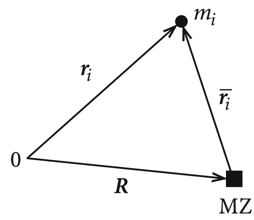


Fig. 3.2 Definition of the mass center

Sometimes the decomposition of the angular momentum into **relative** and **center of mass contributions** appears reasonable:

$$\mathbf{L} = \sum_i m_i \left(\mathbf{r}_i \times \dot{\mathbf{r}}_i\right) = \sum_i m_i \left[\left(\mathbf{R} + \bar{\mathbf{r}}_i\right) \times \left(\dot{\mathbf{R}} + \dot{\bar{\mathbf{r}}}_i\right)\right] =$$
$$= \sum_i m_i \left[\left(\mathbf{R} \times \dot{\mathbf{R}}\right) + \left(\mathbf{R} \times \dot{\bar{\mathbf{r}}}_i\right) + \left(\bar{\mathbf{r}}_i \times \dot{\mathbf{R}}\right) + \left(\bar{\mathbf{r}}_i \times \dot{\bar{\mathbf{r}}}_i\right)\right] .$$

Now it is (Fig. 3.2)

$$\sum_i m_i \bar{\mathbf{r}}_i = \sum_i m_i \mathbf{r}_i - \sum_i m_i \mathbf{R} = M\,\mathbf{R} - M\,\mathbf{R} = 0 .$$

Therewith the angular momentum of the N particle system reads:

$$\mathbf{L} = (\mathbf{R} \times \mathbf{P}) + \sum_{i=1}^{N} \left(\bar{\mathbf{r}}_i \times \bar{\mathbf{p}}_i\right) = \mathbf{L}_\mathrm{s} + \mathbf{L}_\mathrm{r} . \tag{3.15}$$

$$\mathbf{L}_\mathrm{s} = (\mathbf{R} \times \mathbf{P}) \; : \quad \textit{angular momentum of the total mass concentrated in the center of mass, with reference to the origin of coordinates} \tag{3.16}$$

$$\mathbf{L}_\mathrm{r} = \sum_{i=1}^{N} \left(\bar{\mathbf{r}}_i \times \bar{\mathbf{p}}_i\right) \; : \quad \textit{total angular momentum of } N \textit{ particles, with reference to the center of mass} \tag{3.17}$$

Unlike the total linear momentum (3.6) the total angular momentum is **not** exclusively expressible only by coordinates of the center of gravity.

3.1.3 Conservation of Energy

We multiply the equation of motion (3.1) for the single particles scalarly by $\dot{\mathbf{r}}_i$ and then sum up over all the N particles:

$$\sum_i m_i\,(\ddot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) = \sum_i \mathbf{F}_i \cdot \dot{\mathbf{r}}_i \,.$$

On the left-hand side we recognize the time derivative of the **total kinetic energy**:

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i^2 \,. \tag{3.18}$$

If $\mathbf{F}_i$ is a conservative force, for which

$$\nabla_i \times \mathbf{F}_i = 0\,; \quad i = 1, 2, \ldots, N \tag{3.19}$$

holds, then a potential V does exist with

$$\mathbf{F}_i = -\nabla_i V\,(\mathbf{r}_1, \ldots, \mathbf{r}_N) \;, \tag{3.20}$$

where the index i indicates that the partial differentiations are only with respect to the coordinates of the i-th particle. But from that we can also argue:

$$\sum_i \mathbf{F}_i \cdot \dot{\mathbf{r}}_i = -\sum_i (\nabla_i V) \cdot \dot{\mathbf{r}}_i = -\frac{dV}{dt} \,. \tag{3.21}$$

We want to generalize the issue, as we previously did for the single mass point, by splitting the actual forces into **conservative** and **dissipative** parts, where of course only for the conservative part the Eq. (3.21) can be used. We then obtain the

Energy Theorem

$$\frac{d}{dt}(T + V) = \sum_{i=1}^{N} \mathbf{F}_i^{(\mathrm{diss})} \cdot \dot{\mathbf{r}}_i \,. \tag{3.22}$$

The rate of time change of the total mechanical energy of a mass-point system is equal to the power of the dissipative forces. If the latter are absent then we get the

Energy Conservation Law

$$T + V = E = \mathrm{const}\,, \quad \text{if } \mathbf{F}_i^{(\mathrm{diss})} \equiv 0 \quad \forall i \,. \tag{3.23}$$

It is useful also to split the potential with respect to *internal* and *external* contributions.

The two-particle forces $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ acting between the particles i and j are to be assumed as conservative in all known physically relevant cases. $\mathbf{F}_{ij}$ is the force which particle j exerts on particle i. If one chooses the current site of the particle j as the origin of coordinates (force center) then $\mathbf{F}_{ij}$ represents a central force which additionally is supposed to be conservative. Consequently, the interaction potential V_{ij} can depend only on the particle distance:

$$r_{ij} = \left|\mathbf{r}_i - \mathbf{r}_j\right| \tag{3.24}$$

That means:

$$\begin{aligned} V_{ij} &= V_{ij}\left(\mathbf{r}_i, \mathbf{r}_j\right) = V_{ij}\left(r_{ij}\right) \\ \Longrightarrow V_{ij} &= V_{ji} \ , \quad V_{ii} = 0 \ . \end{aligned} \tag{3.25}$$

With

$$\begin{aligned} \nabla_i &= \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial z_i}\right) \ , \\ \nabla_{ij} &= \left(\frac{\partial}{\partial x_{ij}}, \frac{\partial}{\partial y_{ij}}, \frac{\partial}{\partial z_{ij}}\right) \ , \\ x_{ij} &= x_i - x_j, \ \ldots \end{aligned}$$

follows:

$$\mathbf{F}_{ij} = -\nabla_i V_{ij} = -\nabla_{ij} V_{ij} = +\nabla_j V_{ij} \stackrel{V_{ij}=V_{ji}}{=} -\mathbf{F}_{ji} \ . \tag{3.26}$$

Therewith we can write:

$$\begin{aligned} \sum_{i,j} \mathbf{F}_{ij} \cdot \dot{\mathbf{r}}_i &= \frac{1}{2} \sum_{i,j} \left(\mathbf{F}_{ij} \cdot \dot{\mathbf{r}}_i + \mathbf{F}_{ji} \cdot \dot{\mathbf{r}}_j\right) = \frac{1}{2} \sum_{i,j} \mathbf{F}_{ij} \cdot \dot{\mathbf{r}}_{ij} = \\ &= -\frac{1}{2} \sum_{i,j} \nabla_{ij} V_{ij} \cdot \dot{\mathbf{r}}_{ij} = -\frac{1}{2} \frac{d}{dt} \sum_{i,j} V_{ij} \ . \end{aligned} \tag{3.27}$$

The external force acting on particle i is of course independent of the coordinates of the other mass points. If it is, additionally, conservative then the associated potential can also depend only on $\mathbf{r}_i$:

$$V_i = V_i\left(\mathbf{r}_i\right) \ ; \quad \mathbf{F}_i^{(\mathrm{ex})} = -\nabla_i V_i \ . \tag{3.28}$$

That means:

$$\sum_i \mathbf{F}_i^{(\mathrm{ex})} \cdot \dot{\mathbf{r}}_i = -\sum_i \nabla_i V_i \cdot \dot{\mathbf{r}}_i = -\frac{d}{dt} \sum_i V_i \ . \tag{3.29}$$

Therewith we finally get for the **total potential** used in (3.22) and (3.23):

$$V(\mathbf{r}_1, \ldots, \mathbf{r}_N) = \sum_{i=1}^{N} V_i(\mathbf{r}_i) + \frac{1}{2} \sum_{i,j} V_{ij}\left(r_{ij}\right) \; . \tag{3.30}$$

3.1.4 Virial Theorem

Due to the particle motion in mass-point systems kinetic energy is steadily converted into potential energy and vice versa. Think of harmonic oscillators which possess at the turning points only potential energy while the kinetic energy becomes maximal when passing through the zero level of potential energy. So the virial theorem is sometimes useful because it provides information about the *time-averaged* contributions of kinetic and potential energy to the total energy. To derive the theorem we at first multiply the equation of motion scalarly by $\mathbf{r}_i$:

$$\sum_i m_i\left(\ddot{\mathbf{r}}_i \cdot \mathbf{r}_i\right) = \sum_i \mathbf{F}_i \cdot \mathbf{r}_i \; .$$

This can be reformulated as follows:

$$\sum_i \frac{d}{dt} m_i\left(\dot{\mathbf{r}}_i \cdot \mathbf{r}_i\right) - \sum_i m_i \dot{r}_i^2 = -\sum_i \nabla_i V \cdot \mathbf{r}_i \; . \tag{3.31}$$

We restrict ourselves here to **conservative forces**.

The **time average** of an arbitrary time function $f(t)$ is defined as follows:

$$\langle f \rangle = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} f(t) dt \; . \tag{3.32}$$

We now apply this prescription to the first summand in (3.31):

$$\left\langle \sum_i \frac{d}{dt}\left[m_i\left(\dot{\mathbf{r}}_i \cdot \mathbf{r}_i\right)\right] \right\rangle = \sum_i \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^{\tau} \frac{d}{dt}\left[m_i\left(\dot{\mathbf{r}}_i \cdot \mathbf{r}_i\right)\right] dt =$$
$$= \lim_{\tau \to \infty} \frac{1}{\tau} \left[\sum_i m_i\left(\dot{\mathbf{r}}_i \cdot \mathbf{r}_i\right)\right]\Bigg|_0^{\tau} \; .$$

If we restrict ourselves to motions which take place in a finite region of space (hyperbolic comet orbits, e.g., are excluded) and the velocities of the particles are always finite, then the right-hand side of this equation vanishes. Hence after averaging what remains of (3.31) is:

$$2\,\langle T\rangle = \left\langle \sum_i \mathbf{r}_i \cdot \nabla_i V \right\rangle \,. \tag{3.33}$$

The right-hand side is the so-called **virial of the forces**. The **virial theorem** (3.33) tells us that under the mentioned assumptions the time average of the kinetic energy is equal to one half of the virial of the system.

Special statements can be derived for **closed** systems

$$V\left(\mathbf{r}_1, \, \ldots \, , \mathbf{r}_N\right) = \frac{1}{2} \sum_{i,j} V_{ij}\left(r_{ij}\right) \tag{3.34}$$

if the *internal* potential V_{ij} can be written as

$$V_{ij} = \alpha_{ij}\, r_{ij}^m \;; \quad m \in \mathbb{Z} \tag{3.35}$$

Then it follows (proof?):

$$\sum_i \mathbf{r}_i \cdot \nabla_i V = m\, V \;, \tag{3.36}$$

whereby the virial theorem simplifies to

$$2\,\langle T\rangle = m\,\langle V\rangle \tag{3.37}$$

Examples

(1) **coupled oscillators**

$$\begin{aligned} V_{ij} &= \frac{1}{2} k_{ij}\, r_{ij}^2 \\ m &= 2 \Longrightarrow \langle T\rangle = \langle V\rangle \;. \end{aligned} \tag{3.38}$$

(2) **Coulomb and gravitational potential**

$$\begin{aligned} V_{ij} &= \frac{\alpha}{r_{ij}} \\ m &= -1 \Longrightarrow 2\,\langle T\rangle = -\,\langle V\rangle \;. \end{aligned} \tag{3.39}$$

So it holds for the total energy:

$$E = \langle T\rangle + \langle V\rangle = -\,\langle T\rangle \;. \tag{3.40}$$

We see that the total energy is always negative as long as the movement is restricted to a finite space region (*bounded* motion).

3.2 Two-Particle Systems

3.2.1 Relative Motion

We now want to discuss a system of **two** mass points as an important special case of our considerations of the last section. For this purpose we introduce according to (3.5) a **center-of-mass coordinate**

$$\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} \tag{3.41}$$

and a **relative coordinate**

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \tag{3.42}$$

The position vectors of the two particles $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively, can be expressed by $\mathbf{r}$ and $\mathbf{R}$:

$$\mathbf{r}_1 = \mathbf{R} + \frac{m_2}{M}\mathbf{r}\,, \tag{3.43}$$

$$\mathbf{r}_2 = \mathbf{R} - \frac{m_1}{M}\mathbf{r}\,. \tag{3.44}$$

We transform the **coupled equations of motion** for $\mathbf{r}_{1,2}$ into those for $\mathbf{r}$ and $\mathbf{R}$. According to the center of mass theorem (3.8) it directly holds:

$$M\ddot{\mathbf{R}} = \mathbf{F}^{(\mathrm{ex})}\,. \tag{3.45}$$

For the relative coordinate we find:

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = \frac{\mathbf{F}_1^{(\mathrm{ex})}}{m_1} - \frac{\mathbf{F}_2^{(\mathrm{ex})}}{m_2} + \frac{\mathbf{F}_{12}}{m_1} - \frac{\mathbf{F}_{21}}{m_2}\,.$$

Definition 3.2.1 **'Reduced mass'**

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2} \Longleftrightarrow \mu = \frac{m_1 m_2}{m_1 + m_2}\,. \tag{3.46}$$

That yields as relative acceleration:

$$\ddot{\mathbf{r}} = \frac{\mathbf{F}_1^{(\mathrm{ex})}}{m_1} - \frac{\mathbf{F}_2^{(\mathrm{ex})}}{m_2} + \frac{1}{\mu}\mathbf{F}_{12} \; . \tag{3.47}$$

In a **closed system** $\left(\mathbf{F}_i^{(\mathrm{ex})} = 0\right)$ the two equations of motion (3.45) and (3.47) completely decouple. Strictly speaking, only for a closed system the splitting into relative and center-of-mass parts makes sense:

$$\mathbf{P} = M\dot{\mathbf{R}} = \mathrm{const} \; , \tag{3.48}$$

$$\mathbf{F}_{12} = \mu\ddot{\mathbf{r}} \propto \mathbf{r} \; . \tag{3.49}$$

Thus the relative motion takes place as if the reduced mass μ moves in the central field $\mathbf{F}_{12}$ which has its origin at $\mathbf{r}_2$ ($\Longrightarrow$ *effective* one-particle problem!).

In a similar manner we can decompose the **kinetic energy** T by use of (3.43) and (3.44) into a relative and a center-of-mass part. One easily finds:

$$\begin{aligned} T &= T_\mathrm{s} + T_\mathrm{r} \; , \\ T_\mathrm{s} &= \frac{1}{2}M\dot{\mathbf{R}}^2 \; , \\ T_\mathrm{r} &= \frac{1}{2}\mu\dot{\mathbf{r}}^2 \; . \end{aligned} \tag{3.50}$$

If we still assume that **all forces are conservative** then we can define as in (3.30) a **potential**:

$$\begin{aligned} V(\mathbf{r}_1, \mathbf{r}_2) &= \sum_{i=1}^{2} V_i(\mathbf{r}_i) + \frac{1}{2}\sum_{i,j=1}^{2} V_{ij}(r) \; , \\ \mathbf{F}_i^{(\mathrm{ex})} &= -\nabla_i V_i(\mathbf{r}_i) \; , \\ \mathbf{F}_{ij} &= -\nabla_i V_{ij} \; . \end{aligned}$$

This yields for the **total energy** E:

$$\begin{aligned} E &= E_\mathrm{s} + E_\mathrm{r} \; , \\ E_\mathrm{s} &= T_\mathrm{s} + V_1 + V_2 \; , \\ E_\mathrm{r} &= T_\mathrm{r} + V_{12} \; , \end{aligned} \tag{3.51}$$

where for closed systems one has to put $V_1 = V_2 = 0$.

Analogously a decomposition of the **angular momentum** is also possible. We had already found in (3.15):

$$\mathbf{L} = \mathbf{L}_{\mathrm{r}} + \mathbf{L}_{\mathrm{s}} \,, \tag{3.52}$$

$$\mathbf{L}_{\mathrm{s}} = (\mathbf{R} \times \mathbf{P}) = M \left(\mathbf{R} \times \dot{\mathbf{R}}\right) \,. \tag{3.53}$$

For the two-particle system we reformulate the relative part that represents the angular momentum of the mass-point system with respect to the center of mass:

$$\begin{aligned}
\mathbf{L}_{\mathrm{r}} &= \sum_i m_i \left(\bar{\mathbf{r}}_i \times \dot{\bar{\mathbf{r}}}_i\right) = \\
&= m_1 \left[\left(\frac{\mu}{m_1}\mathbf{r}\right) \times \left(\frac{\mu}{m_1}\dot{\mathbf{r}}\right)\right] + m_2 \left[\left(-\frac{\mu}{m_2}\mathbf{r}\right) \times \left(-\frac{\mu}{m_2}\dot{\mathbf{r}}\right)\right] = \\
&= \mu^2 \left(\mathbf{r} \times \dot{\mathbf{r}}\right) \left(\frac{1}{m_1} + \frac{1}{m_2}\right) \,.
\end{aligned}$$

That results in:

$$\mathbf{L}_{\mathrm{r}} = \mu \, (\mathbf{r} \times \dot{\mathbf{r}}) \,. \tag{3.54}$$

In a closed system all relevant quantities are thus decomposable into relative and center-of-mass portions. The original two-body problem has changed into two *effective* one-particle problems.

3.2.2 Two-Body Collision

By **collision** or **scattering** one denotes the interaction of two mass points m_1 and m_2, which together represent a closed system. As to the interaction we assume that the potential between them depends only on the separation of the particles and is **sufficiently short-range**. For large distances between the particles the interaction potential V becomes ineffective. Details concerning the immediate region of interaction are normally not available. Nevertheless it is possible to derive statements about the movement of the bodies after the collision since the (in general rather involved) internal forces do not influence the center-of-mass motion. Outside the interaction zone both bodies perform a force-free and therefore uniformly straight-line motion.

We assume that the initial momenta $\mathbf{p}_1,\ \mathbf{p}_2$ are known. We look for general statements about the final momenta $\mathbf{p}_1',\ \mathbf{p}_2'$. The number and the masses of the particles are assumed to remain unchanged during the collision process (**non-reactive** collisions).

For the investigation of the collision processes one uses two different reference systems. Experiments are done in the

lab(oratory) system Σ_L;

theoretically better tractable system is often the

center-of-gravity (mass) system Σ_S,

in which the center of mass is assumed to be stationary. The conversion between the two systems is simple:

$$\dot{\mathbf{r}}_i,\ \dot{\mathbf{r}}_i' \ : \ \text{velocities in } \Sigma_L \ ,$$

$$\dot{\bar{\mathbf{r}}}_i,\ \dot{\bar{\mathbf{r}}}_i' \ : \ \text{velocities in } \Sigma_S \ .$$

It holds the connection (Fig. 3.3):

$$\dot{\mathbf{r}}_i - \dot{\bar{\mathbf{r}}}_i = \dot{\mathbf{r}}_i' - \dot{\bar{\mathbf{r}}}_i' = \dot{\mathbf{R}}_L \ , \tag{3.55}$$

$$\dot{\mathbf{R}}_S = 0 \ ; \quad \mathbf{R}_S = 0 \ . \tag{3.56}$$

Since we presume a closed system Σ_S is an inertial system (Fig. 3.4). Decisive support for the study of collisions is given by the energy and momentum theorems which we now want to formulate for both the reference systems:

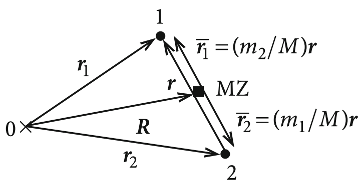


Fig. 3.3 Center-of-mass and relative coordinates of two mass points

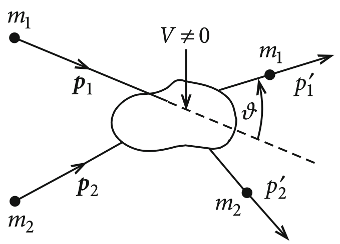


Fig. 3.4 Schematic sketch of the collision process between two mass points

(a) Momentum Conservation

The conservation of the linear momentum holds in both reference systems since Σ_S represents an inertial system:

$$\begin{aligned}
\Sigma_L &: \mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_1' + \mathbf{p}_2' = \mathbf{P} = \text{const} \,, \\
\Sigma_S &: \bar{\mathbf{p}}_1 + \bar{\mathbf{p}}_2 = \bar{\mathbf{p}}_1' + \bar{\mathbf{p}}_2' = 0 \quad \left(\bar{\mathbf{p}}_i = m_i \dot{\bar{\mathbf{r}}}_i\right) \,.
\end{aligned} \tag{3.57}$$

This results in:

$$\bar{\mathbf{p}}_1 = -\bar{\mathbf{p}}_2 \,; \quad \bar{\mathbf{p}}_1' = -\bar{\mathbf{p}}_2' \,. \tag{3.58}$$

The momentum theorem thus provides three equations for the determination of the six unknowns $\mathbf{p}_1', \ \mathbf{p}_2'$.

(b) Energy Conservation

$$\Sigma_L \ : \ \sum_{i=1}^{2} \frac{\mathbf{p}_i^2}{2m_i} = \sum_{i=1}^{2} \frac{\mathbf{p}_i'^2}{2m_i} + Q \,, \tag{3.59}$$

$$\Sigma_S \ : \ \sum_{i=1}^{2} \frac{\bar{\mathbf{p}}_i^2}{2m_i} = \sum_{i=1}^{2} \frac{\bar{\mathbf{p}}_i'^2}{2m_i} + \overline{Q} \,. \tag{3.60}$$

The quantities Q and $\overline{Q}$ include the conversion of mechanical energy into other forms of energy during the collision process. We show at first that $Q = \overline{Q}$ must hold:

$$\begin{aligned}
Q &= \sum_i \frac{1}{2m_i}\left(\mathbf{p}_i^2 - \mathbf{p}_i'^2\right) = \frac{1}{2}\sum_i m_i \left(\dot{\mathbf{r}}_i^2 - \dot{\mathbf{r}}_i'^2\right) = \\
&= \frac{1}{2}\sum_i m_i \left[\left(\dot{\bar{\mathbf{r}}}_i + \dot{\mathbf{R}}_L\right)^2 - \left(\dot{\bar{\mathbf{r}}}_i' + \dot{\mathbf{R}}_L\right)^2\right] = \\
&= \overline{Q} + \sum_i m_i \left(\dot{\bar{\mathbf{r}}}_i - \dot{\bar{\mathbf{r}}}_i'\right) \cdot \dot{\mathbf{R}}_L = \\
&= \overline{Q} \,, \quad \text{since } \sum_i m_i \, \bar{\mathbf{r}}_i = \sum_i m_i \, \bar{\mathbf{r}}_i' = 0 \,.
\end{aligned}$$

$Q = 0$: elastic collision ,
$Q > 0$: inelastic (endothermal) collision; kinetic energy is converted into internal energy of the collision partners (*excitation* of the collision partners) ,
$Q < 0$: inelastic (exothermal) collision; internal energy is converted into kinetic translational energy (*de-excitation* of the collision partners) .

From (3.58) follows:

$$\bar{\mathbf{p}}_1^2 = \bar{\mathbf{p}}_2^2 \; ; \quad \bar{\mathbf{p}}_1'^2 = \bar{\mathbf{p}}_2'^2 \; , \tag{3.61}$$

so that the energy theorem (3.60) in the center-of-gravity system can also be brought into the form

$$T_\mathrm{r} = \frac{\bar{\mathbf{p}}_i^2}{2\mu} = \frac{\bar{\mathbf{p}}_i'^2}{2\mu} + Q = T_\mathrm{r}' + Q \tag{3.62}$$

That holds both for $i = 1$ and $i = 2$. Therewith the energy theorem provides one further parameter. It fixes the magnitude of $\bar{\mathbf{p}}_i'$:

$$\bar{p}_i' = \sqrt{\bar{p}_i^2 - 2\mu Q} \; . \tag{3.63}$$

The direction is still free, i.e. two further parameters are lacking. These are available only in case of a more detailed knowledge of the collision process.

In Fig. 3.5 $\bar{\vartheta} = \sphericalangle\left(\bar{\mathbf{p}}_1, \bar{\mathbf{p}}_1'\right)$ denotes the **scattering angle** in Σ_S, which can assume arbitrary values:

$$0 \le \overline{\vartheta} \le \pi \; .$$

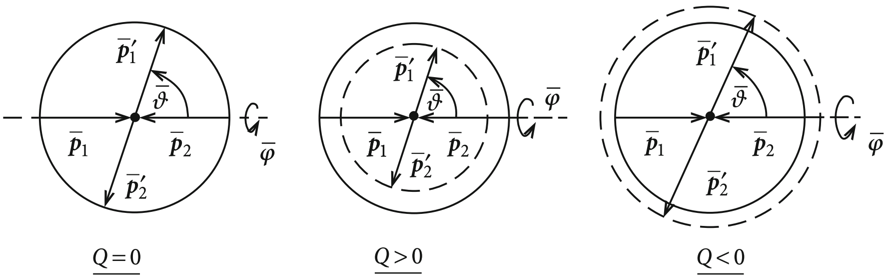


Fig. 3.5 Scattering angle and momenta before and after the elastic (*left*), the inelastic endothermal (*middle*), inelastic exothermal (*right*) collision of two mass points

Furthermore, we have to take into consideration that the $\mathbf{p}_1, \mathbf{p}_2$- and the $\mathbf{p}_1', \mathbf{p}_2'$-planes of course do not necessarily coincide in Σ_L. The azimuthal angle $\overline{\varphi}$ therefore also is undetermined.

3.2.3 Elastic Collision

Let us investigate the special case $Q = 0$ a bit more in detail. For that we start from the usual assumption that one of the collision partners does not move before the collision (*stationary target*):

$$\mathbf{r}_2 = 0\,; \quad \dot{\mathbf{r}}_2 = 0\,.$$

That means:

$$\mathbf{p}_1 = \mathbf{P}\,; \quad \mathbf{p}_2 = 0\,. \tag{3.64}$$

With (3.55) we have for the momenta after the collision:

$$\mathbf{p}_1' = m_1\dot{\mathbf{R}}_L + \bar{\mathbf{p}}_1' = \frac{m_1}{M}\,\mathbf{p}_1 + \bar{\mathbf{p}}_1'\,. \tag{3.65}$$

The momentum theorem (3.57) still delivers:

$$\mathbf{p}_2' = \mathbf{p}_1 - \mathbf{p}_1' = \frac{m_2}{M}\,\mathbf{p}_1 - \bar{\mathbf{p}}_1'\,. \tag{3.66}$$

Therewith the momenta $\mathbf{p}_{1,2}'$ are fixed except for the summand $\bar{\mathbf{p}}_1'$. So there are still lacking three unknowns. For $\bar{\mathbf{p}}_1'$ we still can find the magnitude with (3.63) : $\bar{p}_1' = \bar{p}_1$. Because of

$$\mathbf{p}_1 = m_1\left(\dot{\bar{\mathbf{r}}}_1 + \dot{\mathbf{R}}_L\right) = \bar{\mathbf{p}}_1 + \frac{m_1}{M}\,\mathbf{p}_1 = \frac{M}{m_2}\,\bar{\mathbf{p}}_1 \tag{3.67}$$

we get

$$\bar{p}_1' = \bar{p}_1 = \frac{m_2}{M}\,p_1\,.$$

Thus, except for the directions of $\bar{\mathbf{p}}_1'$, given by $\overline{\vartheta}, \overline{\varphi}$, the momenta after the collision are fixed in the lab system.

In the special case discussed here of a stationary target (3.64) all involved momenta lie in the same plane (*scattering plane*). Thus there is no need to look for the azimuth ($\varphi = \overline{\varphi}$). The relation between the scattering angles ϑ and $\overline{\vartheta}$ is determined as follows:

$$x = \bar{p}_1' \sin \overline{\vartheta} \; ; \quad y = \bar{p}_1' \cos \overline{\vartheta}$$

$$\Longrightarrow \tan \vartheta = \frac{x}{y + \frac{m_1}{M} p_1} = \frac{\sin \overline{\vartheta}}{\cos \overline{\vartheta} + \gamma} \; . \tag{3.68}$$

where we have defined:

$$\gamma = \frac{m_1}{M} \frac{p_1}{\bar{p}_1'} = \frac{m_1}{m_2} \; . \tag{3.69}$$

(a) $\boldsymbol{\gamma > 1 : \; m_1 > m_2}$

This case is represented in Fig. 3.6:

$$\bar{p}_1' = \frac{m_2}{M} p_1 < \frac{m_1}{M} p_1 \; .$$

Obviously there exists a maximal scattering angle $\vartheta_{\max}$:

$$\sin \vartheta_{\max} = \frac{\bar{p}_1'}{\frac{m_1}{M} p_1} = \frac{m_2}{m_1} = \frac{1}{\gamma} < 1$$

$$\Longrightarrow 0 \le \vartheta \le \vartheta_{\max} < \frac{\pi}{2} \; . \tag{3.70}$$

Thus the scattering takes place independently of the type of particle interaction only in the forward direction. For each $\vartheta < \vartheta_{\max}$ **two** scattering angles $\overline{\vartheta}$ and therewith **two** pairs of final momenta exist in Σ_s.

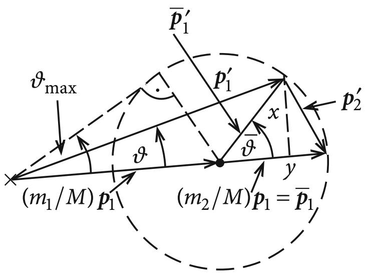


Fig. 3.6 Momenta and scattering angles in the scattering plane for the elastic collision of two mass points ($m_1 > m_2$, m_2 at rest before the collision)

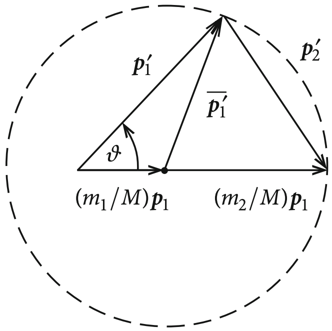


Fig. 3.7 Momenta and scattering angles in the scattering plane for the elastic collision between two mass points ($m_1 < m_2$, m_2 at rest before the collision)

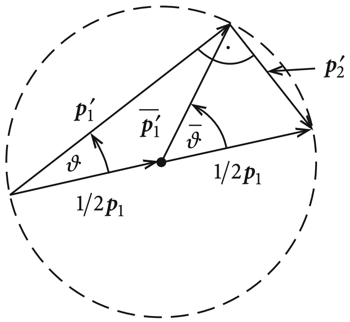


Fig. 3.8 Momenta and scattering angles within the scattering plane for the elastic collision of two mass points ($m_1 = m_2$, m_2 *at rest before the collision)*

(b) $\gamma < 1$: $m_1 < m_2$

According to (3.67) it is now

$$\bar{p}_1' > \frac{m_1}{M} p_1 \ .$$

That means that ultimately all scattering angles ϑ between 0 and π are possible (Fig. 3.7).

(c) $\gamma = 1$: $m_1 = m_2$

After the Thales theorem, in this special case the angle between the two final momenta just amounts to $\pi/2$. The two particles are always scattered into directions which are perpendicular to each other, forming a right angle, and that again independent of the actual interaction (Fig. 3.8).

The **central collision** defined by

$$\overline{\vartheta} = \pi \tag{3.71}$$

constitutes an exception. In this case it is obviously $\mathbf{p}_2' = \mathbf{p}_1$; $\mathbf{p}_1' = 0$. Hence, particle 1 is at rest after the collision while particle 2, which was at rest before the collision, now takes over the whole momentum.

Let us ask ourselves at the end about the **energy transfer** for the elastic collision in the laboratory system:

before the collision: $T = T_1 = \frac{p_1^2}{2m_1}$, $T_2 = 0$,

after the collision: $T' = T_1' + T_2' = \frac{p_1'^2}{2m_1} + \frac{p_2'^2}{2m_2}$.

Definition 3.2.2 'energy transfer'

$$\eta = \frac{T_2'}{T_1} . \tag{3.72}$$

For this one finds:

$$\begin{aligned}\eta &= \frac{m_1}{m_2}\frac{p_2'^2}{p_1^2} \overset{(3.66)}{=} \frac{m_1}{m_2}\frac{1}{p_1^2}\left(\frac{m_2^2}{M^2}p_1^2 - \frac{2m_2}{M}\,\mathbf{p}_1\cdot\bar{\mathbf{p}}_1' + \bar{p}_1'^2\right) = \\ &= \frac{m_1}{m_2}\left(\frac{m_2^2}{M^2} - 2\frac{m_2}{M}\frac{m_2}{M}\cos\overline{\vartheta} + \frac{m_2^2}{M^2}\right) = 2\frac{m_1 m_2}{M^2}\left(1-\cos\overline{\vartheta}\right) = \\ &= 2\frac{\mu}{M}\left(1-\cos\overline{\vartheta}\right) .\end{aligned} \tag{3.73}$$

Obviously the energy transfer is maximal for the central collision $\overline{\vartheta} = \pi$:

$$\eta\left(\overline{\vartheta} = \pi\right) = 4\,\frac{m_1 m_2}{M^2} . \tag{3.74}$$

In case of equal masses $m_1 = m_2$ it is $\eta = 1$, i.e., particle 2 takes the whole kinetic energy from particle 1 after the collision.

3.2.4 Inelastic Collision

The inelastic collision is defined by $Q \neq 0$. The total kinetic energy after the collision is therefore different from that before. In principle the same considerations are applicable as for the elastic collision. The momentum relations (3.57) and (3.58) do not change, i.e. the motion of the center of gravity remains unaffected. However it now holds (3.63):

$$\bar{p}_i' = \sqrt{\bar{p}_i^2 - 2\mu\, Q} \neq \bar{p}_i . \tag{3.75}$$

Because of (3.67)

$$\mathbf{p}_1 = \frac{M}{m_2}\,\bar{\mathbf{p}}_1 \tag{3.76}$$

we get, e.g., for the quantity γ in (3.69):

$$\gamma = \frac{m_1}{M}\frac{p_1}{\bar{p}_1'} = \frac{m_1}{m_2}\frac{\bar{p}_1}{\sqrt{\bar{p}_1^2 - 2\mu\,Q}} = \frac{m_1}{m_2}\sqrt{\frac{T_\mathrm{r}}{T_\mathrm{r} - Q}}\,. \tag{3.77}$$

With this expression for γ, slightly different compared to (3.69), the case-by-case analysis of the last section has to be repeated which will not be done here in detail. $\gamma \gtrless 1$ is now realizable not only by $m_1 \gtrless m_2$ but is also determined by Q.

(1) Capture Reaction

By this we understand the case where the two particles move on as a unit after the collision excluding therewith any relative motion:

$$\Longrightarrow T_\mathrm{r}' = 0 \Longleftrightarrow Q = T_\mathrm{r} \Longrightarrow \gamma = \infty\,. \tag{3.78}$$

For the velocities in the laboratory system then holds:

$$\begin{aligned} \dot{\mathbf{r}}_1' = \dot{\mathbf{r}}_2' = \dot{\mathbf{R}}_\mathrm{L} = \frac{m_1}{m_1 + m_2}\dot{\mathbf{r}}_1 \Longrightarrow \mathbf{p}_1' = \frac{\mu}{m_2}\mathbf{p}_1\,,\\ \mathbf{p}_2' = \frac{\mu}{m_1}\mathbf{p}_1 \Longrightarrow \vartheta = 0\,. \end{aligned} \tag{3.79}$$

Hence, no change of direction takes place.

(2) Particle Decay

Both particles are at first bound to as a single unit, the energy before the collision is zero:

$$\dot{\mathbf{r}}_1 = \dot{\mathbf{r}}_2 = 0\,.$$

But that means also:

$$\dot{\mathbf{R}}_\mathrm{L} = 0 \quad \text{and} \quad \mathbf{p}_1' = -\mathbf{p}_2'\,.$$

With (3.55) it follows $\dot{\mathbf{r}}_i' = \dot{\bar{\mathbf{r}}}_i'$; so the velocities in Σ_L and Σ_S are equal:

$$\mathbf{p}_i' = \bar{\mathbf{p}}_i' \; ; \quad \mathbf{p}_i = \bar{\mathbf{p}}_i = 0 \; .$$

Furthermore, according to (3.62) it must be:

$$\frac{\bar{p}_i'^2}{2\mu} = -Q = \frac{p_i'^2}{2\mu}$$

That leads to:

$$p_1' = p_2' = \sqrt{-2\mu\, Q} \; . \tag{3.80}$$

The two particles thus fly apart in opposite directions with velocity magnitudes according to:

$$\left(\dot{\mathbf{r}}_1'\right)^2 = -\frac{m_2}{m_1 + m_2}\frac{2Q}{m_1} = \left(\frac{m_2}{m_1}\right)^2 \left(\dot{\mathbf{r}}_2'\right)^2 \; , \tag{3.81}$$

$$\frac{\left|\dot{\mathbf{r}}_1'\right|}{\left|\dot{\mathbf{r}}_2'\right|} = \frac{m_2}{m_1} \tag{3.82}$$

3.2.5 *Planetary Motion as a Two-Particle Problem*

We have already extensively discussed the planetary motion in Sect. 2.5 as a one-body problem by assuming a space-fixed center of force thereby neglecting its co-movement. Strictly speaking, that is an approximation. We want to show in this section that this simplification is allowed as long as the masses of the interacting bodies (sun–planet, earth–satellite) are of different orders of magnitude, however, it is not allowed if they are of the same order of magnitude.

Between two masses $m_1,\ m_2$ with position vectors $\mathbf{r}_1,\ \mathbf{r}_2$, if only the gravitational force acts, the potential is given as in (2.261) by

$$V(\mathbf{r}_1, \mathbf{r}_2) = -\gamma \frac{m_1 m_2}{r_{12}} \tag{3.83}$$

This corresponds to the *internal* force:

$$\begin{aligned}\mathbf{F}_{12} &= -\nabla_1 V = -\nabla_{12} V(r_{12}) = -\frac{d}{dr_{12}} V(\mathbf{r}_{12})\, \nabla_{12} r_{12} = \\ &= -\gamma \frac{m_1 m_2}{r_{12}^3}\, \mathbf{r}_{12} \; .\end{aligned} \tag{3.84}$$

We assume that there are no *external* forces so that the center of gravity performs a uniform straight-line motion:

$$\mathbf{P} = \text{const} . \tag{3.85}$$

We can therefore concentrate ourselves according to (3.49) exclusively on the **relative motion**,

$$\mathbf{F}_{12} = \mu \, \ddot{\mathbf{r}}_{12} \sim \ddot{\mathbf{r}}_{12} \; ; \quad \mu = \frac{m_1 m_2}{m_1 + m_2} \; , \tag{3.86}$$

which represents an *effective* one-particle central problem. It is to solve the following equation of motion:

$$\mu \ddot{\mathbf{r}}_{12} = -\gamma \mu M \frac{\mathbf{r}_{12}}{r_{12}^3} \; . \tag{3.87}$$

The mathematical scope of work is formally the same as in Sect. 2.5. It corresponds to the movement of a mass μ in the gravitational field of a motionless force center of the mass $M = m_1 + m_2$, so that in particular according to (2.260) conservation of relative energy and relative angular momentum hold:

$$E_\mathrm{r} = \frac{\mu}{2} \dot{r}_{12}^2 + \frac{L_\mathrm{r}^2}{2\mu \, r_{12}^2} + V\left(r_{12}\right) \; , \tag{3.88}$$

$$\mathbf{L}_\mathrm{r} = \mu \, \left(\mathbf{r}_{12} \times \dot{\mathbf{r}}_{12}\right) \; . \tag{3.89}$$

The relative movement takes place in a fixed plane. With the same procedure as in Sect. 3.2.5, we find that the solutions of the differential equation (3.88) represent **conic sections**:

$$\frac{1}{r_{12}} = \frac{1}{k_\mathrm{r}} (1 + \varepsilon \cos \varphi) \; , \tag{3.90}$$

$$k_\mathrm{r} = \frac{L_\mathrm{r}^2}{\gamma \, M \, \mu^2} \tag{3.91}$$

($\varepsilon < 1$: ellipse; $\varepsilon = 1$: parabola; $\varepsilon > 1$: hyperbola).

The vector $\mathbf{r}_{12}$ thus describes, for instance, an ellipse for $\varepsilon < 1$. Now we can easily find expressions for the space coordinates $\mathbf{r}_1, \mathbf{r}_2$ of the two interacting bodies. If we put the zero of the coordinates into the center of gravity, $\mathbf{R} = 0$ (center-of-mass system), then it holds according to (3.43) and (3.44):

$$\mathbf{r}_1 = \frac{m_2}{M} \mathbf{r}_{12} \; ; \quad \mathbf{r}_2 = -\frac{m_1}{M} \mathbf{r}_{12} \; . \tag{3.92}$$

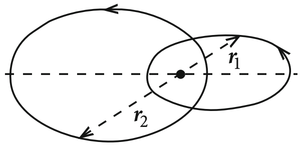


Fig. 3.9 Relative motion of two masses under the influence of the gravitational force

The two masses thus move on geometrically similar and equidirectionally placed ellipses around the common center of gravity which coincides with one of the two focal points of each ellipse (Fig. 3.9). For the semi-major axis a_r of the ellipse of the relative movement one finds according to (2.270):

$$a_\mathrm{r} = -\frac{\gamma\,\mu\,M}{2E_\mathrm{r}} \; .$$

The paths of the two masses m_1 and m_2 are then ellipses with semi-major axes:

$$a_1 = -\gamma\frac{\mu\,m_2}{2E_\mathrm{r}} \; ; \quad a_2 = -\gamma\frac{\mu\,m_1}{2E_\mathrm{r}} \; .$$

That means:

$$\frac{a_1}{a_2} = \frac{m_2}{m_1} \; . \tag{3.93}$$

Just as $\mathbf{L}_\mathrm{r}$, the angular momenta of the two masses are also constants of motion:

$$\mathbf{L}_i = m_i\,(\mathbf{r}_i \times \dot{\mathbf{r}}_i) = \frac{\mu}{m_i}\mathbf{L}_\mathrm{r} \; ; \quad i = 1, 2 \; . \tag{3.94}$$

The orbital periods are of course identical!

If the mass of one of the bodies is very much larger than that of the other (e.g. mass of the sun $\gg$ mass of a planet),

$$m_1 \gg m_2 \; ,$$

then one can assume

$$\mu \approx m_2 \; , \quad a_1 \ll a_2 \; ,$$

so that the co-movement of the mass m_1 can be neglected to a good approximation. Then the results of Sect. 2.5 become valid.

3.2.6 Coupled Oscillations

As a further example of a two-particle system we now consider a pair of mass points which are connected with each other by springs and with two fixed walls (Fig. 3.10). Thereby the masses shall perform only a one-dimensional motion along the x axis. This is a simple system with both *internal* and *external* forces which are all conservative:

$$F_1^{(\mathrm{ex})} = -k_1\,(x_1 - x_{01}) \Longleftrightarrow V_1\,(x_1) = \frac{k_1}{2}\,(x_1 - x_{01})^2\ ,$$

$$F_2^{(\mathrm{ex})} = -k_2\,(x_2 - x_{02}) \Longleftrightarrow V_2\,(x_2) = \frac{k_2}{2}\,(x_2 - x_{02})^2\ , \tag{3.95}$$

$$F_{12} = -k_{12}\,[(x_1 - x_{01}) - (x_2 - x_{02})] = -F_{21} \tag{3.96}$$

$$\Longleftrightarrow V_{12}\,(x_{12}) = \frac{k_{12}}{2}\,[(x_1 - x_{01}) - (x_2 - x_{02})]^2\ .$$

x_{01} and x_{02} are the equilibrium positions of the two masses. Since all forces present are conservative the **energy conservation law** holds:

$$E = \frac{m_1}{2}\dot{x}_1^2 + \frac{m_2}{2}\dot{x}_2^2 + V_1(x_1) + V_2(x_2) + V_{12}(x_{12}) \stackrel{!}{=} \text{const}\ . \tag{3.97}$$

It is convenient to introduce new coordinates:

$$y_i = x_i - x_{0i}\ ; \quad i = 1, 2$$

Then we have to solve the following system of **coupled equations of motion**:

$$\begin{aligned} m_1\ddot{y}_1 &= -k_1 y_1 - k_{12}\,(y_1 - y_2)\ , \\ m_2\ddot{y}_2 &= -k_2 y_2 + k_{12}\,(y_1 - y_2)\ . \end{aligned} \tag{3.98}$$

We seek the solution using the **ansatz**:

$$y_i = \alpha_i \cos\omega t\ ; \quad i = 1, 2\ . \tag{3.99}$$

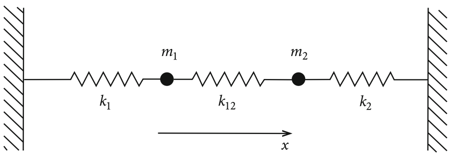


Fig. 3.10 Coupled oscillation of two mass points under the influence of spring forces

It results in the following **homogeneous system of equations**:

$$\begin{pmatrix} k_1 + k_{12} - m_1\omega^2 & -k_{12} \\ -k_{12} & k_2 + k_{12} - m_2\omega^2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} . \tag{3.100}$$

Condition for a non-trivial solution is according to (1.352) that the determinant of the (2×2)-matrix of coefficients (*secular (characteristic) determinant*) vanishes:

$$0 \stackrel{!}{=} \left(k_1 + k_{12} - m_1\omega^2\right)\left(k_2 + k_{12} - m_2\omega^2\right) - k_{12}^2 .$$

This is a quadratic equation for ω^2 the solution of which yields the following two **eigen frequencies**:

$$\begin{aligned} \omega_\pm^2 = \frac{1}{2}\Bigg\{ & \frac{1}{m_1}(k_1 + k_{12}) + \frac{1}{m_2}(k_2 + k_{12}) \pm \\ & \pm \sqrt{\left[\frac{1}{m_1}(k_1 + k_{12}) - \frac{1}{m_2}(k_2 + k_{12})\right]^2 + \frac{4k_{12}^2}{m_1 m_2}}\Bigg\} . \end{aligned} \tag{3.101}$$

In case of a *switched off* inter particle interaction $k_{12} = 0$ and one gets the eigen frequencies of two independent oscillators:

$$\omega_+^{(0)2} = \frac{k_1}{m_1} ; \quad \omega_-^{(0)2} = \frac{k_2}{m_2} .$$

The interaction obviously modifies the eigen frequencies. For the **amplitude ratio** of our solution ansatz (3.99) one finds:

$$\frac{\alpha_2^{(\pm)}}{\alpha_1^{(\pm)}} = \frac{1}{k_{12}}\left(k_1 + k_{12} - m_1\omega_\pm^2\right) = k_{12}\left(k_2 + k_{12} - m_2\omega_\pm^2\right)^{-1} . \tag{3.102}$$

For clarity let us focus the following discussion on a **symmetric system of coupled oscillators**, i.e. we assume:

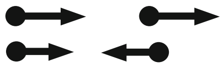
$$m_1 = m_2 = m ; \quad k_1 = k_2 = k \tag{3.103}$$

Then the eigen frequencies simplify to

$$\omega_+^2 = \frac{k + 2k_{12}}{m} ; \quad \omega_-^2 = \frac{k}{m} , \tag{3.104}$$

and for the associated amplitudes follows:

$$\begin{aligned} \alpha_1^{(-)} &= \alpha_2^{(-)} \,, \\ \alpha_1^{(+)} &= -\alpha_2^{(+)} \,. \end{aligned} \tag{3.105}$$



In the first case the two masses are oscillating synchronously with identical amplitudes in the same direction. The *inner* spring is thereby neither stretched nor compressed playing therewith no active role. That explains why ω_- agrees with the eigen frequency of the uncoupled oscillators. In the second case the two masses are oscillating against each other with equal amplitudes. That affects of course the *inner* spring; k_{12} therefore appears explicitly in ω_+.

Hence we have found the two **special** solutions

$$\begin{aligned} y_1^{(-)}(t) &= \alpha \cos \omega_- t = y_2^{(-)}(t) \,, \\ y_1^{(+)}(t) &= \beta \cos \omega_+ t = -y_2^{(+)}(t) \,. \end{aligned} \tag{3.106}$$

The **general** solution can then be written as linear combination and represents a superposition of two harmonic oscillations with different frequencies:

$$\begin{aligned} x_1(t) &= x_{01} + \alpha \cos\left(\omega_- t + \varphi_{(-)}\right) + \beta \cos\left(\omega_+ t + \varphi_{(+)}\right) \,, \\ x_2(t) &= x_{02} + \alpha \cos\left(\omega_- t + \varphi_{(-)}\right) - \beta \cos\left(\omega_+ t + \varphi_{(+)}\right) \,. \end{aligned} \tag{3.107}$$

$\alpha, \beta, \varphi_+, \varphi_-$ are to be fixed by initial conditions.

3.3 Exercises

Exercise 3.3.1 Two masses m_1 and m_2 are connected by springs with each other and with two fixed walls. The movement takes place in x direction; x_{01} and x_{02} are the equilibrium positions of the masses and $k_1,\ k_2,\ k_{12}$ are the spring constants (Fig. 3.11). We choose:

$$m_1 = \frac{1}{2} m_2 = m \,,$$

$$k_{12} = \frac{1}{5} k_2 = \frac{1}{2} k_1 = k \,.$$

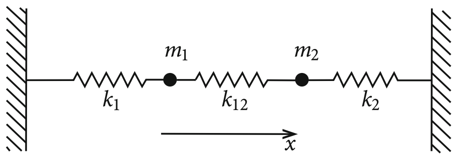


Fig. 3.11 Coupled oscillation of two mass points under the influence of spring forces

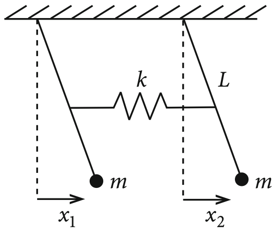


Fig. 3.12 Planar oscillations of a coupled thread pendulum

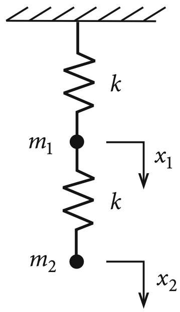


Fig. 3.13 One-dimensional movement of two coupled thread pendula

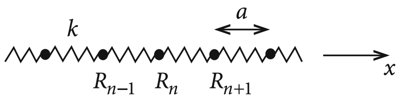


Fig. 3.14 Linear chain as one-dimensional model of a crystal for investigating lattice vibrations

1. Which forces act on the two masses?
2. Write down the equations of motion!
3. Calculate the *eigen frequencies* ω of the coupled oscillation!

Exercise 3.3.2 Two simple thread pendula of equal lengths L are coupled by a spring k (Fig. 3.12). The pendular oscillation takes place in a fixed plane. Discuss the motion for small oscillations about the equilibrium positions. Use the initial conditions: $x_1(0) = 0,\ x_2(0) = x_0,\ \dot{x}_1(0) = \dot{x}_2(0) = 0$.

Exercise 3.3.3 Two masses m_1 and m_2 are connected with each other by springs and with a fixed wall (Fig. 3.13). The two spring constants are equal. Write down and solve the equations of motion! Determine the oscillation frequencies!

Exercise 3.3.4 *'One-dimensional model of a crystal'*

1. The masses $m_n = m$ with $n = 0, \pm 1, \pm 2, \ldots$ are able to move along the x axis. Springs (spring constants k) take care for restoring forces (Hooke's law) between neighboring masses therewith defining equilibrium positions $R_n = n \cdot a$. The length a is the lattice constant of this infinite linear chain (Fig. 3.14). The

Fig. 3.15 Linear chain with alternating spring constants

displacements out of the equilibrium are denoted by

$$u_n(t) = x_n(t) - R_n \ .$$

Solve the equations of motion with the ansatz:

$$u_n(t) = A e^{i(qR_n - \omega t)} \ .$$

Determine the *'dispersion relation'*

$$\omega = \omega(q) \ .$$

State a reason why the *'wave number'* q can be restricted to the region

$$-\frac{\pi}{a} \leq q \leq +\frac{\pi}{a} \ .$$

2. Investigate the same problem for the case where the spring constants possess alternatingly the values f_1 and $f_2 \neq f_1$ (Fig. 3.15).
   How should the ansatz for $u_n(t)$ from part 1. be modified?

Exercise 3.3.5 Two masses $m_1, \ m_2$ are connected with each other by a *'mass-less'* bar of length l. The dumbbell, being in the earth's gravitational field, is thrown from the origin of coordinates in an arbitrary direction.

1. Write down the equation of motion of the center of mass!
2. Which path does the center of mass follow if the initial velocity is $\mathbf{v}_0$?
3. Decompose the total angular momentum into a relative and a center-of-mass part $\mathbf{L}_\mathrm{r}$ and $\mathbf{L}_\mathrm{s}$. Calculate $\mathbf{L}_\mathrm{s}$.
4. Formulate the equation of motion for the relative motion. What can be said about the relative angular momentum $\mathbf{L}_\mathrm{r}$?
5. Show that the masses m_1 and m_2 describe circular paths around the center of mass with constant angular velocity. What can be said about the radii?

Exercise 3.3.6 A particle of mass m with momentum $\mathbf{p}$ hits a particle of the same mass which is at rest in the laboratory system (see Fig. 3.16).

1. Derive a relation between $|\mathbf{p}|$ and the angles α and β.
2. Discuss the case $\alpha = \beta$. How large is α if the collision is elastic ($Q = 0$)? Which part of the kinetic energy can be maximally lost during an inelastic collision ($Q > 0$)?

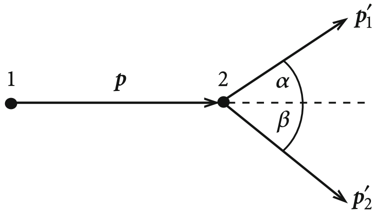


Fig. 3.16 Collision between two particles of the same mass

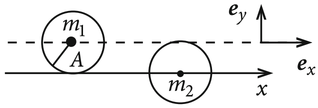


Fig. 3.17 Collision of two billiard balls with the same radius *A and a center-of-mass distance A*

Exercise 3.3.7 Consider the elastic collision between two hard spheres (billiard balls) with masses m_1, m_2 and equal radii A (see Fig. 3.17). In the laboratory system sphere 2 is at rest with its center on the x axis. Sphere 1 moves before the collision with constant momentum $\mathbf{p}_1 = p_1\mathbf{e}_x$ ($p_1 > 0$). The path of the center of gravity is a parallel line to the x axis at a distance A.

1. Which momenta $\mathbf{p}_1', \mathbf{p}_2'$ are found in the laboratory system after the collision? (No friction effects during the collision!)
2. What are the momenta $\bar{\mathbf{p}}_{1,2}, \bar{\mathbf{p}}_{1,2}'$ in the center-of-mass system?

3.4 Self-Examination Questions

To Section 3.1

1. What do we understand by internal and external forces acting on a mass-point system? When is such a mass-point system denoted as closed?
2. What is the definition of the center of mass?
3. Formulate and describe by examples the center of mass theorem!
4. Which information is given by the angular-momentum law?
5. Decompose the total angular momentum of a mass-point system in relative and center of mass contributions L_r and L_s. Which are the reference points of L_r and L_s?
6. Which information is given by the energy theorem?
7. What is the physical statement of the virial theorem?

To Section 3.2

1. How are center-of-mass coordinate and relative coordinate defined for a two-particle system?
2. How is the reduced mass defined?
3. How do the relative parts of angular momentum and energy look like in a two-particle system?

4. What does one understand by the *collision* of two mass points?
5. What are elastic and inelastic collisions?
6. Under which special conditions are two particles scattered into mutually perpendicular directions, independent of the actual interaction during the collision process?
7. Discuss the *central collision*!
8. Describe qualitatively two-particle collisions as the *capture reaction* and the *particle decay* as special cases!
9. Discuss qualitatively the planetary motion as a two-body problem! What are the trajectories of the moving sun and planet? In case of elliptical paths what can be said about the semi axes and the orbital periods?
10. Formulate the (one-dimensional) equations of motion for the coupled oscillation of a pair of mass points being connected with each other by springs and with two walls!
11. Determine the eigen frequencies of the above coupled oscillation for the special case when the two masses and also the two spring constants, which connect the two masses with the external walls, are identical!

Chapter 4
The Rigid Body

4.1 Model of a Rigid Body

Up to now we have discussed the phenomena of Classical Mechanics for the single mass point and for systems of mass points. Thereby, the respective physical problem was always considered to be solved as soon as the path line $\mathbf{r}_i(t)$ of each mass point had been derived from given force equations. For a macroscopic solid body with its particle number in the range of 10^{23} per cm^3 the mass-point concept of course becomes questionable. On the other hand, however, it is to be reflected whether one is really interested in the detailed microscopic particle motions. From a macroscopic point of view the solid appears as a continuum. Observables such as

1. displacements, translations,
2. rotations,
3. deformations

are applicable to microscopic particle paths only to a very limited degree. Thus we would rather treat the body as a whole, as a macroscopic unit. This fact allows for drastic idealizations (*'models'*) which, on their part, are often necessary to make a mathematical treatment of the problem feasible in the first place. The construction of

theoretical models

is typical for (theoretical) physics. In a certain sense a *model* can be compared with a **caricature** which tries to emphasize the essentials of the current problem while all the *'unnecessary ballast'* is dumped. That means, as a down side, normally a model can be valid only in a restricted, well-defined context; outside that it is either useless or even misleading.

W. Nolting, *Theoretical Physics 1*, DOI 10.1007/978-3-319-40108-9_4

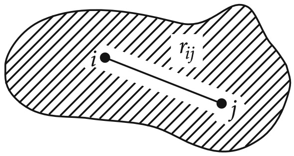


Fig. 4.1 Model of the rigid body (fixed particle distances r_{ij})

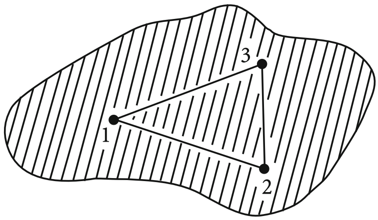


Fig. 4.2 To the determination of the degrees of freedom of the rigid body

The *'model of a rigid body'* is a system of N mass points such that the distances between the mass points are fixed for ever (Fig. 4.1)

$$r_{ij} = |\mathbf{r}_i - \mathbf{r}_j| = c_{ij} = \text{const} . \tag{4.1}$$

Hence the rigid body is *by definition not deformable*. Investigations concerning deformations, typical for elasticity theory, hydrodynamics, ..., are excluded from the very beginning.

Let us first try to find out the **number of degrees of freedom** of a rigid body. For this purpose we pick out three non-collinear (Fig. 4.2). For the description we need for each of them three Cartesian coordinates. These are at first nine parameters, which, however, have to fulfill, because of (4.1), three constraints:

$$r_{12} = c_{12} \; , \quad r_{13} = c_{13} \; , \quad r_{23} = c_{23}$$

So there are only six independent parameters. Each additional mass point of the rigid body introduces three more new coordinates, but also three more new constraints,

$$r_{j1} = c_{j1} \; , \quad r_{j2} = c_{j2} \; , \quad r_{j3} = c_{j3} \; ,$$

so that no additional free parameters come into play. The rigid body has therefore only

six degrees of freedom.

For a complete description of a rigid body one therefore needs only six independent quantities. Normally, however, one does not choose for this purpose the coordinates of three arbitrarily selected points of the body, but prefers to describe the movement as a whole in space:

1. By the **translation** of a special point S which is very often, but not necessarily always, the center of gravity of the body. It must be a point which is fixedly

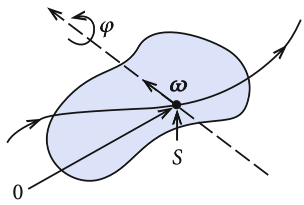


Fig. 4.3 Translation and rotation of the rigid body

connected with the body, which, however, need not necessarily lie within the body. That means we then have three degrees of freedom for the translation of the body (Fig. 4.3).

2. By the **rotation** around an axis through the point S. The axis does not need to be body- or space-fixed, it must only go through the point S. That yields three more degrees of freedom due to the rotation, namely **two** specifications of angles for specifying the rotation axis and **one** for the rotation angle.

For a general motion of the rigid body translation and rotation are coupled in a rather complicated manner. The translation, however, we have elaborately discussed as *mechanics of the free mass point* in Chap. 2. Therefore we will concentrate ourselves here primarily on two **special cases**:

(a) **(spinning) top:**
The rigid body is fixed at one point (no translation) therewith being left with only three degrees of freedom,

(b) **physical pendulum:**
The rigid body can rotate only around a fixed axis being therefore left with only one degree of freedom, namely the rotation angle.

For later applications an essential complication will arise, e.g., in the fact that rotations around different axis are **not** commutable.

We have introduced in Sect. 3.1 for the N-particle systems some important quantities which are of significance for the **total system**, e.g.:

$$\text{total mass:} \quad M = \sum_i m_i \ ,$$

$$\text{center of gravity:} \quad \mathbf{R} = \frac{1}{M} \sum_i m_i \, \mathbf{r}_i \ ,$$

$$\text{total momentum:} \quad \mathbf{P} = \sum_i m_i \dot{\mathbf{r}}_i \ ,$$

$$\text{total angular momentum:} \quad \mathbf{L} = \sum_i m_i \left(\mathbf{r}_i \times \dot{\mathbf{r}}_i\right), \ldots \ ,$$

They are given by summation over the respective single-particle quantities. How are these terms now calculated for the **continuum**? We explain the procedure by the

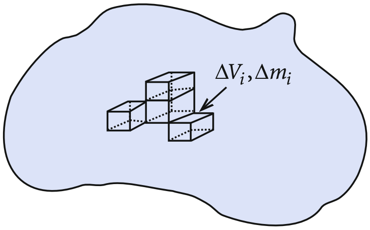


Fig. 4.4 Volume decomposition for continuum integrations

example of the total mass: One first decomposes the rigid body into small partial volume elements $\Delta V_i(\mathbf{r}_i)$, each of which contains a mass $\Delta m_i(\mathbf{r}_i)$. $\mathbf{r}_i$ is the position vector of a certain point in the i-th volume element (Fig. 4.4). Then it holds of course:

$$M = \sum_i \Delta m_i = \sum_i \frac{\Delta m_i}{\Delta V_i} \Delta V_i \; .$$

In a limiting process we now let the volumes ΔV_i become smaller and smaller ($\Delta V_i \to 0 \Longrightarrow \Delta m_i \to 0$) finding therewith the definition of the

$$\textbf{mass density}: \quad \rho(\mathbf{r}) = \lim_{\Delta V \to 0} \frac{\Delta m(\mathbf{r})}{\Delta V(\mathbf{r})} \; . \tag{4.2}$$

Since both Δm and ΔV are *quantity terms* (*extensive* quantities) this limiting value will in general be unequal zero. It is then:

$$\begin{aligned} \rho(\mathbf{r}) d^3 r &= \text{mass of the volume element} \\ d^3 r &= dx\,dy\,dz \text{ at } \mathbf{r} = (x, y, z) \; . \end{aligned} \tag{4.3}$$

The sum over all volume elements now becomes in the familiar *Riemann's sense* a so-called **volume (triple) integral** introduced in Sect. 1.2.5:

$$M = \int d^3 r \, \rho(\mathbf{r}) \; , \tag{4.4}$$

$$\mathbf{R} = \frac{1}{M} \int d^3 r \, \rho(\mathbf{r}) \mathbf{r} \; , \tag{4.5}$$

$$\mathbf{P} = \int d^3 r \, \rho(\mathbf{r}) \mathbf{v}(\mathbf{r}), \ldots \tag{4.6}$$

The integration is formally done over the entire space where, however, finite contributions come only from the space region occupied by the rigid body.

4.2 Rotation Around an Axis

We investigate at first a special form of motion of the rigid body, namely the rotation around a fixed axis. The system then possesses only one degree of freedom, that is the **rotation angle** around the axis. We will see in the following that energy theorem, angular-momentum law and center-of-mass theorem are sufficient to write down the equations of motions which in principle can be solved.

4.2.1 Conservation of Energy

We presume that all external forces are conservative thus possessing a potential. So the energy conservation law (2.231) holds. For its evaluation we first discuss the **kinetic energy**

$$T = \sum_i \frac{m_i}{2} \dot{\mathbf{r}}_i^2$$

of the rigid body. We assume a space-fixed axis and choose the z axis of the system of coordinates in such a way that it coincides with the rotation axis (Fig. 4.5). For the angular velocity $\boldsymbol{\omega}$ then holds:

$$\boldsymbol{\omega} = (0, 0, \omega)\,; \quad \omega = \dot{\varphi}\ . \tag{4.7}$$

Each point of the rigid body performs a circular motion, the linear velocity of which results according to (2.40) in

$$\dot{\mathbf{r}}_i = (\boldsymbol{\omega} \times \mathbf{r}_i) = \omega\,(-y_i, x_i, 0)\ . \tag{4.8}$$

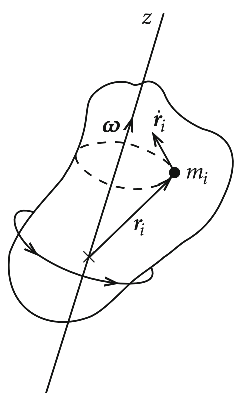


Fig. 4.5 Rotation of a rigid body around a space-fixed axis

Therewith we can specify the kinetic energy:

$$T = \frac{1}{2}\sum_i m_i \left(x_i^2 + y_i^2\right)\omega^2 = \frac{1}{2} J\,\omega^2 \,. \tag{4.9}$$

This equation defines the

Moment of Inertia

$$J = \sum_i m_i \left(x_i^2 + y_i^2\right) \tag{4.10}$$

as the sum of the products of the masses with the square of their distances from the rotation axis. J is a temporally constant scalar quantity which depends on the position and the direction of the axis within the rigid body. For concrete calculations one in general goes over from the discrete summation to an integration:

$$J = \int \rho(x,y,z)\left(x^2+y^2\right)dx\,dy\,dz = \int d^3r\,\rho(\mathbf{r})(\mathbf{n}\times\mathbf{r})^2 \,, \tag{4.11}$$

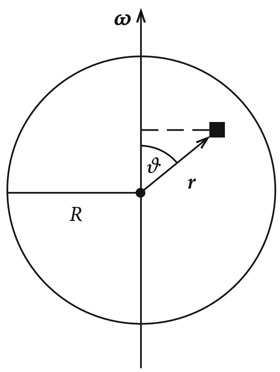
where $\mathbf{n} = \boldsymbol{\omega}/\omega$.

Examples

(1) Sphere with homogeneous mass distribution
The axis runs through the center of gravity (center of the sphere) but, apart from that, having an arbitrary direction (Fig. 4.6). For the mass density holds in this case:

$$\rho(\mathbf{r}) = \begin{cases} \rho_0 \,, & \text{for } r \le R \\ 0 \,, & \text{otherwise}\,. \end{cases}$$

Fig. 4.6 For the calculation of the moment of inertia of a sphere with homogeneous mass distribution



That yields with (4.11):

$$J = \int d^3r\,\rho(\mathbf{r})r^2 \sin^2\vartheta = \rho_0 \int\limits_0^R \int\limits_0^\pi \int\limits_0^{2\pi} r^4 dr \sin^3\vartheta\, d\vartheta\, d\varphi =$$

$$= 2\pi\,\rho_0 \int\limits_0^R r^4 dr \int\limits_{-1}^{+1} \left(1-\cos^2\vartheta\right)\, d\cos\vartheta = \frac{2\pi}{5}\rho_0 R^5 \left(2-\frac{2}{3}\right) =$$

$$= \left(\frac{4\pi}{3}R^3\rho_0\right)\frac{2}{5}R^2 = \frac{2}{5}MR^2\ . \tag{4.12}$$

(2) Cylinder with homogeneous mass distribution
As axis we choose the symmetry axis of the cylinder (length L, radius R, see Fig. 4.7). It is recommendable to use the cylindrical coordinates (ρ, φ, z) (coordinate ρ **not** to be confused with the density ρ!) for the calculation:

$$J = \int d^3r\,\rho(\mathbf{r})\rho^2 \overset{(1.382)}{=} \rho_0 \int\limits_0^R \int\limits_0^{2\pi} \int\limits_{-\frac{L}{2}}^{+\frac{L}{2}} \rho^3 d\rho\, d\varphi\, dz =$$

$$= 2\pi\,L\,\rho_0\frac{R^4}{4} = \frac{1}{2}MR^2\ . \tag{4.13}$$

Let us now come back to the energy theorem, for the formulation of which the potential energy is still lacking. Since the body has only one rotational degree of freedom the potential V can depend only on the rotation angle φ: $V = V(\varphi)$. The

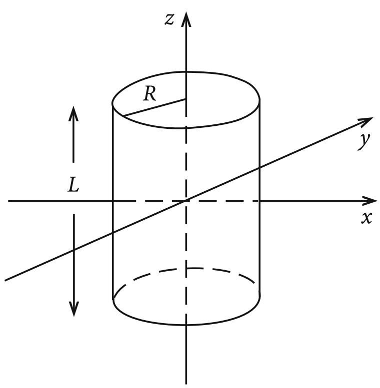


Fig. 4.7 For the calculation of the moment of inertia of a cylinder with homogeneous mass distribution

energy conservation law

$$E = T + V = \frac{1}{2} J \omega^2 + V(\varphi) = \frac{1}{2} J \dot{\varphi}^2 + V(\varphi) \tag{4.14}$$

has then mathematically the same structure as that for the one-dimensional motion (2.202). Hence it can be integrated in the same manner by **separation of variables**:

$$t - t_0 = \int\limits_{\varphi_0}^{\varphi} \frac{d\varphi'}{\sqrt{\frac{2}{J}\left(E - V\left(\varphi'\right)\right)}} \; . \tag{4.15}$$

The function $t = t(\varphi)$, therewith in principle deduced, or its inverse $\varphi = \varphi(t)$ determine uniquely and completely the motion of a rigid body which is rotatable around a fixed axis. This we will demonstrate with an example in section after the next.

4.2.2 Angular-Momentum Law

Only in special cases, e.g. for rotationally symmetric mass distributions, the angular momentum $\mathbf{L}$ is parallel to $\boldsymbol{\omega}$. We will therefore be interested here only in the component parallel to $\boldsymbol{\omega}$, i.e. the z component of the angular momentum:

$$\begin{aligned} L_\omega &= \mathbf{L} \cdot \mathbf{n} = \sum_i m_i \left(\mathbf{r}_i \times \dot{\mathbf{r}}_i\right) \cdot \mathbf{n} = \sum_i m_i \left(\mathbf{n} \times \mathbf{r}_i\right) \cdot \dot{\mathbf{r}}_i = \\ &= \sum_i m_i \left(\mathbf{n} \times \mathbf{r}_i\right) \cdot \left(\boldsymbol{\omega} \times \mathbf{r}_i\right) = \left(\sum_i m_i \left(\mathbf{n} \times \mathbf{r}_i\right)^2\right) \omega \; , \\ &\Longrightarrow L_\omega = \mathbf{L} \cdot \mathbf{n} = J\,\omega = J \dot{\varphi} \; . \end{aligned} \tag{4.16}$$

From that we can construct an equation of motion where we exploit the general angular-momentum law (3.13):

$$\frac{d}{dt}\mathbf{L} = \sum_i \left(\mathbf{r}_i \times \mathbf{F}_i^{(\mathrm{ex})}\right) = \sum_i \mathbf{M}_i^{(\mathrm{ex})} = \mathbf{M}^{(\mathrm{ex})} \; ,$$

At first it follows for the component along the rotation axis:

$$J \ddot{\varphi} = J \dot{\omega} = \sum_i \left(\mathbf{r}_i \times \mathbf{F}_i^{(\mathrm{ex})}\right) \cdot \mathbf{n} = M_\omega^{(\mathrm{ex})} \; . \tag{4.17}$$

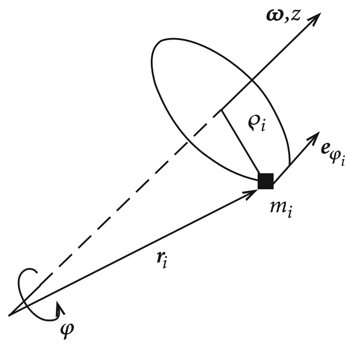


Fig. 4.8 Parameters for the calculation of the paraxial component of the angular momentum of a rigid body

The right-hand side can further be rewritten:

$$\frac{1}{\omega}\sum_i\left(\mathbf{r}_i\times\mathbf{F}_i^{(\mathrm{ex})}\right)\cdot\boldsymbol{\omega}=\frac{1}{\omega}\sum_i(\boldsymbol{\omega}\times\mathbf{r}_i)\cdot\mathbf{F}_i^{(\mathrm{ex})}=\sum_i\rho_i\left(\mathbf{F}_i^{(\mathrm{ex})}\cdot\mathbf{e}_{\varphi_i}\right)\ .$$

Hence the equation of motion reads:

$$J\ddot{\varphi}=\sum_i\rho_i\left(\mathbf{F}_i^{(\mathrm{ex})}\cdot\mathbf{e}_{\varphi_i}\right)\ . \tag{4.18}$$

$\mathbf{e}_{\varphi_i}$ is the azimuthal unit vector (1.392) for the i-th mass element:

$$\mathbf{e}_{\varphi_i}=(-\sin\varphi_i,\ \cos\varphi_i,\ 0)\ ;\quad \varphi_i=\varphi_{i0}+\varphi\ .$$

In case of a vanishing external torque $\mathbf{M}^{(\mathrm{ex})}$, we know that $\boldsymbol{\omega}=$ const. This means according to (4.9) the kinetic energy of the rotation is a conserved quantity:

$$\mathbf{M}_\omega^{(\mathrm{ex})}=\mathbf{M}^{(\mathrm{ex})}\cdot\mathbf{n}=0\Longrightarrow\omega=\mathrm{const}\Longrightarrow T=\mathrm{const}\ . \tag{4.19}$$

It is clear from (4.16) that then the paraxial component of the angular momentum is also constant (Fig. 4.8).

4.2.3 Physical Pendulum

By the term *'physical pendulum'* one understands a rigid body which is situated in the homogeneous earth's gravitational field and is rotatable around a horizontal axis (Fig. 4.9). The latter is again assumed to coincide with the z axis (4.7):

$$\mathbf{F}_i^{(\mathrm{ex})}=(m_i g,0,0)\ . \tag{4.20}$$

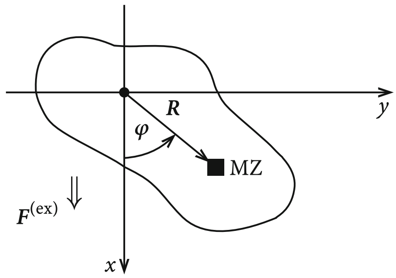


Fig. 4.9 Rigid body as physical pendulum

With (4.17) it then holds:

$$J\ddot{\varphi} = -\sum_i m_i y_i g = -Mg\,R_y\ . \tag{4.21}$$

R_y is the y component of the position vector of the center-of-gravity. If we choose the zero on the rotation axis such that

$$\mathbf{R} = \left(R_x, R_y, 0\right) = R(\cos\varphi, \sin\varphi, 0)\ ,$$

then it follows from (4.21) for the **pendular motion** $\varphi = \varphi(t)$ a **non-linear differential equation of second order**:

$$J\ddot{\varphi} + Mg\,R\sin\varphi = 0\ . \tag{4.22}$$

Therewith we have derived the equation of motion from the angular-momentum law. The comparison with the equation of motion (2.124) of the thread (simple) pendulum (*'mathematical pendulum'*),

$$\ddot{\varphi} + \frac{g}{l}\sin\varphi = 0\ ,$$

shows that the physical pendulum oscillates just like the mathematical pendulum with a thread length of

$$l = \frac{J}{M\,R}\ . \tag{4.23}$$

With this substitution we can thus adopt all the statements of Sect. 2.3.4. For small amplitudes we can approximate $\sin\varphi \approx \varphi$. Then (4.22) is solvable with the ansatz:

$$\varphi(t) = A\sin\overline{\omega}t + B\cos\overline{\omega}t$$

and the **angular frequency**

$$\overline{\omega} = \sqrt{\frac{Mg\,R}{J}}\ . \tag{4.24}$$

A and B are fixed by the necessary initial conditions.

The equation of motion (4.22) can also be derived via the energy theorem. For the potential of the mass m_i in the gravitational field holds (2.210)

$$V_i = -m_i g\, x_i\ . \tag{4.25}$$

The total potential of the external forces is then given by:

$$V = \sum_i V_i = -g \sum_i m_i x_i = -Mg\,R_x\ ,$$

$$V = -Mg\,R\cos\varphi = V(\varphi)\ . \tag{4.26}$$

That yields the **energy conservation law** of the physical pendulum:

$$E = \frac{1}{2} J\dot{\varphi}^2 - Mg\,R\cos\varphi = \text{const}\ . \tag{4.27}$$

After differentiating this expression once more with respect to time we indeed get again the equation of motion (4.22).

4.2.4 Steiner's Theorem

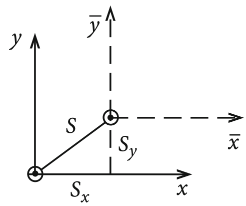
The **moment of inertia** J defined in (4.11) is an important characteristic parameter of the rotary motion of a rigid body which depends on both the direction and the actual position of the rotation axis. According to Steiner's theorem the moment of inertia about a given axis can be determined in a simple manner if the moment of inertia J_s with regard to an axis through the center of gravity and parallel to the given one is known.

The moment of inertia J about an arbitrary rotation axis is additively composed by the moment of inertia J_s about a parallel axis through the center of gravity and the moment of inertia for the total mass M concentrated in the center of gravity about the original axis:

$$J = J_\mathrm{s} + M\,S^2 \tag{4.28}$$

(S = perpendicular distance of the center of gravity from the rotation axis, i.e. '*distance between the axes*').

Fig. 4.10 Illustration for the derivation of Steiner's theorem



Proof Without loss of generality we can as usual assume that the rotation axis defines the z axis. Then the moment of inertia about the actual rotation axis is

$$J = \sum_i m_i \left(x_i^2 + y_i^2\right)$$

and that about the parallel axis through the center of gravity (Fig. 4.10):

$$J_\mathrm{s} = \sum_i m_i \left(\bar{x}_i^2 + \bar{y}_i^2\right) \ .$$

From Fig. 4.10 we have:

$$x_i = \bar{x}_i + S_x \ , \quad y_i = \bar{y}_i + S_y \ .$$

Therewith follows:

$$\begin{aligned} J &= \sum_i m_i \left[(\bar{x}_i + S_x)^2 + \left(\bar{y}_i + S_y\right)^2\right] = \\ &= \sum_i m_i \left(\bar{x}_i^2 + \bar{y}_i^2\right) + \left(S_x^2 + S_y^2\right) \sum_i m_i + 2S_x \sum_i m_i \bar{x}_i + 2S_y \sum_i m_i \bar{y}_i = \\ &= J_\mathrm{s} + M\,S^2 + 2S_x M\,R_{\bar{x}} + 2S_y\,M\,R_{\bar{y}} \ . \end{aligned}$$

With $R_{\bar{x}} = R_{\bar{y}} = 0$ (x and y components of the center of gravity in a coordinate system in which the center of gravity lies on the z axis) it results:

$$J = J_\mathrm{s} + M\,S^2.$$

As a special detail one reads off from (4.28) that out of an ensemble of parallel axes the one through the center of gravity always yields the smallest moment of inertia.

4.2.5 *Rolling Motion*

As a further important example of a rigid body with only **one** rotational degree of freedom we consider the

homogeneous cylinder rolling off an inclined plane

Though the rotation axis is again body-fixed it is not space-fixed. It is shifting in parallel to itself (Fig. 4.11). The velocity of each of the cylinder points is composed by two contributions, a **rotational contribution** due to the rotation around the cylinder axis during the rolling motion and a **translational contribution** which is the same for all points of the cylinder and happens in s direction:

$$\dot{\mathbf{r}}_i = \dot{\mathbf{r}}_{iR} + \dot{\mathbf{r}}_{iT} \ . \tag{4.29}$$

The rotational contribution we have already calculated in (4.8):

$$\dot{\mathbf{r}}_{iR} = (\boldsymbol{\omega} \times \bar{\mathbf{r}}_i) \ . \tag{4.30}$$

The translational contribution is obtained from the **rolling off condition**

$$\Delta s = R\,\Delta\varphi \Longrightarrow |\dot{\mathbf{r}}_{iT}| = |\dot{\mathbf{s}}| = R\,|\dot{\varphi}| \ . \tag{4.31}$$

The cylinder shall **roll, not slide**.

(a) Kinetic Energy

$$T = \frac{1}{2}\sum_i m_i \dot{\mathbf{r}}_i^2 = \frac{1}{2}\sum_i m_i \left[(\boldsymbol{\omega} \times \bar{\mathbf{r}}_i)^2 + 2\dot{\mathbf{s}} \cdot (\boldsymbol{\omega} \times \bar{\mathbf{r}}_i) + \dot{s}^2\right] \ .$$

The mixed term disappears because in a homogeneous cylinder two volume elements located diametrally opposite to the rotation axis have the same mass but

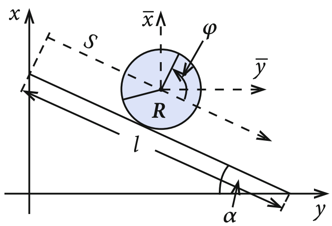


Fig. 4.11 Rolling cylinder on an inclined plane under the influence of the gravitational force

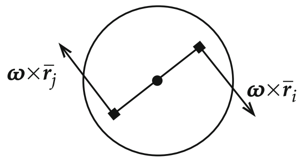


Fig. 4.12 Sketch for the derivation of the kinetic energy of a cylinder rolling on an inclined plane

rotation velocities are in opposite directions (Fig. 4.12). The sum over all elements is therefore zero. It can of course be shown also by a direct calculation that

$$\sum_i m_i\,(\boldsymbol{\omega}\times\bar{\mathbf{r}}_i)=0$$

must hold.

The first term is the kinetic energy of the rotational motion as we have found in (4.9). Thus it remains if one exploits (4.9), (4.13), and (4.30):

$$T=\frac{1}{2}J\,\omega^2+\frac{1}{2}M\dot{s}^2=\frac{1}{2}\left(\frac{1}{2}MR^2\right)\left(\frac{1}{R^2}\dot{s}^2\right)+\frac{1}{2}M\dot{s}^2\ .$$

This results in the simple expression:

$$T=\frac{3}{4}M\dot{s}^2\ . \tag{4.32}$$

(b) Potential Energy

The gravitational force acts on the cylinder:

$$V=\sum_i V_i=\sum_i m_i g\,x_i=Mg\,R_x\ . \tag{4.33}$$

R_x is the x-component of the center of gravity of the cylinder. By use of (4.5) it can be shown that the center of gravity of the homogeneous cylinder lies at the mid-point of the axis. So it is (Fig. 4.11)

$$R_x=(l-s)\sin\alpha$$

and therewith

$$V=Mg(l-s)\sin\alpha\ . \tag{4.34}$$

Hence the total potential agrees with the potential of the total mass concentrated at the center of gravity.

(c) Energy Theorem

Since only conservative forces act the total energy E is a conserved quantity:

$$E = T + V = \frac{3}{4} M\dot{s}^2 + (l - s)\, Mg \sin\alpha = \text{const} . \tag{4.35}$$

Differentiating this relation with respect to time and then dividing it by $(3/2)\, M\dot{s}$ leads to the **equation of motion**

$$\ddot{s} = \frac{2}{3} g \sin\alpha . \tag{4.36}$$

In case of a frictionless **sliding** of the body the acceleration on the inclined plane would be

$$\ddot{s} = g \sin\alpha$$

as can easily be demonstrated with Fig. 4.11. The acceleration of the rolling off cylinder thus amounts to only two-thirds of this value.

4.2.6 Analogy Between Translational and Rotational Motion

We have discovered in the preceding sections a strong analogy between the rotational motion around a body-fixed axis and the one-dimensional particle motion which, finally, we want to gather once more at the end of this section:

particle	**rotator**
position: x	rotation angle: φ
mass: m	moment of inertia: J
velocity: $v = \dot{x}$	angular velocity: $\omega = \dot{\varphi}$
momentum: $p = m\, v$	angular momentum: $L_\omega = J\, \omega$
force: F	torque: $M_\omega^{(\mathrm{ex})}$
kinetic energy: $T = (m/2)v^2$	kinetic energy: $T = (1/2)J\, \omega^2$
equation of motion: $F = m\ddot{x}$	equation of motion: $M_\omega^{(\mathrm{ex})} = J\ddot{\varphi}$

4.3 Inertial Tensor

In Sect. 4.2 we discussed the motion of a rigid body around a fixed axis. Thereby it was found that the moment of inertia J about a rotation axis is the fundamental quantity for the rotational movement. If the rotation axis has a temporally changing direction,

$$\mathbf{n}(t) = \frac{\boldsymbol{\omega}(t)}{\omega(t)} , \tag{4.37}$$

then the moment of inertia, too, will become a time-dependent quantity. Problems of this kind are dealt with by the introduction of the **inertia tensor**. To understand this some preparations are necessary.

4.3.1 Kinematics of the Rigid Body

In our introductory Sect. 4.1 we had already decomposed the **general motion of a rigid body** into

1. the **translation** of an arbitrarily chosen point S of the body
   and
2. the **rotation** around an axis through this point S .

We now introduce two reference systems which are initially both Cartesian:
$\widehat{\Sigma}$: **space-fixed** reference system with a space-fixed origin of coordinates $\mathcal{O}$. It is assumed to be an inertial system. Axis : $\hat{\mathbf{e}}_\alpha, \alpha = 1, 2, 3$.
Σ: **body-fixed** reference system with the body-fixed origin S. Axes: $\mathbf{e}_\alpha(t)$, $\alpha = 1, 2, 3$.

The point S has the position vector $\mathbf{r}_0(t)$ as seen from $\widehat{\Sigma}$. Then it holds for the points of the rigid body:

$$\hat{\mathbf{r}}_i(t) = \sum_{\alpha=1}^{3} \hat{x}_{i\alpha}(t)\hat{\mathbf{e}}_\alpha \qquad (\text{in } \widehat{\Sigma}) , \tag{4.38}$$

$$\mathbf{r}_i(t) = \sum_{\alpha=1}^{3} x_{i\alpha}\mathbf{e}_\alpha(t) \qquad (\text{in } \Sigma) \tag{4.39}$$

with the obvious relation:

$$\hat{\mathbf{r}}_i(t) = \mathbf{r}_0(t) + \mathbf{r}_i(t) . \tag{4.40}$$

The coordinates $x_{i\alpha}$ in the body-fixed system Σ are by the definition of the rigid body time-independent quantities. The position of the rigid body is therewith completely given by the position of Σ relative to $\widehat{\Sigma}$.

We are now interested in the **velocities** of the mass points of the rigid body (Fig. 4.13). These we find rather easily with the general theory of arbitrarily relative to each other moving reference systems that we derived in Sect. 2.2.5. The full time derivative of a vector represented in Σ seen from $\widehat{\Sigma}$ can be written as the **operator**

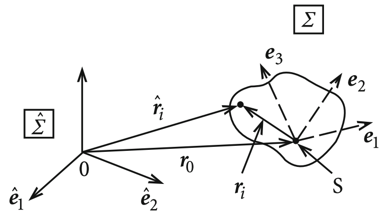


Fig. 4.13 Sketch for the calculation of the velocity of a mass point in a rigid body

identity (2.75):

$$\frac{\hat{d}}{dt} = \frac{d}{dt} + \boldsymbol{\omega} \times \quad .$$

deri- vative in $\widehat{\Sigma}$

derivative in Σ, which only concerns the components

influence of the rotation of Σ relative to $\widehat{\Sigma}$

The first term on the right-hand side plays by definition no role for the rigid body. Thus it remains:

$$\dot{\mathbf{r}}_i = (\boldsymbol{\omega} \times \mathbf{r}_i) \tag{4.41}$$

or with (4.40):

$$\dot{\hat{\mathbf{r}}}_i(t) = \dot{\mathbf{r}}_0(t) + (\boldsymbol{\omega} \times \mathbf{r}_i) \quad . \tag{4.42}$$

This is an important result. It signifies that at any moment of time the motion of a rigid body can be resolved into the translational motion $\mathbf{r}_0(t)$ of the origin of the body-fixed system and the rotation around the momentary rotation axis $\boldsymbol{\omega}(t)$ where the latter always passes through the origin S of the body-fixed system.

4.3.2 Kinetic Energy of the Rigid Body

We start from the definition of the kinetic energy T,

$$T = \frac{1}{2} \sum_i m_i \dot{\hat{\mathbf{r}}}_i^2 \; ,$$

and insert the expression (4.42) for the velocity:

$$T = \frac{1}{2}\sum_i m_i \dot{\mathbf{r}}_0^2 + \frac{1}{2}\sum_i m_i \left(\boldsymbol{\omega} \times \mathbf{r}_i\right)^2 + \sum_i m_i \left(\boldsymbol{\omega} \times \mathbf{r}_i\right) \cdot \dot{\mathbf{r}}_0 \ . \tag{4.43}$$

The third term is a scalar triple product and can therefore be rewritten as follows:

$$\sum_i m_i \, \mathbf{r}_i \cdot \left(\dot{\mathbf{r}}_0 \times \boldsymbol{\omega}\right) \ .$$

There are two typical cases for the discussion of the rigid body:

1. One point of the body remains space-fixed, while the body rotates with the angular velocity $\boldsymbol{\omega}$. Then it appears absolutely reasonable to choose this point as the origin S of Σ and in general also as the origin of $\widehat{\Sigma}$. One then speaks of a **spinning top** for which holds:

$$\mathbf{r}_0 = \mathbf{0} \ , \quad \dot{\mathbf{r}}_0 = \mathbf{0}$$

2. If no point is space-fixed one usually chooses the origin S at the center of mass and that means:

$$\sum_i m_i \mathbf{r}_i = \mathbf{0} \ .$$

We see that these two cases, the only relevant ones, both let the third term in (4.43) disappear. We therefore apply from the beginning the kinetic energy in the form:

$$T = \frac{1}{2} M \dot{\mathbf{r}}_0^2 + \frac{1}{2}\sum_i m_i \left(\boldsymbol{\omega} \times \mathbf{r}_i\right)^2 = T_T + T_{\mathrm{R}} \ . \tag{4.44}$$

Hence we have a clear separation of the kinetic energy into a rotational and translational part where we are interested mainly in the rotational part. We will inspect its dependence on the angular momentum a bit more in detail. The translational energy appears only in the case 2. stated above being then identical to the kinetic energy of the total mass concentrated at the center of mass.

It holds according to (1.201),

$$(\mathbf{a} \times \mathbf{b})^2 = a^2 b^2 - (\mathbf{a} \cdot \mathbf{b})^2 \ ,$$

and therewith

$$\begin{aligned}(\boldsymbol{\omega} \times \mathbf{r}_i)^2 &= \omega^2 r_i^2 - (\boldsymbol{\omega} \cdot \mathbf{r}_i)^2 = \left(\omega_1^2 + \omega_2^2 + \omega_3^2\right)\left(x_{i1}^2 + x_{i2}^2 + x_{i3}^2\right) - \\ &\quad - \left(\omega_1 x_{i1} + \omega_2 x_{i2} + \omega_3 x_{i3}\right)^2 \ .\end{aligned}$$

Insertion into (4.44) and arranging according to the components of $\boldsymbol{\omega}$ yields:

$$\begin{aligned}2T_{\mathrm{R}} =& \omega_1^2 \sum m_i \left(x_{i2}^2 + x_{i3}^2\right) - \omega_1\omega_2 \sum m_i x_{i1} x_{i2} - \omega_1\omega_3 \sum m_i x_{i1} x_{i3} - \\ &- \omega_2\omega_1 \sum m_i x_{i2} x_{i1} + \omega_2^2 \sum m_i \left(x_{i1}^2 + x_{i3}^2\right) - \omega_2\omega_3 \sum m_i x_{i2} x_{i3} - \\ &- \omega_3\omega_1 \sum m_i x_{i3} x_{i1} - \omega_3\omega_2 \sum m_i x_{i3} x_{i2} + \omega_3^2 \sum m_i \left(x_{i1}^2 + x_{i2}^2\right) \ .\end{aligned}$$

We define as

Components of the Inertial Tensor

$$J_{lm} = \sum_i m_i \left(\mathbf{r}_i^2 \delta_{lm} - x_{il} x_{im}\right) \ ; \quad l, m = 1, 2, 3 \ . \tag{4.45}$$

Therewith we can abbreviate and write as **rotational kinetic energy**:

$$T_{\mathrm{R}} = \frac{1}{2} \sum_{l,m=1}^{3} J_{lm} \omega_l \omega_m \ ; \quad \boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3) \ . \tag{4.46}$$

We see that T_{R} is **homogeneously quadratic** with respect to the components of the angular velocity. That means:

$$\frac{\partial T_{\mathrm{R}}}{\partial \omega_1} \omega_1 + \frac{\partial T_{\mathrm{R}}}{\partial \omega_2} \omega_2 + \frac{\partial T_{\mathrm{R}}}{\partial \omega_3} \omega_3 = 2T_{\mathrm{R}} \ .$$

The ensemble of coefficients is called

Inertial Tensor

$$\underline{\mathbf{J}} = (J_{lm}) = \begin{pmatrix} \sum\limits_i m_i \left(x_{i2}^2 + x_{i3}^2\right) & -\sum\limits_i m_i x_{i1} x_{i2} & -\sum\limits_i m_i x_{i1} x_{i3} \\ -\sum\limits_i m_i x_{i2} x_{i1} & \sum\limits_i m_i \left(x_{i1}^2 + x_{i3}^2\right) & -\sum\limits_i m_i x_{i2} x_{i3} \\ -\sum\limits_i m_i x_{i3} x_{i1} & -\sum\limits_i m_i x_{i3} x_{i2} & \sum\limits_i m_i \left(x_{i1}^2 + x_{i2}^2\right) \end{pmatrix} . \tag{4.47}$$

With a given system of coordinates the elements of the inertia tensor are uniquely fixed by the mass distribution of the rigid body. If the mass is continuously distributed with a known mass density $\rho(\mathbf{r})$ then one can switch for the actual calculation of the elements from the discrete summation to a continuous integration:

$$J_{lm} = \int d^3r \, \rho(\mathbf{r}) (r^2 \delta_{lm} - x_l x_m) \ . \tag{4.48}$$

Before proceeding with the physical discussion let us first inspect in the next section some of the most important tensor properties.

4.3.3 Properties of the Inertial Tensor

(1) What Is a Tensor?

Strictly speaking it is nothing other than a proper extension of the term *'vector'*. By a

tensor of k-th rank in an n-dimensional space

one understands an n^k number of elements

$$(F_{i_1,i_2,\ldots,i_k}) \ ; \quad i_j = 1,\ldots,n \ ,$$

which for coordinate rotations transform linearly satisfying certain rules. The elements are called the **components of the tensor**. They carry k indexes each of which runs from 1 to n. The rules are chosen just so that the *'normal'* vectors are first-rank tensors. One requires that in connection with coordinate rotations a tensor of k-th rank transforms itself with respect to all k indexes like a *'normal'* vector. According to our underlying physical problems of course only the cases $n = 1, 2, 3$ are interesting. Furthermore, in physics we can restrict ourselves to $k = 0, 1, 2$.
$\mathbf{k = 0}$: scalar: $\bar{x} = x$
$\mathbf{k = 1}$: vector, $n = 3$ components (in the three-dimensional space), for which, according to (1.309), it holds after a coordinate rotation:

$$\bar{x}_i = \sum_j d_{ij} x_j$$

(d_{ij}: components of the rotation matrix (1.307)),
$\mathbf{k = 2}$: $(F_{ij})_{i,j=1,2,3}$: $n^2 = 9$ components with

$$\bar{F}_{ij} = \sum_{l,m} d_{il} d_{jm} F_{lm} \tag{4.49}$$

and so on.
Second-rank tensors can always be written as square matrices. However, in contrast to normal matrices which are represented by collections of elements (numbers), which may behave arbitrarily with coordinate transformations, the above-mentioned **transformation behavior** is absolutely mandatory for the elements of a tensor.

Why is it necessary that the system of coefficients (4.47) does exhibit tensor properties? The components of the inertial tensor in a **given** system of coordinates are uniquely determined by the mass distribution of the rigid body. But with a rotation of the system of coordinates the components will change. Furthermore, of course also the components of the angular velocity $\boldsymbol{\omega}$ will undergo a change. However, it is clear that a rotation of the coordinate system should **not** influence the (measurable) rotational kinetic energy T_{R}. Equation (4.46) shows that this is then and only then the case when $\underline{\mathbf{J}}$ exhibits the transformation properties of a second-rank tensor:

$$\overline{T}_{\mathrm{R}} = \frac{1}{2}\sum_{l,m}\overline{J}_{lm}\overline{\omega}_l\overline{\omega}_m = \frac{1}{2}\sum_{l,m}\sum_{i,j} d_{li}d_{mj}J_{ij}\sum_{s,t} d_{ls}d_{mt}\omega_{\mathrm{s}}\omega_t =$$

$$= \frac{1}{2}\sum_{i,j}\sum_{s,t} J_{ij}\omega_{\mathrm{s}}\omega_t\delta_{is}\delta_{jt} = \frac{1}{2}\sum_{i,j} J_{ij}\omega_i\omega_j = T_{\mathrm{R}} \; .$$

In the penultimate step we have exploited the orthonormality relations (1.316) for rows and columns of the rotation matrix.

(2) Connection Between Moment of Inertia and Inertial Tensor

For the case of a fixed axis we had introduced the **moment of inertia** by the relation (4.9)

$$T_{\mathrm{R}} = \frac{1}{2}J\,\omega^2$$

With the components n_1, n_2, n_3 of the unit vector in the direction of the rotation axis

$$\mathbf{n} = \frac{\boldsymbol{\omega}}{\omega}$$

we can alternatively write for (4.46):

$$T_{\mathrm{R}} = \frac{1}{2}\left(\sum_{l,m} J_{lm}n_l n_m\right)\omega^2 \; .$$

The comparison yields the following important relationship between the **moment of inertia**, related to a fixed axis, and the **inertial tensor**:

$$J = \sum_{l,m} J_{lm}n_l n_m \; . \tag{4.50}$$

So we see that from a known inertial tensor it is rather easy to calculate the moment of inertia related to an arbitrary axis $\mathbf{n}$. The terms on the principal diagonal of the inertial tensor are then obviously the moments of inertia along the Cartesian coordinate axes since it holds for these rotation axes $\mathbf{n} = (1,0,0)\;,\;(0,1,0)\;,\;(0,0,1)$. In general one can say that by the inertia tensor $\underline{\mathbf{J}}$ there is assigned to each space-direction $\mathbf{n}$ a moment of inertia $J_{\mathbf{n}}$.

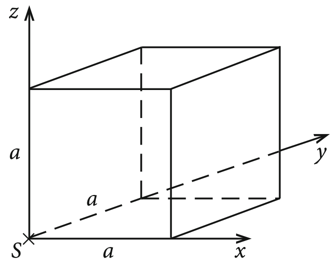


Fig. 4.14 For the calculation of the inertial tensor of a cube of edge length a with homogeneous mass density

(3) Example

We calculate the inertial tensor of a cube with homogeneous mass density. The point of reference S shall be in the bottom left corner of the cube (Fig. 4.14)

$$J_{11} = \rho_0 \iiint\limits_0^a dx\,dy\,dz\left(y^2 + z^2\right) = \rho_0 a^2\left(\frac{a^3}{3} + \frac{a^3}{3}\right) = \frac{2}{3}Ma^2\ ,$$

$$J_{13} = -\rho_0 \iiint\limits_0^a dx\,dy\,dz\,xz = -\rho_0\frac{a^2}{2}a\frac{a^2}{2} = -\frac{1}{4}M\,a^2\ .$$

The other elements are determined analogously:

$$\underline{\mathbf{J}} = M\,a^2\begin{pmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{pmatrix}\ . \tag{4.51}$$

(4) Principal Axes of Inertia

The inertial tensor $\underline{\mathbf{J}}$ is

symmetric $(J_{lm} = J_{ml})$ **and real** $\left(J_{lm} = J^*_{lm}\right)$**.**

For such a tensor it can generally be shown that for a fixed origin of coordinates there does exist a special rotation of the reference system so that all the off-diagonal elements disappear:

$$\underline{\mathbf{J}} = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}\ . \tag{4.52}$$

One speaks of a **'principal axes transformation'** and denotes the respective coordinate axes as **principal axes of inertia**. A, B, C are the **principal moments of inertia**. Later we will show how to determine the principal moments of inertia in practical applications.

(5) Inertial Ellipsoid

The inertial ellipsoid is introduced to illustrate the connection between moment of inertia and inertial tensor. Starting from the relation (4.50) between these two terms one ascribes to $\underline{\mathbf{J}}$ an area in the three-dimensional space, and that by the equation:

$$1 = \sum_{l,m} J_{lm} x_l x_m = J_{11} x_1^2 + J_{22} x_2^2 + J_{33} x_3^2 + \\ + 2J_{12} x_1 x_2 + 2J_{13} x_1 x_3 + 2J_{23} x_2 x_3 \, . \tag{4.53}$$

It is the equation of an **ellipsoid**.

If we insert into the picture of the ellipsoid (Fig. 4.15) an arbitrary axis defined by the unit vector **n**, then we can read off the coordinates of the intersection point P. Because of (4.50) it must hold:

$$P: \; x_i = \frac{n_i}{\sqrt{J}} \, . \tag{4.54}$$

The distance ρ between this point and the origin of coordinates S

$$\rho = \sqrt{\sum_i x_i^2} = \sqrt{\frac{1}{J} \left(n_1^2 + n_2^2 + n_3^2\right)} = \frac{1}{\sqrt{J}} \, , \tag{4.55}$$

delivers immediately the moment of inertia J with respect to the axis **n**. If the inertial ellipsoid is known then J can very easily determined for arbitrary directions of the axis.

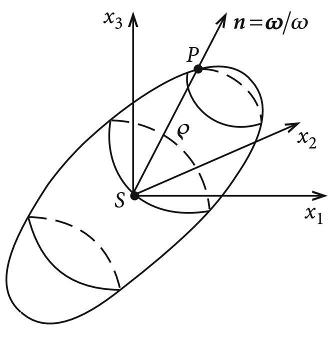


Fig. 4.15 Representation of the inertial ellipsoid of a rigid body rotating around the axis **n**

Every ellipsoid can be brought by a proper rotation of the coordinate system into its **'normal form'** for which the coordinate axes coincide with the symmetry axes so that the mixed terms disappear. That corresponds to the principal axes transformation mentioned under point (4). One denotes these special coordinate axes by

$$\xi, \eta, \zeta \ ,$$

for which then holds with (4.52) and (4.53):

$$1 = A\xi^2 + B\eta^2 + C\zeta^2 \ . \tag{4.56}$$

The inertial ellipsoid thus has the **edge lengths**

$$1/\sqrt{A}\ , \quad 1/\sqrt{B}\ , \quad 1/\sqrt{C}\ .$$

The rotational kinetic energy adopts in the principal axes system ξ, η, ζ according to (4.46) the simple form:

$$T_{\mathrm{R}} = \frac{1}{2}\left(A\omega_{\xi}^2 + B\omega_{\eta}^2 + C\omega_{\zeta}^2\right) \ . \tag{4.57}$$

The **symmetric** inertial tensor $\underline{\mathbf{J}}$ contains six independent elements being therefore characterized by six independent quantities. We can consider the three principal moments of inertia A, B, C and the three angles which fix the spatial orientation of the principal axes of inertia ζ, η, ξ as the six independent quantities. Later we will see that these are just the so-called Euler's angle to be discussed in a forthcoming section.

(6) Denotations

asymmetric spinning top:		$A \neq B \neq C$
symmetric spinning top:		$A = B \neq C$
	or	$A = C \neq B$
	or	$B = C \neq A$
spherical spinning top:		$A = B = C$.

4.3.4 Angular Momentum of the Rigid Body

In this section we want to find out the connection between the angular momentum and the inertial tensor of a rigid body. For the rotation around a fixed axis we found the relatively simple expression (4.16) for the paraxial angular-momentum component

$$L_\omega = J\omega \ .$$

Via the general relation for the angular momentum

$$\widehat{\mathbf{L}} = \sum_i m_i \left(\hat{\mathbf{r}}_i \times \dot{\hat{\mathbf{r}}}_i \right)$$

one gets by insertion of (4.40) for $\hat{\mathbf{r}}_i$ and (4.42) for $\dot{\hat{\mathbf{r}}}_i$:

$$\begin{aligned}\widehat{\mathbf{L}} = &\sum_i m_i \, \mathbf{r}_0 \times \dot{\mathbf{r}}_0(t) + \sum_i m_i \, \mathbf{r}_0 \times (\boldsymbol{\omega} \times \mathbf{r}_i) + \\ &+ \sum_i m_i \, \mathbf{r}_i \times \dot{\mathbf{r}}_0(t) + \sum_i m_i \, \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) \ .\end{aligned}$$

The second and the third summand vanish since we had agreed upon in Sect. 4.3.2 to choose as origin S in Σ a point in the rigid body which is fixed in space if such a point exists ($\mathbf{r}_0 = 0$, $\dot{\mathbf{r}}_0 = 0$) or, if it does not exist, to identify the center of gravity with S $\left(\sum_i m_i \, \mathbf{r}_i = \mathbf{0}\right)$:

$$\widehat{\mathbf{L}} = M \, \mathbf{r}_0(t) \times \dot{\mathbf{r}}_0(t) + \sum_i m_i \, \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) = \mathbf{L}_\mathrm{s} + \mathbf{L} \ . \tag{4.58}$$

The first summand is zero, when S as space-fixed point is simultaneously the origin in both Σ and $\widehat{\Sigma}$, otherwise it represents the angular momentum of the total mass concentrated in the center of gravity and therefore is relatively uninteresting. Hence we can restrict our considerations to the body's own angular momentum which refers to the origin S in Σ:

$$\begin{aligned}\mathbf{L} &= \sum_i m_i \, \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) \\ &= \sum_i m_i \left[r_i^2 \, \boldsymbol{\omega} - (\mathbf{r}_i \cdot \boldsymbol{\omega}) \; \mathbf{r}_i \right] \ .\end{aligned} \tag{4.59}$$

Multiplying this expression scalarly with $\boldsymbol{\omega}$ leads to:

$$\boldsymbol{\omega} \cdot \mathbf{L} = \sum_i m_i \left[r_i^2 \omega^2 - (\mathbf{r}_i \cdot \boldsymbol{\omega})^2 \right] = \sum_i m_i \, (\mathbf{r}_i \times \boldsymbol{\omega})^2 \ .$$

The comparison with (4.44) shows that between angular momentum and rotational kinetic energy the following relation exists:

$$T_{\mathrm{R}} = \frac{1}{2} (\boldsymbol{\omega} \cdot \mathbf{L}) \ . \tag{4.60}$$

As will be shown later, in general $\mathbf{L}$ does not have the same direction as $\boldsymbol{\omega}$. However, since T_{R} is definitely a positive number we can conclude from (4.60) that $\boldsymbol{\omega}$ and $\mathbf{L}$ will always enclose an acute angle.

Let us explicitly write down according to (4.59) the components of $\mathbf{L}$:

$$\begin{aligned}
L_1 &= \omega_1 \sum_i m_i \left(x_{i2}^2 + x_{i3}^2 \right) - \omega_2 \sum_i m_i x_{i1} x_{i2} - \omega_3 \sum_i m_i x_{i1} x_{i3} \ , \\
L_2 &= -\omega_1 \sum_i m_i x_{i2} x_{i1} + \omega_2 \sum_i m_i \left(x_{i1}^2 + x_{i3}^2 \right) - \omega_3 \sum_i m_i x_{i2} x_{i3} \ , \\
L_3 &= -\omega_1 \sum_i m_i x_{i3} x_{i1} - \omega_2 \sum_i m_i x_{i3} x_{i2} + \omega_3 \sum_i m_i \left(x_{i1}^2 + x_{i2}^2 \right) \ .
\end{aligned}$$

In view of (4.47) the following relationship between angular momentum and angular velocity is found:

$$L_l = \sum_{m=1}^{3} J_{lm} \omega_m \iff \mathbf{L} = \underline{\mathbf{J}} \, \boldsymbol{\omega} \ . \tag{4.61}$$

The components of the angular momentum are thus linear functions of the angular-velocity components. In the principal axes system the relations become especially simple:

$$\mathbf{L} = \left(A\omega_\xi, B\omega_\eta, C\omega_\zeta \right) \ . \tag{4.62}$$

The connection between angular momentum and angular velocity can also be demonstrated graphically by the use of the inertial ellipsoid. For the surface of the inertial ellipsoid (4.56) holds $F(\xi, \eta, \zeta) = 1$ with

$$F = A\xi^2 + B\eta^2 + C\zeta^2 = F(\xi, \eta, \zeta) \ .$$

From (1.271) we know that the gradient of F is orthogonal to the area $F = \text{const}$:

$$\nabla F = (2A\xi, \ 2B\eta, \ 2C\zeta) \ .$$

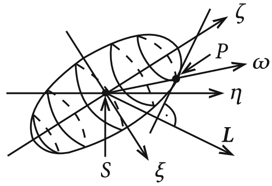


Fig. 4.16 Angular momentum and inertial ellipsoid in the principal axes system of a rigid body

For the components of the intersection point P of the rotation axis with the ellipsoid surface (Fig. 4.16) one gets because of (4.54):

$$\xi_p = \frac{n_\xi}{\sqrt{J}} \; ; \quad \eta_p = \frac{n_\eta}{\sqrt{J}} \; ; \quad \zeta_p = \frac{n_\zeta}{\sqrt{J}} \; . \tag{4.63}$$

Therewith follows:

$$\nabla F|_p = \frac{2}{\sqrt{J}} \left(A n_\xi, \, B n_\eta, \, C n_\zeta \right) = \frac{2}{\omega\sqrt{J}} \mathbf{L} \; .$$

The angular-momentum vector therefore stands perpendicularly on the tangential plane constructed at the intersection point P of the rotation axis with the inertial ellipsoid (Fig. 4.16). Furthermore, $\mathbf{L}$ is of course related to the origin S of Σ. Figure 4.16 illustrates that $\boldsymbol{\omega}$ and $\mathbf{L}$ are parallel then and only then when the rotation is carried out around one of the principal axes of inertia. Then only one component in (4.62) is different from zero and the proportionality of $\boldsymbol{\omega}$ and $\mathbf{L}$ becomes obvious.

This last fact can be exploited to determine the principal axes and the principal moments of inertia. We assume an arbitrary body-fixed system of coordinates. The angular velocity $\overline{\boldsymbol{\omega}}$ may have the direction of one of the principal axes of inertia. Then it must hold:

$$\mathbf{L} = \underline{\mathbf{J}}\,\overline{\boldsymbol{\omega}} = \overline{J}\,\overline{\boldsymbol{\omega}} \; . \tag{4.64}$$

That is a so-called **'eigenvalue equation'** of the matrix $\underline{\mathbf{J}}$. Unknowns are the scalar $\overline{J}$, which is named the **eigenvalue** of the matrix $\underline{\mathbf{J}}$, and the corresponding **eigenvector** of the matrix $\overline{\boldsymbol{\omega}}$. Equation (4.64) is equivalent to the following homogeneous system of equations:

$$\begin{aligned}
\left(J_{11} - \overline{J}\right) \overline{\omega}_1 + J_{12}\,\overline{\omega}_2 + J_{13}\,\overline{\omega}_3 &= 0 \; , \\
J_{21}\,\overline{\omega}_1 + \left(J_{22} - \overline{J}\right) \overline{\omega}_2 + J_{23}\,\overline{\omega}_3 &= 0 \; , \\
J_{31}\,\overline{\omega}_1 + J_{32}\,\overline{\omega}_2 + \left(J_{33} - \overline{J}\right) \overline{\omega}_3 &= 0 \; .
\end{aligned} \tag{4.65}$$

this homogeneous system of equations has non-trivial solutions according to (1.352) only when the determinant of the coefficient matrix vanishes:

$$\det\begin{pmatrix} J_{11}-\overline{J} & J_{12} & J_{13} \\ J_{21} & J_{22}-\overline{J} & J_{23} \\ J_{31} & J_{32} & J_{33}-\overline{J} \end{pmatrix} = \det\left(\underline{\mathbf{J}}-\overline{J}\cdot E\right) \stackrel{!}{=} 0\,. \tag{4.66}$$

If we evaluate this equation by the use of the **Sarrus'rule** (1.326) then it results in a polynomial of third degree for the unknown moment $\overline{J}$, which is called

characteristic (secular) equation.

Such an equation has **three** solutions:

$$\overline{J}_1 = A\,,\quad \overline{J}_2 = B\,,\quad \overline{J}_3 = C\,,$$

Since $\underline{\mathbf{J}}$ is symmetric and real each of the three solutions is real. They are just the principal moments of inertia.

Inserting the solutions for $\overline{J}$ one after another into the system of equations (4.65) leads to conditional equations for the three components of the angular velocity in direction of the respective principal axis of inertia. The rank of the coefficient matrix is according to (1.353) smaller than three so that always only the ratios $\overline{\omega}_1^{(i)} : \overline{\omega}_2^{(i)} : \overline{\omega}_3^{(i)}$ of the components of the eigenvector $\overline{\boldsymbol{\omega}}^{(i)}, i = 1,2,3$ are determinable. That, however, turns out to be sufficient to fix the directions of the $\overline{\boldsymbol{\omega}}^{(i)}$, which as per the ansatz (4.64) do agree with the principal axes of inertia.

4.4 Theory of the Spinning Top

From now on we assume that the rigid body possesses one space-fixed point which we take as the origin S of the body-fixed system of coordinates Σ.

4.4.1 Euler's Equations

We exploit the angular-momentum law (3.13)

$$\frac{d}{dt}\mathbf{L} = \mathbf{M}\,, \tag{4.67}$$

in order to derive equations of motion for the spinning top. $\mathbf{M}$ is the external torque where, for simplicity, we leave out from now on the superscript *ex*. In this form, however, the angular-momentum law holds only in the **inertial system** $\widehat{\Sigma}$. In this system, however, not only the components of the angular velocity but also the elements of the inertial tensor turn out to be time-dependent. It appears therefore not very reasonable for $\mathbf{L}$ to work with the result (4.61) of the last section.

It is more advisable to formulate the angular-momentum law in the co-rotating body-fixed reference system Σ where we choose as coordinate axes just the principal axes of inertia. Following convention we denote from now on the components of the angular velocity by p, q, r:

$$\boldsymbol{\omega} = p\,\mathbf{e}_\xi + q\,\mathbf{e}_\eta + r\,\mathbf{e}_\zeta \;, \tag{4.68}$$

$$\mathbf{L} = A\,p\,\mathbf{e}_\xi + B\,q\,\mathbf{e}_\eta + C\,r\,\mathbf{e}_\zeta \;. \tag{4.69}$$

For the time differentiation required in (4.67) we now apply again the operator identity (Sect. 4.2.1)

$$\left(\frac{d}{dt}\right)_{\widehat{\Sigma}} = \left(\frac{d}{dt}\right)_{\Sigma} + \boldsymbol{\omega}\times \;, \tag{4.70}$$

by which we are led to the following angular-momentum law:

$$\mathbf{M} = \dot{\mathbf{L}} + (\boldsymbol{\omega}\times\mathbf{L}) \;. \tag{4.71}$$

The time differentiation on the right-hand side has now to be performed in the body-fixed system for which the components A, B, C of the inertial tensor are time-independent:

$$\mathbf{M} = A\,\dot{p}\,\mathbf{e}_\xi + B\,\dot{q}\,\mathbf{e}_\eta + C\,\dot{r}\,\mathbf{e}_\zeta + \begin{vmatrix} \mathbf{e}_\xi & \mathbf{e}_\eta & \mathbf{e}_\zeta \\ p & q & r \\ A\,p & B\,q & C\,r \end{vmatrix} \;.$$

In detail that means:

$$\begin{aligned} M_\xi &= A\,\dot{p} + \;(C-B)\,q\,r \;, \\ M_\eta &= B\,\dot{q} + \;(A-C)\,r\,p \;, \\ M_\zeta &= C\,\dot{r} + \;(B-A)\,p\,q \;. \end{aligned} \tag{4.72}$$

These equations are called **Euler's equations** which for known components of the torque $\mathbf{M}$ in the body-fixed principal axes system represent a coupled system of differential equations for the components p, q, r of the angular velocity $\boldsymbol{\omega}$. They are the **equations of motion** for the rotational motion of the rigid body.

For the concrete evaluation of the system of equations one needs the components of the torque $\mathbf{M}$ with respect to the principal axes of inertia. Since $\mathbf{M}$ is caused by **external** forces there will appear on the left-hand side of (4.72) therefore also quantities which are defined in the space-fixed system $\widehat{\Sigma}$. Thus we have to establish relations between space-fixed and body-fixed reference systems. Of course we also need them in order to be able to find the actual position of the rigid body in the space-fixed system $\widehat{\Sigma}$ from the solutions p, q, r of Euler's equations.

4.4.2 Euler's Angles

Euler's angles indicate how a body-fixed co-rotating system is oriented with respect to a space-fixed system.

The space-fixed system of coordinates $\widehat{\Sigma}$ may be defined by the coordinates $\hat{x}, \hat{y}, \hat{z}$, the body-fixed system by x, y, z. As **line of nodes** K one denotes the intersection line of both the $(\hat{x}, \hat{y})$- and (x, y) equatorial planes perpendicular to $\hat{z}$ and z, respectively. There appear the following angles (Fig. 4.17):

$$\varphi = \sphericalangle(\hat{x} \text{ axis, line of nodes}) \; ,$$

$$\vartheta = \sphericalangle(\hat{z} \text{ axis, } z \text{ axis}) \; ,$$

$$\psi = \sphericalangle(\text{line of nodes, } x \text{ axis}) \; .$$

We can make the two systems of coordinates $\widehat{\Sigma}$ and Σ coincide with each other by **three single rotations**. At first we perform a rotation of the space-fixed initial system around the $\hat{z}$ axis in the mathematically positive sense by the angle φ; the $\hat{x}$ axis then coincides with line of nodes. In the next step we rotate the system around this line by the angle ϑ; the $\hat{z}$ axis therewith becomes the new z axis. Around that we finally rotate the reference system by the angle ψ in order to get the new x axis. The order of the various rotations is very important. Rotations by finite angles are normally not commutable. For given **Euler's angles** φ, ϑ, ψ we therefore are always able to rotate the space-fixed axis system in such a way that it coincides with the body-fixed system. That means that for known $\varphi = \varphi(t)$, $\vartheta = \vartheta(t)$, and $\psi = \psi(t)$ the position of the spinning top is determinable for all times.

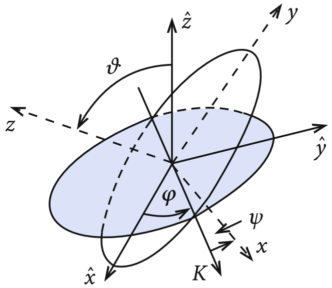


Fig. 4.17 Demonstration of Euler's angles for the motion of a spinning top

We now need the time-derivatives of Euler's angles and the components of the angular momentum. Rotation means changing the angles ϑ, φ, ψ:

$$\dot{\vartheta} \Longrightarrow \text{rotation around the line of nodes ,}$$
$$\dot{\varphi} \Longrightarrow \text{rotation around the } \hat{z} \text{ axis ,}$$
$$\dot{\psi} \Longrightarrow \text{rotation around the } z \text{ axis .}$$

We can treat these **partial rotations as vectors** along the respective directions and decompose them into components along the body-fixed axes:

$$\begin{aligned}
\dot{\vartheta}\mathbf{e}_\mathrm{K} &= \dot{\vartheta}\cos\psi\,\mathbf{e}_x - \dot{\vartheta}\sin\psi\,\mathbf{e}_y\ ,\\
\dot{\varphi}\,\hat{\mathbf{e}}_z &= \dot{\varphi}\sin\vartheta\sin\psi\,\mathbf{e}_x + \dot{\varphi}\sin\vartheta\cos\psi\,\mathbf{e}_y + \dot{\varphi}\cos\vartheta\,\mathbf{e}_z\ ,\\
\dot{\psi}\,\mathbf{e}_z &= \dot{\psi}\,\mathbf{e}_z\ .
\end{aligned}$$

The total angular momentum is then the vector sum of these three contributions. If we choose the body-fixed system as the principal axis system

$$\mathbf{e}_x = \mathbf{e}_\xi\ , \quad \mathbf{e}_y = \mathbf{e}_\eta\ , \quad \mathbf{e}_z = \mathbf{e}_\zeta$$

then the comparison with (4.68) gives us the components of the angular velocity:

$$\begin{aligned}
p &= \dot{\varphi}\sin\vartheta\sin\psi + \dot{\vartheta}\cos\psi\ ,\\
q &= \dot{\varphi}\sin\vartheta\cos\psi - \dot{\vartheta}\sin\psi\ ,\\
r &= \dot{\varphi}\cos\vartheta + \dot{\psi}\ .
\end{aligned} \tag{4.73}$$

As soon as one has determined p, q, r as solutions of Euler's equations (4.72) then via (4.73) the equations of motion for Euler's angles are known, by which the position of the rigid body relative to the space-fixed system can be finally found. This is the general procedure which will now be tested by some relevant special cases.

4.4.3 Rotations Around Free Axes

If we assume at first that the external torques vanish so that we get from (4.72) the **equations of the force-free spinning top**:

$$\begin{aligned}
A\,\dot{p} + (C-B)\,q\,r &= 0\ ,\\
B\,\dot{q} + (A-C)\,r\,p &= 0\ ,\\
C\,\dot{r} + (B-A)\,p\,q &= 0\ .
\end{aligned} \tag{4.74}$$

Multiplying the first equation by p, the second by q, the third by r and then adding up the three equations leads to:

$$\frac{d}{dt}\frac{1}{2}\left(A\,p^2 + B\,q^2 + C\,r^2\right) \overset{(4.57)}{=} \frac{d}{dt}T_{\mathrm{R}} = 0\ . \tag{4.75}$$

This is the **energy conservation law** for the body-fixed system.

Now we multiply the first equation in (4.74) by $A\,p$, the second by $B\,q$, the third by $C\,r$, and add them together to get:

$$\frac{d}{dt}\frac{1}{2}\left(A^2p^2 + B^2q^2 + C^2r^2\right) \overset{(4.62)}{=} \frac{d}{dt}\frac{1}{2}|\mathbf{L}|^2 = 0\ . \tag{4.76}$$

The **magnitude** of $\mathbf{L}$ is thus a conserved quantity in the body-fixed system, provided that the torque vanishes. If the direction of $\mathbf{L}$ is also to be constant, then according to (4.69) we get $A\dot{p} = B\dot{q} = C\dot{r} = 0$. So it follows from (4.74):

$$(C - B)\,q\,r = (A - C)\,r\,p = (B - A)\,p\,q = 0\ . \tag{4.77}$$

If we assume that the principal moments of inertia A, B, C are pairwise different then necessarily two of the components q, p, r must be zero. $\boldsymbol{\omega}$ has therewith the direction of one of the principal axes of inertia, i.e. $\mathbf{L}$ and $\boldsymbol{\omega}$ are parallel. Since $\mathbf{L}$ is constant with respect to both direction and magnitude in the space-fixed system, too, the same must also hold for $\boldsymbol{\omega}$. Therewith the direction of the rotation axis is constant in the body-fixed as well as in the space-fixed system. One calls such axes **'free axes'**. A rigid body rotating around a *free axis* does not *swerve from side to side*.

Whether or not such a rotation represents a really stable state of motion one finds out by inspecting the **influence of a small perturbation**. The rotation may take place, e.g., around an axis close to the ξ axis, i.e. close to the axis belonging to the principal moment of inertia A. Then we have

$$p = \omega_\xi = p_0 + \Delta p_0\ ; \quad p_0 = \text{const}$$

with a small correction Δp_0. The other components

$$q \Longrightarrow \Delta q\ ; \quad r \Longrightarrow \Delta r$$

are then also small. That we insert into (4.74) and neglect the terms of second order in the corrections:

$$\begin{aligned}
&A\,\Delta\dot{p}_0 = 0 \Longrightarrow \Delta p_0 = \text{const}\ ,\\
&B\,\Delta\dot{q} + (A - C)\,p_0\,\Delta r = 0\ ,\\
&C\,\Delta\dot{r} + (B - A)\,p_0\,\Delta q = 0\ .
\end{aligned}$$

We differentiate once more with respect to time:

$$B\,\Delta\ddot{q} + (A-C)\,p_0\,\Delta\dot{r} = B\,\Delta\ddot{q} - \frac{(A-C)(B-A)}{C}\,p_0^2\,\Delta q = 0\ ,$$

$$C\,\Delta\ddot{r} + (B-A)\,p_0\,\Delta\dot{q} = C\,\Delta\,\ddot{r} - \frac{(B-A)(A-C)}{B}\,p_0^2\,\Delta r = 0\ .$$

With the definition

$$D^2 = \frac{p_0^2}{BC}\,(A-C)(A-B) \tag{4.78}$$

we find the differential equations

$$\Delta\ddot{q} + D^2\,\Delta q = 0\ ,$$
$$\Delta\ddot{r} + D^2\,\Delta r = 0\ , \tag{4.79}$$

which can easily be solved. One gets oscillations in case of $D^2 > 0$. If the quantities Δq and Δr are small at the beginning they remain small for ever. The axis is therefore stable. However, in case of $D^2 < 0$ there result exponentially decreasing and increasing solutions of the type

$$\Delta q = \Delta q_0\,e^{\pm|D|t}\ ,$$
$$\Delta r = \Delta r_0\,e^{\pm|D|t}\ . \tag{4.80}$$

The initial state is therefore not stable. The axis is unstable. $D^2 > 0$ holds if $A > C,\ A > B$ or $A < C,\ A < B$. Rotations around the axis with the largest and smallest, respectively, principal moment of inertia are thus stable. The rotation around the axis with the intermediate principal moment of inertia ($C < A < B$ or $B < A < C$) is unstable because $D^2 < 0$. Already very small deviations of the rotation axis from the ξ direction increase exponentially according to (4.80).

4.4.4 Force-Free Symmetric Spinning Top

We speak of a **symmetric spinning top** if two of the principal moments of inertia are equal, for instance:

$$A = B \neq C\ . \tag{4.81}$$

In such a case the direction of the rotation axis, i.e. $\boldsymbol{\omega}$, cannot remain fixed. One denotes the distinguished third axis (here: ζ axis) as the

body axis

of the rigid body. The force-freeness can always be realized for a rigid body by choosing the center of gravity as fixpoint S because then the total torque due to the gravitational field disappears:

$$\mathbf{M} = \sum_i \mathbf{r}_i \times m_i\, \mathbf{g} = M\,(\mathbf{R} \times \mathbf{g}) = 0 \quad \text{for } \mathbf{R} = \mathbf{0}\,.$$

Under the precondition (4.81) the equations of motion (4.74) of the force-free spinning top simplify as follows:

$$\begin{aligned} A\,\dot{p} + (C-A)\,q\,r &= 0\,,\\ A\,\dot{q} + (A-C)\,r\,p &= 0\,,\\ C\,\dot{r} \qquad\qquad\quad &= 0\,. \end{aligned} \tag{4.82}$$

The solution for $r = \omega_\zeta$ comes out immediately:

$$r = r_0 = \text{const}\,. \tag{4.83}$$

We can always choose the ζ direction so that r_0 is positive. Then

$$\Omega = \frac{A-C}{A}\,r_0 \tag{4.84}$$

becomes positive for $A > C$ and negative for $A < C$. From (4.82) we find with (4.83) and (4.84):

$$\dot{p} - \Omega\,q = 0\,; \quad \dot{q} + \Omega\,p = 0\,. \tag{4.85}$$

We differentiate once more with respect to the time:

$$\begin{aligned} \ddot{p} - \Omega\dot{q} &= \ddot{p} + \Omega^2\,p = 0\,,\\ \ddot{q} + \Omega\dot{p} &= \ddot{q} + \Omega^2\,q = 0\,. \end{aligned} \tag{4.86}$$

These are again **oscillation equations**. The solutions which simultaneously satisfy (4.85) are:

$$\begin{aligned} p &= \alpha\sin(\Omega\,t + \beta)\,,\\ q &= \alpha\cos(\Omega\,t + \beta)\,. \end{aligned} \tag{4.87}$$

α, β are integration constants. From (4.83) and (4.87) we draw the following conclusions:

1. The ζ component r of the angular velocity $\boldsymbol{\omega}$, i.e. the projection on the body axis is constant,
2. $\omega = |\boldsymbol{\omega}|$ is constant,
3. the projection of $\boldsymbol{\omega}$ on the ξ, η plane, which corresponds to the p, q components, describes a circle of radius α.

Conclusion 1. is just the statement (4.83), conclusion 3. results from (4.87) and conclusion 2. holds because of:

$$\omega^2 = r^2 + p^2 + q^2 = r_0^2 + \alpha^2 = \text{const} . \tag{4.88}$$

$\boldsymbol{\omega}$ thus describes a circular cone around the body axis with the aperture angle γ:

$$\tan\gamma = \frac{\alpha}{r_0} . \tag{4.89}$$

One calls this cone the **pole cone**. $\boldsymbol{\omega}$ moves on the pole-cone mantle with the angular velocity Ω (4.84).

Example The earth is an oblate ellipsoid of revolution, thus to good approximation a symmetric spinning top (Fig. 4.18). During the rotation, the body axis (*geometric north pole*) and the rotation axis $\boldsymbol{\omega}$ (*kinematic north pole*) do not exactly coincide. $\boldsymbol{\omega}$ moves on a cone around the body axis. The kinematic north pole describes a circle with a radius of about 10 m around the geometric north pole with a period of about 433 days (**Chandler's period**).
Up to now we have discussed the movement of the symmetric force-free spinning top in the body-fixed (ξ, η, ζ) system. We still have to transform it to the space-fixed system. For this purpose we determine Euler's angles as functions of time. For a start we have with (4.73):

$$\begin{aligned}
p &= \alpha\sin(\Omega\, t + \beta) = \dot{\varphi}\sin\vartheta\sin\psi + \dot{\vartheta}\cos\psi \ , \\
q &= \alpha\cos(\Omega\, t + \beta) = \dot{\varphi}\sin\vartheta\cos\psi - \dot{\vartheta}\sin\psi \ , \\
r &= r_0 = \dot{\varphi}\cos\vartheta + \dot{\psi} \ .
\end{aligned} \tag{4.90}$$

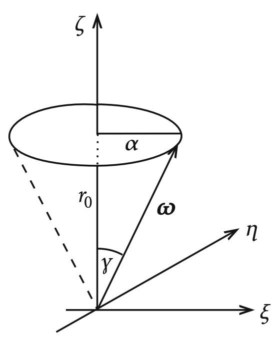


Fig. 4.18 Movement of the rotation axis around the body axis for the case of a force-free symmetric spinning top

Since the motion is force-free the angular momentum $\mathbf{L}$ in the space-fixed system $\widehat{\Sigma}$ is a constant with respect to both direction and magnitude. We can then always place the $\hat{z}$ axis so that:

$$\mathbf{L} = L\,\hat{\mathbf{e}}_z \tag{4.91}$$

In the body-fixed system $(\xi,\ \eta,\ \zeta)$ the unit vector $\hat{\mathbf{e}}_z$ has the components:

$$\begin{aligned}(\hat{\mathbf{e}}_z)_\xi &= \sin\vartheta\,\sin\psi\ ,\\ (\hat{\mathbf{e}}_z)_\eta &= \sin\vartheta\,\cos\psi\ ,\\ (\hat{\mathbf{e}}_z)_\zeta &= \cos\vartheta\ .\end{aligned} \tag{4.92}$$

This leads with (4.90) to the following system of equations:

$$\begin{aligned}L_\xi &= A\,p = A\dot{\varphi}\sin\vartheta\,\sin\psi + A\dot{\vartheta}\cos\psi \overset{!}{=} L\sin\vartheta\,\sin\psi\ ,\\ L_\eta &= A\,q = A\dot{\varphi}\sin\vartheta\,\cos\psi - A\dot{\vartheta}\sin\psi \overset{!}{=} L\sin\vartheta\,\cos\psi\ ,\\ L_\zeta &= C\,r = C\dot{\varphi}\cos\vartheta + C\dot{\psi} \overset{!}{=} L\cos\vartheta\ .\end{aligned}$$

This system of equations can be solved only with

$$\vartheta = \vartheta_0 = \text{const}\ , \quad \dot{\varphi} = \text{const} \tag{4.93}$$

Therewith (4.90) reads:

$$\begin{aligned}\alpha\sin(\Omega\,t+\beta) &= \dot{\varphi}\sin\vartheta_0\sin\psi\ ,\\ \alpha\cos(\Omega\,t+\beta) &= \dot{\varphi}\sin\vartheta_0\cos\psi\ ,\\ r_0 &= \dot{\varphi}\cos\vartheta_0 + \dot{\psi}\ .\end{aligned} \tag{4.94}$$

The ratio of the first two equations yields:

$$\psi = \Omega\,t + \beta = \frac{A-C}{A}\,r_0 t + \beta\ . \tag{4.95}$$

If one inserts this for instance into the first equation it follows $\alpha = \dot{\varphi}\sin\vartheta_0$ and therewith

$$\varphi = \frac{\alpha}{\sin\vartheta_0}t + \varphi_0\ . \tag{4.96}$$

The third equation in (4.94) then still leads to

$$\vartheta = \vartheta_0\,; \quad \tan\vartheta_0 = \frac{\alpha\,A}{r_0 C}\ . \tag{4.97}$$

Equations (4.95)–(4.97) represent the full solution of the equation of motion of the force-free symmetric spinning top. We are left with four independent integration constants $\alpha, \beta, \varphi_0, r_0$. In principle it should be six but two of them we have already implicitly used for fixing the $\hat{z}$ direction!

4.4.4.1 Discussion of the Spinning Top Motion

(a) ϑ: Angle between space-fixed $\hat{z}$ axis and body-fixed z axis. The $\hat{z}$ axis is given according to (4.91) by the direction of the angular momentum **L**. The z axis is the body axis (ζ axis). From that it follows:
The body axis moves with constant aperture angle $\vartheta = \vartheta_0$ and with constant angular velocity $\dot{\varphi}$ around the direction of the angular momentum. The cone described by the body axis is called **'nutation cone'**.
(b) $\dot{\psi}$: Angular velocity by which the body (more strictly the body-fixed η, ξ plane) rotates around the body axis.
(c) $\boldsymbol{\omega}$: The angular velocity $\boldsymbol{\omega}$ is equal to the vector sum of $\dot{\boldsymbol{\varphi}}$ and $\dot{\boldsymbol{\psi}}$. It always lies in the $\hat{z}, \zeta$ plane, thus rotates together with the body axis around the direction of the angular momentum ($\hat{z}$ axis) enclosing with the body axis the angle γ (4.89). The **momentary rotation axis** defined by $\boldsymbol{\omega}$ moves therefore on the so-called **space cone** around the space-fixed angular-momentum direction.

The pole cone rolls off with its *outside* mantle on the space-fixed space cone and therefore directs the body axis on the nutation cone.

For $A > C$ it is $\dot{\boldsymbol{\psi}} \uparrow\uparrow \mathbf{e}_\zeta$. Then the outside area of the pole cone rolls off the space-cone mantle (Fig. 4.19).

For $A < C$ it is $\dot{\boldsymbol{\psi}} \uparrow\downarrow \mathbf{e}_\zeta$. The pole cone rolls off with its inside area on the space-fixed space cone where again the body axis is directed on the nutation cone (Fig. 4.20).

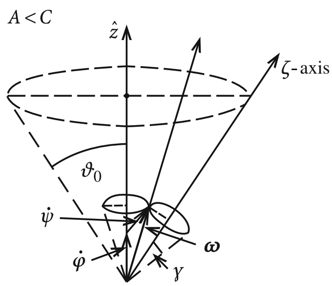


Fig. 4.19 Course of motion for the force-free symmetric spinning top with principal moments of inertia $A = B > C$

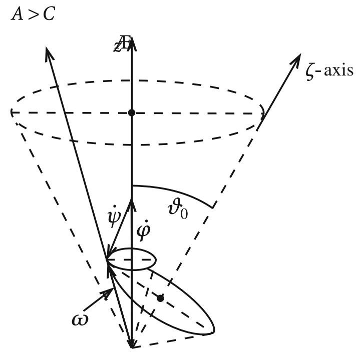


Fig. 4.20 Course of motion for the force-free symmetric spinning top with principal moments of inertia $A = B < C$

4.5 Exercises

Exercise 4.5.1 Calculate the moment of inertia

1. of a homogeneous spherical shell (outer radius R, thickness $d \ll R$, mass M) with respect to a rotation axis through the center,
2. of a cube with homogeneous mass density (edge length a, mass M) with respect to one of the cube edges as rotation axis,
3. of a cylinder with the mass M and radius R with respect to the symmetry axis. The mass distribution is such that the mass density increases outward linearly with the radius starting with zero at the symmetry axis.

Exercise 4.5.2 The cube from part 2. of Exercise 4.5.1 is hanging on one of its edges vertically down in the earth's gravitational field (Fig. 4.21). It performs small oscillations around this axis. Write down the equation of motion and find the oscillation period and the angular frequency. What would be the length of an equivalent thread pendulum?

Exercise 4.5.3 A thin-walled hollow cylinder (radius R, mass M) is rolling down an inclined plane. It starts to roll at the time $t = 0$, where $v(t)$ is the velocity of a point on its axis.

1. Formulate the energy theorem and express the total kinetic energy by $v(t)$.
2. Calculate $v(t)$!

Exercise 4.5.4 Two homogeneous cylinders with masses M_1, M_2, radii R_1, R_2 are wrapped by a thread and therewith connected to each other (Fig. 4.22). The axis of first cylinder is tightly horizontally pivoted. However, it can be rotated frictionlessly. The second cylinder falls in the earth's gravitational field in x direction where on both the cylinders the thread is unrolling. Formulate by use of the angular-momentum law the equation of motion and determine in particular the thread tensions $\mathbf{F}_1$ and $\mathbf{F}_2$!

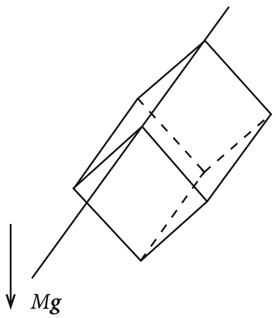


Fig. 4.21 Rotation of a cube with homogeneous mass density around one of the edges in the earth's gravitational field

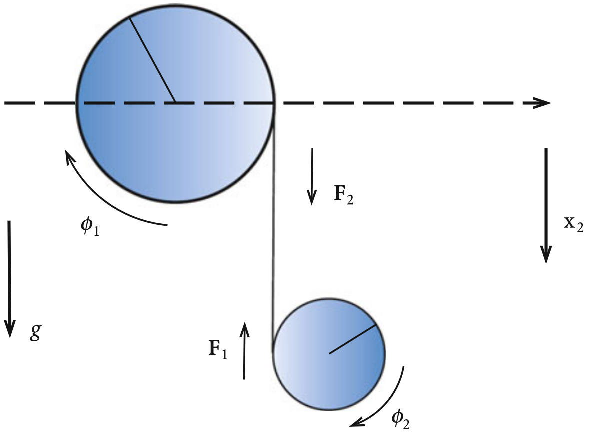


Fig. 4.22 Two frictionlessly rotatable cylinders coupled by a thread

Exercise 4.5.5 A torque $\mathbf{M}_{\text{ex}}$ acts on a rigid body. It is oriented always perpendicularly to the angular momentum $\mathbf{L}$ and is also perpendicular to a given fixed axis in $\mathbf{n}$ direction:

$$\mathbf{M}_{\text{ex}} = M(\mathbf{n} \times \mathbf{l}) \;\; ; \;\; \mathbf{l} = \frac{\mathbf{L}}{L} \, .$$

1. Verify that

$$|\mathbf{L}| = \text{const} \;\; ; \;\; \mathbf{L} \cdot \mathbf{n} = \text{const} \, .$$

2. Describe the time dependence of the angular-momentum vector $\mathbf{L}$. For this purpose investigate $\frac{d\mathbf{L}}{dt}$!

Exercise 4.5.6

1. A rigid body possesses an inertial tensor $\underline{\mathbf{J}} = (J_{ij})$, where this tensor is related to a body-fixed system of coordinates Σ the origin of which coincides with the center of gravity. How does the inertia tensor alter for a system of coordinates Σ'

that compared to Σ has parallel axes but is shifted by the vector $\mathbf{a}$ (generalized Steiner's theorem)?

2. Show that the inertial tensor transforms as follows as a consequence of a rotation of the body-fixed system of coordinates:

$$J'_{nm} = \sum_{ij} d_{ni} d_{mj} J_{ij} \ .$$

Here d_{ij} are the elements of the orthogonal rotation matrix.

Exercise 4.5.7 Inspect a cuboid with the edge lengths a, b, c and homogeneous mass density ρ_0.

1. Let $\widehat{\Sigma}$ be a body-fixed Cartesian system of coordinates with its origin in the lower left corner of the cuboid and with axes along the cuboid edges. Determine the inertia tensor $\underline{\widehat{\mathbf{J}}}$!
2. The cuboid rotates with the angular velocity ω around its space diagonal. Calculate via the inertial tensor the moment of inertia with respect to this axis!
3. Let $\overline{\Sigma}$ also be a body-fixed Cartesian system of coordinates with axes parallel to those from 1. The origin, however, is now at the center of gravity of the cuboid. Determine the inertial tensor $\underline{\overline{\mathbf{J}}}$! What is now the moment of inertia for the rotation around the space diagonal?
4. Use the inertial tensors calculated in 1. and 3. to fix the moments of inertia related, respectively, to a rotation axis which coincides with a cuboid edge, e.g. in y direction, and to an axis parallel to the former through the center of gravity of the cuboid. Verify Steiner's theorem!

Exercise 4.5.8 Use the solution in the first part of Exercise 4.5.7 in order to find the inertial tensor of a cube with an edge length a and with homogeneous mass density for a Cartesian system of coordinates, the origin of which lies in one of the cube corners while the axes coincide with the cube edges. Calculate the principal moments and axes of inertia!

4.6 Self-Examination Questions

To Section 4.1

1. Describe the model of the rigid body!
2. How many degrees of freedom does the rigid body have?
3. What do we understand by *mass density*?
4. What is a spinning top?

To Section 4.2

1. Define for the rotation around a given axis the moment of inertia of the rigid body! Which parameters do play a role?
2. What is a *physical pendulum*? Which relation does it have to the mathematical pendulum?
3. Formulate and interpret Steiner's theorem!
4. How many degrees of freedom has a cylinder rolling down an inclined plane? Find its equation of motion!
5. Which analogies do exist between translational and rotational motion?

To Section 4.3

1. How are the components of the inertial tensor defined?
2. How does the rotational kinetic energy depend on the components of the angular velocity? What does these components fix for a given system of coordinates?
3. Explain the term tensor! When is a square matrix a tensor of second rank?
4. What is the relation between the moment of inertia with respect to a fixed axis and the inertial tensor?
5. What do you understand by principal axes transformation? Explain the concepts principal axes of inertia and principal moments of inertia?
6. How can one use the inertial ellipsoid to get the moment of inertia with respect to a given axis?
7. What are the differences between symmetric, asymmetric and spherical spinning top?
8. Express the angular momentum of a rigid body by its inertial tensor!
9. Demonstrate with the inertial ellipsoid the relation between angular momentum $\mathbf{L}$ and angular velocity $\boldsymbol{\omega}$. When are $\boldsymbol{\omega}$ and $\mathbf{L}$ parallel?

To Section 4.4

1. What information is provided by Euler's equations?
2. Define Euler' angles!
3. What are the equations of motion of the force-free spinning top?
4. What is meant by *free axes*?
5. Explain the terms body axis, pole cone, nutation cone, and space cone for the spinning-top motion!

Appendix A
Solutions of the Exercises

Section 1.1

Solution 1.1.1 We use the rules (1.19)–(1.22) and solve the exercises by tracing back the various terms to null sequences.

1.
$$a_n = \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}} \xrightarrow{n \to \infty} 0$$

2.
$$a_n = \frac{n^3 + 1}{2n^3 + n^2 + n} = \frac{1 + \frac{1}{n^3}}{2 + \frac{1}{n} + \frac{1}{n^2}} \xrightarrow{n \to \infty} \frac{1}{2}$$

3.
$$a_n = \frac{n^2 - 1}{(n+1)^2} + 5 = \frac{1 - \frac{1}{n^2}}{1 + \frac{2}{n} + \frac{1}{n^2}} + 5 \xrightarrow{n \to \infty} 1 + 5 = 6\,.$$

Solution 1.1.2

1. We use (1.29):

$$S_3 = \sum_{m=1}^{3} 3 \left(\frac{1}{2}\right)^m = 3 \cdot \frac{1}{2} \sum_{m=1}^{3} \left(\frac{1}{2}\right)^{m-1} = \frac{3}{2} \frac{1 - \left(\frac{1}{2}\right)^3}{1 - \frac{1}{2}} = \frac{3}{2} \frac{\frac{7}{8}}{\frac{1}{2}} = \frac{21}{8}\,.$$

W. Nolting, *Theoretical Physics 1*, DOI 10.1007/978-3-319-40108-9

With (1.30) follows:

$$S = \sum_{m=1}^{\infty} 3\left(\frac{1}{2}\right)^m = \frac{3}{2}\sum_{m=1}^{\infty}\left(\frac{1}{2}\right)^{m-1} = \frac{3}{2}\,\frac{1}{1-\frac{1}{2}} = 3\,.$$

2. The answer is yes because:

$$1{,}111\ldots = 1 + \frac{1}{10} + \frac{1}{100} + \cdots = \sum_{m=1}^{\infty}\left(\frac{1}{10}\right)^{m-1} = \frac{1}{1-\frac{1}{10}} = \frac{10}{9}$$

Solution 1.1.3 It obviously holds for the harmonic series (1.27):

$$S_{2n} - S_n = \sum_{k=n+1}^{2n}\frac{1}{k} \geq \sum_{k=n+1}^{2n}\frac{1}{2n} = n\cdot\frac{1}{2n} = \frac{1}{2}\,.$$

Thus it is

$$\lim_{n\to\infty}(S_{2n}-S_n) \geq \frac{1}{2}\,.$$

Therefore the series cannot converge!

Solution 1.1.4

- $$\cos^2\varphi\cdot\tan^2\varphi + \cos^2\varphi = \sin^2\varphi + \cos^2\varphi = 1$$

- $$\frac{1-\cos^2\varphi}{\sin\varphi\cdot\cos\varphi} = \frac{\sin^2\varphi}{\sin\varphi\cdot\cos\varphi} = \frac{\sin\varphi}{\cos\varphi} = \tan\varphi$$

- $$1 - \frac{1}{\cos^2\varphi} = \frac{\cos^2\varphi - 1}{\cos^2\varphi} = -\frac{\sin^2\varphi}{\cos^2\varphi} = -\tan^2\varphi$$

- $$\frac{1}{1-\sin\varphi} + \frac{1}{1+\sin\varphi} = \frac{1+\sin\varphi+1-\sin\varphi}{1-\sin^2\varphi} = \frac{2}{\cos^2\varphi}$$

- $$\frac{\sin(\varphi_1+\varphi_2)+\sin(\varphi_1-\varphi_2)}{\cos(\varphi_1+\varphi_2)+\cos(\varphi_1-\varphi_2)}=\frac{2\sin\varphi_1\cdot\cos\varphi_2}{2\cos\varphi_1\cdot\cos\varphi_2}=\tan\varphi_1\,.$$

  Here we have exploited the addition theorems (1.60) and (1.61).

- $$\frac{\cos^2\varphi}{\sin 2\varphi}\overset{(1.60)}{=}\frac{\cos^2\varphi}{2\sin\varphi\cdot\cos\varphi}=\frac{1}{2}\,\frac{\cos\varphi}{\sin\varphi}=\frac{1}{2}\,\cot\varphi\,.$$

Solution 1.1.5 The derivation is easily found with the addition theorem (1.61):

$$1-\cos\varphi=1-\cos^2\frac{\varphi}{2}+\sin^2\frac{\varphi}{2}\overset{(1.47)}{=}\sin^2\frac{\varphi}{2}+\sin^2\frac{\varphi}{2}=2\,\sin^2\frac{\varphi}{2}\,.$$

Solution 1.1.6

1.

$$y=f(x)=\cos x\quad\Longrightarrow\quad f'(x)=-\sin x\,.$$

The reasoning is completely analogous to that for the derivative of the sine in (1.80).

$$\begin{aligned}\frac{\Delta y}{\Delta x}&=\frac{\cos(x+\Delta x)-\cos x}{\Delta x}\\&=\frac{\cos x\,\cos\Delta x-\sin x\,\sin\Delta x-\cos x}{\Delta x}\\&=\frac{\cos x(\cos\Delta x-1)}{\Delta x}-\sin x\,\frac{\sin\Delta x}{\Delta x}\\&=-\cos x\,\sin\frac{\Delta x}{2}\cdot\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}-\sin x\,\frac{\sin\Delta x}{\Delta x}\,.\end{aligned}$$

In the next to last step we have applied the formula from Solution 1.1.5. For the remaining limiting process $\Delta x\to 0$ we can exploit (1.50):

$$f'(x)=\lim_{\Delta x\to 0}\left(-\cos x\,\sin\frac{\Delta x}{2}\cdot\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}-\sin x\,\frac{\sin\Delta x}{\Delta x}\right)=-\sin x\,.$$

2. With the quotient-rule (1.86) one calculates

$$\begin{aligned}\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) &= \frac{\cos x \sin' x - \cos' x \sin x}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x}\,.\end{aligned}$$

3. It follows again with quotient-rule (1.86):

$$\begin{aligned}\frac{d}{dx}\cot x = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) &= \frac{\cos' x \sin x - \cos x \sin' x}{\sin^2 x} \\ &= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} \\ &= -\frac{1}{\sin^2 x}\end{aligned}$$

Solution 1.1.7 We apply the rules of differentiation from Sect. 1.1.9:

1.

$$f_1'(x) = 15x^4$$

2.

$$f_2'(x) = 21x^2 - 6x^{\frac{1}{2}}$$

3.

$$f_3'(x) = \frac{\left(3x^2 - 2\right)5x^2 - 10x\left(x^3 - 2x\right)}{25x^4} = \frac{5x^4 + 10x^2}{25x^4} = \frac{x^2 + 2}{5x^2}$$

4.

$$f_4'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$$

5.

$$f_5'(x) = \frac{1}{2}\left(1 + x^2\right)^{-\frac{1}{2}} \cdot 2x = \frac{x}{\sqrt{1 + x^2}}$$

6.

$$f_6'(x) = 3(-\sin(6x)) \cdot 6 = -18\sin(6x)$$

7.

$$f_7'(x) = \cos x^2 \cdot 2x = 2x\cos x^2$$

8.

$$f_8'(x) = 6x^2 \exp\left(2x^3 - 4\right)$$

9.

$$f_9'(x) = \frac{2}{2x+1}\,.$$

Solution 1.1.8 The derivatives of the trigonometric functions needed in the following have been derived in (1.80) and in Solution 1.1.6.

1. Let

$$y = \sin x$$

so that

$$f^{-1}(y) = \arcsin y = x\,.$$

Then it holds with (1.91):

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{\cos x} = \frac{1}{\sqrt{1-\sin^2 x}}\,.$$

That yields the assertion:

$$\frac{d}{dy}\arcsin y = \frac{1}{\sqrt{1-y^2}}\,.$$

2. We now choose

$$y = \cos x$$

and therewith

$$f^{-1}(y) = \arccos y = x\,.$$

It is then:

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{-\sin x} = -\frac{1}{\sqrt{1-\cos^2 x}} .$$

That proves the assertion:

$$\frac{d}{dy} \arccos y = -\frac{1}{\sqrt{1-y^2}} .$$

3. Next we investigate

$$y = \tan x$$

and therewith

$$f^{-1}(y) = \arctan y = x .$$

It then holds:

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{\tan' x} = \cos^2 x = \frac{1}{1+\tan^2 x} .$$

That yields:

$$\frac{d}{dy} \arctan y = \frac{1}{1+y^2} .$$

4. Finally we inspect

$$y = \cot x$$

and therewith

$$f^{-1}(y) = \operatorname{arccot} y = x .$$

It then holds:

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{\cot' x} = -\sin^2 x = -\frac{1}{1+\cot^2 x} .$$

That proves the assertion:

$$\frac{d}{dy} \operatorname{arccot} y = -\frac{1}{1+y^2} .$$

Solution 1.1.9

1.

$$\begin{aligned} a_0 &= f(0) \\ a_1 &= f'(0) \\ a_2 &= \frac{1}{2} f''(0) \\ \cdots &= \cdots \\ a_n &= \frac{1}{n!} \left. \frac{d^n f}{dx^n} \right|_{x=0} \end{aligned}$$

$$\curvearrowright \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \; .$$

2. We substitute

$$u = x - x_0 \quad ; \quad g(u) \equiv f(u + x_0) \equiv f(x) \; .$$

Then it holds with part 1.:

$$g(u) = \sum_{n=0}^{\infty} \frac{g^{(n)}(0)}{n!} u^n \; .$$

Because of

$$g^{(n)}(0) = f^{(n)}(x_0)$$

Immediately it follows the assertion.

Solution 1.1.10 The reasoning works with the aid of the Taylor expansion (1.92) which we perform around $x = 0$.

$$\begin{aligned} f(x) &= (1+x)^n & &\curvearrowright \; f(0) = 1 \\ f'(x) &= n\,(1+x)^{n-1} & &\curvearrowright \; f'(0) = n \\ f''(x) &= n(n-1)\,(1+x)^{n-2} & &\curvearrowright \; f''(0) = n(n-1) \\ f'''(x) &= n(n-1)(n-2)\,(1+x)^{n-3} & &\curvearrowright \; f'''(0) = n(n-1)(n-2) \\ \cdots &= \cdots \; . \end{aligned}$$

That yields in (1.92):

$$(1+x)^n = 1 + n\,x + \frac{n(n-1)}{2!}\,x^2 + \frac{n(n-1)(n-2)}{3!}\,x^3 + \ldots$$

If $x \ll 1$ then the series can be cut off with sufficient accuracy after a finite number of terms.

Solution 1.1.11 We use the Taylor expansion (1.92) for $f(x) = \ln(1+x)$ at $x = 0$:

$$\begin{aligned}
f(x) &= \ln(1+x) && \curvearrowright\ f(0) = \ln(1) = 0\\
f'(x) &= \frac{1}{(1+x)} && \curvearrowright\ f'(0) = 1\\
f''(x) &= \frac{-1}{(1+x)^2} && \curvearrowright\ f''(0) = -1\\
f'''(x) &= \frac{+2}{(1+x)^3} && \curvearrowright\ f'''(0) = 2\\
\cdots &= && \cdots\,.
\end{aligned}$$

We show that

$$\frac{d^n}{dx^n}\ln(1+x) = \frac{(n-1)!(-1)^{n-1}}{(1+x)^n} \qquad (n \geq 1)\,.$$

holds. For $n = 1, 2, 3$ this relation is obviously correct. By iterated induction we conclude from n to $n+1$, i.e. we show that if the expression is correct for n then it is also correct for $n+1$:

$$\frac{d^{n+1}}{dx^{n+1}}\ln(1+x) = \frac{d}{dx}\frac{(n-1)!(-1)^{n-1}}{(1+x)^n} = -n\frac{(n-1)!(-1)^{n-1}}{(1+x)^{n+1}} = \frac{(n)!(-1)^n}{(1+x)^{n+1}}\,.$$

Therewith:

$$f^{(n)}(0) = (n-1)!(-1)^{n-1}\,.$$

Taylor expansion:

$$\ln(1+x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}\,x^n = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\,x^n\,.$$

That is the assertion. We remark that nothing is said as yet about the convergence of the series. Considerations, which we cannot follow here, show up that convergence is guaranteed for $|x| < 1$.

Solution 1.1.12

$$f(x) = \cos x \ ; \ x_0 = 0 \, .$$

For the Taylor expansion (1.92)we use:

$$\begin{aligned} &f(0) = 1 \ ; f'(0) = -\sin(0) = 0 \ ; f''(0) = -\cos(0) = -1 \ ; \\ &f'''(0) = \sin(0) = 0 \ ; f^{(4)} = \cos(0) = 1 \ ; \ \dots \\ \curvearrowright \ &f^{(2n)}(0) = (-1)^n \ ; f^{(2n+1)}(0) = 0 \, . \end{aligned}$$

That means:

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \, .$$

Solution 1.1.13 We write

$$f(x) = \frac{f_1(x)}{f_2(x)} \, .$$

With the series expansions (1.94) and (1.51) we find:

$$\begin{aligned} f_1(x) &= x - \sin x \\ &= x - \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \dots\right) \\ &= \frac{x^3}{3!} - x^5\left(\frac{1}{5!} - \frac{x^2}{7!} + \dots\right) \end{aligned}$$

$$\begin{aligned} f_2(x) &= e^x + e^{-x} - 2 \\ &= \left(1 + x + \frac{x^2}{2!} + \dots\right) + \left(1 - x + \frac{x^2}{2!} - \dots\right) - 2 \\ &= x^2 + 2x^4\left(\frac{1}{4!} + \frac{x^2}{6!} + \dots\right) \end{aligned}$$

Therewith it follows:

$$f(x) = \frac{\frac{x^3}{3!} - x^5(\dots)}{x^2 + 2x^4(\dots)} = \frac{\frac{x}{3!} - x^3(\dots)}{1 + 2x^2(\dots)} \xrightarrow{x \to 0} \frac{0 - 0}{1 + 0} = 0 \, .$$

We now investigate the same expression by applying l'Hospital's rule:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{f_1'(x)}{f_2'(x)} = \lim_{x\to 0} \frac{1-\cos x}{e^x - e^{-x}} = \frac{1-1}{1-1}\,.$$

This expression is still indeterminate. Therefore, we have to proceed one step further:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{f_1''(x)}{f_2''(x)} = \lim_{x\to 0} \frac{\sin x}{e^x + e^{-x}} = \frac{0}{2} = 0\,.$$

Solution 1.1.14

1.

$$f(x) = 2x^4 - 8x^2\,.$$

Zeros:

$$0 \overset{!}{=} 2x^4 - 8x^2 = 2x^2\left(x^2-4\right) = 2x^2(x+2)(x-2)$$
$$\curvearrowright \quad x_1 = x_2 = 0 \;\;;\;\; x_3 = -2 \;\;;\;\; x_4 = +2 \qquad .$$

Extreme values:

$$f'(x) \overset{!}{=} 0 = 8x^3 - 16x = 8x\left(x^2-2\right)$$
$$\curvearrowright \quad x_a = 0 \;\;;\;\; x_b = \sqrt{2} \;\;;\;\; x_c = -\sqrt{2}\,.$$

From

$$f''(x) = 24x^2 - 16 = 8\left(3x^2-2\right)$$

follows

$$f''(x_a) = -16 < 0 \qquad \curvearrowright \quad x_a: \quad \text{maximum}$$
$$f''(x_b) = f''(x_c) = 32 > 0 \qquad \curvearrowright \quad x_{b,c}: \quad \text{minima}$$

with

$$f(x_a) = 0 \;\;;\;\; f(x_b) = f(x_c) = -8\,.$$

2.

$$g(x) = \sin\left(\frac{1}{2}\,x\right)\,.$$

Zeros:

$$x_m = 2m\pi \qquad m \in \mathbb{Z}\,.$$

Extreme values:

$$0 \overset{!}{=} g'(x) = \frac{1}{2}\cos\left(\frac{1}{2}\,x\right) \qquad \curvearrowright x_n = (2n+1)\pi\ ;\ \ n \in \mathbb{Z}\,.$$

One decomposes the zeros conveniently according to $n = 2n_1$ and $n = 2n_2 + 1$; $n_{1,2} \in \mathbb{N}$:

$$x^{\pm}_{n_1} = \pm\,(4n_1+1)\,\pi\ \ ;\ \ x^{\pm}_{n_2} = \pm\,(4n_2+3)\,\pi\ .$$

That results in

$$\begin{aligned}
g''\left(x^{+}_{n_1}\right) &= -\frac{1}{4}\,\sin\left((4n_1+1)\,\frac{\pi}{2}\right) = -\frac{1}{4} &&\curvearrowright \text{ maxima}\\
g''\left(x^{-}_{n_1}\right) &= +\frac{1}{4}\,\sin\left((4n_1+1)\,\frac{\pi}{2}\right) = +\frac{1}{4} &&\curvearrowright \text{ minima}\\
g''\left(x^{+}_{n_2}\right) &= -\frac{1}{4}\,\sin\left((4n_2+3)\,\frac{\pi}{2}\right) = +\frac{1}{4} &&\curvearrowright \text{ minima}\\
g''\left(x^{-}_{n_2}\right) &= +\frac{1}{4}\,\sin\left((4n_2+3)\,\frac{\pi}{2}\right) = -\frac{1}{4} &&\curvearrowright \text{ maxima}
\end{aligned}$$

with

$$g(x^{+}_{n_1}) = g(x^{-}_{n_2}) = +1\ \ ;\ \ g\left(x^{-}_{n_1}\right) = g\left(x^{+}_{n_2}\right) = -1\ .$$

Section 1.2

Solution 1.2.1

1. Choose

$$\begin{aligned}
f_1(x) &= \cos x \quad ; \quad f_2'(x) = \cos x\\
\curvearrowright\ f_1'(x) &= -\sin x \quad ; \quad f_2(x) = \sin x\ .
\end{aligned}$$

Therewith we can write:

$$\begin{aligned}\int \cos^2 x dx &= \cos x \sin x - \int (-\sin^2 x) dx + c \\ &= \cos x \sin x + \int (1 - \cos^2 x) dx + c \\ &= \cos x \sin x + x - \int \cos^2 x dx + \hat{c} \\ &= \frac{1}{2} \cos x \sin x + \frac{x}{2} + \hat{\hat{c}} \; .\end{aligned}$$

2. Choose

$$\begin{aligned} & f_1(x) = x^2 \quad ; \quad f_2'(x) = \cos^2 x \\ \curvearrowright \; & f_1'(x) = 2x \quad ; \quad f_2(x) = \frac{1}{2} \cos x \sin x + \frac{x}{2} \; .\end{aligned}$$

Here we have already exploited the result from part 1. Therewith we get:

$$\begin{aligned}\int x^2 \cos^2 x dx &= \frac{1}{2} x^2 (\cos x \sin x + x) - \int (x \cos x \sin x + x^2) dx \\ &= \frac{1}{2} x^2 (\cos x \sin x + x) - \frac{x^3}{3} - A \\ A = \int x \cos x \sin x dx &= \frac{1}{2} \int x \frac{d}{dx} \sin^2 x dx \\ &= \frac{x}{2} \sin^2 x - \frac{1}{2} \int \sin^2 x dx + c_1 \\ &= \frac{x}{2} \sin^2 x - \frac{x}{2} + \frac{1}{2} \int \cos^2 x dx + c_2 \\ &= \frac{x}{2} \sin^2 x - \frac{x}{2} + \frac{1}{4} (\cos x \sin x + x) + c_3 \; .\end{aligned}$$

So the result is:

$$\int x^2 \cos^2 x dx = \frac{1}{2} (\cos x \sin x + x) \left(x^2 - \frac{1}{2} \right) + \frac{x}{2} \cos^2 x - \frac{x^3}{3} + c \; .$$

3. Choose

$$\begin{aligned} & f_1(x) = x \quad ; \quad f_2'(x) = \sin x \\ \curvearrowright \; & f_1'(x) = 1 \quad ; \quad f_2(x) = -\cos x \; .\end{aligned}$$

Then it follows immediately:

$$\int x \sin x dx = -x \cos x + \int \cos x dx = -x \cos x + \sin x + c \ .$$

4. Choose

$$f_1(x) = \ln x \quad ; \quad f_2'(x) = x$$
$$\curvearrowright \ f_1'(x) = \frac{1}{x} \quad ; \quad f_2(x) = \frac{1}{2}x^2 \ .$$

Thus it is to calculate:

$$\int x \ln x dx = \frac{1}{2}x^2 \ln x - \int \frac{x}{2} dx = \frac{1}{2}x^2 \left(\ln x - \frac{1}{2}\right) + c \ .$$

Solution 1.2.2

1. We substitute

$$u = 5x - 4 \ \curvearrowright \ du = 5dx \ \curvearrowright \ dx = \frac{1}{5}du \ \curvearrowright \ u(x=1) = 1; u(x=0) = -4 \ .$$

It follows:

$$\int_0^1 (5x-4)^3 \, dx = \frac{1}{5}\int_{-4}^1 u^3 \, du = \frac{1}{20} u^4\Big|_{-4}^1 = \frac{1}{20}(1 - 4^4) = \frac{1}{20}(-255)$$
$$= -\frac{51}{4} \ .$$

2. We substitute

$$u = \pi x + \frac{5\pi}{2} \ \curvearrowright \ du = \pi dx \ \curvearrowright \ dx = \frac{1}{\pi}du \ \curvearrowright \ u(x=1) = \frac{7\pi}{2}; u\left(x = \frac{3}{2}\right) = 4\pi \ .$$

It follows:

$$\int_1^{\frac{3}{2}} \sin\left(\pi x + \frac{5\pi}{2}\right) dx = \frac{1}{\pi}\int_{\frac{7\pi}{2}}^{4\pi} \sin u \, du = -\frac{1}{\pi} \cos u\Big|_{\frac{7\pi}{2}}^{4\pi} = -\frac{1}{\pi}(1-0)$$
$$= -\frac{1}{\pi}$$

3. We substitute

$$u = 7 - 3x \ \curvearrowright \ du = -3dx \ \curvearrowright \ dx = -\frac{1}{3}du \ \curvearrowright \ u(x=1) = 4; u(x=2) = 1 \ .$$

It follows:

$$\int_1^2 \frac{dx}{\sqrt{7-3x}} = -\frac{1}{3}\int_4^1 \frac{du}{\sqrt{u}} = -\frac{2}{3}u^{\frac{1}{2}}\,|_4^1 = -\frac{2}{3}(1-2)$$
$$= \frac{2}{3}\,.$$

4. We substitute

$$u = 2x^3 + 4 \curvearrowright du = 6x^2 dx \curvearrowright dx = \frac{1}{6x^2}du \curvearrowright u(x=-1) = 2; u(x=+1) = 6\,.$$

It follows:

$$\int_{-1}^{+1} x^2\sqrt{2x^3+4}\,dx = \frac{1}{6}\int_2^6 \sqrt{u}\,du = \frac{2}{18}u^{\frac{3}{2}}\,|_2^6 = \frac{1}{9}\left(6^{\frac{3}{2}} - 2^{\frac{3}{2}}\right)$$
$$= \frac{2}{9}\sqrt{2}\left(3\sqrt{3}-1\right) = 1,32\,.$$

Solution 1.2.3

1.

$$\int_{x=0}^1\int_{y=0}^2 x^2\,dxdy = \int_0^1 x^2 dx\int_0^2 dy = \int_0^1 x^2 dx \cdot y\Big|_0^2$$
$$= 2\int_0^1 x^2 dx = \frac{2}{3}x^3\Big|_0^1$$
$$= \frac{2}{3}$$

2.

$$\int_{x=0}^{\pi}\int_{y=\frac{1}{2}\pi}^{\pi} \sin x \cdot \sin y\,dxdy = \int_0^{\pi}\sin x\,dx\int_{\frac{1}{2}\pi}^{\pi}\sin y\,dy$$
$$= (-\cos x)\Big|_0^{\pi}\cdot(-\cos y)\Big|_{\frac{1}{2}\pi}^{\pi} = (+1-(-1))(+1-0)$$
$$= 2$$

3.

$$\begin{aligned}\int_{x=0}^{2}\int_{y=x-1}^{3x} x^2\,dxdy &= \int_0^2 x^2\,dx\int_{x-1}^{3x} dy = \int_0^2 x^2\,dx\,(3x-x+1)\\ &= \int_0^2 (2x^3+x^2)\,dx = \left(\frac{1}{2}x^4+\frac{1}{3}x^3\right)\Big|_0^2 = 8\left(1+\frac{1}{3}\right)\\ &= \frac{32}{3}\end{aligned}$$

4.

$$\begin{aligned}\int_{x=0}^{1}\int_{y=0}^{2x}\int_{z=0}^{x+y} dxdydz = &\quad \int_0^1 dx\int_0^{2x} dy\int_0^{x+y} dz = \int_0^1 dx\int_0^{2x} dy\,(x+y)\\ = &\quad \int_0^1 dx\left(xy+\frac{1}{2}y^2\right)\Big|_0^{2x} = 4\int_0^1 dx\,x^2 = \frac{4}{3}x^3\Big|_0^1\\ = &\quad \frac{4}{3}\end{aligned}$$

Section 1.3

Solution 1.3.1

1. With the orthogonality relation

$$\mathbf{e}_i\cdot\mathbf{e}_j = \delta_{ij}$$

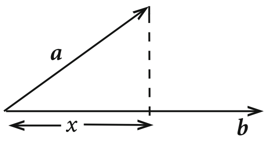
it follows immediately:

$$\begin{gathered}\mathbf{e}_3\cdot(\mathbf{e}_1+\mathbf{e}_2) = \mathbf{e}_3\cdot\mathbf{e}_1+\mathbf{e}_3\cdot\mathbf{e}_2 = 0\ ,\\ (5\mathbf{e}_1+3\mathbf{e}_2)\cdot(7\mathbf{e}_1-16\mathbf{e}_3) = 35\ ,\\ (\mathbf{e}_1+7\mathbf{e}_2-3\mathbf{e}_3)\cdot(12\mathbf{e}_1-3\mathbf{e}_2-4\mathbf{e}_3) = 12-21+12 = 3\ .\end{gathered}$$

2. Requirement: $\mathbf{a}\cdot\mathbf{b} \overset{!}{=} 0$.

$$\mathbf{a}\cdot\mathbf{b} = -3-12-3\alpha \Longrightarrow \alpha = -5\ .$$

Fig. A.1



3. Projection of **a** on the direction of **b** (Fig. A.1):

$$x = a\cos\sphericalangle(\mathbf{a}\,,\mathbf{b}) = \frac{1}{b}(\mathbf{a}\cdot\mathbf{b})\ ,$$

$$b^2 = \mathbf{b}\cdot\mathbf{b} = 16+9 = 25 \Longrightarrow b = 5\ ,$$

$$\mathbf{a}\cdot\mathbf{b} = 4-12 = -8 \Longrightarrow x = -\frac{8}{5}\ .$$

4. $\mathbf{e}_b$: Unit vector in **b**-direction.

$$b = \frac{1}{\sqrt{3}} \Longrightarrow \mathbf{e}_b = \frac{1}{\sqrt{3}}(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)\ .$$

$$\mathbf{e}_b\cdot\mathbf{a} = \frac{1}{\sqrt{3}}(1-2+3) = \frac{2}{\sqrt{3}}\ .$$

Therewith one finds:

$$\mathbf{a}_{\parallel} = \mathbf{e}_b\cdot(\mathbf{e}_b\cdot\mathbf{a}) = \frac{2}{3}(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)\ ,$$

$$\mathbf{a}_{\perp} = \mathbf{a}-\mathbf{a}_{\parallel} = \frac{1}{3}(\mathbf{e}_1-8\mathbf{e}_2+7\mathbf{e}_3)\ .$$

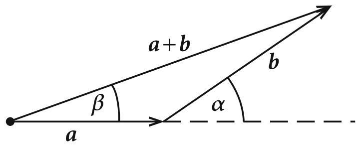
Test: $\mathbf{a}_{\parallel}\cdot\mathbf{a}_{\perp} = \frac{2}{9}(1-8+7) = 0$.

5. $\cos(\sphericalangle\mathbf{a},\mathbf{b}) = \frac{1}{a\,b}(\mathbf{a}\cdot\mathbf{b})$.

$$a = b = \sqrt{1+(2+\sqrt{3})^2} = \sqrt{8+4\sqrt{3}} = 2\sqrt{2+\sqrt{3}}\,.$$

$$\mathbf{a}\cdot\mathbf{b} = 2\,(2+\sqrt{3}) \Longrightarrow \cos(\sphericalangle\mathbf{a},\mathbf{b}) = \frac{1}{2} \Longrightarrow \sphericalangle(\mathbf{a},\mathbf{b}) = 60^\circ\ .$$

Fig. A.2



Solution 1.3.2

1. Cosine rule (Fig. A.2):

$$(\mathbf{a}+\mathbf{b})^2 = a^2 + 2\,\mathbf{a}\cdot\mathbf{b} + b^2 \;.$$

$$|\mathbf{a}+\mathbf{b}| = \sqrt{a^2+b^2+2ab\cos\alpha}\;.$$

Insertion of the given numbers:

$$|\mathbf{a}+\mathbf{b}| = \sqrt{117+108\cos\alpha}\;,$$

$$\cos(0)=1\;,\quad \cos(60^\circ)=\frac{1}{2}\;,\quad \cos(90^\circ)=0\;,$$

$$\cos(150^\circ) = -\frac{1}{2}\sqrt{3}\;,\quad \cos(180^\circ)=-1\;.$$

$$\begin{aligned}
\Longrightarrow |\mathbf{a}+\mathbf{b}| &= \sqrt{225}\,\text{cm} = 15\,\text{cm} &&\Longleftrightarrow \alpha = 0^\circ\;,\\
&= \sqrt{171}\,\text{cm} = 13.1\,\text{cm} &&\Longleftrightarrow \alpha = 60^\circ\;,\\
&= \sqrt{117}\,\text{cm} = 10.8\,\text{cm} &&\Longleftrightarrow \alpha = 90^\circ\;,\\
&= \sqrt{117-54\sqrt{3}}\,\text{cm} = 4.8\,\text{cm} &&\Longleftrightarrow \alpha = 150^\circ\;,\\
&= \sqrt{9}\,\text{cm} = 3\,\text{cm} &&\Longleftrightarrow \alpha = 180^\circ\;.
\end{aligned}$$

For the angle β holds:

$$\cos\beta = \frac{(\mathbf{a}+\mathbf{b})\cdot\mathbf{a}}{a|\mathbf{a}+\mathbf{b}|} = \frac{a+b\cos\alpha}{|\mathbf{a}+\mathbf{b}|} = \frac{6}{|\mathbf{a}+\mathbf{b}|} + 9\frac{\cos\alpha}{|\mathbf{a}+\mathbf{b}|}\;.$$

$\alpha = 0$:

$$\cos\beta = 1 \quad\Longrightarrow\quad \beta = 0\;,$$

$\alpha = 60^\circ$:

$$\cos\beta = \frac{10.5}{13.1} = 0.8 \quad\Longrightarrow\quad \beta = 36.87^\circ\;,$$

$\alpha = 90°$:

$$\cos\beta = \frac{6}{10,8} = 0.56 \quad \Longrightarrow \quad \beta = 55.94° \ ,$$

$\alpha = 150°$:

$$\cos\beta = \frac{6 - 4.5\sqrt{3}}{4.8} = -0.37 \quad \Longrightarrow \quad \beta = 111.95° \ ,$$

$\alpha = 180°$:

$$\cos\beta = \frac{6 - 9}{3} = -1 \quad \Longrightarrow \quad \beta = 180° \ .$$

2. See Fig. A.3

$$\begin{aligned}
\vartheta &= \sphericalangle(\mathbf{a}, \mathbf{b}) = 180° - 36° = 144° \ , \\
\cos\vartheta &= -0.809 \ . \\
|\mathbf{a} + \mathbf{b}|^2 &= a^2 + b^2 + 2\mathbf{a}\cdot\mathbf{b} = 36 + 49 + 84\cos\vartheta \\
&\Longrightarrow |\mathbf{a} + \mathbf{b}| = 4.13\,\mathrm{cm} \ , \\
|\mathbf{a} - \mathbf{b}|^2 &= a^2 + b^2 - 2\mathbf{a}\cdot\mathbf{b} = 36 + 49 - 84\cos\vartheta \\
&\Longrightarrow |\mathbf{a} - \mathbf{b}| = 12.37\,\mathrm{cm} \ , \\
\cos[\sphericalangle(\mathbf{a} + \mathbf{b}, \mathbf{e}_1)] &= \frac{(\mathbf{a} + \mathbf{b})\cdot\mathbf{e}_1}{|\mathbf{a} + \mathbf{b}|} \\
&= \frac{6\cdot\cos 36° + 7\cdot\cos 180°}{4.13} = -0.520 \\
&\Longrightarrow \sphericalangle(\mathbf{a} + \mathbf{b}, \mathbf{e}_1) = 121.32° \ , \\
\cos[\sphericalangle(\mathbf{a} - \mathbf{b}, \mathbf{e}_1)] &= \frac{(\mathbf{a} - \mathbf{b})\cdot\mathbf{e}_1}{|\mathbf{a} - \mathbf{b}|} = \frac{6\cos 36° + 7}{12.37} = 0.958 \\
&\Longrightarrow \sphericalangle(\mathbf{a} - \mathbf{b}, \mathbf{e}_1) = 16.61° \ .
\end{aligned}$$

Fig. A.3

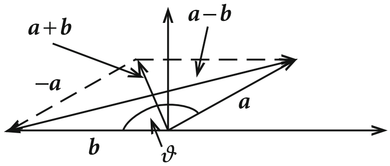
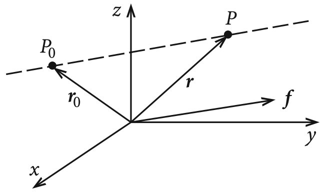


Fig. A.4



3. $\mathbf{P_0P} = \mathbf{r} - \mathbf{r}_0 = \alpha\mathbf{f}$.

   An arbitrary point P on the sought-after straight-line has then the following position vector (Fig. A.4):

$$\mathbf{r} = \mathbf{r}_0 + \alpha\mathbf{f} = (x_0 + \alpha a)\mathbf{e}_1 + (y_0 + \alpha b)\mathbf{e}_2 + (z_0 + \alpha c)\mathbf{e}_3$$

   $(x_0, y_0, z_0,\ a, b, c$ are known, $\alpha \in \mathbb{R})$.

Solution 1.3.3

1.

$$(\mathbf{a} \times \mathbf{b})^2 = a^2b^2 \sin^2[\sphericalangle(\mathbf{a}, \mathbf{b})] \,,$$
$$(\mathbf{a} \cdot \mathbf{b})^2 = a^2b^2 \cos^2[\sphericalangle(\mathbf{a}, \mathbf{b})] \,.$$

   Since $\sin^2 x + \cos^2 x = 1$ it follows:

$$(\mathbf{a} \times \mathbf{b})^2 + (\mathbf{a} \cdot \mathbf{b})^2 = a^2b^2 \,.$$

2.

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) &= \mathbf{c} \cdot [\mathbf{d} \times (\mathbf{a} \times \mathbf{b})] = \quad \text{(scalar triple product)}\\ &= \mathbf{c} \cdot [\mathbf{a}(\mathbf{d} \cdot \mathbf{b}) - \mathbf{b}(\mathbf{d} \cdot \mathbf{a})] = \quad \text{(double vector product)}\\ &= (\mathbf{a} \cdot \mathbf{c})(\mathbf{d} \cdot \mathbf{b}) - (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{d}) \,.\end{aligned}$$

3.

$$
\begin{aligned}
(\mathbf{a}\times\mathbf{b})\cdot[(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})] &= (\mathbf{a}\times\mathbf{b})\cdot\{\mathbf{c}\,[(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}]-\mathbf{a}\,[(\mathbf{b}\times\mathbf{c})\cdot\mathbf{c}]\} = \\
&= [(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}]\,[(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}] = \quad \text{(double vector product)} \\
&= [\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]^2 \quad \text{(scalar triple product)}\ .
\end{aligned}
$$

Solution 1.3.4

1.

$$\mathbf{a} = (2,4,2)\ ;\quad \mathbf{b} = (3,-2,-7) \Longrightarrow a = \sqrt{24}\ ;\quad b = \sqrt{62}\ .$$

$$(\mathbf{a}+\mathbf{b}) = (5,2,-5) \Longrightarrow |\mathbf{a}+\mathbf{b}| = 3\sqrt{6}\ ,$$

$$(\mathbf{a}-\mathbf{b}) = (-1,6,9) \Longrightarrow |\mathbf{a}-\mathbf{b}| = \sqrt{118}\ ,$$

$$(-\mathbf{a}) = (-2,-4,-2) \Longrightarrow |-\mathbf{a}| = \sqrt{24} = 2\sqrt{6}\ ,$$

$$6(2\mathbf{a}-3\mathbf{b}) = (-30,84,150) \Longrightarrow 6|2\mathbf{a}-3\mathbf{b}| = 18\sqrt{94}\ .$$

Test of the triangle inequality:

$$|\mathbf{a}+\mathbf{b}| = 3\cdot\sqrt{6} \le a+b = 2\sqrt{6}+\sqrt{62}\ .$$

This is obviously correct since $\sqrt{6} \le \sqrt{62}$.

2.

$$
\begin{aligned}
(\mathbf{a}\times\mathbf{b}) &= (a_2b_3-a_3b_2,\ a_3b_1-a_1b_3,\ a_1b_2-a_2b_1) \\
&= (-28+4, 6+14, -4-12) = 4(-6,5,-4)\ , \\
(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b}) &= -2(\mathbf{a}\times\mathbf{b}) = 8(6,-5,4)\ , \\
\mathbf{a}\cdot(\mathbf{a}-\mathbf{b}) &= 24-(6-8-14) = 40\ .
\end{aligned}
$$

3. Area of the parallelogram:

$$|\mathbf{a}\times\mathbf{b}| = 4\sqrt{77}.$$

$$\text{unit vector}:\ \mathbf{e} = \frac{\mathbf{a}\times\mathbf{b}}{|\mathbf{a}\times\mathbf{b}|} = -\frac{1}{\sqrt{77}}(6,-5,4).$$

Solution 1.3.5 Thales theorem: *'The angle in the semicircle is a right one'*. According to Fig. A.5 it is to show:

$$(\mathbf{a}+\mathbf{b})\cdot(\mathbf{b}-\mathbf{a}) \stackrel{!}{=} 0\ .$$

Fig. A.5

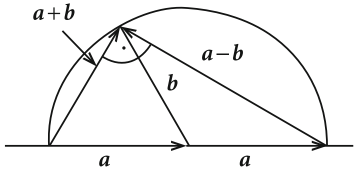
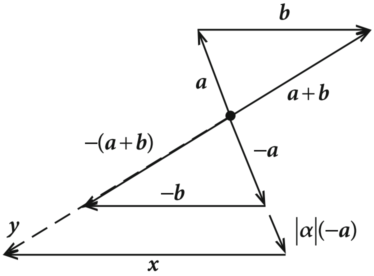


Fig. A.6



This is exactly then the case if

$$\mathbf{a} \cdot \mathbf{b} - a^2 + b^2 - \mathbf{b} \cdot \mathbf{a} \stackrel{!}{=} 0 \quad \Longleftrightarrow \quad a^2 = b^2 \ .$$

That is obviously fulfilled.

Solution 1.3.6 It holds:

$$(-\mathbf{a}) + (-\mathbf{b}) = -(\mathbf{a} + \mathbf{b}) \ .$$

We recognize in Fig. A.6:

$$\mathbf{x} + |\alpha| \, (-\mathbf{a}) = \mathbf{y} \ ,$$
$$\mathbf{x} = \hat{\alpha}(-\mathbf{b}) \ ,$$
$$\mathbf{y} = \overline{\alpha} \, [-(\mathbf{a} + \mathbf{b})] \ .$$

The first intercept theorem yields:

$$\frac{|\mathbf{y}|}{|-(\mathbf{a}+\mathbf{b})|} = \frac{|\alpha||-\mathbf{a}|}{|-\mathbf{a}|} = |\alpha| \Longrightarrow \overline{\alpha} = |\alpha| \ .$$

The second intercept theorem leads to:

$$\frac{|\mathbf{x}|}{|-\mathbf{b}|} = \frac{|\alpha||-\mathbf{a}|}{|-\mathbf{a}|} = |\alpha| \Longrightarrow \hat{\alpha} = |\alpha| \ .$$

Thus it holds

$$\hat{\alpha} = \overline{\alpha} = |\alpha|$$

and therewith

$$\mathbf{x} = -|\alpha|\,\mathbf{b}\;; \quad \mathbf{y} = -|\alpha|\,(\mathbf{a}+\mathbf{b})\;,$$

so that finally the assertion is proven:

$$-|\alpha|\,(\mathbf{a}+\mathbf{b}) = -|\alpha|\,\mathbf{b} - |\alpha|\,\mathbf{a}$$

Solution 1.3.7 Apply the expansion rule for the double vector product:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})\;.$$

For $\mathbf{a} = \mathbf{c}$ holds:

$$\begin{aligned}
&\mathbf{a} \times (\mathbf{b} \times \mathbf{a}) = a^2\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{a} \\
&\Longrightarrow \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{a^2}\,\mathbf{a} + \frac{1}{a^2}\,[\mathbf{a} \times (\mathbf{b} \times \mathbf{a})] = \mathbf{b}_{\parallel} + \mathbf{b}_{\perp}\;.
\end{aligned}$$

Solution 1.3.8

$$\begin{aligned}
(\mathbf{a}-\mathbf{b}) \cdot [(\mathbf{a}+\mathbf{b}) \times \mathbf{c}] &= (\mathbf{a}-\mathbf{b})(\mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}) = \\
&= \mathbf{a} \cdot (\mathbf{a} \times \mathbf{c}) + \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) - \mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) - \mathbf{b} \cdot (\mathbf{b} \times \mathbf{c}) = \\
&= 2\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})\;.
\end{aligned}$$

Solution 1.3.9

$$(\mathbf{a} \times \mathbf{b}) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ -1 & 2 & -3 \\ 3 & -1 & 5 \end{vmatrix} = (7, -4, -5)\;,$$

$$(\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 3 & -1 & 5 \\ -1 & 0 & 2 \end{vmatrix} = (-2, -11, -1)\;.$$

Therewith one easily finds:

$$\begin{aligned}
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) &= (-1, 2, -3) \cdot (-2, -11, -1) = -17\;, \\
(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} &= (7, -4, -5) \cdot (-1, 0, 2) = -17
\end{aligned}$$

Note the cyclic invariance of the scalar triple product!

$$|(\mathbf{a}\times\mathbf{b})\times\mathbf{c}| = \left\|\begin{matrix}\mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 7 & -4 & -5 \\ -1 & 0 & 2\end{matrix}\right\| = |(-8,-9,-4)| = \sqrt{161}\ ,$$

$$|\mathbf{a}\times(\mathbf{b}\times\mathbf{c})| = \left\|\begin{matrix}\mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ -1 & 2 & -3 \\ -2 & -11 & -1\end{matrix}\right\| = |(-35,5,15)| = 5\cdot\sqrt{59}$$

Note that the vector product is *not* associative!

$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix}\mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 7 & -4 & -5 \\ -2 & -11 & -1\end{vmatrix} = (-51,17,-85)\ ,$$

$$(\mathbf{a}\times\mathbf{b})(\mathbf{b}\cdot\mathbf{c}) = 7(7,-4,-5)\ .$$

Solution 1.3.10 In Exercise 1.3.3 we showed

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{d}\cdot\mathbf{b}) - (\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d})\ .$$

Therewith one finds:

$$\begin{aligned}
&(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) + (\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{d}) + (\mathbf{c}\times\mathbf{a})\cdot(\mathbf{b}\times\mathbf{d}) = \\
&= (\mathbf{a}\cdot\mathbf{c})(\mathbf{d}\cdot\mathbf{b}) - (\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d}) + (\mathbf{b}\cdot\mathbf{a})(\mathbf{c}\cdot\mathbf{d}) - (\mathbf{c}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{d}) + \\
&\quad + (\mathbf{c}\cdot\mathbf{b})(\mathbf{a}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{b})(\mathbf{c}\cdot\mathbf{d}) = 0\ .
\end{aligned}$$

Solution 1.3.11 Decomposition of $\mathbf{y}$ relative to $\mathbf{a}$:

$$\mathbf{y} = \mathbf{y}_\parallel + \mathbf{y}_\perp$$

$\mathbf{y}_\parallel \uparrow\uparrow \mathbf{a}$ and distributivity of the vector product

$$\Longrightarrow\ \mathbf{a}\times\mathbf{y} = \mathbf{a}\times\mathbf{y}_\perp \overset{!}{=} \mathbf{b}$$

$\mathbf{y}_\perp$ can be determined

- Magnitude

$$|\mathbf{a}\times\mathbf{y}_\perp| = a y_\perp \sin\frac{\pi}{2} = a y_\perp \overset{!}{=} b$$

$$\Longrightarrow\ y_\perp = \frac{b}{a}$$

- Direction

$$\begin{aligned}
\mathbf{a}\times\mathbf{y}_\perp=\mathbf{b} \;&\Rightarrow\; \mathbf{a},\mathbf{y}_\perp,\mathbf{b}\;:\quad \text{right-handed system}\\
&\Rightarrow\; \mathbf{b},\mathbf{a},\mathbf{y}_\perp\;:\quad \text{also right-handed system}\\
&\Rightarrow\; \mathbf{y}_\perp\uparrow\uparrow\mathbf{b}\times\mathbf{a}
\end{aligned}$$

Therewith it follows:

$$\mathbf{y}_\perp=\frac{b}{a}\frac{\mathbf{b}\times\mathbf{a}}{|\mathbf{b}\times\mathbf{a}|}\overset{\mathbf{b}\perp\mathbf{a}}{=}\frac{1}{a^2}(\mathbf{b}\times\mathbf{a})$$

However $y_\parallel$ completely undetermined $\Longrightarrow$ *no unique* solution $\mathbf{y}=\mathbf{y}_\parallel+\mathbf{y}_\perp$!

Solution 1.3.12 The proof succeeds by direct exploitation of the expansion rule for the double vector product:

$$\begin{aligned}
&\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{b}\times(\mathbf{c}\times\mathbf{a})+\mathbf{c}\times(\mathbf{a}\times\mathbf{b})=\\
&=\mathbf{b}(\mathbf{a}\cdot\mathbf{c})-\mathbf{c}(\mathbf{a}\cdot\mathbf{b})+\mathbf{c}(\mathbf{b}\cdot\mathbf{a})-\mathbf{a}(\mathbf{b}\cdot\mathbf{c})+\mathbf{a}(\mathbf{c}\cdot\mathbf{b})-\mathbf{b}(\mathbf{c}\cdot\mathbf{a})=0\;.
\end{aligned}$$

Solution 1.3.13

1. Take $V=\mathbf{a}_1\cdot(\mathbf{a}_2\times\mathbf{a}_3)$.

   (a)

$$\begin{aligned}
&\mathbf{b}_1\perp\mathbf{a}_2,\mathbf{a}_3\\
&\Longrightarrow\mathbf{b}_1\cdot\mathbf{a}_i=0\quad\text{für } i=2,3\\
&\Longrightarrow\mathbf{a}_1\cdot\mathbf{b}_1=\frac{1}{V}\mathbf{a}_1\cdot(\mathbf{a}_2\times\mathbf{a}_3)=1\;.
\end{aligned}$$

   (b)

$$\begin{aligned}
&\mathbf{b}_2\perp\mathbf{a}_1,\mathbf{a}_3\\
&\Longrightarrow\mathbf{b}_2\cdot\mathbf{a}_i=0\quad\text{für } i=1,3\\
&\Longrightarrow\mathbf{a}_2\cdot\mathbf{b}_2=\frac{1}{V}\mathbf{a}_2\cdot(\mathbf{a}_3\times\mathbf{a}_1)=1\;.
\end{aligned}$$

   (c)

$$\begin{aligned}
&\mathbf{b}_3\perp\mathbf{a}_1,\mathbf{a}_2\\
&\Longrightarrow\mathbf{b}_3\cdot\mathbf{a}_i=0\quad\text{für } i=1,2\\
&\Longrightarrow\mathbf{a}_3\cdot\mathbf{b}_3=\frac{1}{V}\mathbf{a}_3\cdot(\mathbf{a}_1\times\mathbf{a}_2)=1\;.
\end{aligned}$$

2.

$$\begin{aligned}\mathbf{b}_2 \times \mathbf{b}_3 &= \frac{1}{V^2}(\mathbf{a}_3 \times \mathbf{a}_1) \times (\mathbf{a}_1 \times \mathbf{a}_2) = \\ &= \frac{1}{V^2}\{\mathbf{a}_1[(\mathbf{a}_3 \times \mathbf{a}_1) \cdot \mathbf{a}_2] - \mathbf{a}_2[(\mathbf{a}_3 \times \mathbf{a}_1) \cdot \mathbf{a}_1]\} = \frac{1}{V}\mathbf{a}_1\end{aligned}$$

$$\Longrightarrow \mathbf{b}_1 \cdot (\mathbf{b}_2 \times \mathbf{b}_3) = \frac{1}{V}\mathbf{b}_1 \cdot \mathbf{a}_1 = \frac{1}{V} = [\mathbf{a}_1 \cdot (\mathbf{a}_2 \times \mathbf{a}_3)]^{-1} \ .$$

3. According to 2. holds:

$$\mathbf{b}_1 \cdot (\mathbf{b}_2 \times \mathbf{b}_3) = \frac{1}{V} \ .$$

That means:

$$(\mathbf{b}_2 \times \mathbf{b}_3) = \frac{1}{V}\mathbf{a}_1 \Longrightarrow \mathbf{a}_1 = V(\mathbf{b}_2 \times \mathbf{b}_3) \ .$$

Therewith:

$$\mathbf{a}_1 = \frac{\mathbf{b}_2 \times \mathbf{b}_3}{\mathbf{b}_1 \cdot (\mathbf{b}_2 \times \mathbf{b}_3)} \ .$$

Analogously we calculate the other $\mathbf{a}_i$!

4.

$$\bar{\mathbf{e}}_1 = \frac{\mathbf{e}_2 \times \mathbf{e}_3}{\mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3)} = \mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1 \ .$$

In the same manner we obtain:

$$\bar{\mathbf{e}}_2 = \mathbf{e}_2 \ ; \quad \bar{\mathbf{e}}_3 = \mathbf{e}_3 \ .$$

Solution 1.3.14 Test the axioms:

1. $\mathbf{a} \cdot \mathbf{b} = 4a_1b_1 - 2a_1b_2 - 2a_2b_1 + 3a_2b_2$.

commutativity:	$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ obviously fulfilled!
distributivity:	$(\mathbf{a} + \mathbf{c}) \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{b} + \mathbf{c} \cdot \mathbf{b}$
	can be verified by insertion!
bilinearity:	$\alpha \in \mathbb{R}$. from the definition follows immediately:
	$(\alpha\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (\alpha\mathbf{b}) = \alpha(\mathbf{a} \cdot \mathbf{b})$.
magnitude:	$\mathbf{a} \cdot \mathbf{a} = 4a_1^2 - 4a_1a_2 + 3a_2^2 = (2a_1 - a_2)^2 + 2a_2^2 \geq 0$
	$\mathbf{a} \cdot \mathbf{a} = 0$ only for $\mathbf{a} = (0, 0)$.

Hence, it is indeed a scalar product!

2. $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_2 b_1 + 2a_1 b_2$.
   It cannot be a scalar product because of $\mathbf{a} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} \neq 0$.

Solution 1.3.15

1. The axioms of the vector space are easily verifiable. They are all fulfilled.

2.a The vectors are linearly independent because from

$$0 = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3$$

follows:

$$\alpha_1 + 7\alpha_2 + 8\alpha_3 = 0 \,,$$
$$-\alpha_2 = 0 \,,$$
$$11\alpha_3 = 0 \,.$$

But that leads to:

$$\alpha_1 = \alpha_2 = \alpha_3 = 0 \,.$$

2.b The vectors are linearly dependent since from

$$0 = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3$$

follows:

$$-18\alpha_1 + 6\alpha_2 = 0 \,,$$
$$3\alpha_2 - \alpha_3 = 0 \,,$$
$$15\alpha_1 - 5\alpha_2 = 0 \,.$$

This means:

$$\alpha_2 = 3\alpha_1 \,, \quad \alpha_3 = 3\alpha_2 \,.$$

Thus the above condition can be fulfilled by

$$\alpha_1 = 1 \,, \quad \alpha_2 = 3 \,, \quad \alpha_3 = 9$$

The α_i are then not necessarily all of them equal to zero.

Section 1.4

Solution 1.4.1

1. We obtain the *new* basis by rotating the *old* system of coordinates (Fig. A.7). The representation becomes *especially simple* for a rotation by the angle 45°.

$$\mathbf{e}_1 = \frac{1}{\sqrt{2}}\left(\mathbf{e}_1' - \mathbf{e}_2'\right) \ ,$$
$$\mathbf{e}_2 = \frac{1}{\sqrt{2}}\left(\mathbf{e}_1' + \mathbf{e}_2'\right) \ .$$

The factor $\frac{1}{\sqrt{2}}$ takes care for the correct normalization:

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = 0 \ ; \quad \mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{e}_2 \cdot \mathbf{e}_2 = 1 \ .$$

Parameter representation of the space curve with ωt in the x, y-system:

$$\mathbf{r}(t) = a_1 \cos \omega t \, \mathbf{e}_1 + a_2 \sin \omega t \, \mathbf{e}_2 \ .$$

2.

$$\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} a_1 \cos \omega t \\ a_2 \sin \omega t \end{pmatrix} \ .$$

That leads to the midpoint equation of an ellipse:

$$\frac{x^2(t)}{a_1^2} + \frac{y^2(t)}{a_2^2} = 1$$

Fig. A.7

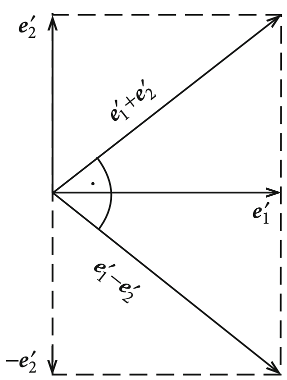
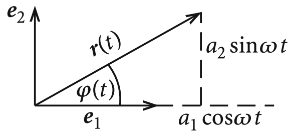


Fig. A.8



3.a

$$\mathbf{e}_1 \cdot \mathbf{r}(t) = |\mathbf{r}(t)| \cos\varphi(t) = a_1 \cos\omega t$$
$$\Longrightarrow \varphi(t) = \arccos\left(\frac{a_1 \cos\omega t}{\sqrt{a_1^2 \cos^2 \omega t + a_2^2 \sin^2 \omega t}}\right) .$$

Geometric interpretation (Fig. A.8):

$$\tan\varphi(t) = \frac{a_2}{a_1} \tan\omega t .$$

Because of

$$\tan^2\varphi = \frac{1}{\cos^2\varphi} - 1$$

this is obviously equivalent to the above result.

3.b Analogously one finds:

$$\psi(t) = \arccos\left(\frac{a_2 \sin\omega t}{\sqrt{a_1^2 \cos^2 \omega t + a_2^2 \sin^2 \omega t}}\right) = \frac{\pi}{2} - \varphi(t) .$$

4.

$$|\mathbf{r}(t)| = \sqrt{a_1^2 \cos^2 \omega t + a_2^2 \sin^2 \omega t} ,$$
$$\mathbf{v}(t) = \dot{\mathbf{r}}(t) = -a_1\omega \sin\omega t \, \mathbf{e}_1 + a_2\omega \cos\omega t \, \mathbf{e}_2$$
$$\Longrightarrow |\mathbf{v}(t)| = \omega\sqrt{a_1^2 \sin^2 \omega t + a_2^2 \cos^2 \omega t} ,$$
$$\mathbf{a}(t) = \ddot{\mathbf{r}}(t) = -\omega^2 \mathbf{r}(t)$$
$$\Longrightarrow |\mathbf{a}(t)| = \omega^2 |\mathbf{r}(t)| .$$

5. Notice that in general:

$$\dot{r}(t) = \frac{d}{dt}|\mathbf{r}(t)| \neq |\dot{\mathbf{r}}(t)|$$

This one sees as follows:

$$\frac{d}{dt}|\mathbf{r}(t)| = \frac{d}{dt}\sqrt{\mathbf{r}(t)\cdot\mathbf{r}(t)} = \frac{\mathbf{r}(t)\cdot\dot{\mathbf{r}}(t)}{|\mathbf{r}(t)|} = |\dot{\mathbf{r}}(t)|\cos[\sphericalangle(\mathbf{r},\dot{\mathbf{r}})]\ .$$

In our case we have:

$$\frac{d}{dt}|\mathbf{r}(t)| = \frac{\omega\left(a_2^2-a_1^2\right)\sin\omega t\cos\omega t}{\sqrt{a_1^2\cos^2\omega t + a_2^2\sin^2\omega t}}\ .$$

6.

$$\cos\alpha(t) = \frac{\mathbf{r}(t)\cdot\dot{\mathbf{r}}(t)}{|\mathbf{r}(t)|\cdot|\dot{\mathbf{r}}(t)|} \stackrel{e)}{=} \frac{\dot{\overline{r(t)}}}{|\dot{\mathbf{r}}(t)|}\ ,$$

$$\alpha(t) = \arccos\left[\frac{\left(a_2^2-a_1^2\right)\sin\omega t\cos\omega t}{\sqrt{\left(a_1^2-a_2^2\right)^2\sin^2\omega t\cos^2\omega t + a_1^2a_2^2}}\right]\ ,$$

$$\gamma(t) = \pi\ , \quad \text{since } \mathbf{a}(t)\sim -\mathbf{r}(t)\ ,$$

$$\beta(t) = \pi - \alpha(t)\ .$$

(Fig. A.9)

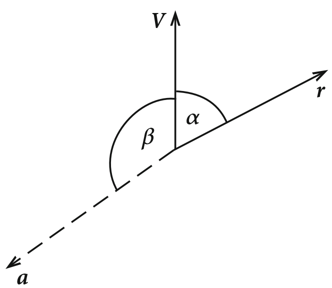
Solution 1.4.2

1. The x_1-component consists of two contributions, a contribution because of the rolling off of the wheel, $R\varphi$, and another one due to the rotation of the wheel, $R\sin\varphi$.

   That means:

$$x_1(\varphi) = R\varphi + R\sin\varphi\ .$$

Fig. A.9



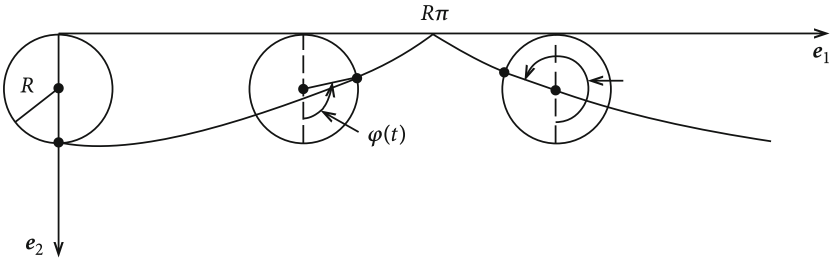


Fig. A.10

For the other component one can read off from Fig. A.10

$$x_2(\varphi) = 2R - (R - R\cos\varphi) = R + R\cos\varphi$$

The full parameter representation for the cycloid that we are looking for is then given by:

$$\mathbf{r}(\varphi) = R(\varphi + \sin\varphi, 1 + \cos\varphi, 0) \ .$$

2.

$$\begin{aligned} x_1(t) &= v \cdot t + l\sin\varphi(t) \ , \\ x_2(t) &= l\cos\varphi(t) \ . \end{aligned}$$

This means:

$$\mathbf{r}(t) = [v\,t + l\sin\varphi(t), l\cos\varphi(t), 0] \ .$$

Solution 1.4.3

1.

$$\begin{aligned} |\mathbf{r}(t)| &= \sqrt{e^{-2\sin t} + \frac{1}{\cot^2 t} + \ln^2(1+t^2)} \\ &\Longrightarrow |\mathbf{r}(t=0)| = 1 \ . \end{aligned}$$

2.

$$\begin{aligned} \dot{\mathbf{r}}(t) &= \left(-\cos t\, e^{-\sin t}, \frac{1}{\cos^2 t}, \frac{2t}{1+t^2}\right) \\ &\Longrightarrow \dot{\mathbf{r}}(t=0) = (-1, 1, 0) \ . \end{aligned}$$

3.

$$|\dot{\mathbf{r}}(t)| = \sqrt{\cos^2 t\, e^{-2\sin t} + \frac{1}{\cos^4 t} + \frac{4t^2}{(1+t^2)^2}}$$
$$\Longrightarrow |\dot{\mathbf{r}}(t=0)| = \sqrt{2}\,.$$

4.

$$\ddot{\mathbf{r}}(t) = \left((\cos^2 t + \sin t)e^{-\sin t}, \frac{2\sin t}{\cos^3 t}, \frac{2(1-t^2)}{(1+t^2)^2}\right)$$
$$\Longrightarrow \ddot{\mathbf{r}}(t=0) = (1,0,2)\,.$$

5.

$$|\ddot{\mathbf{r}}(t)| = \left[(\cos^2 t + \sin t)^2\, e^{-2\sin t} + \frac{4\sin^2 t}{\cos^6 t} + 4\frac{(1-t^2)^2}{(1+t^2)^4}\right]^{1/2}$$
$$\Longrightarrow |\ddot{\mathbf{r}}(t=0)| = \sqrt{5}\,.$$

Solution 1.4.4

1.

$$\begin{aligned}\frac{d}{dt}\left[\mathbf{a}(t)\cdot\mathbf{b}(t)\right] &= \frac{d}{dt}\sum_{i,j} a_i(t)b_j(t)(\mathbf{e}_i\cdot\mathbf{e}_j) = \\ &= \sum_i \left[\dot{a}_i(t)b_i(t) + a_i(t)\dot{b}_i(t)\right] = \\ &= \dot{\mathbf{a}}(t)\cdot\mathbf{b}(t) + \mathbf{a}(t)\cdot\dot{\mathbf{b}}(t)\,.\end{aligned}$$

2. We calculate the k-th component:

$$\begin{aligned}\frac{d}{dt}\left[\mathbf{a}(t)\times\mathbf{b}(t)\right]_k &= \frac{d}{dt}\sum_{i,j}\varepsilon_{ijk}a_i(t)b_j(t) = \\ &= \sum_{i,j}\varepsilon_{ijk}\left[\dot{a}_i(t)b_j(t) + a_i(t)\dot{b}_j(t)\right] = \\ &= \left[\dot{\mathbf{a}}(t)\times\mathbf{b}(t)\right]_k + \left[\mathbf{a}(t)\times\dot{\mathbf{b}}(t)\right]_k\,.\end{aligned}$$

This holds for $k = 1, 2, 3$.

3. The definition of the scalar product yields at first:

$$\mathbf{a}(t)\cdot\dot{\mathbf{a}}(t) = \sum_{i,j} a_i(t)\dot{a}_j(t)(\mathbf{e}_i\cdot\mathbf{e}_j) = \sum_i a_i(t)\dot{a}_i(t)\ .$$

Otherwise it holds:

$$|\mathbf{a}(t)|\cdot\frac{d}{dt}|\mathbf{a}(t)| = \sqrt{\sum_i a_i^2(t)}\ \frac{\sum_j a_j(t)\dot{a}_j(t)}{\sqrt{\sum_j a_j^2(t)}} =$$

$$= \sum_j a_j(t)\dot{a}_j(t)\ .$$

Hence the two expression are equal!

Solution 1.4.5

1. We need at first:

$$\frac{d\mathbf{r}}{dt} = \frac{1}{t_0}\left(3\cos\frac{t}{t_0}, 4, -3\sin\frac{t}{t_0}\right) \Longrightarrow \left|\frac{d\mathbf{r}}{dt}\right| = \frac{5}{t_0}\ .$$

Therewith the arc length is calculated with $s(t=0)=0$ to:

$$s(t) = \int_0^t \left|\frac{d\mathbf{r}(t')}{dt'}\right| dt' = 5\frac{t}{t_0}\ .$$

2. With

$$t(s) = \frac{t_0}{5}s$$

we find the *natural parametrization*:

$$\mathbf{r}(s) = \left(3\sin\frac{s}{5}, \frac{4}{5}s, 3\cos\frac{s}{5}\right)\ .$$

Hence it follows for the tangent-unit vector:

$$\hat{\mathbf{t}}(s) = \frac{dr(s)}{ds} = \frac{1}{5}\left(3\cos\frac{s}{5}, 4, -3\sin\frac{s}{5}\right)\ .$$

3.

$$\frac{d\hat{\mathbf{t}}(s)}{ds} = \frac{3}{25}\left(-\sin\frac{s}{5}, 0, -\cos\frac{s}{5}\right) ,$$

curvature: $\kappa = \left|\frac{d\hat{t}}{ds}\right| = \frac{3}{25}$, radius of curvature: $\rho = \kappa^{-1} = \frac{25}{3}$.

4. The normal-unit vector is determined with the preceding results:

$$\hat{\mathbf{n}} = \rho\frac{d\hat{\mathbf{t}}}{ds} = \left(-\sin\frac{s}{5}\ , 0, -\cos\frac{s}{5}\right) .$$

5. For the complete derivation of the moving trihedron we still need the binormal-unit vector:

$$\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}} = \frac{1}{5}\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 3\cos\frac{s}{5} & 4 & -3\sin\frac{s}{5} \\ -\sin\frac{s}{5} & 0 & -\cos\frac{s}{5} \end{vmatrix} =$$

$$= \frac{1}{5}\left(-4\cos\frac{s}{5}, 3, 4\sin\frac{s}{5}\right) .$$

The point at time $t = 5\pi\, t_0$ means $s = 25\pi$:

$$\hat{\mathbf{t}}(25\pi) = \frac{1}{5}(-3, 4, 0)\ , \quad \hat{\mathbf{n}}(25\pi) = (0, 0, 1)\ , \quad \hat{\mathbf{b}}(25\pi) = \frac{1}{5}(4, 3, 0)\ .$$

6. For the torsion of the space curve we first calculate:

$$\frac{d\hat{\mathbf{b}}}{ds} = \frac{1}{25}\left(4\sin\frac{s}{5}, 0, 4\cos\frac{s}{5}\right) \stackrel{!}{=} -\tau\hat{\mathbf{n}}\ .$$

The comparison with 4. yields:

$$\tau = \frac{4}{25}\ .$$

Solution 1.4.6 Acceleration and velocity of the particle motion:

$$\dot{\mathbf{r}}(t) = \mathbf{v}(t) = v(t)\cdot\hat{\mathbf{t}} \qquad (\hat{\mathbf{t}}, \hat{\mathbf{n}}, \hat{\mathbf{b}}) \quad \text{moving trihedron}$$

$$v(t) = |\dot{\mathbf{r}}(t)| = \dot{s} \qquad s: \quad \text{arc length}$$

According to (1.245) it holds:

$$\ddot{\mathbf{r}}(t) = \mathbf{a}(t) = a_t \cdot \hat{\mathbf{t}} + a_n \cdot \hat{\mathbf{n}}$$

$$a_t = \dot{v} \qquad \text{tangential acceleration}$$

$$a_n = \frac{v^2}{\rho} = \kappa v^2 \qquad \text{centripetal acceleration}$$

Therewith it follows:

$$\begin{aligned}
\mathbf{v} \times \mathbf{a} &= v\dot{v} \underbrace{\hat{\mathbf{t}} \times \hat{\mathbf{t}}}_{=0} + \kappa v^3 \underbrace{\hat{\mathbf{t}} \times \hat{\mathbf{n}}}_{=\hat{\mathbf{b}}} \\
&\Longrightarrow \mathbf{v} \times \mathbf{a} = \kappa v^3 \hat{\mathbf{b}} \\
&\Longrightarrow |\mathbf{v} \times \mathbf{a}| = \kappa v^3 \\
&\Longrightarrow \kappa = \frac{1}{v^3} |\mathbf{v} \times \mathbf{a}|
\end{aligned}$$

Solution 1.4.7 For a rearranging we use Frenet's formulae (1.231):

$$\begin{aligned}
\frac{d\mathbf{r}}{ds} \cdot \left(\frac{d^2\mathbf{r}}{ds^2} \times \frac{d^3\mathbf{r}}{ds^3} \right) &= \hat{\mathbf{t}} \cdot \left(\frac{d\hat{\mathbf{t}}}{ds} \times \frac{d^2\hat{\mathbf{t}}}{ds^2} \right) = \\
&= \kappa^2 \hat{\mathbf{t}} \cdot \left(\mathbf{n} \times \frac{d\hat{\mathbf{n}}}{ds} \right) = \kappa^2 \hat{\mathbf{t}} \cdot \left(\mathbf{n} \times (\tau \hat{\mathbf{b}} - \kappa \hat{\mathbf{t}}) \right) = \\
&= \kappa^2 \tau \, \hat{\mathbf{t}} \cdot (\mathbf{n} \times \hat{\mathbf{b}}) = \kappa^2 \tau \ .
\end{aligned}$$

Solution 1.4.8

1.

$$\dot{\mathbf{r}}(t) = (1, 2t, 2t^2) \Longrightarrow |\dot{\mathbf{r}}| = (1 + 2t^2) \ .$$

With $s(t = 0) = 0$ one obtains:

$$s(t) = \int_0^t (1 + 2\,t'^2)\, dt' = t + \frac{2}{3} t^3 \ .$$

2. Actually $\hat{\mathbf{t}}$ is defined as a function of the arc length s. However, $\hat{\mathbf{t}}$ is here sought for as function of the time t:

$$\hat{\mathbf{t}} = \frac{d\mathbf{r}(t)}{dt} \frac{dt}{ds} = \frac{\dot{\mathbf{r}}(t)}{|\dot{\mathbf{r}}(t)|} \Longrightarrow \hat{\mathbf{t}} = \frac{1}{1 + 2t^2} (1, 2t, 2t^2) \ .$$

3.

$$\kappa = \left|\frac{d\hat{\mathbf{t}}}{ds}\right| = \left|\frac{d\hat{\mathbf{t}}}{dt}\frac{dt}{ds}\right| = \left|\frac{d\hat{\mathbf{t}}}{dt}\right|\frac{1}{|\dot{\mathbf{r}}(t)|} = \frac{2}{(1+2t^2)^2} \; .$$

4.

$$\hat{\mathbf{n}} = \frac{1}{\kappa}\frac{d\hat{\mathbf{t}}}{ds} = \frac{1}{\kappa}\frac{1}{|\dot{\mathbf{r}}(t)|}\frac{d\hat{\mathbf{t}}}{dt} = \frac{1}{1+2t^2}(-2t, 1-2t^2, 2t) \; ,$$

$$\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}} = \frac{1}{1+2t^2}(2t^2, -2t, 1) \; .$$

5.

$$\frac{d\hat{\mathbf{b}}}{ds} = \frac{d\hat{\mathbf{b}}}{dt}\frac{dt}{ds} = \frac{2}{(1+2t^2)^3}(2t, 2t^2-1, -2t) \stackrel{!}{=} -\tau\hat{\mathbf{n}} \; .$$

From this it follows that:

$$\tau = \frac{2}{(1+2t^2)^2} \; .$$

Solution 1.4.9

1. According to Exercise 1.4.2 it is about the parameter representation of the cycloid with the tangent-unit vector:

$$\hat{\mathbf{t}} = \frac{d\mathbf{r}}{d\varphi} \cdot \frac{d\varphi}{ds}$$

It holds:

$$\frac{d\mathbf{r}}{d\varphi} = R\,(1+\cos\varphi, -\sin\varphi, 0)$$

Therewith:

$$\begin{aligned}\left|\frac{d\mathbf{r}}{d\varphi}\right| &= R\sqrt{(1+\cos\varphi)^2 + \sin^2\varphi} \\ &= R\sqrt{1 + 2\cos\varphi + \cos^2\varphi + \sin^2\varphi} \\ &= R\sqrt{2(1+\cos\varphi)} \stackrel{!}{=} \frac{ds}{d\varphi} \\ \Longrightarrow \quad \hat{\mathbf{t}} &= \frac{1}{R\sqrt{2(1+\cos\varphi)}} \cdot \frac{d\mathbf{r}}{d\varphi}\end{aligned}$$

$$= \frac{1}{\sqrt{2(1+\cos\varphi)}}(\underbrace{1+\cos\varphi}_{2\cos^2\frac{\varphi}{2}}, \underbrace{-\sin\varphi}_{-2\sin\frac{\varphi}{2}\cos\frac{\varphi}{2}}, 0)$$

$$= \left(\cos\frac{\varphi}{2}, -\sin\frac{\varphi}{2}, 0\right)$$

curvature:

$$\frac{d\hat{\mathbf{t}}}{ds} = \frac{d\hat{\mathbf{t}}}{d\varphi}\cdot\frac{d\varphi}{ds} = \frac{1}{R\sqrt{2(1+\cos\varphi)}}\cdot\frac{1}{2}\left(-\sin\frac{\varphi}{2}, -\cos\frac{\varphi}{2}, 0\right)$$

$$= \frac{-1}{4R\cos\frac{\varphi}{2}}\left(\sin\frac{\varphi}{2}, \cos\frac{\varphi}{2}, 0\right)$$

$$\Longrightarrow \quad \kappa = \left|\frac{d\hat{\mathbf{t}}}{ds}\right| = \frac{1}{4R\cos\frac{\varphi}{2}}$$

normal-unit vector:

$$\hat{\mathbf{n}} = \frac{1}{\kappa}\frac{d\hat{\mathbf{t}}}{ds} = -\left(\sin\frac{\varphi}{2}, \cos\frac{\varphi}{2}, 0\right)$$

binormal-unit vector:

$$\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}}$$

$$= \mathbf{e}_1(0-0) + \mathbf{e}_2(0-0) + \mathbf{e}_3\left(-\cos^2\frac{\varphi}{2} - \sin^2\frac{\varphi}{2}\right)$$

$$= -\mathbf{e}_3 = (0, 0, -1)$$

That means:

$$\frac{d\hat{\mathbf{b}}}{ds} = 0 \stackrel{!}{=} -\tau\hat{\mathbf{n}} \implies \tau = 0$$

moving trihedron:

$$\hat{\mathbf{t}} = \left(\cos\frac{\varphi}{2}, -\sin\frac{\varphi}{2}, 0\right)$$

$$\hat{\mathbf{n}} = \left(-\sin\frac{\varphi}{2}, -\cos\frac{\varphi}{2}, 0\right)$$

$$\hat{\mathbf{b}} = (0, 0, -1)$$

2.

$$\mathbf{r}(\varphi) = (\varphi, f(\varphi), 0)$$

arc length:

$$\frac{d\mathbf{r}}{d\varphi} = \big(1, f'(\varphi), 0\big) \implies \left|\frac{d\mathbf{r}}{d\varphi}\right| = \sqrt{1 + f'^2(\varphi)}$$

$$s(\varphi) = \int_{\varphi_0}^{\varphi} d\varphi' \sqrt{1 + f'^2(\varphi')} \implies \varphi = \varphi(s)$$

$$\implies \frac{ds}{d\varphi} = \sqrt{1 + f'^2(\varphi)}$$

tangent-unit vector:

$$\hat{\mathbf{t}} = \frac{\partial \mathbf{r}}{\partial \varphi} \cdot \frac{d\varphi}{ds} = \frac{1}{\sqrt{1 + f'^2(\varphi)}} \big(1, f'(\varphi), 0\big)$$

$$\begin{aligned}
\implies \frac{\partial \hat{\mathbf{t}}}{\partial \varphi} &= \frac{-f' \cdot f''}{(1 + f'^2)^{\frac{3}{2}}}(1, f', 0) + \frac{1}{(1 + f'^2)^{\frac{1}{2}}}(0, f'', 0) \\
&= \frac{1}{(1 + f'^2)^{\frac{3}{2}}}(-f'f'', f'', 0)
\end{aligned}$$

$$\implies \frac{\partial \hat{\mathbf{t}}}{\partial s} = \frac{\partial \hat{\mathbf{t}}}{\partial \varphi} \cdot \frac{\partial \varphi}{\partial s} = \frac{f''}{(1 + f'^2)^2}(-f', 1, 0)$$

curvature:

$$\kappa = \left|\frac{d}{ds}\hat{\mathbf{t}}(s)\right| = \left|\frac{f''(\varphi)}{(1 + f'^2)^{\frac{3}{2}}}\right|$$

alternatively:

$$\begin{aligned}
\varphi &= \varphi(s) \\
\hat{\mathbf{t}} &= \frac{\partial \mathbf{r}}{\partial s} = \left(\frac{d\varphi}{ds}, \frac{df}{ds}, 0\right) \\
\implies \frac{d\hat{\mathbf{t}}}{ds} &= \left(\frac{d^2\varphi}{ds^2}, \frac{d^2 f}{ds^2}, 0\right) \\
\implies \kappa &= \sqrt{\left(\frac{d^2\varphi}{ds^2}\right)^2 + \left(\frac{d^2 f}{ds^2}\right)^2}
\end{aligned}$$

Example: *'circle'*

$$x = \varphi$$

$$y = \begin{cases} +\sqrt{R^2 - \varphi^2} & \text{upper half-plane} \\ -\sqrt{R^2 - \varphi^2} & \text{lower half-plane} \end{cases}$$

$$\Longrightarrow \; f(\varphi) = \pm\sqrt{R^2 - \varphi^2}$$

$$f'(\varphi) = \pm\frac{-\varphi}{\sqrt{R^2 - \varphi^2}} = \frac{-\varphi}{f(\varphi)}$$

$$f''(\varphi) = -\frac{1}{f} + \frac{\varphi f'}{f^2} = -\frac{1}{f}(1 + f'^2)$$

$$\Longrightarrow \; \frac{f''}{(1 + f'^2)^{\frac{3}{2}}} = \frac{-1}{f(1 + f'^2)^{\frac{1}{2}}} = \frac{-1}{f\left(1 + \frac{\varphi^2}{f^2}\right)^{\frac{1}{2}}}$$

$$= \frac{-1}{(f^2 + \varphi^2)^{\frac{1}{2}}} = -\frac{1}{R}$$

$$\Longrightarrow \; \kappa = \left|\frac{f''}{(1 + f'^2)^{\frac{3}{2}}}\right| = \frac{1}{R}$$

Section 1.5

Solution 1.5.1

1.a

$$\mathbf{a}(\mathbf{r}) = \frac{1}{r}(\boldsymbol{\omega} \times \mathbf{r}) = \frac{\omega_0}{r}(-x_2, x_1, 0) \; .$$

In the $x_3 = 0$-plane it holds $r = \sqrt{x_1^2 + x_2^2}$. This means:

$$|\mathbf{a}(\mathbf{r})|_{x_3 = 0} = \omega_0 \; .$$

The field lines represent arrows of constant length ω_0 being perpendicular to $\mathbf{r}$ and perpendicular to $\mathbf{e}_3$. Thus they are located tangentially to a circle around the origin of coordinates with radius ω_0 (Fig. A.11).

1.b

$$\mathbf{a}(\mathbf{r}) = \alpha\mathbf{r} \; ; \quad \alpha < 0.$$

Fig. A.11

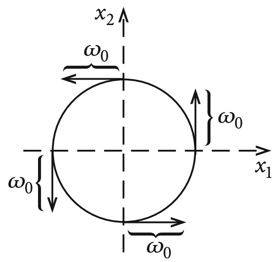
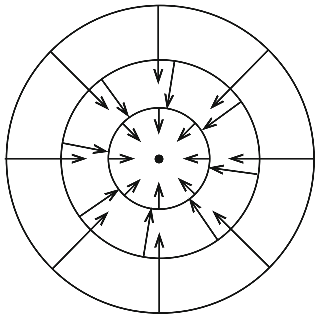


Fig. A.12



The contour lines

$$|\mathbf{a}(\mathbf{r})| = |\alpha| r$$

are equally spaced concentric circles. The field is characterized by arrows of the length $|\alpha|r$, which because of $\alpha < 0$ are pointing radially towards the origin of coordinates (Fig. A.12).

1.c

$$\mathbf{a}(\mathbf{r}) = \alpha(x_1 + x_2)\,\mathbf{e}_1 + \alpha(x_2 - x_1)\,\mathbf{e}_2; \quad \alpha > 0.$$

The contour lines

$$|\mathbf{a}(\mathbf{r})|_{x_3=0} = \alpha\sqrt{(x_1 + x_2)^2 + (x_2 - x_1)^2} = \sqrt{2}\,\alpha r$$

are same as in part 1.b concentric circles with from line to line constant radius change. The arrow lengths are radially increasing according to $\sqrt{2}\,\alpha r$. Their directions are exhibited in Fig. A.13.

1.d

$$\mathbf{a}(\mathbf{r}) = \frac{\alpha}{x_2^2 + x_3^2 + \beta^2}\mathbf{e}_1 \; ; \quad \alpha, \beta > 0 \; .$$

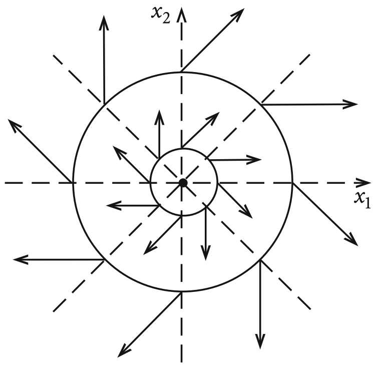


Fig. A.13

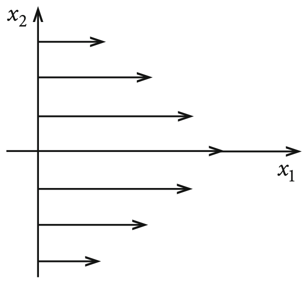


Fig. A.14

The contour lines follow from

$$|\mathbf{a}(\mathbf{r})|_{x_3=0} = \frac{\alpha}{x_2^2 + \beta^2} \; .$$

The field-line arrows are parallel to the x_1-axis. Their lengths decrease with increasing x_2 values (Fig. A.14).

2.a

$$\mathbf{a}(\mathbf{r}) = \frac{\omega_0}{r}(-x_2, x_1, 0).$$

With

$$\frac{\partial}{\partial x_i}\frac{1}{r} = -\frac{x_i}{r^3}$$

follows:

$$\frac{\partial}{\partial x_1}\mathbf{a} = \frac{\omega_0}{r^3}\left(x_1 x_2, r^2 - x_1^2, 0\right) \; ,$$

$$\frac{\partial}{\partial x_2}\mathbf{a} = \frac{\omega_0}{r^3}\left(x_2^2 - r^2, -x_1 x_2, 0\right) \; ,$$

$$\frac{\partial}{\partial x_3}\mathbf{a} = \frac{\omega_0}{r^3}(x_2x_3, -x_1x_3, 0)\ .$$

2.b

$$\mathbf{a}(\mathbf{r}) = \alpha(x_1, x_2, x_3)\ ,$$

$$\frac{\partial}{\partial x_i}\mathbf{a}(\mathbf{r}) = \alpha\mathbf{e}_i\ ; \quad i = 1, 2, 3\ .$$

2.c

$$\mathbf{a}(\mathbf{r}) = \alpha(x_1 + x_2, x_2 - x_1, 0)\ ,$$

$$\frac{\partial}{\partial x_1}\mathbf{a}(\mathbf{r}) = \alpha(1, -1, 0)\ ,$$

$$\frac{\partial}{\partial x_2}\mathbf{a}(\mathbf{r}) = \alpha(1, 1, 0)\ ,$$

$$\frac{\partial}{\partial x_3}\mathbf{a}(\mathbf{r}) = \alpha(0, 0, 0)\ .$$

2.d

$$\mathbf{a}(\mathbf{r}) = \frac{\alpha}{x_2^2 + x_3^2 + \beta^2}\mathbf{e}_1\ ,$$

$$\frac{\partial}{\partial x_1}\mathbf{a} = \mathbf{0}\ ,$$

$$\frac{\partial}{\partial x_2}\mathbf{a} = \frac{-2\alpha x_2}{\left(x_2^2 + x_3^2 + \beta^2\right)^2}(1, 0, 0)\ ,$$

$$\frac{\partial}{\partial x_3}\mathbf{a} = \frac{-2\alpha x_3}{\left(x_2^2 + x_3^2 + \beta^2\right)^2}(1, 0, 0)\ .$$

3.a

$$\nabla \cdot \mathbf{a} = \sum_{j=1}^{3} \frac{\partial a_j}{\partial x_j} = \frac{\omega_0}{r^3}(x_1x_2 - x_2x_1) = 0\ ,$$

$$\nabla \times \mathbf{a} = \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}, \frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1}, \frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right) =$$

$$= \frac{\omega_0}{r^3}\left(x_1x_3, x_2x_3, r^2 + x_3^2\right)\ .$$

3.b

$$\nabla \cdot \mathbf{a} = 3\alpha \ ,$$
$$\nabla \times \mathbf{a} = 0 \ , \quad \text{since} \quad \frac{\partial a_i}{\partial x_j} = 0 \quad \text{for} \quad i \neq j \ .$$

3.c

$$\nabla \cdot \mathbf{a} = 2\alpha \ ,$$
$$\nabla \times \mathbf{a} = \alpha(0-0, 0-0, -1-1) = -2\alpha(0,0,1) \ .$$

3.d

$$\nabla \cdot \mathbf{a} = 0 \ ,$$
$$\nabla \times \mathbf{a} = -\frac{2\alpha}{\left(x_2^2 + x_3^2 + \beta^2\right)^2}(0, x_3, -x_2) \ .$$

Solution 1.5.2

1.

$$\frac{d}{dr}\frac{e^{-\alpha r}}{r} = \left(-\frac{1}{r^2} - \frac{\alpha}{r}\right) e^{-\alpha r} \ ,$$
$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r} \ .$$

The partial derivatives of the potential φ thus read:

$$\frac{\partial}{\partial x_i}\varphi(\mathbf{r}) = -\frac{q}{4\pi\varepsilon_0} x_i (1 + \alpha r) \frac{e^{-\alpha r}}{r^3} \ .$$

That yields for the gradient field:

$$\nabla\varphi(\mathbf{r}) = -\frac{q}{4\pi\varepsilon_0} \cdot \frac{1 + \alpha r}{r^2} e^{-\alpha r} \mathbf{e}_r \ .$$

2.

$$\frac{\partial^2 \varphi}{\partial x_i^2} = -\frac{q}{4\pi\varepsilon_0} e^{-\alpha r} \left[\frac{1+\alpha r}{r^3} + x_i \alpha \frac{x_i}{r}\frac{1}{r^3} + x_i(1+\alpha r)\left(-\frac{3}{r^4}\frac{x_i}{r} - \frac{\alpha}{r^3}\frac{x_i}{r}\right)\right] =$$
$$= -\frac{q}{4\pi\varepsilon_0} \cdot \frac{e^{-\alpha r}}{r^5}\left[r^2 + \alpha r^3 + \alpha x_i^2 r - x_i^2(1+\alpha r)(3+\alpha r)\right] \ .$$

With $\sum_i x_i^2 = r^2$ eventually follows:

$$\Delta\varphi = \alpha^2 \frac{q}{4\pi\varepsilon_0} \cdot \frac{e^{-\alpha r}}{r} = \alpha^2 \varphi(\mathbf{r}) \ .$$

Solution 1.5.3

1. We define the scalar field

$$\varphi(x_1, x_2, x_3) = \frac{x_1^2}{a^2} + \frac{x_2^2}{a^2} + \frac{x_3^2}{b^2}$$

knowing then that the gradient field $\nabla\varphi(\mathbf{r})$ has a direction perpendicularly to the plane $\varphi = \text{const}$. The sought-after surface-normal vector $\mathbf{n}$ follows from that as:

$$\mathbf{n} = \frac{\nabla\varphi}{|\nabla\varphi|} \ .$$

One finds very easily:

$$\nabla\varphi = 2\left(\frac{x_1}{a^2}, \frac{x_2}{a^2}, \frac{x_3}{b^2}\right) \ .$$

This yields for $\mathbf{n}$:

$$\mathbf{n} = \frac{\left(\frac{x_1}{a^2}, \frac{x_2}{a^2}, \frac{x_3}{b^2}\right)}{\sqrt{\frac{1}{a^4}\left(x_1^2 + x_2^2\right) + \frac{1}{b^4}x_3^2}} \ .$$

Thereby x_1, x_2, x_3 are to be chosen so that the following relation is fulfilled:

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} = 1$$

2.a $\mathbf{n} = \frac{1}{\sqrt{2}}(1, 1, 0)$,

2.b $\mathbf{n} = \frac{1}{\sqrt{2 + \frac{a^2}{b^2}}}\left(1, 1, \frac{a}{b}\right)$,

2.c $\mathbf{n} = \frac{1}{\sqrt{3 + \frac{a^2}{b^2}}}\left(-1, \sqrt{2}, -\frac{a}{b}\right)$,

2.d $\mathbf{n} = (0, 0, 1)$,

2.e $\mathbf{n} = (0, -1, 0)$.

Solution 1.5.4

1.

$$\begin{aligned}
\frac{\partial}{\partial x_i}\varphi_1(\mathbf{r}) &= -\alpha_i \sin(\boldsymbol{\alpha}\cdot\mathbf{r})\ ; \quad i=1,2,3\\
\Longrightarrow \nabla\varphi_1(\mathbf{r}) &= -\boldsymbol{\alpha}\sin(\boldsymbol{\alpha}\cdot\mathbf{r})\ ,\\
\frac{\partial^2}{\partial x_i^2}\varphi_1(\mathbf{r}) &= -\alpha_i^2\cos(\boldsymbol{\alpha}\cdot\mathbf{r})\\
\Longrightarrow \Delta\varphi_1(\mathbf{r}) &= -|\boldsymbol{\alpha}|^2\varphi_1(\mathbf{r})\ .
\end{aligned}$$

The calculation for $\varphi_2(\mathbf{r})$ runs analogously:

$$\begin{aligned}
\frac{\partial}{\partial x_i}\varphi_2(\mathbf{r}) &= -2\gamma\, r\frac{x_i}{r}e^{-\gamma\, r^2}\ ; \quad i=1,2,3\ ,\\
\frac{\partial^2}{\partial x_i^2}\varphi_2(\mathbf{r}) &= e^{-\gamma\, r^2}\left(-2\gamma+\gamma^2x_i^2\right)\\
\Longrightarrow \nabla\varphi_2(\mathbf{r}) &= -2\gamma\, e^{-\gamma r^2}\mathbf{r}\ .\\
\Delta\varphi_2(\mathbf{r}) &= 2\gamma(2\gamma r^2-3)e^{-\gamma r^2}\ .
\end{aligned}$$

2.

$$\frac{\partial}{\partial x_i}\left(\frac{x_i}{r}\right)=\frac{1}{r}-\frac{x_i^2}{r^3}\Longrightarrow \nabla\mathbf{e}_r=\frac{2}{r}\ .$$

3. We seek the conditions for

$$\nabla\cdot\mathbf{a}(\mathbf{r})\stackrel{!}{=}0\ .$$

Because of

$$\frac{\partial a_i}{\partial x_i}=\frac{\partial}{\partial x_i}f(r)x_i=f(r)+\frac{df(r)}{dr}\frac{x_i^2}{r}$$

it holds

$$\nabla\cdot\mathbf{a}(\mathbf{r})=3f(r)+r\frac{df(r)}{dr}\ ,$$

so that the condition for a source-free field reads:

$$\frac{df}{dr}=-\frac{3}{r}f(r)$$

Hence, if $f(r)$ is of the form

$$f(r) = \frac{\alpha}{r^3} \quad (\alpha \text{ arbitrary})$$

then the divergence of the field $\mathbf{a}(r)$ vanishes.

4. For the k-th component of the vector field it holds:

$$(\nabla\varphi_1 \times \nabla\varphi_2)_k = \sum_{i,j} \varepsilon_{ijk} \frac{\partial\varphi_1}{\partial x_i} \frac{\partial\varphi_2}{\partial x_j} \; .$$

Therewith it follows:

$$\frac{\partial}{\partial x_k} a_k(\mathbf{r}) = \sum_{i,j} \varepsilon_{ijk} \left(\frac{\partial^2\varphi_1}{\partial x_k \partial x_i} \frac{\partial\varphi_2}{\partial x_j} + \frac{\partial\varphi_1}{\partial x_i} \frac{\partial^2\varphi_2}{\partial x_k \partial x_j} \right) \; .$$

This helps to calculate the divergence:

$$\begin{aligned}
\nabla \cdot \mathbf{a}(\mathbf{r}) &= \sum_{i,j,k} \varepsilon_{ijk} \left(\frac{\partial^2\varphi_1}{\partial x_k \partial x_i} \frac{\partial\varphi_2}{\partial x_j} + \frac{\partial\varphi_1}{\partial x_i} \frac{\partial^2\varphi_2}{\partial x_k \partial x_j} \right) = \\
&= \frac{1}{2} \sum_{i,j,k} \frac{\partial\varphi_2}{\partial x_j} \left(\varepsilon_{ijk} \frac{\partial^2\varphi_1}{\partial x_k \partial x_i} + \varepsilon_{kji} \frac{\partial^2\varphi_1}{\partial x_i \partial x_k} \right) + \\
&\quad + \frac{1}{2} \sum_{i,j,k} \frac{\partial\varphi_1}{\partial x_i} \left(\varepsilon_{ijk} \frac{\partial^2\varphi_2}{\partial x_k \partial x_j} + \varepsilon_{ikj} \frac{\partial^2\varphi_2}{\partial x_j \partial x_k} \right) \; .
\end{aligned}$$

In the brackets of the second summand we have simply interchanged the summation indexes i and k as well as j and k. φ_1 and φ_2 are two times continuously differentiable so that the sequence of the partial differentiations is arbitrary:

$$\begin{aligned}
\nabla \cdot \mathbf{a}(\mathbf{r}) &= \frac{1}{2} \sum_{i,j,k} \frac{\partial\varphi_2}{\partial x_j} \frac{\partial^2\varphi_1}{\partial x_k \partial x_i} \underbrace{\left(\varepsilon_{ijk} + \varepsilon_{kji}\right)}_{0} + \\
&\quad + \frac{1}{2} \sum_{i,j,k} \frac{\partial\varphi_1}{\partial x_i} \frac{\partial^2\varphi_2}{\partial x_k \partial x_j} \underbrace{\left(\varepsilon_{ijk} + \varepsilon_{ikj}\right)}_{0} = 0 \; .
\end{aligned}$$

5.

$$\nabla \cdot (\varphi\mathbf{a}) = \sum_{j=1}^{3} \frac{\partial}{\partial x_j} (\varphi\, a_j) = \sum_{j=1}^{3} \varphi \frac{\partial a_j}{\partial x_j} + \sum_{j=1}^{3} a_j \frac{\partial\varphi}{\partial x_j} = \varphi\nabla \cdot \mathbf{a} + \mathbf{a} \cdot \nabla\varphi \; .$$

Solution 1.5.5

$$\nabla\times\ \mathbf{a}(\mathbf{r}) = \mathbf{e}_1\left(\frac{\partial}{\partial x_2}a_3 - \frac{\partial}{\partial x_3}a_2\right) + \mathbf{e}_2\left(\frac{\partial}{\partial x_3}a_1 - \frac{\partial}{\partial x_1}a_3\right) + \mathbf{e}_3\left(\frac{\partial}{\partial x_1}a_2 - \frac{\partial}{\partial x_2}a_1\right)$$

It is to inspect:

$$\mathbf{a}(\mathbf{r}) = \left(\gamma x_1 x_2 - x_3^3, (\gamma-2)x_1^2, (1-\gamma)x_1 x_3^2\right)$$

$$\begin{aligned}
\frac{\partial}{\partial x_2}a_1 &= \gamma x_1\ ; & \frac{\partial}{\partial x_3}a_1 &= -3x_3^2\\
\frac{\partial}{\partial x_1}a_2 &= 2(\gamma-2)x_1\ ; & \frac{\partial}{\partial x_3}a_2 &= 0\\
\frac{\partial}{\partial x_1}a_3 &= (1-\gamma)x_3^2\ ; & \frac{\partial}{\partial x_2}a_3 &= 0
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow\quad \nabla\times\ \mathbf{a}(\mathbf{r}) &= \left(0-0, -3x_3^2-(1-\gamma)x_3^2, 2(\gamma-2)x_1-\gamma x_1\right)\\
&= \left(0, -(4-\gamma)x_3^2, (\gamma-4)x_1\right)
\end{aligned}$$

'curl-free' ($\nabla\times\mathbf{a}=0$) if $\gamma=4$!

$$\begin{aligned}
\nabla\cdot\ \mathbf{a}(\mathbf{r}) &= \sum_{j=1}^{3}\frac{\partial}{\partial x_j}a_j\\
&= \gamma x_2 + 0 + 2(1-\gamma)x_1 x_3\\
&\stackrel{?}{=} 0\\
\Longleftrightarrow\quad 0 &= 2x_1x_3 - \gamma(2x_1x_3 - x_2)\\
\Longleftrightarrow\quad \gamma &= \frac{2x_1x_3}{2x_1x_3 - x_2} \neq \text{const}
\end{aligned}$$

$\Longrightarrow\quad \nabla\cdot\mathbf{a}(\mathbf{r}) = 0$ *can not* be realized with $\gamma = $ const!

Solution 1.5.6

1.

$$\mathbf{b}(\mathbf{r}) = \left(x_2x_3 + 12x_1x_2,\ x_1x_3 - 8x_2x_3^3 + 6x_1^2,\ x_1x_2 - 12x_2^2x_3^2\right)$$

$$\begin{aligned}
(\nabla\times\mathbf{b})_1 &= \frac{\partial}{\partial x_2}b_3 - \frac{\partial}{\partial x_3}b_2 \\
&= x_1 - 24x_2x_3^2 - x_1 + 24x_2x_3^2 = 0 \\
(\nabla\times\mathbf{b})_2 &= \frac{\partial}{\partial x_3}b_1 - \frac{\partial}{\partial x_1}b_3 \\
&= x_2 - x_2 = 0 \\
(\nabla\times\mathbf{b})_3 &= \frac{\partial}{\partial x_1}b_2 - \frac{\partial}{\partial x_2}b_1 \\
&= x_3 - x_3 = 0
\end{aligned}$$

$$\Longrightarrow\ \nabla\times\mathbf{b} = 0 \iff \text{'curl-free'}\ .$$

2.

$$\nabla\varphi(\mathbf{r}) = \mathbf{b}(\mathbf{r})$$

(a)

$$\begin{aligned}
\frac{\partial\varphi}{\partial x_1} &\overset{!}{=} b_1 = x_2x_3 + 12x_1x_2 \\
\Longrightarrow\ \varphi(\mathbf{r}) &= x_1x_2x_3 + 6x_1^2x_2 + f(x_2, x_3)
\end{aligned}$$

(b)

$$\begin{aligned}
&\frac{\partial\varphi}{\partial x_2} \overset{!}{=} b_2 = x_1x_3 - 8x_2x_3^2 + 6x_1^2 \overset{!}{=} x_1x_3 + 6x_1^2 + \frac{\partial f}{\partial x_2} \\
\Longrightarrow\ &\frac{\partial f}{\partial x_2} = -8x_2x_3^2 \Longrightarrow f(x_2, x_3) = g(x_3) - 4x_2^2x_3^2 \\
\Longrightarrow\ &\varphi(\mathbf{r}) = x_1x_2x_3 - 4x_2^2x_3^2 + 6x_1^2x_2 + g(x_3)
\end{aligned}$$

(c)

$$\frac{\partial\varphi}{\partial x_3} \overset{!}{=} b_3 = x_1x_2 - 12x_2^2x_3^2 \overset{!}{=} x_1x_2 - 8x_2^2x_3 + \frac{\partial g}{\partial x_3}$$

$$\Longrightarrow \frac{\partial g}{\partial x_3} = 8x_2^2x_3 - 12x_2^2x_3^2 \Longrightarrow g(x_3) = 4x_2^2x_3^2 - 4x_2^2x_3^3 + c$$

$$\Longrightarrow \varphi(\mathbf{r}) = x_1x_2x_3 - 4x_2^2x_3^2 + 6x_1^2x_2 + 4x_2^2x_3^2 - 4x_2^2x_3^3 + c$$
$$= x_1x_2x_3 + 6x_1^2x_2 - 4x_2^2x_3^3 + c \qquad (c = \text{const}).$$

Test:

$$\frac{\partial\varphi}{\partial x_1} = x_2x_3 + 12x_1x_2 = b_1$$
$$\frac{\partial\varphi}{\partial x_2} = x_1x_3 + 6x_1^2 - 8x_2x_3^3 = b_2$$
$$\frac{\partial\varphi}{\partial x_3} = x_1x_2 - 12x_2^2x_3^2 = b_3 \qquad \text{q.e.d.}$$

Solution 1.5.7

1. The proof is carried out by directly exploiting the definition:

$$\nabla\times[f(r)\mathbf{r}] = \left(\frac{\partial}{\partial x_2}fx_3 - \frac{\partial}{\partial x_3}fx_2, \frac{\partial}{\partial x_3}fx_1 - \frac{\partial}{\partial x_1}fx_3, \frac{\partial}{\partial x_1}fx_2 - \frac{\partial}{\partial x_2}fx_1\right) =$$
$$= \frac{df}{dr}\left(\frac{x_2}{r}x_3 - \frac{x_3}{r}x_2, \frac{x_3}{r}x_1 - \frac{x_1}{r}x_3, \frac{x_1}{r}x_2 - \frac{x_2}{r}x_1\right) = 0 \ .$$

2.

$$\nabla\times(\varphi\mathbf{a}) = \mathbf{e}_1\left(\frac{\partial}{\partial x_2}\varphi a_3 - \frac{\partial}{\partial x_3}\varphi a_2\right) + \mathbf{e}_2\left(\frac{\partial}{\partial x_3}\varphi a_1 - \frac{\partial}{\partial x_1}\varphi a_3\right) +$$
$$+\mathbf{e}_3\left(\frac{\partial}{\partial x_1}\varphi a_2 - \frac{\partial}{\partial x_2}\varphi a_1\right) =$$
$$= \varphi\left[\mathbf{e}_1\left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}\right) + \mathbf{e}_2\left(\frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1}\right) +\right.$$
$$\left.+\mathbf{e}_3\left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right)\right] + \mathbf{e}_1\left(a_3\frac{\partial\varphi}{\partial x_2} - a_2\frac{\partial\varphi}{\partial x_3}\right) +$$
$$+\mathbf{e}_2\left(a_1\frac{\partial\varphi}{\partial x_3} - a_3\frac{\partial\varphi}{\partial x_1}\right) + \mathbf{e}_3\left(a_2\frac{\partial\varphi}{\partial x_1} - a_1\frac{\partial\varphi}{\partial x_2}\right) =$$
$$= \varphi\nabla\times\mathbf{a} + (\nabla\varphi)\times\mathbf{a} \ .$$

3. We verify the relation representatively for the 1-component:

$$\begin{aligned}(\nabla\times(\nabla\times\mathbf{a}))_1 &= \frac{\partial}{\partial x_2}(\nabla\times\mathbf{a})_3 - \frac{\partial}{\partial x_3}(\nabla\times\mathbf{a})_2 = \\ &= \frac{\partial}{\partial x_2}\left(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\right) - \frac{\partial}{\partial x_3}\left(\frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1}\right) = \\ &= -\Delta a_1 + \frac{\partial^2}{\partial x_1^2}a_1 + \frac{\partial^2 a_2}{\partial x_2 \partial x_1} + \frac{\partial^2 a_3}{\partial x_3 \partial x_1} = \\ &= -\Delta a_1 + \frac{\partial}{\partial x_1}\nabla\cdot\mathbf{a}\ .\end{aligned}$$

With the respective analogous calculation for the other components the assertion is proven.

4.

$$\begin{aligned}\left(\nabla\times\left(\frac{1}{2}\boldsymbol{\alpha}\times\mathbf{r}\right)\right)_1 &= \frac{1}{2}\left[\frac{\partial}{\partial x_2}(\boldsymbol{\alpha}\times\mathbf{r})_3 - \frac{\partial}{\partial x_3}(\boldsymbol{\alpha}\times\mathbf{r})_2\right] = \\ &= \frac{1}{2}\frac{\partial}{\partial x_2}\sum_{i,j}\varepsilon_{ij3}\alpha_i x_j - \frac{1}{2}\frac{\partial}{\partial x_3}\sum_{i,j}\varepsilon_{ij2}\alpha_i x_j = \\ &= \frac{1}{2}\sum_i\left(\varepsilon_{i23} - \varepsilon_{i32}\right)\alpha_i = \alpha_1\ .\end{aligned}$$

Analogously the two other components are found. It follows:

$$\nabla\times\left(\frac{1}{2}\boldsymbol{\alpha}\times\mathbf{r}\right) = \boldsymbol{\alpha}\ .$$

Solution 1.5.8

1. $\mathbf{a}(\mathbf{r}), \mathbf{b}(\mathbf{r})$ are vector fields with $\mathbf{r} = (x_1, x_2, x_3)$
   Vector product:

$$\mathbf{a}\times\mathbf{b} = \sum_{ijk}\varepsilon_{ijk}a_i b_j \mathbf{e}_k\ , \qquad \text{where} \qquad \{\mathbf{e}_k\}\ \text{complete and orthonormal}$$

$$\begin{aligned}\frac{\partial}{\partial x_m}(\mathbf{a}\times\mathbf{b}) &= \sum_{ijk}\varepsilon_{ijk}\frac{\partial}{\partial x_m}(a_i b_j)\mathbf{e}_k \\ &= \sum_{ijk}\varepsilon_{ijk}\left(\frac{\partial a_i}{\partial x_m}b_j + a_i\frac{\partial b_j}{\partial x_m}\right)\mathbf{e}_k \\ &= \left(\frac{\partial}{\partial x_m}\mathbf{a}\right)\times\mathbf{b} + \mathbf{a}\times\left(\frac{\partial}{\partial x_m}\mathbf{b}\right)\end{aligned}$$

2. $\varphi_{1,2}(\mathbf{r})$: scalar fields

$$\nabla(\varphi_1 \cdot \varphi_2) = \sum_{j=1}^{3} \mathbf{e}_j \underbrace{\frac{\partial}{\partial x_j}(\varphi_1 \cdot \varphi_2)}_{\frac{\partial \varphi_1}{\partial x_j}\varphi_2 + \varphi_1 \frac{\partial \varphi_2}{\partial x_j}}$$

$$= \varphi_2 \sum_{j=1}^{3} \mathbf{e}_j \frac{\partial \varphi_1}{\partial x_j} + \varphi_1 \sum_{j=1}^{3} \mathbf{e}_j \frac{\partial \varphi_2}{\partial x_j}$$

$$= \varphi_2 \nabla \varphi_1 + \varphi_1 \nabla \varphi_2$$

3. According to (1.195) it holds:

$$\mathbf{a} \times \mathbf{b} = \sum_{ijk} \varepsilon_{ijk} a_i b_j \mathbf{e}_k$$

$$\nabla \times \mathbf{a} = \sum_{ijk} \varepsilon_{ijk} \left(\frac{\partial}{\partial x_i} a_j \right) \mathbf{e}_k$$

Therewith:

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \sum_{k=1}^{3} \frac{\partial}{\partial x_k} (\mathbf{a} \times \mathbf{b})_k$$

$$= \sum_{k} \frac{\partial}{\partial x_k} \sum_{i,j} \varepsilon_{ijk} a_i b_j$$

$$= \sum_{ijk} \varepsilon_{ijk} \left(\frac{\partial a_i}{\partial x_k} b_j + a_i \frac{\partial b_j}{\partial x_k} \right)$$

$$= \sum_{j} b_j \sum_{i,k} \underbrace{\varepsilon_{ijk}}_{=\varepsilon_{kij}} \frac{\partial a_i}{\partial x_k} + \sum_{i} a_i \sum_{j,k} \underbrace{\varepsilon_{ijk}}_{=-\varepsilon_{kji}} \frac{\partial b_j}{\partial x_k}$$

$$= \sum_{j} b_j (\nabla \times \mathbf{a})_j - \sum_{i} a_i (\nabla \times \mathbf{b})_i$$

$$= \mathbf{b} \cdot \nabla \times \mathbf{a} - \mathbf{a} \cdot \nabla \times \mathbf{b}$$

4.

$$\mathbf{d}(\mathbf{r}) = \nabla \varphi_1 \times \nabla \varphi_2$$

With part 3.:

$$\begin{aligned}\nabla \cdot \mathbf{d}(\mathbf{r}) &= \nabla\varphi_2 \cdot \underbrace{(\nabla \times (\nabla\varphi_1))}_{=0 \text{ (see (1.290))}} - \nabla\varphi_1 \cdot \underbrace{(\nabla \times (\nabla\varphi_2))}_{=0} \\ &= 0\end{aligned}$$

Section 1.6

Solution 1.6.1

$$A \cdot B = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 0 & 4 \\ 0 & 0 & 5 \end{pmatrix} \; ; \quad B \cdot A = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 1 & 6 \\ 0 & 0 & 5 \end{pmatrix} \; .$$

Solution 1.6.2

1.

$$(A \cdot B)^T = B^T A^T$$

That appears at least reasonably because $A^T \cdot B^T$ would not be defined for $m \neq r$; $B^T \cdot A^T$, however, is defined.

$$\begin{aligned} C = A \cdot B &= \left(c_{ij}\right) \\ c_{ij} &= \sum_{k=1}^{n} a_{ik} b_{kj} \\ C^T &= \left(c_{ij}^T\right) = \left(c_{ji}\right) \\ c_{ji} &= \sum_{k=1}^{n} a_{jk} b_{ki} = \sum_{k=1}^{n} b_{ik}^T a_{kj}^T = \left(B^T \cdot A^T\right)_{ij} \\ \Longrightarrow \quad (A \cdot B)^T &= B^T A^T \end{aligned}$$

2. For $m = n$ A is a square matrix.
   Obviously it holds $E^T = E$

$$\begin{aligned} E = \left(A^{-1} A\right)^T &\stackrel{1.}{=} A^T \cdot \left(A^{-1}\right)^T \\ &= A^T \cdot \left(A^T\right)^{-1} \\ \Longrightarrow \quad \left(A^{-1}\right)^T &= \left(A^T\right)^{-1} \end{aligned}$$

3. $m = n = r$
   Determine the matrix C so that

$$
\begin{aligned}
& C \cdot (A \cdot B) = E \\
\Longrightarrow\ & C \cdot (A \cdot B) \cdot B^{-1} = C \cdot A = E \cdot B^{-1} = B^{-1} \\
\Longrightarrow\ & C \cdot (A \cdot B) \cdot B^{-1} \cdot A^{-1} = C \cdot A \cdot A^{-1} = C \\
& = B^{-1} \cdot A^{-1} \\
\Longrightarrow\ & C = B^{-1} A^{-1} = (A \cdot B)^{-1}
\end{aligned}
$$

Solution 1.6.3

1. Sarrus-rule:

$$\det A = 0 - 15 + 4 + 0 + 8 - 6 = -9 \,.$$

2. $\det A = 0$, since the fourth row can be written as sum of the first and the second row.
3. An expansion with respect to the third column suggests itself:

$$
\begin{aligned}
\det A &= -8 \ \det \begin{pmatrix} 4 & 3 & 1 \\ 0 & 1 & 7 \\ 3 & -4 & 6 \end{pmatrix} \\
&= -8(24 + 63 - 3 + 112) = -1568 \,.
\end{aligned}
$$

Solution 1.6.4

1. Complete induction:

$$n = 1: \quad \det A = |a_{11}| = a_{11} = \det A^T$$

$$n = 2: \quad \det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\det A^T = \begin{vmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

$$\Longrightarrow \quad \det A = \det A^T$$

$n \ \rightarrow \ n+1 \ :$ Assume A^T to be an $((n+1) \times (n+1))$ matrix.

$$\det A^T = \sum_{j=1}^{n} a_{ij}^T \cdot \underbrace{U_{ij}\left(A^T\right)}_{\substack{\text{determinant of an} \\ (n \times n) \text{ matrix}}}$$

With the induction hypothesis it follows:

$$\det A^T = \sum_{j=1}^{n} a_{ij}^T U_{ji}(A)$$
$$= \sum_{j=1}^{n} a_{ji} U_{ji}(A)$$

That holds for *all* i:

$$\det A^T = \frac{1}{n} \sum_{i,j} a_{ji} U_{ji}(A)$$
$$= \frac{1}{n} \sum_{i,j} a_{ij} U_{ij}(A)$$
$$= \frac{1}{n} \sum_{i} \det A$$
$$= \det A$$

2. B antisymmetric $(n \times n)$ matrix

$$\Longrightarrow \quad B^T = \left(b_{ij}^T\right) \quad \text{with} \quad b_{ij}^T = b_{ji} = -b_{ij}$$
$$\Longrightarrow \quad B^T = -B$$
$$\Longrightarrow \quad \det B^T = (-1)^n \det B \stackrel{1.}{=} \det B$$
$$\Longrightarrow \quad \text{with } n \text{ odd:}$$

$$-\det B = \det B \implies \det B = 0$$

Solution 1.6.5

$$A \cdot A^T = \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix} \cdot \begin{pmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{pmatrix}$$
$$= \begin{pmatrix} (a^2 + b^2 + c^2 + d^2) & & 0 \\ & \ddots & \\ 0 & & (a^2 + b^2 + c^2 + d^2) \end{pmatrix} .$$

The determinant of the product matrix is by direct reading:

$$\det(A \cdot A^T) = (a^2 + b^2 + c^2 + d^2)^4 \ .$$

On the other hand it also holds:

$$\det(A \cdot A^T) = \det A \cdot \det A^T = (\det A)^2 \ .$$

Therewith it follows:

$$\det A = \left(a^2 + b^2 + c^2 + d^2\right)^2 \ .$$

Solution 1.6.6

1. The matrix of coefficients A ,

$$A \equiv \begin{pmatrix} 2 & 1 & 5 \\ 1 & 5 & 2 \\ 5 & 2 & 1 \end{pmatrix} \ ,$$

has a non-vanishing determinant:

$$\det A = -104 \ .$$

The system of equations is therefore uniquely solvable:

$$\det A_1 = \begin{vmatrix} -21 & 1 & 5 \\ 19 & 5 & 2 \\ 2 & 2 & 1 \end{vmatrix} = 104 \ ,$$

$$\det A_2 = \begin{vmatrix} 2 & -21 & 5 \\ 1 & 19 & 2 \\ 5 & 2 & 1 \end{vmatrix} = -624 \ ,$$

$$\det A_3 = \begin{vmatrix} 2 & 1 & -21 \\ 1 & 5 & 19 \\ 5 & 2 & 2 \end{vmatrix} = 520 \ .$$

Hence, according to Cramer's rule the system of equations has the following solutions:

$$x_1 = \frac{104}{-104} = -1 \ ; \quad x_2 = \frac{-624}{-104} = 6 \ ; \quad x_3 = \frac{520}{-104} = -5 \ .$$

2. The second and the third equation are linearly dependent. It thus remains only

$$x_1 - x_2 = 4 - 3x_3 \; ,$$
$$3x_1 + x_2 = -1 + 4x_3$$

with the *'solutions'*:

$$x_1 = \frac{1}{4}(3 + x_3) \; ; \quad x_2 = \frac{13}{4}(x_3 - 1) \; .$$

3. The matrix of coefficients A ,

$$A = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix} \; ,$$

of the homogeneous system of equations possesses a non-vanishing determinant:

$$\det A = 4 \; .$$

Therefore only the trivial solution is possible:

$$x_1 = x_2 = x_3 = 0 \; .$$

4. The determinant of the coefficient matrix is zero:

$$\det A = \begin{vmatrix} 2 & -3 & 1 \\ 4 & 4 & -1 \\ 1 & -\frac{3}{2} & \frac{1}{2} \end{vmatrix} = 0 \; .$$

Non-trivial solutions we can get as follows:

$$2x_1 - 3x_2 = -x_3 \; ,$$
$$4x_1 + 4x_2 = x_3 \; .$$

$$\det A' = 20 \; ; \quad \det A_1 = -x_3 \; ; \quad \det A_2 = 6x_3$$
$$\Longrightarrow x_1 = -\frac{1}{20}x_3 \; ; \quad x_2 = \frac{3}{10}x_3 \; .$$

Solution 1.6.7

1. A represents a rotation since
   (a) rows and columns are orthonormalized,
   (b) $\det A = 1$.

   It represents a rotation around the 2-axis by the angle $\varphi = 135°$ (Fig. A.15):

$$A = \begin{pmatrix} \cos\varphi & 0 & -\sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{pmatrix} .$$

2.

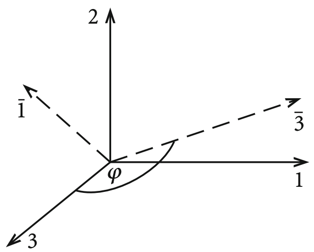
$$\begin{pmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \bar{a}_3 \end{pmatrix} = A \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\sqrt{2} \\ -2 \\ -\frac{1}{2}\sqrt{2} \end{pmatrix} ,$$

$$\begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \bar{b}_3 \end{pmatrix} = A \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\sqrt{2} \\ 5 \\ \frac{7}{2}\sqrt{2} \end{pmatrix} .$$

The scalar product does not change with the rotation:

$$\mathbf{a} \cdot \mathbf{b} = \bar{\mathbf{a}} \cdot \bar{\mathbf{b}} = -14 .$$

Fig. A.15



Solution 1.6.8

1.

$$AB = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \frac{1}{2} & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & -2 \\ 0 & \frac{1}{2}\sqrt{2} & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

$$BA = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \frac{1}{2} & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & -2 \\ 0 & \frac{1}{2}\sqrt{2} & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

$$\Longrightarrow \quad AB = BA$$

2.

$$\det A = \left(\frac{1}{\sqrt{2}}\right)^3 \begin{vmatrix} -1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & -1 \end{vmatrix}$$

$$= \left(\frac{1}{\sqrt{2}}\right)^3 \left(\sqrt{2} + \sqrt{2}\right) = +1$$

$$\det B = \begin{vmatrix} 1 & 0 & 1 \\ 0 & \frac{1}{2} & 0 \\ -1 & 0 & 1 \end{vmatrix} = \frac{1}{2} + \frac{1}{2} = +1$$

$$\det(A \cdot B) = \det A \cdot \det B = \det(B \cdot A) = +1$$

3. Conditions for a rotation matrix:

   (a) rows and columns orthonormal
   (b) $\det D = +1$

   $\det A = \det B = +1$

   A: rows and columns orthonormal $\Longrightarrow$ A is rotation matrix
   B: rows and columns orthogonal but **not** normalized $\Longrightarrow$ B is **not** a rotation matrix

4. A rotation matrix

$$\Longrightarrow \; A^{-1} = A^T$$

$$\Longrightarrow \; A^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & -1 \end{pmatrix}$$

Solution 1.6.9

1.

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \; ; \quad \bar{\mathbf{a}} = \begin{pmatrix} \bar{a}_1 \\ \bar{a}_2 \\ \bar{a}_3 \end{pmatrix} \; .$$

$\bar{\mathbf{a}}$ may be a vector which is related to $\mathbf{a}$ by rotation. Then it holds (1.310):

$$\bar{a}_i = \sum_{j=1}^{3} d_{ij} a_j \; .$$

d_{ij} are the elements of the rotation matrix:

$$\sum_i \bar{a}_i^2 = \sum_i \sum_{j,k} d_{ij} d_{ik} a_j a_k = \sum_{j,k} \Big(\sum_i d_{ij} d_{ik} \Big) a_j a_k \; .$$

Since the columns of the rotation matrix are orthonormalized the bracket is just δ_{jk}:

$$\sum_i \bar{a}_i^2 = \sum_{j,k} \delta_{jk} a_j a_k = \sum_j a_j^2 \qquad \text{q. e. d.}$$

2. $\Sigma, \overline{\Sigma}$ shall be two right-handed systems emerging from each other by a rotation:

$$\begin{aligned} \mathbf{e}_i &= \big(\mathbf{e}_j \times \mathbf{e}_k \big) \; , \\ \bar{\mathbf{e}}_i &= \big(\bar{\mathbf{e}}_j \times \bar{\mathbf{e}}_k \big) \; ; \quad (i,j,k) = (1,2,3) \text{ and cyclic} \; . \end{aligned}$$

It holds the mapping:

$$\bar{\mathbf{e}}_m = \sum_l d_{ml} \mathbf{e}_l \; .$$

This we insert into the above scalar triple product of the unit vectors:

$$\sum_l d_{il}\mathbf{e}_l = \sum_{m,n} d_{jm}d_{kn}(\mathbf{e}_m \times \mathbf{e}_n)\ .$$

We multiply this equation scalarly by $\mathbf{e}_r$:

$$d_{ir} = \sum_{m,n} \varepsilon_{rmn} d_{jm} d_{kn}\ .$$

Evaluation for $i = 1$:

$$\begin{aligned}
d_{1r} &= \sum_{m,n} \varepsilon_{rmn} d_{2m} d_{3n} \\
\Longrightarrow d_{11} &= d_{22}d_{33} - d_{23}d_{32} = \begin{vmatrix} d_{22} & d_{23} \\ d_{32} & d_{33} \end{vmatrix} = A_{11}\ , \\
d_{12} &= d_{23}d_{31} - d_{21}d_{33} = -\begin{vmatrix} d_{21} & d_{23} \\ d_{31} & d_{33} \end{vmatrix} = -A_{12}\ , \\
d_{13} &= d_{21}d_{32} - d_{22}d_{31} = \begin{vmatrix} d_{21} & d_{22} \\ d_{31} & d_{32} \end{vmatrix} = A_{13}\ .
\end{aligned}$$

That means on the whole:

$$d_{1r} = (-1)^{1+r} A_{1r} = U_{1r}\ .$$

Analogously one verifies:

$$\begin{aligned}
d_{2r} &= (-1)^{2+r} A_{2r} = U_{2r}\ , \\
d_{3r} &= (-1)^{3+r} A_{3r} = U_{3r}\ .
\end{aligned}$$

The proof can of course be carried out in a very much shorter way: Because of (1.315), (1.338), and (1.344) it holds:

$$d_{ij} = \left(d^{-1}\right)_{ji} = \frac{U_{ij}}{\det D} = U_{ij}\ .$$

Solution 1.6.10 With (1.315) we can use:

$$D_1^{-1} = D_1^T\ ; \quad D_2^{-1} = D_2^T$$

Exercise 1.6.2, part 1.:

$$(AB)^T = B^T A^T\ ; \quad (AB)^{-1} = B^{-1}A^{-1}$$

This means:

$$D^{-1} = (D_1 \cdot D_2)^{-1} = D_2^{-1} D_1^{-1} = D_2^T D_1^T$$
$$= (D_1 \cdot D_2)^T = D^T$$

Furthermore:

$$\det D = \det(D_1 D_2) = \det D_1 \ \det D_2$$
$$= (+1) \cdot (+1)$$
$$= 1$$

$\Longrightarrow$ D is a rotation matrix
$\Longrightarrow$ rows and columns are orthonormal.

Section 1.7

Exercise 1.7.1

1.

$$\frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} = \begin{vmatrix} \left(\frac{\partial x_1}{\partial y_1}\right)_{y_2} & \left(\frac{\partial x_1}{\partial y_2}\right)_{y_1} \\ \left(\frac{\partial x_2}{\partial y_1}\right)_{y_2} & \left(\frac{\partial x_2}{\partial y_2}\right)_{y_1} \end{vmatrix} .$$

One recognizes immediately:

$$\frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} \overset{(\alpha)}{=} -\frac{\partial(x_1, x_2)}{\partial(y_2, y_1)} \overset{(\beta)}{=} \frac{\partial(x_2, x_1)}{\partial(y_2, y_1)} ,$$

(α): interchange of two columns of the Jacobian determinant, (β): subsequent interchange of two rows of the Jacobian determinant.

2. The first example concerns the identical transformation:

$$(x_1, x_2) \Longrightarrow (x_1, x_2) .$$

$$\frac{\partial(x_1, x_2)}{\partial(x_1, x_2)} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 .$$

The second example concerns the transformation:

$$x_1 = x_1(y_1, y_2) \; ; \quad x_2 = y_2 \; .$$

$$\frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} = \frac{\partial(x_1, y_2)}{\partial(y_1, y_2)} = \begin{vmatrix} \left(\frac{\partial x_1}{\partial y_1}\right)_{y_2} & \left(\frac{\partial x_1}{\partial y_2}\right)_{y_1} \\ 0 & 1 \end{vmatrix}$$

$$\Longrightarrow \frac{\partial(x_1, y_2)}{\partial(y_1, y_2)} = \left(\frac{\partial x_1}{\partial y_1}\right)_{y_2} \; .$$

Solution 1.7.2 With (1.366) it firstly holds:

$$\frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} = \left[\frac{\partial(y_1, y_2)}{\partial(x_1, x_2)}\right]^{-1} \; .$$

According to Exercise 1.7.1 this has the special consequence:

$$\left(\frac{\partial x}{\partial y}\right)_z = \frac{\partial(x, z)}{\partial(y, z)} = \left[\frac{\partial(y, z)}{\partial(x, z)}\right]^{-1} = \left[\left(\frac{\partial y}{\partial x}\right)_z\right]^{-1} \; .$$

For the second part of the exercise we exploit (1.365):

$$\frac{\partial(x, z)}{\partial(y, z)} \cdot \frac{\partial(y, z)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(x, z)} = 1 \; .$$

That agrees with

$$\left(\frac{\partial x}{\partial y}\right)_z \cdot \left[-\left(\frac{\partial z}{\partial x}\right)_y\right] \cdot \left(\frac{\partial y}{\partial z}\right)_x = 1 \; ,$$

and therewith directly follows the assertion!

Solution 1.7.3

1.

$$\frac{\partial(x_1, x_2, x_3)}{\partial(u, v, z)} = \begin{vmatrix} u & -v & 0 \\ v & u & 0 \\ 0 & 0 & 1 \end{vmatrix} = u^2 + v^2 \; .$$

Thus the transformation is everywhere locally reversible except for $(u = 0, \; v = 0, \; z)$.

2.

$$dV = dx_1 dx_2 dx_3 = \frac{\partial(x_1, x_2, x_3)}{\partial(u, v, z)} du\, dv\, dz$$

$$\Longrightarrow dV = (u^2 + v^2) du\, dv\, dz\ .$$

3. Position vector:

$$\mathbf{r} = \left(\frac{1}{2}(u^2 - v^2), uv, z\right)\ .$$

$$\frac{\partial \mathbf{r}}{\partial u} = (u, v, 0) \Longrightarrow b_u = \sqrt{u^2 + v^2}\ ,$$

$$\frac{\partial \mathbf{r}}{\partial v} = (-v, u, 0) \Longrightarrow b_v = b_u\ ,$$

$$\frac{\partial \mathbf{r}}{\partial z} = (0, 0, 1) \Longrightarrow b_z = 1\ .$$

Therewith we get the following curvilinear-orthogonal unit vectors:

$$\mathbf{e}_u = \frac{1}{\sqrt{u^2 + v^2}}(u, v, 0)\ ,$$

$$\mathbf{e}_v = \frac{1}{\sqrt{u^2 + v^2}}(-v, u, 0)\ ,$$

$$\mathbf{e}_z = (0, 0, 1)\ .$$

u-coordinate lines: $x_1 = \frac{1}{2v^2}x_2^2 - \frac{1}{2}v^2$ ($v = \text{const}$) (parabola about the x_1 axis),

v-coordinate lines: $x_1 = -\frac{1}{2u^2}x_2^2 + \frac{1}{2}u^2$ ($u = \text{const}$) (parabola about the negative x_1 axis),

z-coordinate lines: parallels to the x_3 axis.

u- and v-coordinate lines intersect at right angles (Fig. A.16).

4. For the differential of the position vector equation (1.373) holds. That yields with the above results:

$$d\mathbf{r} = \sqrt{u^2 + v^2}\, du\, \mathbf{e}_u + \sqrt{u^2 + v^2}\, dv\, \mathbf{e}_v + dz\, \mathbf{e}_z\ .$$

To get the nabla-operator we apply the general relation (1.376):

$$\nabla = \mathbf{e}_u \frac{1}{\sqrt{u^2 + v^2}} \frac{\partial}{\partial u} + \mathbf{e}_v \frac{1}{\sqrt{u^2 + v^2}} \frac{\partial}{\partial v} + \mathbf{e}_z \frac{\partial}{\partial z}\ .$$

Fig. A.16

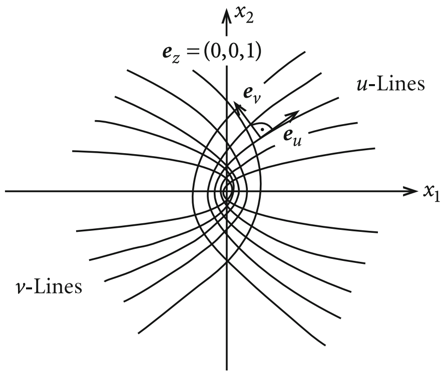
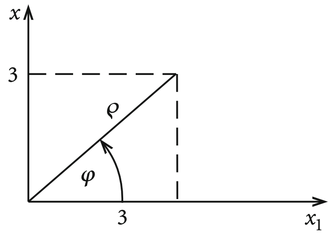


Fig. A.17



Solution 1.7.4 See Fig. A.17

$$\tan\varphi = 1 \Longrightarrow \varphi = \frac{\pi}{4}\ ; \quad \rho = 3\sqrt{2}\ .$$

Solution 1.7.5

1. Spherical coordinates:

   $P_1\ :\ (1,0,1)$

$$\begin{aligned} r\sin\vartheta\cos\varphi &= 1 \\ r\sin\vartheta\sin\varphi &= 0 \\ r\cos\vartheta &= 1 \end{aligned}$$

$$\begin{aligned} \Longrightarrow\ r = \sqrt{2}\,; \quad \cos\vartheta &= \frac{1}{\sqrt{2}} \quad \Longrightarrow\ \vartheta = \frac{\pi}{4} \\ \cos\varphi &= 1 \quad \Longrightarrow\ \varphi = 0 \\ &\Longrightarrow\ P_1\ :\quad (\sqrt{2}, \frac{\pi}{4}, 0) \end{aligned}$$

$P_2\ :\ (0, 1, -1)$

$$\Longrightarrow\ r = \sqrt{2}\,; \cos\vartheta = -\frac{1}{\sqrt{2}} \quad \Longrightarrow\ \vartheta = \frac{3\pi}{4}$$

$$\Longrightarrow\ \varphi = \frac{\pi}{2}$$

$$\Longrightarrow\ P_2\ :\ \left(\sqrt{2}, \frac{3\pi}{4}, \frac{\pi}{2}\right)$$

$P_3\ :\ (0, -3, 0)$

$$\Longrightarrow\ r = 3\,; \cos\vartheta = 0 \quad \Longrightarrow\ \vartheta = \frac{\pi}{2}$$

$$\sin\vartheta\sin\varphi = -1 \quad \Longrightarrow\ \varphi = \frac{3\pi}{2}$$

$$\Longrightarrow\ P_3\ :\ \left(3, \frac{\pi}{2}, \frac{3\pi}{2}\right)$$

2. Cylindrical coordinates:

$P_1\ :\ (1, 0, 1)$

$$\begin{aligned}\rho\cos\varphi &= 1\\ \rho\sin\varphi &= 0\\ z &= 1 \ \Longrightarrow\ \rho = 1\,; \quad \varphi = 0\,; \quad z = 1\end{aligned}$$

$$\Longrightarrow\ P_1 \ :\ (1, 0, 1)$$

$P_2\ :\ (0, 1, -1)$

$$\Longrightarrow\ \rho = 1\,; \qquad \varphi = \frac{\pi}{2}\,; \quad z = -1$$

$$\Longrightarrow\ P_2\ :\ \left(1, \frac{\pi}{2}, -1\right)$$

$P_3\ :\ (0, -3, 0)$

$$\Longrightarrow\ \rho = 3\,; \quad z = 0\,; \qquad \cos\varphi = 0$$

$$\sin\varphi = -1\ \Longrightarrow\ \varphi = \frac{3\pi}{2}$$

$$\Longrightarrow\ P_3\ :\ \left(3, \frac{3\pi}{2}, 0\right)$$

Solution 1.7.6 Cartesian coordinates: $R^2 = x_1^2 + x_2^2$, planar polar coordinates: $R = \rho$.

Solution 1.7.7

1. Vector field in cylindrical coordinates:

$$\mathbf{a} = a_\rho \mathbf{e}_\rho + a_\varphi \mathbf{e}_\varphi + a_z \mathbf{e}_z \ .$$

We have to determine a_ρ, a_φ, a_z! The unit vectors are:

$$\begin{aligned} \mathbf{e}_\rho &= \cos\varphi\, \mathbf{e}_1 + \sin\varphi\, \mathbf{e}_2 \ , \\ \mathbf{e}_\varphi &= -\sin\varphi\, \mathbf{e}_1 + \cos\varphi\, \mathbf{e}_2 \ , \\ \mathbf{e}_z &= \mathbf{e}_3 \ . \end{aligned}$$

The reversal reads:

$$\begin{aligned} \mathbf{e}_1 &= \cos\varphi\, \mathbf{e}_\rho - \sin\varphi\, \mathbf{e}_\varphi \ , \\ \mathbf{e}_2 &= \sin\varphi\, \mathbf{e}_\rho + \cos\varphi\, \mathbf{e}_\varphi \ , \\ \mathbf{e}_3 &= \mathbf{e}_z \ . \end{aligned}$$

With the transformation formulae

$$x_1 = \rho\cos\varphi \ ; \quad x_2 = \rho\sin\varphi \ ; \quad x_3 = z$$

we then obtain by insertion:

$$\mathbf{a} = z(\cos\varphi\, \mathbf{e}_\rho - \sin\varphi\, \mathbf{e}_\varphi) + 2\rho\cos\varphi(\sin\varphi\, \mathbf{e}_\rho + \cos\varphi\, \mathbf{e}_\varphi) + \rho\sin\varphi\, \mathbf{e}_z \ .$$

Finally it follows by comparison:

$$\begin{aligned} a_\rho &= z\cos\varphi + 2\rho\sin\varphi\cos\varphi \ , \\ a_\varphi &= -z\sin\varphi + 2\rho\cos^2\varphi \ , \\ a_z &= \rho\sin\varphi \ . \end{aligned}$$

2. Vector field in spherical coordinates:

$$\mathbf{a} = a_r\, \mathbf{e}_r + a_\vartheta \mathbf{e}_\vartheta + a_\varphi\, \mathbf{e}_\varphi \ .$$

With

$$x_1 = r\sin\vartheta\cos\varphi \ ; \quad x_2 = r\sin\vartheta\sin\varphi \ ; \quad x_3 = r\cos\vartheta$$

and

$$\begin{aligned}
\mathbf{e}_1 &= \cos\varphi\sin\vartheta\,\mathbf{e}_r + \cos\varphi\cos\vartheta\,\mathbf{e}_\vartheta - \sin\varphi\,\mathbf{e}_\varphi\ ,\\
\mathbf{e}_2 &= \sin\varphi\sin\vartheta\,\mathbf{e}_r + \sin\varphi\cos\vartheta\,\mathbf{e}_\vartheta + \cos\varphi\,\mathbf{e}_\varphi\ ,\\
\mathbf{e}_3 &= \cos\vartheta\,\mathbf{e}_r - \sin\vartheta\,\mathbf{e}_\vartheta
\end{aligned}$$

we now have:

$$\begin{aligned}
a_r &= \cos\varphi + r\sin\vartheta\cos\vartheta\sin\varphi\ ,\\
a_\vartheta &= r\cos\varphi\cos^2\vartheta + 2r\sin\vartheta\cos\vartheta\sin\varphi\cos\varphi - r\sin^2\vartheta\sin\varphi\ ,\\
a_\varphi &= -r\cos\vartheta\sin\varphi + 2r\sin\vartheta\cos^2\varphi\ .
\end{aligned}$$

Solution 1.7.8

1. (a) When using Cartesian coordinates for the calculation of the circular area S we have to take the condition $x^2 + y^2 = R^2$ into account for fixing the integration limits:

$$\begin{aligned}
S &= \int_{-R}^{+R} dy \int_{-\sqrt{R^2-y^2}}^{+\sqrt{R^2-y^2}} dx = 2\int_{-R}^{+R} dy\sqrt{R^2-y^2}\\
&= 2\left(\frac{R^2}{2}\arcsin\frac{y}{R} + \frac{y}{2}\sqrt{R^2-y^2}\right)\Bigg|_{-R}^{+R} = 2\frac{1}{2}R^2\pi\\
&= \pi R^2\ .
\end{aligned}$$

(b) With planar polar coordinates the surface element reads:

$$dx\,dy = \frac{\partial(x,y)}{\partial(\rho,\varphi)}\,d\rho d\varphi = \rho\,d\rho d\varphi\ .$$

Therewith it follows immediately:

$$S = \int_0^R\int_0^{2\pi}\rho d\rho d\varphi = 2\pi\int_0^R \rho d\rho = \pi R^2\ .$$

The application of plane polar coordinates obviously means a substantial advantage.

2. It is clear that for the calculation of the volume of a sphere spherical coordinates (r,ϑ,φ) are highly recommendable:

$$dV = \frac{\partial(x,y,z)}{\partial(r,\vartheta,\varphi)} = r^2\sin\vartheta\,drd\vartheta d\varphi\ .$$

So it is to evaluate:

$$\begin{aligned}
V &= \int dV = \int_0^R \int_0^\pi \int_0^{2\pi} r^2 \sin\vartheta \, dr d\vartheta \, d\varphi \\
&= 2\pi \int_0^R r^2 dr \, (-\cos\vartheta)\Big|_0^\pi = 4\pi \, \frac{r^3}{3}\Big|_0^R \\
&= \frac{4\pi}{3} R^3 \; .
\end{aligned}$$

3. One uses conveniently cylindrical coordinates (ρ, φ, z) for the calculation of the multiple integral. For the volume element it holds according to (1.382):

$$\begin{aligned}
V &= \int_{R_1}^{R_2} \rho d\rho \int_0^{\frac{\pi}{2}} d\varphi \int_0^{z_0} dz = z_0 \int_{R_1}^{R_2} \rho d\rho \int_0^{\frac{\pi}{2}} d\varphi \\
&= \frac{1}{2}\pi z_0 \int_{R_1}^{R_2} \rho d\rho = \frac{1}{2}\pi z_0 \, \frac{1}{2}\rho^2\Big|_{R_1}^{R_2} \\
&= \frac{1}{4}\pi z_0 \left(R_2^2 - R_1^2\right) \; .
\end{aligned}$$

Section 2.1

Solution 2.1.1

1. The velocity magnitude v does not change so that it holds with the cosine rule (1.149) (Figs. A.18 and A.19):

$$\begin{aligned}
\Delta v &= \sqrt{v^2 + v^2 - 2v^2 \cos 60°} \\
\Longrightarrow v &= 50 \, \mathrm{cm\, s^{-1}} \; .
\end{aligned}$$

Fig. A.18

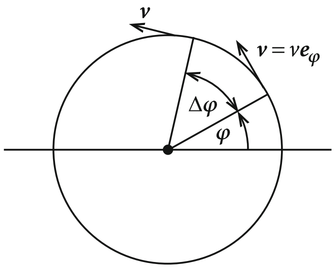
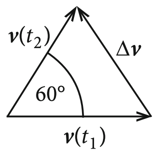


Fig. A.19



2. For the centripetal acceleration we need according to (2.36):

$$\mathbf{a}_r = -R\omega^2 \mathbf{e}_r \; ,$$

$$\omega = \frac{2\pi \frac{60}{360}}{2\,s} = \frac{\pi}{6}\ \mathrm{s}^{-1} \; .$$

From $v = R\,\omega$ it follows then:

$$R = \frac{300}{\pi}\,\mathrm{cm}$$

and therewith:

$$|\mathbf{a}_r| = R\,\omega^2 = \pi \cdot \frac{50}{6}\,\mathrm{cm\,s}^{-2} \; .$$

Solution 2.1.2

1.

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}_p = (2, 3, -4) \; .$$

2. $\boldsymbol{\omega}$ remains unchanged.

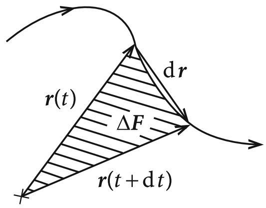
$$\begin{aligned} \mathbf{v}' &= \left[\boldsymbol{\omega} \times (\mathbf{r}_p - \mathbf{a})\right] = [(-1, 2, 1) \times (1, -1, 0)] \\ \Longrightarrow \mathbf{v}' &= (1, 1, -1) \; . \end{aligned}$$

Solution 2.1.3

1.

$$\begin{aligned} \ddot{\mathbf{r}}(t) &= -\mathbf{g} = -(0, 0, g) \; , \\ \dot{\mathbf{r}}(t) &= -\mathbf{g}\,t + \mathbf{v}_0 \quad [\dot{\mathbf{r}}(t=0) = \mathbf{v}_0] \; , \\ \mathbf{r}(t) &= -\frac{1}{2}\mathbf{g}\,t^2 + \mathbf{v}_0 t \quad [\mathbf{r}(t=0) = \mathbf{0}] \; . \end{aligned}$$

Fig. A.20

2. 

2. The *'orbital plane'* is the plane which is spanned by the vectors $\mathbf{r}$ and $\dot{\mathbf{r}}$. So it holds for $\Delta\mathbf{F}$ (Fig. A.20):

$$\Delta\mathbf{F} = \frac{1}{2}(\mathbf{r} \times \dot{\mathbf{r}})dt \, .$$

With 1. we can calculate the vector product $\mathbf{r} \times \dot{\mathbf{r}}$:

$$\mathbf{r} \times \dot{\mathbf{r}} = \left(-\frac{1}{2}t^2\mathbf{g} + t\,\mathbf{v}_0\right) \times (-t\,\mathbf{g} + \mathbf{v}_0) =$$

$$= -\frac{1}{2}t^2(\mathbf{g} \times \mathbf{v}_0) - t^2(\mathbf{v}_0 \times \mathbf{g}) = \frac{1}{2}t^2(\mathbf{g} \times \mathbf{v}_0) \, .$$

One recognizes that, although the vector product $\mathbf{r} \times \dot{\mathbf{r}}$ is time-dependent, its direction is fixed. The surface normal is always parallel to $(\mathbf{g} \times \mathbf{v}_0)$.

3.

$$\mathbf{e}_1' = \frac{1}{v_0}(v_{01}, v_{02}, v_{03}) \, .$$

For the unit vector $\mathbf{e}_2'$ three conditions are to be fulfilled:

(a) $\mathbf{e}_2'$ lies in the orbital plane: $\mathbf{e}_2' \perp \mathbf{g} \times \mathbf{v}_0$,
(b) $\mathbf{e}_2'$ is orthogonal to $\mathbf{e}_1'$: $\mathbf{e}_2' \cdot \mathbf{e}_1' = 0$,
(c) $\mathbf{e}_2'$ is normalized: $\mathbf{e}_2' \cdot \mathbf{e}_2' = 1$.

With

$$\mathbf{e}_2' = (x_1, x_2, x_3)$$

and

$$\mathbf{g} \times \mathbf{v}_0 = (-v_{02}g,\ v_{01}g,\ 0)$$

condition (a) leads to the conditional equation:

$$g\,(x_1 v_{02} - x_2 v_{01}) = 0 \, .$$

From condition (b) it follows

$$\frac{1}{v_0}\left(x_1 v_{01} + x_2 v_{02} + x_3 v_{03}\right) = 0 \; ,$$

while condition (c) means:

$$x_1^2 + x_2^2 + x_3^2 = 1$$

These are three equations for the unknowns x_1, x_2, x_3:

$$\mathbf{e}_2' = \frac{\pm 1}{v_0\sqrt{v_{01}^2 + v_{02}^2}}\left(-v_{01}v_{03},\; -v_{02}v_{03},\; v_{01}^2 + v_{02}^2\right) \; .$$

The sign remains free.

4.

$$\mathbf{e}_3' = \mathbf{e}_1' \times \mathbf{e}_2' = \frac{\pm 1}{\sqrt{v_{01}^2 + v_{02}^2}}\left(v_{02} \; , \; -v_{01} \; , \; 0\right) \; .$$

Solution 2.1.4

1. Spherical coordinates.

$$\begin{aligned}
\mathbf{r}(t) &= r\mathbf{e}_r \\
\dot{\mathbf{r}}(t) &= \dot{r}\mathbf{e}_r + r\dot{\vartheta}\mathbf{e}_\vartheta + r\sin\vartheta\,\dot{\varphi}\mathbf{e}_\varphi \\
\ddot{\mathbf{r}}(t) &= a_r\mathbf{e}_r + a_\vartheta\mathbf{e}_\vartheta + a_\varphi\mathbf{e}_\varphi
\end{aligned}$$

with

$$\begin{aligned}
a_r &= \ddot{r} - r\dot{\vartheta}^2 - r\sin^2\vartheta\,\dot{\varphi}^2 \\
a_\vartheta &= r\ddot{\vartheta} + 2\dot{r}\dot{\vartheta} - r\sin\vartheta\,\cos\vartheta\,\dot{\varphi}^2 \\
a_\varphi &= r\sin\vartheta\,\ddot{\varphi} + 2\sin\vartheta\,\dot{r}\dot{\varphi} + 2r\cos\vartheta\,\dot{\vartheta}\dot{\varphi} \; .
\end{aligned}$$

The given acceleration,

$$\begin{aligned}
\mathbf{a}(t) &= -\frac{\gamma}{r^2}\mathbf{e}_r - \alpha(r)\dot{\mathbf{r}} \\
&= \left(-\frac{\gamma}{r^2} - \alpha(r)\dot{r}\right)\mathbf{e}_r + \left(-\alpha(r)r\dot{\vartheta}\right)\mathbf{e}_\vartheta + \left(-\alpha(r)r\sin\vartheta\,\dot{\varphi}\right)\mathbf{e}_\varphi \; ,
\end{aligned}$$

leads to the conditional equations:

$$\begin{aligned}
\ddot{r} - r\dot{\vartheta}^2 - r\sin^2\vartheta\,\dot{\varphi}^2 &\stackrel{!}{=} -\frac{\gamma}{r^2} - \alpha(r)\dot{r} \\
r\ddot{\vartheta} + 2\dot{r}\dot{\vartheta} - r\sin\vartheta\,\cos\vartheta\,\dot{\varphi}^2 &\stackrel{!}{=} -\alpha(r)r\dot{\vartheta} \\
r\sin\vartheta\,\ddot{\varphi} + 2\sin\vartheta\,\dot{r}\dot{\varphi} + 2r\cos\vartheta\,\dot{\vartheta}\dot{\varphi} &\stackrel{!}{=} -\alpha(r)r\sin\vartheta\,\dot{\varphi}\ .
\end{aligned}$$

2. The third equation is trivially fulfilled ($\varphi = \text{const}$), the two others can be written as:

$$\begin{aligned}
\ddot{r} - r\dot{\vartheta}^2 &\stackrel{!}{=} -\frac{\gamma}{r^2} - \alpha(r)\dot{r} \\
r\ddot{\vartheta} + 2\dot{r}\dot{\vartheta} &\stackrel{!}{=} -\alpha(r)r\dot{\vartheta}\ .
\end{aligned}$$

The input requirement of the exercise yields:

$$\begin{aligned}
\dot{r} &= -\frac{2\beta}{3}r_0(1-\beta t)^{-1/3} = -\frac{2\beta}{3}r_0\sqrt{\frac{r_0}{r}} \\
\ddot{r} &= -\frac{2\beta^2}{9}r_0(1-\beta t)^{-4/3} = -\frac{2\beta^2}{9}\frac{r_0^3}{r^2} \\
\dot{\vartheta} &= \frac{2}{3}\vartheta_0\beta\frac{1}{1-\beta t} = \frac{2\beta}{3}\vartheta_0\left(\frac{r_0}{r}\right)^{3/2} \\
\ddot{\vartheta} &= \frac{2}{3}\vartheta_0\beta^2\frac{1}{(1-\beta t)^2} = \frac{2\beta^2}{3}\vartheta_0\left(\frac{r_0}{r}\right)^3\ .
\end{aligned}$$

Insertion into the above conditional equations leads to:

(i)

$$\begin{aligned}
-\frac{2\beta^2}{9}\frac{r_0^3}{r^2} - r\frac{4\beta^2}{9}\vartheta_0^2\left(\frac{r_0}{r}\right)^3 &= -\frac{\gamma}{r^2} + \alpha(r)\frac{2\beta}{3}r_0\sqrt{\frac{r_0}{r}} \\
\Longleftrightarrow\ -\frac{2\beta^2}{9}\frac{r_0^2}{r^2}(r_0 + 2\vartheta_0^2 r_0) &= -\frac{\gamma}{r^2} + \alpha(r)\frac{2\beta}{3}r_0\sqrt{\frac{r_0}{r}}
\end{aligned}$$

(ii)

$$
\begin{aligned}
&\frac{2\beta^2}{3}\vartheta_0 r\left(\frac{r_0}{r}\right)^3+2\left(-\frac{2\beta}{3}r_0\sqrt{\frac{r_0}{r}}\right)\frac{2\beta}{3}\vartheta_0\left(\frac{r_0}{r}\right)^{3/2}=-\alpha(r)r\frac{2\beta}{3}\vartheta_0\left(\frac{r_0}{r}\right)^{3/2}\\
&\iff \beta r\left(\frac{r_0}{r}\right)^{3/2}-\frac{4\beta}{3}r_0\sqrt{\frac{r_0}{r}}=-\frac{\beta}{3}r\left(\frac{r_0}{r}\right)^{3/2}=-\alpha(r)r\\
&\qquad\qquad\qquad\qquad\qquad\Longrightarrow\ \alpha(r)=\frac{\beta}{3}\left(\frac{r_0}{r}\right)^{3/2}\ .
\end{aligned}
$$

Because of $\alpha > 0$ it follows immediately $\beta > 0$. Insertion of α into (i):

$$
\begin{aligned}
-\frac{2\beta^2}{9}\frac{r_0^2}{r^2}\left(r_0+2\vartheta_0^2 r_0\right)&=-\frac{\gamma}{r^2}+\frac{2\beta^2}{9}r_0\left(\frac{r_0}{r}\right)^2\\
\Longrightarrow\ \beta^2&=\frac{9}{4}\frac{1}{r_0^3}\gamma\frac{1}{1+\vartheta_0^2} \qquad (*)\\
\Longrightarrow\ \beta&=\frac{3}{2}\frac{1}{r_0}\sqrt{\frac{\gamma}{r_0\left(1+\vartheta_0^2\right)}}\ .
\end{aligned}
$$

Trajectory:

$$
\begin{aligned}
\vartheta(t)=-\vartheta_0\ln(1-\beta t)^{2/3}&=-\vartheta_0\ln\left(\frac{r}{r_0}\right)\\
\Longrightarrow\ r(\vartheta)&=r_0\,e^{-\vartheta/\vartheta_0}
\end{aligned}
$$

3. Square of velocity

$$
\begin{aligned}
\mathbf{v}^2&=\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}=\dot{r}^2+r^2\dot{\vartheta}^2\\
&=\frac{4\beta^2}{9}r_0^2\frac{r_0}{r}+r^2\frac{4\beta^2}{9}\vartheta_0^2\left(\frac{r_0}{r}\right)^3=\frac{4\beta^2}{9}\frac{r_0^3}{r}\underbrace{\left(1+\vartheta_0^2\right)}_{\text{from }(*)}=\frac{\gamma}{r}\ .
\end{aligned}
$$

In spite of friction the velocity of the satellite increases when it approaches the earth's surface $\Longrightarrow$ ansatz for the friction may not be realistic enough $\Longrightarrow$ better (?): $F_R \sim \mathbf{v}^2$.

Section 2.2

Solution 2.2.1

1. According to Fig. A.21:

$$\mathbf{r}_0(t) = \mathbf{r} - \bar{\mathbf{r}} =$$
$$= (-7\alpha_2 t, -11\alpha_5, 3\alpha_4 - 4\alpha_6 t) \; .$$

   Relative velocity:

$$\dot{\mathbf{r}}_0(t) = (-7\alpha_2, 0, -4\alpha_6) equiv \, \dot{\mathbf{r}}_0 \; .$$

2.

$$\ddot{\mathbf{r}}(t) = (12\alpha_1, -18\alpha_3 t, 0) \; ,$$
$$\ddot{\bar{\mathbf{r}}}(t) = (12\alpha_1, -18\alpha_3 t, 0) \; .$$

3. If Σ is an inertial system then $\overline{\Sigma}$ is also an inertial system, because $\ddot{\mathbf{r}}_0 = 0$ and $\ddot{\mathbf{r}} = \ddot{\bar{\mathbf{r}}}$, respectively. If a force-less body moves uniformly in a straight line in Σ then this is also the case in $\overline{\Sigma}$.

Solution 2.2.2 For the actual velocity $v = \text{const}$ it holds:

$$v = \frac{dx}{dt} = \frac{dx}{dt'} \frac{dt'}{dt} \; .$$

Therewith it follows for the actual acceleration:

$$a = \frac{d^2x}{dt^2} = \frac{d^2x}{dt'^2} \left(\frac{dt'}{dt}\right)^2 + \frac{dx}{dt'} \frac{d^2t'}{dt^2}$$
$$= a' \left(\frac{dt'}{dt}\right)^2 + v \frac{d^2t'}{dt^2} \left(\frac{dt'}{dt}\right)^{-1} \; .$$

Fig. A.21

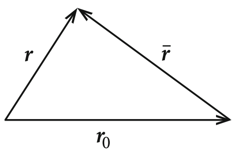
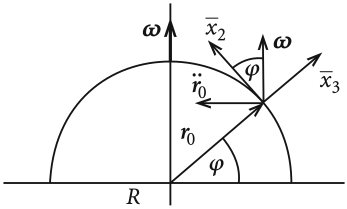


Fig. A.22



Force-free movement means $a = 0$. Therefore it must be

$$a' = -v \, \frac{d^2t'}{dt^2} \left(\frac{dt'}{dt}\right)^{-3} .$$

With

$$\frac{dt'}{dt} = 1 + \dot{\alpha}(t) \;\; ; \;\; \frac{d^2t'}{dt^2} = \ddot{\alpha}(t)$$

one obtains:

$$F' = m\,a' = -mv \, \frac{\ddot{\alpha}(t)}{(1 + \dot{\alpha}(t))^3} .$$

Solution 2.2.3

1. We introduce two systems of coordinates (Fig. A.22):

   Σ: System of coordinates fixed at the earth's center, which does **not** follow the rotation being therefore an inertial system.
   $\overline{\Sigma}$: Co-rotating Cartesian system of coordinates at the earth's surface.
   $\mathbf{r}_0$: Position vector of the origin of $\overline{\Sigma}$ as seen from Σ.
   $\bar{\mathbf{r}}$: Position vector of the mass point in $\overline{\Sigma}$.

   With $\dot{\omega} = 0$ the equation of motion reads according to (2.77):

$$m\ddot{\bar{\mathbf{r}}} = -m\,\mathbf{g} - m\ddot{\mathbf{r}}_0 - m[\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \bar{\mathbf{r}})] - 2m(\boldsymbol{\omega} \times \dot{\bar{\mathbf{r}}}) ,$$
$$\bar{\mathbf{F}}_c = -2m(\boldsymbol{\omega} \times \dot{\bar{\mathbf{r}}}) \; ; \quad \text{(Coriolis force)} ,$$
$$\bar{\mathbf{F}}_z = -m\,[\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \bar{\mathbf{r}})] \; ; \quad \text{(centrifugal force)} .$$

   $\bar{\mathbf{F}}_z$ is here negligible since ω^2 and also the distance $\bar{r}$ from the earth's surface can be assumed to be small. Approximately it is left as equation of motion:

$$\ddot{\bar{\mathbf{r}}} \approx -\mathbf{g} - \ddot{\mathbf{r}}_0 - 2(\boldsymbol{\omega} \times \dot{\bar{\mathbf{r}}}) .$$

2. The origin of $\overline{\Sigma}$ moves on a circle with radius $R\cos\varphi$ around the $\boldsymbol{\omega}$-axis. That means after (2.33):

$$|\ddot{\mathbf{r}}_0| = \omega^2 R\cos\varphi \ ,$$
$$\ddot{\mathbf{r}}_0 = \omega^2 R\cos\varphi\,(\sin\varphi\,\bar{\mathbf{e}}_2 - \cos\varphi\,\bar{\mathbf{e}}_3) \ .$$

3. The force stemming from $\ddot{\mathbf{r}}_0$ is also to be taken into consideration:

$$\hat{\mathbf{g}} = \mathbf{g} + \ddot{\mathbf{r}}_0 = \left(0, \omega^2 R\cos\varphi\sin\varphi, -\omega^2 R\cos^2\varphi + g\right) \ .$$

Liquids orient their surfaces always perpendicular to $\hat{\mathbf{g}}$, not to $\mathbf{g}$. $\hat{\mathbf{g}}$ determines the vertical, which deviates a bit from the radial direction. $\hat{\mathbf{g}}$ is dependent on the geographical latitude. The real earth's surface is perpendicular to $\hat{g}$ (*'flattening of the earth'*).

4.

$$\boldsymbol{\omega} = (0, \omega\cos\varphi\ , \omega\sin\varphi) \ .$$

According to 1. it then holds for the Coriolis force:

$$\bar{\mathbf{F}}_c = -2m(\boldsymbol{\omega}\times\dot{\bar{\mathbf{r}}}) = -2m\,\omega(\dot{\bar{x}}_3\cos\varphi - \dot{\bar{x}}_2\sin\varphi, \dot{\bar{x}}_1\sin\varphi, -\dot{\bar{x}}_1\cos\varphi) \ .$$

5. Equations of motion:

$$\begin{aligned} m\ddot{\bar{x}}_1 &= -2m\,\omega(\dot{\bar{x}}_3\cos\varphi - \dot{\bar{x}}_2\sin\varphi) \ , \\ m\ddot{\bar{x}}_2 &= -2m\,\omega\dot{\bar{x}}_1\sin\varphi \ , \\ m\ddot{\bar{x}}_3 &= -m\,\hat{g} + 2m\,\omega\dot{\bar{x}}_1\cos\varphi \ . \end{aligned}$$

$\hat{g}$ is the measured earth's acceleration.

6. After precondition one can assume that $\dot{\bar{x}}_1 \approx 0$, $\dot{\bar{x}}_2 \approx 0$ holds during the fall time. We then have to solve the following system of equations of motion:

$$\begin{aligned} \ddot{\bar{x}}_1 &= -2\omega\,\dot{\bar{x}}_3\cos\varphi \ , \\ \ddot{\bar{x}}_2 &= 0 \ , \\ \ddot{\bar{x}}_3 &= -\hat{g} \ . \end{aligned}$$

With the initial conditions

$$\bar{\mathbf{r}}(t=0) = (0,0,H) \ ; \quad \dot{\bar{\mathbf{r}}}(t=0) = (0,0,0)$$

one gets the solution:

$$\bar{\mathbf{r}}(t) = \left(\frac{1}{3}\,\omega\cos\varphi\,\hat{g}\,t^3, 0, -\frac{1}{2}\,\hat{g}\,t^2 + H\right) .$$

The fall time t_F results from

$$\bar{x}_3(t = t_F) \stackrel{!}{=} 0$$

to

$$t_F = \sqrt{\frac{2H}{\hat{g}}} .$$

That yields the east-deviation:

$$\bar{x}_1(t_F) = \frac{1}{3}\,\omega\cos\varphi\,\hat{g}\left(\frac{2H}{\hat{g}}\right)^{3/2} .$$

As $\cos\varphi$ is always positive the earth's rotation ($\omega \neq 0$) provokes an east-deviation on **both** hemispheres of the earth.

Section 2.3

Solution 2.3.1

1. (a) Presumption:

$$W(x_1, x_2, x_3; t) \neq 0 .$$

It may hold:

$$\sum_{i=1}^{3} \gamma_i x_i(t) = 0 \quad \gamma_i \in \mathbb{R} , \quad \forall t . \qquad (*)$$

Differentiating gives in addition:

$$\sum_{i=1}^{3} \gamma_i \dot{x}_i(t) = 0 ; \quad \sum_{i=1}^{3} \gamma_i \ddot{x}_i(t) = 0 .$$

Let us consider these three equations as linear homogeneous system of equations for the γ_i:

$\Longrightarrow$ $W(x_1, x_2, x_3; t)$ corresponds to the coefficient-determinant of this system of equations

$\Longrightarrow$ non-trivial solutions only for $W = 0$, but W unequal zero as per assumption!

$\Longrightarrow$ $(*)$ satisfiable only for $\gamma_1 = \gamma_2 = \gamma_3 = 0$!

$\Longrightarrow$ $x_i(t)$ $(i = 1, 2, 3)$ linearly independent!

(b) Presumption:
$x_1(t), x_2(t), x_3(t)$: Three linearly independent solutions of the linear homogeneous differential equation of third order.
If for an arbitrary t_0

$$W(x_1, x_2, x_3; t_0) = 0$$

holds then the above system of equations will have for $t = t_0$ a non-trivial solution: $\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3$.
So

$$f(t) \equiv \sum_{i=1}^{3} \hat{\gamma}_i x_i(t)$$

is as linear combination of the solutions $x_i(t)$ also a solution of the differential equation with

$$f(t_0) = 0\,; \quad \dot{f}(t_0) = 0\,; \quad \ddot{f}(t_0) = 0\,.$$

Now it is

$$\bar{f}(t) \equiv 0$$

also a solution of the differential equation with **the same** initial conditions for $t = t_0$. From the uniqueness theorem follows

$$\bar{f}(t) \equiv f(t) \iff 0 = \sum_{i=1}^{3} \hat{\gamma}_i x_i(t)\ .$$

Not all $\hat{\gamma}_i$ are equal to zero. Therefore the x_1, x_2, x_3 should be linearly dependent contrary to the presumption

$$\Longrightarrow \quad W(x_1, x_2, x_3; t) \neq 0 \quad \forall t\ .$$

2.

$$\dddot{x}(t) - \frac{6}{t^2}\dot{x}(t) + \frac{12}{t^3}x(t) = 0 \quad (t \neq 0!)$$

Special solutions:

(a)

$$\begin{aligned} x_1(t) = \frac{1}{t^2} \implies \dot{x}_1(t) &= -\frac{2}{t^3} \\ \ddot{x}_1(t) &= +\frac{6}{t^4} \\ \dddot{x}_1(t) &= -\frac{24}{t^5} \end{aligned}$$

$$\implies -\frac{24}{t^5} + \frac{12}{t^5} + \frac{12}{t^5} = 0$$

(b)

$$\begin{aligned} x_2(t) = t^2 \implies \dot{x}_2(t) &= 2t \\ \ddot{x}_2(t) &= 2 \\ \dddot{x}_2(t) &\equiv 0 \end{aligned}$$

$$\implies 0 - \frac{12}{t} + \frac{12}{t} = 0$$

(c)

$$\begin{aligned} x_3(t) = t^3 \implies \dot{x}_3(t) &= 3t^2 \\ \ddot{x}_3(t) &= 6t \\ \dddot{x}_3(t) &\equiv 6 \end{aligned}$$

$$\implies 6 - 18 + 12 = 0$$

Wronski-determinant:

$$W(x_1,x_2,x_3;t) = \begin{vmatrix} \frac{1}{t^2} & t^2 & t^3 \\ -\frac{2}{t^3} & 2t & 3t^2 \\ +\frac{6}{t^4} & 2 & 6t \end{vmatrix}$$

$$= 12 + 18 - 4 - 12 - 6 + 12$$

$$= 20 \neq 0$$

$\Longrightarrow$ linearly independent
General solution:

$$f(t) = \gamma_1 \frac{1}{t^2} + \gamma_2 t^2 + \gamma_3 t^3 \quad (t \neq 0)$$

Solution 2.3.2

1.

$$\mathbf{v}_0 = v_0 \mathbf{e}_z \ .$$

It is a one-dimensional problem:

$$\ddot{z} = -g \Longrightarrow \dot{z}(t) - \dot{z}(t_s) = -g(t - t_s) \, ; \quad (\, t_s\text{: start time}) \ .$$

$$\dot{z}(t_s) = v_0 \Longrightarrow \dot{z}(t) = v_0 - g(t - t_s) \ .$$

$$z(t_s) = 0 \quad \text{(ground)} \Longrightarrow z(t) = v_0(t - t_s) - \frac{1}{2} g (t - t_s)^2 \ .$$

2. 1. stone: $t_s = 0 \Longrightarrow z_1(t) = v_0 t - \frac{1}{2} g\, t^2$.
   2. stone: $t_s = t_0 \Longrightarrow z_2(t) = v_0(t - t_0) - \frac{1}{2} g (t - t_0)^2$.
      The two stones meet at the time t_x,

$$z_1(t_x) = z_2(t_x),$$

i.e.

$$t_x = \frac{v_0}{g} + \frac{1}{2} t_0$$

3.

$$\dot{z}_1(t_x) = v_0 - g\, t_x = -\frac{1}{2} g\, t_0 \quad \text{(downward motion)} \ ,$$

$$\dot{z}_2(t_x) = v_0 - g(t_x - t_0) = +\frac{1}{2} g\, t_0 \quad \text{(upward motion)} \ .$$

Solution 2.3.3

1. Equations of motion:

$$m_1\ddot{x}_1 = m_1 g + S_1 \ ,$$
$$m_2\ddot{x}_2 = m_2 g + S_2 \ ,$$

   thread tension: $S_1 = S_2 = S$,
   constraint: $x_1 + x_2 =$ length of the thread $\Longrightarrow \quad \ddot{x}_1 = -\ddot{x}_2$. So one gets:

$$m_1\ddot{x}_1 = m_1 g + S \ ,$$
$$-m_2\ddot{x}_1 = m_2 g + S \ .$$

2.

$$\ddot{x}_1 = \frac{m_1 - m_2}{m_1 + m_2}\, g = -\ddot{x}_2 \ .$$

   It represents the *retarded* free fall. Equilibrium happens for $m_1 = m_2$.
3. The thread tension

$$S = m_1(\ddot{x}_1 - g) = -\frac{2m_1 m_2}{m_1 + m_2} g$$

   is maximal at the equilibrium.

Solution 2.3.4

1. These are one-dimensional motions:

$$m_1\ddot{z}_1 = m_1 g \sin\alpha + S \ ,$$
$$m_2\ddot{z}_2 = m_2 g \sin\beta + S$$

   (S: thread tension).
2. The constant length of the thread brings about:

$$\ddot{z}_1 = -\ddot{z}_2 \ .$$

   By subtraction of the two equations of motion in 1. we obtain the accelerations:

$$\ddot{z}_1 = \frac{m_1 \sin\alpha - m_2 \sin\beta}{m_1 + m_2} g = -\ddot{z}_2 \ .$$

   That is the *retarded* free fall.

3. The thread tension S arises out of 1. and 2.:

$$S = m_1(\ddot{z}_1 - g\sin\alpha) = -m_1 m_2\, g\,\frac{\sin\alpha + \sin\beta}{m_1 + m_2}\ .$$

4.

$$\ddot{z}_1 = 0 = \ddot{z}_2 \iff m_1\sin\alpha = m_2\sin\beta\ .$$

Solution 2.3.5

1. The forces of the resting piece of the rope are compensated by the base. On the overhanging piece of the length x the force

$$F = m\frac{x}{l}g\ .$$

is acting. That yields the equation of motion:

$$m\ddot{x} = m\frac{x}{l}g\ .$$

2. Ansatz for the solution:

$$x \propto e^{\alpha t}\ .$$

The equation of motion is fulfilled if one chooses

$$\alpha^2 = \frac{g}{l} \iff \alpha_{1,2} = \pm\sqrt{\frac{g}{l}}$$

The general solution of the homogeneous differential equation of second order therewith reads:

$$x(t) = A_+ e^{\sqrt{g/l}\,t} + A_-\, e^{-\sqrt{g/l}\,t}\ .$$

The initial conditions

$$x(0) = x_0\ ; \quad \dot{x}(0) = 0$$

fix $A_\pm$:

$$A_+ = A_- = \frac{1}{2}x_0\ .$$

It follows:

$$x(t) = x_0 \cosh\left(\sqrt{\frac{g}{l}}t\right) ,$$
$$\dot{x}(t) = x_0\sqrt{\frac{g}{l}} \sinh\left(\sqrt{\frac{g}{l}}t\right) .$$

3. At the time t_e the end of the rope is just at the edge of the base:

$$x(t_\mathrm{e}) = l = x_0 \cosh\left(\sqrt{\frac{g}{l}}t_\mathrm{e}\right) ,$$
$$\dot{x}(t_\mathrm{e}) = x_0\sqrt{\frac{g}{l}} \sinh\left(\sqrt{\frac{g}{l}}t_\mathrm{e}\right) .$$

Squaring the last equation leads to:

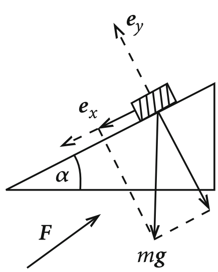
$$\begin{aligned}
[\dot{x}(t_\mathrm{e})]^2 &= x_0^2\frac{g}{l}\sinh^2\left(\sqrt{\frac{g}{l}}t_\mathrm{e}\right) = \\
&= x_0^2\frac{g}{l}\left[\cosh^2\left(\sqrt{\frac{g}{l}}t_\mathrm{e}\right) - 1\right] = \\
&= \frac{g}{l}\left(l^2 - x_0^2\right) \\
\Longrightarrow \quad \dot{x}\,(t_\mathrm{e}) &= \sqrt{\frac{g}{l}(l^2 - x_0^2)} \,.
\end{aligned}$$

Solution 2.3.6

1. Let $\mathbf{F}$ be the force which the scales contribute to the equilibrium. Its direction is at first undetermined (Fig. A.23). For equilibrium it must hold:

$$m(\mathbf{g} - \ddot{\mathbf{x}}) + \mathbf{F} = 0 \,.$$

Fig. A.23



(a) fixed mass:

$$\ddot{x} = 0 \Longrightarrow \mathbf{F} = -m\,\mathbf{g} \;.$$

The force $\mathbf{F}$ fully compensates the gravitational force being parallel to $\mathbf{g}$. Weight-display:

$$F_{\parallel} = -\frac{1}{g}(\mathbf{F}\cdot\mathbf{g}) = m\,g \;.$$

(b) mobile mass:

$$\begin{aligned} \mathbf{F} &= -m(\mathbf{g} - \ddot{\mathbf{x}}) \;, \\ m\,\ddot{\mathbf{x}} &= m\,g\sin\alpha\,\mathbf{e}_x \\ \Longrightarrow \mathbf{F} &= m\,g\cos\alpha\,\mathbf{e}_y \;. \end{aligned}$$

Weight-display:

$$F_{\parallel} = -(\mathbf{F}\cdot\mathbf{g})\frac{1}{g} = -m\,g\cos\alpha\frac{1}{g}(\mathbf{e}_y\cdot\mathbf{g}) = m\,g\cos^2\alpha \;.$$

As long as the mass is in motion the scales exhibit a smaller amount. The limiting case $\alpha = \frac{\pi}{2}$ corresponds to the free fall. The scales then show the value zero.

2. The contact force is in both cases the same:

(a) $F_y = \mathbf{F}\cdot\mathbf{e}_y = -m\,(\mathbf{g}\cdot\mathbf{e}_y) = m\,g\cos\alpha$,
(b) $F_y = m\,g\cos\alpha\,(\mathbf{e}_y\cdot\mathbf{e}_y) = m\,g\cos\alpha$.

Solution 2.3.7

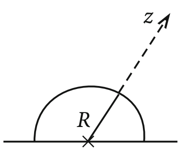
1. The vertical throw (Fig. A.24) represents a one-dimensional motion:

$$m\ddot{z} = -\gamma\frac{m\,M}{z^2} \;.$$

Especially at the earth's surface it holds:

$$m\,g = \gamma\frac{m\,M}{R^2} \Longleftrightarrow \gamma\,M = g\,R^2 \;.$$

Fig. A.24



The above equation of motion can therefore also be written as follows:

$$\ddot{z} = -g\frac{R^2}{z^2} \ .$$

This we rewrite furthermore by use of the chain rule ($v = \dot{z}$):

$$\ddot{z} = \frac{dv}{dt} = \frac{dv}{dz}v = -\gamma\frac{M}{z^2} \ .$$

Separation of variables leads to:

$$\int_{v_0}^{v} v'\,dv' = -\gamma\,M\int_{R}^{z}\frac{dz'}{z'^2}$$

$$\Longrightarrow \frac{1}{2}\left(v^2 - v_0^2\right) = \gamma\,M\left(\frac{1}{z} - \frac{1}{R}\right) \ .$$

That yields the distance-dependence of the velocity:

$$v(z) = \sqrt{v_0^2 + 2\gamma\,M\frac{R-z}{Rz}} \ .$$

2.

$$v(z \to \infty) = \sqrt{v_0^2 - \frac{2\gamma\,M}{R}} \ .$$

In order that the particle can leave the gravitational region it must necessarily be:

$$v(z \to \infty) \geq 0$$

That is possible only if

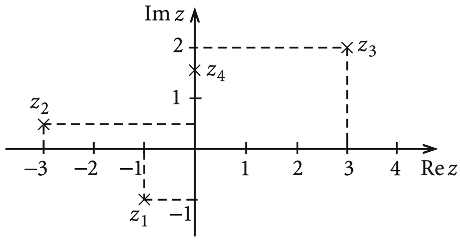
$$v_0 \geq \sqrt{\frac{2\gamma\,M}{R}} \ .$$

Numerical values:

$$\gamma = 6.67 \cdot 10^{-11}\mathrm{N\,m^2\,kg^{-2}} \ , \quad M = 5.98 \cdot 10^{24}\,\mathrm{kg} \ , \quad R = 6.37 \cdot 10^6\,\mathrm{m} \ :$$

$$v_0 \geq 11.2\,\mathrm{km\,s^{-1}} \ .$$

Fig. A.25



Solution 2.3.8

1.

$$(-i)^3 = i \,, \quad i^{15} = -i \,, \quad \sqrt{4(-25)} = 10\,i \,, \quad \ln(1+i) = \ln\sqrt{2} + i\frac{\pi}{4} \,,$$

$$e^{i(\pi/3)} = \frac{1}{2} + \frac{1}{2}\sqrt{3}\,i \,, \quad e^{i(\pi/2)} = i \,.$$

2. (a) $z = 2$,
   (b) $z = 23 + 2i$.
3. See Fig. A.25!
4.

$$z_1 = \sqrt{2}\,e^{i(3\pi/4)} \,, \quad z_2 = \sqrt{2}\,e^{i(5\pi/4)} \,, \quad z_3 = e^3 \cdot e^{2i} \,,$$

$$z_4 = e^{i(\pi/6)} \,, \quad z_5 = e^{-i(\pi/2)} \,.$$

5.

$$z_1 = -\sqrt{e} \,, \quad z_2 = -i\,e^{-1} \,,$$

$$z_3 = (e^3 \cos 1) - i\,(e^3 \sin 1) \,.$$

6. (a) $\mathrm{Re}\, e^{z(t)} = e^{-t}\cos 2\pi\, t \,; \quad \tau = 1 \,,$
   (b) $\mathrm{Re}\, e^{z(t)} = e^{2t}\cos\left(\frac{3}{2}t\right) \,; \quad \tau = \frac{4\pi}{3} \,.$

Solution 2.3.9 The proof turns out very simple by use of Euler's formula:

$$\begin{aligned}
exp(i(\alpha+\beta)) &= \cos(\alpha+\beta) + i\,\sin(\alpha+\beta) \\
&= exp(i\,\alpha)\,exp(i\,\beta) \\
&= \big(\cos\alpha + i\,\sin\alpha\big)\,(\cos\beta + i\,\sin\beta) \\
&= (\cos\alpha\,\cos\beta - \sin\alpha\,\sin\beta) + i\,(\sin\alpha\,\cos\beta + \cos\alpha\,\sin\beta) \,.
\end{aligned}$$

Real and imaginary parts of complex numbers are independent of each other. Therefore the comparison yields directly the assertion:

$$\sin(\alpha+\beta) = \sin\alpha\,\cos\beta+\sin\beta\,\cos\alpha \quad ; \quad \cos(\alpha+\beta) = \cos\alpha\,\cos\beta-\sin\alpha\,\sin\beta\ .$$

Solution 2.3.10

1.

$$7\ddot{x} - 4\dot{x} - 3x = 6\ .$$

For the respective homogeneous equation,

$$7\ddot{x} - 4\dot{x} - 3x = 0\ ,$$

it is convenient to use the ansatz:

$$x = e^{\gamma t}\ .$$

Insertion provides a conditional equation for γ,

$$7\gamma^2 - 4\gamma - 3 = 0\ ,$$

which is solved by

$$\gamma_1 = 1 \quad \text{and} \quad \gamma_2 = -\frac{3}{7}$$

The general solution of the homogeneous differential equation therefore reads:

$$x_{\text{hom}}(t) = a_1\, e^t + a_2\, e^{-(3/7)t}\ .$$

It is easy to *guess* a special solution of the inhomogeneous equation:

$$x_s(t) = -2\ .$$

Therewith the general solution is determined:

$$x(t) = a_1\, e^t + a_2\, e^{-(3/7)t} - 2\ .$$

2.

$$\ddot{z} - 10\,\dot{z} + 9\,z = 9\,t\ .$$

A special solution can easily be *guessed*:

$$z_s(t) = t + \frac{10}{9} \; .$$

The respective homogeneous differential equation

$$\ddot{z} - 10\,\dot{z} + \;9\,z = 0$$

is solved by

$$z(t) = e^{\gamma t}$$

if

$$\gamma^2 - 10\,\gamma + 9 = 0$$

is fulfilled. That is the case for

$$\begin{aligned} &\gamma_1 = 1 \quad \text{and} \quad \gamma_2 = 9 \\ &\Longrightarrow z_{\text{hom}}(t) = \alpha_1\, e^t + \alpha_2\, e^{9t} \; . \end{aligned}$$

The general solution of the inhomogeneous differential equation eventually reads:

$$z(t) = \alpha_1 e^t + \alpha_2\, e^{9t} + t + \frac{10}{9} \; .$$

Solution 2.3.11

1.

$$\ddot{y} + \dot{y} + y = 2t + 3 \; .$$

There should exist a special solution which is linear in t (why?)!

$$y(t) = 2t + \alpha \; .$$

Insertion yields:

$$2 + 2t + \alpha = 2t + 3 \Longrightarrow \alpha = 1 \Longrightarrow y_s(t) = 2t + 1 \; .$$

2.

$$4\ddot{y} + 2\dot{y} + 3y = -2t + 5 \; .$$

Here, too, there should exist a special solution being linear in t:

$$y(t) = \alpha \cdot t + \beta \ .$$

Insertion yields now:

$$2\alpha + 3\alpha\, t + 3\beta = -2t + 5 \Longrightarrow \alpha = -\frac{2}{3} \ ; \quad \beta = \frac{19}{9}$$
$$\Longrightarrow y_s(t) = -\frac{2}{3}t + \frac{19}{9} \ .$$

Solution 2.3.12 The homogeneous differential equation

$$\ddot{z} + 4z = 0$$

is solved by the ansatz

$$z(t) = e^{\gamma\, t}$$

if

$$e^{\gamma\, t}(\gamma^2 + 4) = 0$$

is fulfilled. That is the case for

$$\gamma_1 = +2\, i \quad \text{and} \quad \gamma_2 = -2\, i$$

The general solution has therefore the form:

$$z(t) = a_1 e^{2i\, t} + a_2 e^{-2i\, t} \ .$$

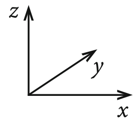
1. Boundary conditions: $z(0) = 0 \ ; \quad z\left(\frac{\pi}{4}\right) = 1$

$$\Longrightarrow a_1 + a_2 = 0 \ ; \quad i(a_1 - a_2) = 1$$
$$\Longrightarrow z(t) = \sin 2t \ .$$

2. Boundary conditions: $z\left(\frac{\pi}{2}\right) = -1 \ ; \quad \dot{z}\left(\frac{\pi}{2}\right) = 1$

$$\Longrightarrow a_1 + a_2 = 1 \ ; \quad 2i(-a_1 + a_2) = 1$$
$$\Longrightarrow z(t) = \cos 2t - \frac{1}{2}\sin 2t \ .$$

Fig. A.26



Solution 2.3.13

1.

$$\mathbf{g} = (0, 0, g) \ .$$

Equation of motion:

$$m\ddot{\mathbf{r}} = -\alpha\, v\, \dot{\mathbf{r}} - m\, \mathbf{g} \ .$$

The first term on the right-hand side is the Newton-version of the friction force ($v = |\dot{\mathbf{r}}|$). Restriction to the vertical motion yields (Fig. A.26):

$$m\ddot{z} = -\alpha\, v\, \dot{z} - m\, g \ .$$

2. The uniform straight-line motion corresponds to the force-free motion. The initial velocity must therefore be chosen so that the friction force compensates the gravitational force:

$$|\dot{z}_0| = \sqrt{\frac{m}{\alpha} g} \ .$$

3. Motion of falling:

$$\dot{z} = -v \leq 0 \ .$$

The equation of motion to be solved reads then as follows:

$$-\frac{d}{dt} v = \frac{\alpha}{m} v^2 - g \ .$$

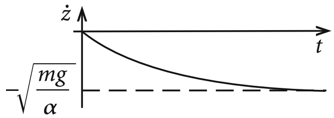
Separation of variables leads to:

$$dt = \frac{dv}{g - \frac{\alpha}{m} v^2} \ .$$

That can easily be integrated [$v(t = 0) = 0$]:

$$t = \frac{1}{g} \int\limits_0^v \frac{dv'}{1 - \frac{\alpha}{m\, g} v'^2} = \sqrt{\frac{m}{\alpha\, g}} \operatorname{arctanh}\left(\sqrt{\frac{\alpha}{m\, g}} \cdot v \right) \ .$$

Fig. A.27



Therewith we have determined the time-dependence of the velocity (Fig. A.27; $(\tanh x_{x\to\infty} \longrightarrow 1)$):

$$\dot{z}(t) = -v(t) = -\sqrt{\frac{mg}{\alpha}} \tanh\left(\sqrt{\frac{\alpha\, g]}{m}} t\right) .$$

4. With

$$\int \tanh x\, dx = \ln(\cosh x) + c_0$$

and the result for $\dot{z}(t)$ in part 3. it follows by integrating once more:

$$z(t) = -\frac{m}{\alpha} \ln\left[\cosh\left(\sqrt{\frac{\alpha g}{m}} t\right)\right] + c_0 ,$$

$$z(t = 0) = H = c_0$$

$$\Longrightarrow z(t) = H - \frac{m}{\alpha} \ln\left[\cosh\left(\sqrt{\frac{\alpha g}{m}} t\right)\right] .$$

We still discuss the limiting case of vanishing friction ($\alpha \to 0$): Because of

$$\cosh x \underset{x \ll 1}{\longrightarrow} \left(1 + \frac{x^2}{2}\right) ,$$

$$\ln(\cosh x) \underset{x \ll 1}{\longrightarrow} \frac{x^2}{2}$$

it follows

$$z(t) \underset{\alpha \to 0}{\longrightarrow} H - \frac{m}{\alpha}\frac{\alpha g}{2m} t^2 = H - \frac{1}{2} g\, t^2 .$$

This is the free fall!

Solution 2.3.14

1. Equation of motion:

$$m\ddot{\mathbf{r}} = -\alpha\dot{\mathbf{r}} - m\,\mathbf{g} ; \quad \mathbf{g} = (0, 0, g) .$$

For the single components it is to solve:

$$m\ddot{x}_i = -\alpha\,\dot{x}_i - mg\,\delta_{i3}\ ; \quad i = 1, 2, 3\ .$$

That is a linear differential equation of second order being homogeneous for $i = 1, 2$ and inhomogeneous for $i = 3$.

2. The solution of the related homogeneous differential equation

$$m\ddot{x}_i + \alpha\,\dot{x}_i = 0$$

succeeds with the ansatz

$$x_i(t) = e^{\gamma t}$$

where γ is determined by

$$(m\,\gamma^2 + \alpha\,\gamma)e^{\gamma t} = 0$$

One sees that the values

$$\gamma_1 = 0 \quad \text{and} \quad \gamma_2 = -\frac{\alpha}{m}$$

are possible. That yields the general solution of the homogeneous equation:

$$x_i(t) = a_i^{(1)} + a_i^{(2)} e^{-(\alpha/m)t}\ .$$

For $i = 3$ we still need a special solution of the respective inhomogeneous differential equation:

$$x_{3s}(t) = -\frac{m}{\alpha} g\, t\ .$$

This solution can be guessed rather easily or can be found more systematically by a physical consideration as in part 2. of Exercise 2.3.13.
Eventually we have found therewith the general solution of the equation of motion:

$$x_i(t) = a_i^{(1)} + a_i^{(2)} e^{-(\alpha/m)t} - \frac{m}{\alpha} g\, t \cdot \delta_{i3}\ .$$

3.

$$\mathbf{r}(t = 0) = (0, 0, 0)\ ; \quad \dot{\mathbf{r}}(t = 0) = (v_0, 0, v_0)\ .$$

4.a

$$\dot{x}_1(t) = -\frac{\alpha}{m} a_1^{(2)} e^{-(\alpha/m)t}$$
$$\Longrightarrow x_1(0) = a_1^{(1)} + a_1^{(2)} = 0 \,,$$
$$\dot{x}_1(0) = -\frac{\alpha}{m} a_1^{(2)} = v_0$$
$$\Longrightarrow x_1(t) = \frac{m\,v_0}{\alpha}\left(1 - e^{-(\alpha/m)t}\right) \,.$$

4.b

$$x_2(0) = a_2^{(1)} + a_2^{(2)} = 0 \,,$$
$$\dot{x}_2(0) = -\frac{\alpha}{m} a_2^{(2)} = 0$$
$$\Longrightarrow x_2(t) \equiv 0 \,.$$

4.c

$$x_3(0) = a_3^{(1)} + a_3^{(2)} = 0 \,,$$
$$\dot{x}_3(0) = -\frac{\alpha}{m} a_3^{(2)} - \frac{m}{\alpha} g = v_0$$
$$\Longrightarrow x_3(t) = \frac{m}{\alpha}\left[\left(\frac{m}{\alpha} g + v_0\right)\left(1 - e^{-(\alpha/m)t}\right) - g\,t\right] \,.$$

5. The maximum flight altitude is given by

$$\dot{x}_3(t_{\mathrm{H}}) \stackrel{!}{=} 0$$

It is reached after the time

$$t_{\mathrm{H}} = -\frac{m}{\alpha} \ln \frac{m\,g}{m\,g + \alpha\,v_0}$$

and amounts to

$$x_3(t_{\mathrm{H}}) = \frac{m}{\alpha}\left(v_0 + \frac{m\,g}{\alpha}\ \ln \frac{m\,g}{m\,g + \alpha\,v_0}\right) \,.$$

Solution 2.3.15

1.

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t\ ,$$
$$\dot{x}(t) = -A\,\omega_0\sin\omega_0 t + B\,\omega_0\cos\omega_0 t\ ,$$
$$\ddot{x}(t) = -\omega_0^2\,x(t)\ .$$

At the maximum deflection it must hold:

$$\dot{x}(t_1) = 0\ ; \quad \ddot{x}(t_1) < 0\ .$$

Therewith it follows:

$$t_1 = \frac{1}{\omega_0}\arctan\frac{B}{A}\ .$$

With

$$\cos x = \frac{1}{\sqrt{1+\tan^2 x}}\ , \quad \sin x = \frac{\tan x}{\sqrt{1+\tan^2 x}}$$

we get by insertion:

$$x_{\max} = x(t_1) = \sqrt{A^2+B^2}\ ,$$
$$\ddot{x}(t_1) = -\omega_0^2\sqrt{A^2+B^2}\ .$$

2. The maximum velocity is fixed by the requirement

$$\ddot{x}(t_2) \overset{!}{=} 0 \quad [\dddot{x}(t_2) < 0]\ .$$

That is equivalent to

$$x(t_2) \overset{!}{=} 0\ .$$

resulting in:

$$t_2 = \frac{1}{\omega_0}\arctan\left(-\frac{A}{B}\right)\ .$$

At this time t_2 the oscillator reaches its maximum velocity

$$\dot{x}_{\max} = \dot{x}(t_2) = \omega_0\sqrt{A^2+B^2} = \omega_0 x_{\max}\ .$$

3. The maximum acceleration implies

$$\dddot{x}(t_3) = 0\ ; \quad x^{(4)}(t_3) < 0$$

Because of

$$\dddot{x}(t) = -\omega_0^2\, \dot{x}(t)\ ,$$
$$x^{(4)}(t) = \omega_0^4\, x(t)\ ,$$

one could suppose from part 1. that $t_3 = t_1$. However, in view of $x(t_1) = x_{\max} > 0$ $\ddot{x}$ has a minimum at t_1. We therefore have to assume

$$t_3 = t_1 + \frac{\pi}{\omega_0}$$

It is then

$$\dddot{x}(t_3) = -\omega_0^2\, \dot{x}(t_3) = 0\ ,$$
$$x^{(4)}(t_3) = \omega_0^4\, x(t_3) = -\omega_0^4\, x(t_1) = -\omega_0^4\, x_{\max} < 0$$
$$\Longrightarrow x(t_3) = -x_{\max}\ ; \quad \dot{x}(t_3) = 0\ ; \quad \ddot{x}(t_3) = \omega_0^2\, x_{\max}\ .$$

Solution 2.3.16

1. $0 \le t \le t_0$:
   Equation of motion:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = \frac{v_0}{t_0}$$

where:

$$\omega_0^2 = \frac{k}{m}\ ;\ \beta = \frac{\alpha}{2m}\ .$$

General solution of the inhomogeneous differential equation of second order:

$$x(t) = x_{\text{hom}}(t) + x_S(t)\ .$$

The general solution of the related homogeneous differential equation has been derived already with (2.173):

$$x_{\text{hom}}(t) = e^{-\beta t}\left(a_+ e^{i\omega t} + a_- e^{-i\omega t}\right) \qquad \text{with} \qquad \omega = \sqrt{\omega_0^2 - \beta^2}\ .$$

Special solution:

$$x_S(t) \equiv \frac{v_0}{t_0\omega_0^2}$$

$\Longrightarrow$ general solution (for $0 \leq t \leq t_0$):

$$x(t) = e^{-\beta t}\left(a_+ e^{i\omega t} + a_- e^{-i\omega t}\right) + \frac{v_0}{t_0\omega_0^2} \ .$$

Initial conditions:

$$\begin{aligned}
&x(0) = 0\ ; \quad \dot{x}(0) = 0 \\
\Longrightarrow\ &0 = (a_+ + a_-) + \frac{v_0}{t_0\omega_0^2} \\
\Longleftrightarrow\ &a_+ + a_- = -\frac{v_0}{t_0\omega_0^2} \\
&0 = -\beta(a_+ + a_-) + i\omega(a_+ - a_-) \\
\Longrightarrow\ &a_+ - a_- = -i\frac{\beta}{\omega}(a_+ + a_-) = i\frac{\beta}{\omega}\frac{v_0}{t_0\omega_0^2}
\end{aligned}$$

$$\begin{aligned}
x(t) &= e^{-\beta t}\left((a_+ + a_-)\cos\omega t + i(a_+ - a_-)\sin\omega t\right) + \frac{v_0}{t_0\omega_0^2} \\
\Longrightarrow\ x(t) &= \frac{v_0}{t_0\omega_0^2}\left\{1 - \left(\cos\omega t + \frac{\beta}{\omega}\sin\omega t\right)e^{-\beta t}\right\} \ .
\end{aligned}$$

2. $t > t_0$:
   Force switched off

$$\Longrightarrow\ \ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \ .$$

The general solution is known:

$$x(t) = e^{-\beta t}\left\{b_+\cos\omega t + b_-\sin\omega t\right\} \ .$$

F makes a **finite** jump at $t = t_0$. The same must then hold for $\ddot{x}$, too, but $\dot{x}(t), x(t)$ are continuous at $t = t_0$! That leads to boundary conditions which fix b_+, b_-.

Continuity of $x(t)$:

$$0 = \gamma - e^{-\beta t_0}\left\{(\gamma + b_+)\cos\omega t_0 + \left(\gamma\frac{\beta}{\omega} + b_-\right)\sin\omega t_0\right\} \quad \text{with} \quad \gamma = \frac{v_0}{t_0\omega_0^2}$$

$$\Longrightarrow \; b_+\cos\omega t_0 + b_-\sin\omega t_0 = \gamma\left(e^{\beta t_0} - \cos\omega t_0 - \frac{\beta}{\omega}\sin\omega t_0\right) \equiv \gamma_1 \,. \qquad (*)$$

Continuity of $\dot{x}(t)$:

$$\gamma\beta e^{-\beta t_0}\left(\cos\omega t_0 + \frac{\beta}{\omega}\sin\omega t_0\right) - \gamma e^{-\beta t_0}\left(-\omega\sin\omega t_0 + \beta\cos\omega t_0\right) \overset{!}{=}$$

$$\overset{!}{=} -\beta e^{-\beta t_0}\left(b_+\cos\omega t_0 + b_-\sin\omega t_0\right) + e^{-\beta t_0}\omega\left(-b_+\sin\omega t_0 + b_-\cos\omega t_0\right)$$

$$\Longleftrightarrow \; b_+\left(\beta\cos\omega t_0 + \omega\sin\omega t_0\right) + b_-\left(\beta\sin\omega t_0 - \omega\cos\omega t_0\right) =$$

$$= -\gamma\left(\frac{\beta^2}{\omega} + \omega\right)\sin\omega t_0 \equiv \gamma_2 \,. \qquad (**)$$

We combine $(*)$ and $(**)$:

$$\underbrace{\begin{pmatrix} \cos\omega t_0 & \sin\omega t_0 \\ \beta\cos\omega t_0 + \omega\sin\omega t_0 & \beta\sin\omega t_0 - \omega\cos\omega t_0 \end{pmatrix}}_{=A}\begin{pmatrix} b_+ \\ b_- \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}$$

$$\begin{aligned}
\Longrightarrow \; \det A &= \beta\sin\omega t_0\cos\omega t_0 - \omega\cos^2\omega t_0 - \beta\cos\omega t_0\sin\omega t_0 - \omega\sin^2\omega t_0 \\
&= -\omega \\
A_+ &= \begin{pmatrix} \gamma_1 & \sin\omega t_0 \\ \gamma_2 & \beta\sin\omega t_0 - \omega\cos\omega t_0 \end{pmatrix} \\
\det A_+ &= \gamma_1(\beta\sin\omega t_0 - \omega\cos\omega t_0) - \gamma_2\sin\omega t_0 \\
&= \gamma\left(e^{\beta t_0} - \cos\omega t_0 - \frac{\beta}{\omega}\sin\omega t_0\right)(\beta\sin\omega t_0 - \omega\cos\omega t_0) + \\
&\quad + \gamma\left(\frac{\beta^2}{\omega} + \omega\right)\sin^2\omega t_0 \\
&= \gamma\left\{e^{\beta t_0}(\beta\sin\omega t_0 - \omega\cos\omega t_0) - \beta\cos\omega t_0\sin\omega t_0 + \omega\cos^2\omega t_0 - \right. \\
&\quad \left. -\frac{\beta^2}{\omega}\sin^2\omega t_0 + \beta\sin\omega t_0\cos\omega t_0 + \left(\frac{\beta^2}{\omega} + \omega\right)\sin^2\omega t_0\right\} \\
&= \gamma\left\{e^{\beta t_0}(\beta\sin\omega t_0 - \omega\cos\omega t_0) + \omega\right\} \,.
\end{aligned}$$

Cramer's rule:

$$b_+ = \frac{\det A_+}{\det A} = \gamma \left\{ e^{\beta t_0} \left(\cos \omega t_0 - \frac{\beta}{\omega} \sin \omega t_0 \right) - 1 \right\}$$

$$A_- = \begin{pmatrix} \cos \omega t_0 & \gamma_1 \\ \beta \cos \omega t_0 + \omega \sin \omega t_0 & \gamma_2 \end{pmatrix}$$

$$\begin{aligned}
\Longrightarrow \quad \det A_- &= \gamma_2 \cos \omega t_0 - \gamma_1 (\beta \cos \omega t_0 + \omega \sin \omega t_0) \\
&= -\gamma \left(\frac{\beta^2}{\omega} + \omega \right) \sin \omega t_0 \cos \omega t_0 - \gamma \left(e^{\beta t_0} - \cos \omega t_0 - \frac{\beta}{\omega} \sin \omega t_0 \right) \cdot \\
&\quad \cdot (\beta \cos \omega t_0 + \omega \sin \omega t_0) \\
&= -\gamma \left(\frac{\beta^2}{\omega} + \omega \right) \sin \omega t_0 \cos \omega t_0 - \gamma e^{\beta t_0} (\beta \cos \omega t_0 + \omega \sin \omega t_0) - \\
&\quad - \gamma \Big(- \beta \cos^2 \omega t_0 - \omega \cos \omega t_0 \sin \omega t_0 - \\
&\qquad - \frac{\beta^2}{\omega} \sin \omega t_0 \cos \omega t_0 - \beta \sin^2 \omega t_0 \Big) \\
&= -\gamma \left(e^{\beta t_0} (\beta \cos \omega t_0 + \omega \sin \omega t_0) - \beta \right) .
\end{aligned}$$

Cramer's rule:

$$b_- = \frac{\det A_-}{\det A} = \gamma \left(e^{\beta t_0} \left(\sin \omega t_0 + \frac{\beta}{\omega} \cos \omega t_0 \right) - \frac{\beta}{\omega} \right)$$

$$\begin{aligned}
\cos \omega t \cdot \cos \omega t_0 + \sin \omega t \cdot \sin \omega t_0 &= \cos(\omega (t - t_0)) \\
- \sin \omega t_0 \cdot \cos \omega t + \sin \omega t \cdot \cos \omega t_0 &= \sin(\omega (t - t_0)) .
\end{aligned}$$

$\Longrightarrow$ solution for $t > t_0$:

$$\begin{aligned}
x(t) = \frac{v_0}{t_0 \omega_0^2} \ &\Big\{ e^{-\beta (t - t_0)} \left(\cos \omega (t - t_0) + \frac{\beta}{\omega} \sin \omega (t - t_0) \right) - \\
&\quad - e^{-\beta t} \left(\cos \omega t + \frac{\beta}{\omega} \sin \omega t \right) \Big\} .
\end{aligned}$$

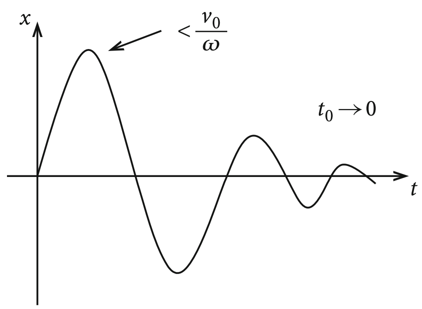
This represents a damped oscillation with $x(t) \overset{t \to \infty}{\longrightarrow} 0$, i.e. transition into the original rest position!

3. *Extremely short impulse:* $t_0 \to 0$
The time interval relevant in part 1. approaches zero. Thus the solution of part 1. does not play a role for the now considered case, only part 2. is decisive. When performing the limiting process $t_0 \to 0$ for the results in part 2. we have to bear in mind that $\gamma \propto t_0^{-1}$. We have therefore to apply l'Hospital's rule (1.96):

$$\begin{aligned}
b_+ &= \lim_{t_0 \to 0} \left\{ \frac{1}{\frac{\omega_0^2}{v_0}} \left(\beta e^{\beta t_0} \left(\cos \omega t_0 - \frac{\beta}{\omega} \sin \omega t_0 \right) + \right.\right. \\
&\quad \left.\left. + e^{\beta t_0} \left(-\omega \sin \omega t_0 - \frac{\beta}{\omega} \omega \cos \omega t_0 \right) \right) \right\} \\
&= \frac{v_0}{\omega_0^2} (\beta(1-0) + 1(-0-\beta)) \\
\Longrightarrow \quad b_+ &= 0 \\
b_- &= \lim_{t_0 \to 0} \frac{1}{\frac{\omega_0^2}{v_0}} \left\{ \beta e^{\beta t_0} \left(\sin \omega t_0 + \frac{\beta}{\omega} \cos \omega t_0 \right) + \right. \\
&\quad \left. + e^{\beta t_0} \left(\omega \cos \omega t_0 - \frac{\beta}{\omega} \omega \sin \omega t_0 \right) \right\} \\
&= \frac{v_0}{\omega_0^2} \left\{ \frac{\beta^2}{\omega} + \omega \right\} = \frac{v_0}{\omega_0^2} \cdot \frac{\beta^2 + \omega^2}{\omega} \\
\Longrightarrow \quad b_- &= \frac{v_0}{\omega} \\
\Longrightarrow \quad x(t) &= \frac{v_0}{\omega} \sin \omega t \, e^{-\beta t}
\end{aligned}$$

The short impulse on the oscillator in its rest position ($x(t = 0) = 0$) brings about the initial velocity $\dot{x}(0) = v_0$ and leads therewith to the known result for the *linear oscillator with friction* (Fig. A.28).

Fig. A.28



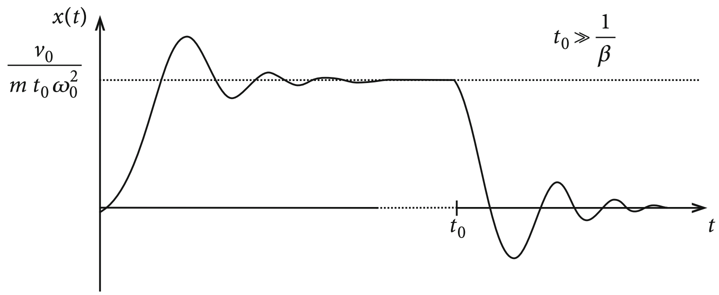


Fig. A.29

4. *Long acting impulse of force:* $t_0 \gg \frac{m}{\alpha}$, i.e. $t_0 \gg \frac{1}{\beta}$
   The solution of part 1. converges to $\frac{v_0}{t_0\omega_0^2}$.
   The solution of part 2. starts at $t = t_0$ with the value $\frac{v_0}{t_0\omega_0^2}$ (horizontal tangent) and is then rapidly damped (Fig. A.29). The second summand in $x(t)$ of part 2.does not play any appreciable role because of $t_0 \gg \frac{1}{\beta}$ and $t \geq t_0$!

Solution 2.3.17 Volume change proportional to the surface:

$$\frac{dV}{dt} \stackrel{!}{=} \gamma 4\pi R^2 \; ; \quad V = \frac{4\pi}{3}R^3$$

$$\Longrightarrow \; 4\pi R^2 \cdot \dot{R} \stackrel{!}{=} \gamma 4\pi R^2$$

$$\Longrightarrow \; \dot{R} = \gamma$$

$$\Longrightarrow \; R(t) = R_0 + \gamma t \; .$$

Time-dependence of the mass (density ρ of the water is constant):

$$m = \rho \cdot \frac{4\pi}{3}R^3$$

$$\Longrightarrow \; \dot{m} = \rho 4\pi R^2 \cdot \dot{R} = \frac{3m}{R}\dot{R} = \frac{3m}{R}\gamma \; .$$

Equation of motion:

$$\frac{d}{dt}(m(t) \cdot \mathbf{v}(t)) = \mathbf{F}_S + \mathbf{F}_R$$

$$\mathbf{v} = (0, 0, v)$$

$$\mathbf{g} = -(0, 0, g)$$

$$\Longrightarrow \; \dot{m}v + m\dot{v} = -mg - \hat{\alpha}R^2 v$$

$$\Longrightarrow \left(\frac{3m\gamma}{R} + \underbrace{\hat{\alpha}R^2}_{\hat{\alpha}\,\frac{3m}{4\pi\rho}\cdot\frac{1}{R}}\right)v = -m\dot{v} - mg$$

$$\Longrightarrow \dot{v} + \frac{v}{R}\underbrace{\left(3\gamma + \frac{3\hat{\alpha}}{4\pi\rho}\right)}_{\equiv\,\varepsilon} = -g$$

$$\Longrightarrow \dot{v} + \frac{\varepsilon}{R}v = -g$$

Rewriting with R as independent variable:

$$\dot{v} = \frac{dv}{dR}\cdot\dot{R} = \gamma v'$$

$$v' + \frac{\varepsilon}{\gamma R}v = -\frac{g}{\gamma}\ .$$

Solution of the homogeneous equation:

$$v_{\text{hom}} = \frac{c}{R^{\frac{\varepsilon}{\gamma}}}\ .$$

Special solution of the inhomogeneous equation:

$$v_S = x\cdot R$$

$$\Longrightarrow x + \frac{\varepsilon}{\gamma}x = -\frac{g}{\gamma}$$

$$\Longrightarrow x = \frac{-\frac{g}{\gamma}}{1+\frac{\varepsilon}{\gamma}}\ .$$

$\Longrightarrow$ general solution:

$$v(R) = \frac{c}{R^{\frac{\varepsilon}{\gamma}}} - \frac{g}{\gamma+\varepsilon}R\ .$$

Initial conditions:

$$v(t=0) = 0\ ;\quad R(t=0) = R_0$$

$$\Longrightarrow 0 = v(R_0) \Longrightarrow c = \frac{g}{\gamma+\varepsilon}R_0^{1+\frac{\varepsilon}{\gamma}}$$

$$\Longrightarrow v(R) = \frac{g}{\gamma+\varepsilon}R_0\left\{\left(\frac{R_0}{R}\right)^{\frac{\varepsilon}{\gamma}} - \frac{R}{R_0}\right\}\ .$$

Finally we still insert $R(t) = R_0 + \gamma t$:

$$v(t) = \frac{gR_0}{\gamma + \varepsilon}\left\{\left(1 + \frac{\gamma}{R_0}t\right)^{-\frac{\varepsilon}{\gamma}} - \left(1 + \frac{\gamma}{R_0}t\right)\right\} .$$

Limiting cases:

1. $\gamma t \ll R_0$:

$$\begin{aligned} v(t) &\approx \frac{gR_0}{\gamma + \varepsilon}\left\{1 - \frac{\varepsilon}{\gamma}\frac{\gamma}{R_0}t + \cdots - 1 - \frac{\gamma}{R_0}t\right\} \\ &= \frac{gR_0}{\gamma + \varepsilon}\left(-\frac{\gamma}{R_0}t\right)\left(1 + \frac{\varepsilon}{\gamma}\right) \end{aligned}$$

$$\Longrightarrow \quad v(t) \approx -gt .$$

The waterdrop is still falling almost freely; v is so small that the momentum change due to the increase of mass persists insignificantly. The friction force, either, does not play a big role.

2. $\gamma t \gg R_0$:

$$v(t) \approx -\frac{gR_0}{\gamma + \varepsilon}\frac{\gamma}{R_0}t = -\frac{g\gamma}{\gamma + \varepsilon} \cdot t .$$

Because of mass increase and friction force the acceleration decreases from g to $g\frac{\gamma}{\gamma+\varepsilon}$.

Solution 2.3.18

1. An electromagnetic field exerts on a particle of mass m and charge q the so-called *'Lorentz force'*:

$$\mathbf{F} = q\,\mathbf{E} + q(\mathbf{v} \times \mathbf{B}) .$$

Here it is assumed that the magnetic induction $\mathbf{B}$ is homogeneous,

$$\mathbf{B} = B\,\mathbf{e}_3 ,$$

and $\mathbf{E} \equiv 0$. The equation of motion therefore reads:

$$m\ddot{\mathbf{r}} = q(\dot{\mathbf{r}} \times \mathbf{B}) .$$

That is equivalent to

$$\frac{d}{dt}\dot{\mathbf{r}} = \omega(\dot{\mathbf{r}} \times \mathbf{e}_3) ; \quad \omega = \frac{q}{m}B .$$

2.

$$
\begin{aligned}
&\dot{\mathbf{r}} \cdot (\dot{\mathbf{r}} \times \mathbf{B}) = 0 \\
&\Longrightarrow \dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} = 0 \\
&\Longrightarrow \frac{d}{dt}(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}) = 0 \\
&\Longrightarrow \frac{d}{dt}|\dot{\mathbf{r}}| = 0 \quad \Longrightarrow \quad |\dot{\mathbf{r}}| = \text{const}
\end{aligned}
$$

3.

$$
\begin{aligned}
&\sphericalangle(\dot{\mathbf{r}}, \mathbf{B}) = \text{const} \\
&\Longleftrightarrow \cos(\dot{\mathbf{r}}, \mathbf{B}) = \text{const} \\
&\Longleftrightarrow \dot{r} \cdot \mathbf{B} = \text{const} \\
&\Longleftrightarrow \frac{d}{dt}(\dot{\mathbf{r}} \cdot \mathbf{B}) = 0 = \ddot{\mathbf{r}} \cdot \mathbf{B} \\
&\Longleftrightarrow \frac{q}{m}(\dot{\mathbf{r}} \times \mathbf{B}) \cdot \mathbf{B} = 0 \,.
\end{aligned}
$$

4. Since **B** is assumed to be time-independent it holds:

$$
q(\dot{\mathbf{r}} \times \mathbf{B}) = q\frac{d}{dt}(\mathbf{r} \times \mathbf{B}) \,.
$$

Therewith the equation of motion in part 1. can immediately be integrated:

$$
m\,\dot{\mathbf{r}} = q(\mathbf{r} \times \mathbf{B}) + \mathbf{c} \,.
$$

The constant vector **c** is fixed by initial conditions:

$$
t = 0\,; \quad m\,\mathbf{v}_0 = q(\mathbf{r}_0 \times \mathbf{B}) + \mathbf{c} \,.
$$

Thus a first intermediate result is found:

$$
m\,\dot{\mathbf{r}} = q(\mathbf{r} \times \mathbf{B}) + [m\,\mathbf{v}_0 - q(\mathbf{r}_0 \times \mathbf{B})] \,.
$$

5.

$$
\dot{\mathbf{r}} = \dot{\mathbf{r}}_{\parallel} + \dot{\mathbf{r}}_{\perp} \,.
$$

The Lorentz force has no field-parallel component. Therefore we expect that $\dot{\mathbf{r}}_{\|}$ is about a constant vector:

$$|\dot{\mathbf{r}}_{\|}| = \dot{\mathbf{r}} \cdot \mathbf{e}_3 = \frac{q}{m}(\mathbf{r} \times \mathbf{B}) \cdot \mathbf{e}_3 + \mathbf{v}_0 \cdot \mathbf{e}_3 - q(\mathbf{r}_0 \times \mathbf{B}) \cdot \mathbf{e}_3 =$$
$$= \mathbf{v}_0 \cdot \mathbf{e}_3 = \text{const} \equiv v_{\|} \ .$$

The magnitude of $\dot{\mathbf{r}}_{\|}$ thus is constant, the direction because of $\mathbf{B} = \text{const}$ anyway:

$$|\dot{\mathbf{r}}|^2 = |\dot{\mathbf{r}}_{\|}|^2 + |\dot{\mathbf{r}}_{\perp}|^2 \Longrightarrow |\dot{\mathbf{r}}_{\perp}| = \text{const} = v_{\perp} \ .$$

It follows that $\dot{\mathbf{r}}_{\perp}$ is a vector with constant magnitude lying in the plane which is perpendicular to $\mathbf{B}$.

6. It is shown in part 5.:

$$\dot{\mathbf{r}} = (v_{\perp} \cos\varphi(t), v_{\perp} \sin\varphi(t), v_{\|})$$
$$\Longrightarrow \ddot{\mathbf{r}} = v_{\perp}\dot{\varphi}(t)(-\sin\varphi(t), \cos\varphi(t), 0) \ .$$

On the other hand, from part 1. it also holds:

$$\ddot{\mathbf{r}} = \omega(\dot{\mathbf{r}} \times \mathbf{e}_3) = \omega(v_{\perp} \sin\varphi(t), -v_{\perp} \cos\varphi(t), 0) \ .$$

Comparison leads to:

$$\dot{\varphi}(t) = -\omega \Longrightarrow \varphi(t) = -\omega\, t + \alpha \ .$$

7. In 5. it was shown:

$$|\dot{\mathbf{r}}_{\perp}| = v_{\perp} = \text{const} \Longrightarrow v_{\perp} = |\mathbf{v}_{0\perp}| = |[\mathbf{e}_3 \times (\mathbf{v}_0 \times \mathbf{e}_3)]| = |(\mathbf{v}_0 \times \mathbf{e}_3)| \ .$$

Hence we can write:

$$\mathbf{e}_2 = \frac{1}{v_{\perp}}[\mathbf{e}_3 \times (\mathbf{v}_0 \times \mathbf{e}_3)] \ ; \quad \mathbf{e}_1 = \frac{1}{v_{\perp}}(\mathbf{v}_0 \times \mathbf{e}_3) \ .$$

Now it is

$$\varphi(t=0) = \sphericalangle(\dot{\mathbf{r}}_{\perp}(t=0), \mathbf{e}_1) = \sphericalangle(\mathbf{v}_{0\perp}, \mathbf{e}_1) = \frac{\pi}{2}$$
$$\Longrightarrow \varphi(t) = -\omega t + \frac{\pi}{2} \ .$$

Therewith the complete solution for $\dot{\mathbf{r}}(t)$ reads:

$$\dot{\mathbf{r}}(t) = (\mathbf{v}_0 \times \mathbf{e}_3)\sin\omega t + [\mathbf{e}_3 \times (\mathbf{v}_0 \times \mathbf{e}_3)]\cos\omega t + (\mathbf{v}_0 \cdot \mathbf{e}_3)\,\mathbf{e}_3 \ .$$

8. A further time-integration yields:

$$\mathbf{r}(t) = -\frac{1}{\omega}\cos\omega t \cdot (\mathbf{v}_0 \times \mathbf{e}_3) + \frac{1}{\omega}\sin\omega t \cdot [\mathbf{e}_3 \times (\mathbf{v}_0 \times \mathbf{e}_3)] + (\mathbf{v}_0 \cdot \mathbf{e}_3)\, t\, \mathbf{e}_3 + \bar{\mathbf{r}}_0 \ .$$

The initial condition

$$\mathbf{r}_0 = \mathbf{r}(t=0) = -\frac{1}{\omega}(\mathbf{v}_0 \times \mathbf{e}_3) + \bar{\mathbf{r}}_0$$

then leads to the complete solution for the trajectory:

$$\mathbf{r}(t) = -\frac{1}{\omega}(\cos\omega t - 1)(\mathbf{v}_0 \times \mathbf{e}_3) + \frac{1}{\omega}\sin\omega t \cdot [\mathbf{e}_3 \times (\mathbf{v}_0 \times \mathbf{e}_3)] + (\mathbf{v}_0 \cdot \mathbf{e}_3)\, t\, \mathbf{e}_3 + \mathbf{r}_0 \ .$$

9. Movement in a plane perpendicular to the field means in the first step:

$$\dot{\mathbf{r}}(t) \perp \mathbf{B} \quad \text{or} \quad v_{\parallel} = 0 \ .$$

According to part 5. this is exactly then the case if

$$\mathbf{v}_0 \perp \mathbf{B}, \mathbf{e}_3 \implies v_{\perp} = v_0 \ .$$

is given because that means:

$$\hat{\mathbf{r}}(t) \equiv \mathbf{r}(t) - \left(\mathbf{r}_0 + \frac{v_0}{\omega}\mathbf{e}_1\right) = \frac{v_0}{\omega}(-\cos\omega t, \sin\omega t, 0)$$

It corresponds to a circular motion in a plane perpendicular to **B** with the frequency

$$\omega = \frac{qB}{m}$$

and the radius

$$R = \frac{v_0}{\omega} = \frac{v_0\, m}{q\, B} \ .$$

10. The general solution in part 8.

$$\hat{\mathbf{r}}(t) = \left(-\frac{v_{\perp}}{\omega}\cos\omega t, \quad \frac{v_{\perp}}{\omega}\sin\omega t, \quad (\mathbf{v}_0 \cdot \mathbf{e}_3)t\right)$$

represents a helical line.

Solution 2.3.19 Antiderivative (*'potential'*) of the force:

$$V(x) \equiv -\int^{x} F(x')dx' = \frac{1}{2}x^2 + \frac{1}{4}\gamma x^4 \quad ; \quad F(x) = -\frac{d}{dx}V(x)\ .$$

We multiply the equation of motion by $\dot{x}$:

$$\begin{aligned} m\ddot{x} \cdot \dot{x} = F(x) \cdot \dot{x} &= -\frac{d}{dx}V(x) \cdot \dot{x} \\ &= -\frac{d}{dt}V(x(t)) = \frac{d}{dt}\left(\frac{m}{2}\dot{x}^2\right) \end{aligned}$$

$$\curvearrowright \quad \frac{d}{dt}\left(\frac{m}{2}\dot{x}^2 + V(x)\right) = 0$$

$$\curvearrowright \quad \frac{m}{2}\dot{x}^2 + V(x) = E\ .$$

E is here a constant of integration (*total energy*).

Separation of variables with subsequent integration:

$$t - t_0 = \int_{x_0}^{x} \frac{dx'}{\sqrt{\frac{2}{m}(E - V(x'))}} = \sqrt{\frac{m}{2E}} \int_{x_0}^{x} \frac{dx'}{\sqrt{1 - \frac{V(x')}{E}}}\ .$$

1. Substitution:

$$\frac{V(x')}{E} = \sin^2\varphi\ .$$

Possible since $V(x) > 0$ and therewith $0 \le \frac{V(x')}{E} \le 1$. That means:

$$\frac{1}{4}\gamma x^4 + \frac{1}{2}kx^2 - E\sin^2\varphi = 0 \quad \curvearrowright \quad x^4 + \frac{2k}{\gamma}x^2 = \frac{4E}{\gamma}\sin^2\varphi\ .$$

Solution:

$$x^2 = -\frac{k}{\gamma} + \sqrt{\frac{k^2}{\gamma^2} + \frac{4E}{\gamma}\sin^2\varphi}\ .$$

x is real, therewith x^2 positive; so only the positive root is relevant.

$$x^2 = -\frac{k}{\gamma}\left(1 - \sqrt{1 + \frac{4E\gamma}{k^2}\sin^2\varphi}\right)\ .$$

Because of $\frac{4E\gamma}{k^2} \ll 1$ the root can be expanded:

$$x^2 \approx -\frac{k}{\gamma}\left(1-\left(1+\frac{2E\gamma}{k^2}\,\sin^2\varphi\right)\right) = \frac{2E}{k}\sin^2\varphi \ .$$

Thus it holds approximately:

$$x \approx \sqrt{\frac{2E}{k}}\,\sin\varphi \ \ ; \ \ dx \approx \sqrt{\frac{2E}{k}}\,\cos\varphi d\varphi \ .$$

Using these expressions we are able to perform the above integrations:

$$t-t_0 = \sqrt{\frac{m}{2E}}\sqrt{\frac{2E}{k}}\int_{\varphi_0}^{\varphi}\frac{\cos\varphi' d\varphi'}{\sqrt{1-\sin^2\varphi'}} = \sqrt{\frac{m}{k}}\int_{\varphi_0}^{\varphi} d\varphi' = \sqrt{\frac{m}{k}}\,(\varphi-\varphi_0) \ .$$

Turning points of the oscillation are given by $\dot{x} = 0$, i.e. by $E = V(x)$ and $\sin^2\varphi = 1$, respectively. That means:

$$\varphi_{1,2} = \pm\frac{\pi}{2} \ .$$

Hence, the oscillation period τ follows from:

$$\frac{1}{2}\tau = \sqrt{\frac{m}{k}}(\varphi_2-\varphi_1) = \pi\,\sqrt{\frac{m}{k}} \ .$$

We recognize that to a first approximation the oscillation period does not at all deviate from that of the purely harmonic oscillator:

$$\tau = 2\pi\,\sqrt{\frac{m}{k}} \ .$$

2. This becomes of course different when we go one step further in the above expansion for x^2:

$$\begin{aligned}
x^2 &= -\frac{k}{\gamma}\left(1-\sqrt{1+\frac{4E\gamma}{k^2}\sin^2\varphi}\right)\\
&\approx -\frac{k}{\gamma}\left(1-\left(1+\frac{2E\gamma}{k^2}\,\sin^2\varphi-\frac{E^2\gamma^2}{2k^4}\sin^4\varphi\right)\right)\\
&= \frac{2E}{k}\,\sin^2\varphi-\frac{E^2\gamma}{2k^3}\sin^4\varphi\\
&= \frac{2E}{k}\,\sin^2\varphi\left(1-\frac{\gamma E}{4k^2}\sin^2\varphi\right)
\end{aligned}$$

$$\curvearrowright \quad x \approx \sqrt{\frac{2E}{k}}\,\sin\varphi\,\sqrt{1-\frac{\gamma E}{4k^2}\sin^2\varphi}$$
$$\approx \sqrt{\frac{2E}{k}}\,\sin\varphi\left(1-\frac{\gamma E}{8k^2}\sin^2\varphi\right)$$

$$\curvearrowright \quad dx \approx \sqrt{\frac{2E}{k}}\left(1-\frac{3\gamma E}{8k^2}\sin^2\varphi\right)\cos\varphi d\varphi\ .$$

In the next step we get with the above used separation of variables:

$$t-t_0 = \sqrt{\frac{m}{2E}}\sqrt{\frac{2E}{k}}\int_{\varphi_0}^{\varphi}\frac{\cos\varphi' d\varphi'}{\sqrt{1-\sin^2\varphi'}}\left(1-\frac{3\gamma E}{8k^2}\sin^2\varphi'\right)$$
$$= \sqrt{\frac{m}{k}}\int_{\varphi_0}^{\varphi} d\varphi'\left(1-\frac{3\gamma E}{8k^2}\sin^2\varphi'\right)\ .$$

The turning points are the same as in part 1.:

$$\frac{1}{2}\tau = \sqrt{\frac{m}{k}}\left(\varphi'-\frac{3\gamma E}{8k^2}\left(-\frac{1}{2}\sin\varphi'\cos\varphi'+\frac{\varphi'}{2}\right)\right)_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}$$
$$= \sqrt{\frac{m}{k}}\left(\pi-\frac{3\gamma E}{16k^2}\pi\right)\ .$$

Finally we get as result for the oscillation period of the (weakly) anharmonic oscillator:

$$\tau = 2\pi\sqrt{\frac{m}{k}}\left(1-\frac{3\gamma E}{16k^2}\right)\ .$$

Section 2.4

Solution 2.4.1

1.

$$(\nabla\times\mathbf{F})_x = \frac{\partial F_z}{\partial y}-\frac{\partial F_y}{\partial z} = 6\alpha_1 xyz^2-6\alpha_1 xyz^2 = 0\ ,$$
$$(\nabla\times\mathbf{F})_y = \frac{\partial F_x}{\partial z}-\frac{\partial F_z}{\partial x} = 3\alpha_1 y^2z^2-12\alpha_2 xz-3\alpha_1 y^2z^2+12\alpha_2 xz = 0\ ,$$
$$(\nabla\times\mathbf{F})_z = \frac{\partial F_y}{\partial x}-\frac{\partial F_x}{\partial y} = 2\alpha_1 yz^3-2\alpha_1 yz^3 = 0$$
$$\Longrightarrow \nabla\times\mathbf{F} = \mathbf{0} \Longrightarrow \mathbf{F}\ \text{conservative}\ .$$

2. Parametrization:

$$\left.\begin{array}{l} C_1 : \ \mathbf{r}(t) = (x_0 t, 0, 0) \ , \\ C_2 : \ \mathbf{r}(t) = (x_0, y_0 t, 0) \ , \\ C_3 : \ \mathbf{r}(t) = (x_0, y_0, z_0 t) \ . \end{array}\right\} \quad 0 \le t \le 1 \ .$$

We calculate the work-contributions executed on the three partial paths:

$$W(C_1) = -\int\limits_{C_1} \mathbf{F}[\mathbf{r}(t)] \cdot \dot{\mathbf{r}}(t) dt = -x_0 \int\limits_0^1 F_x(x_0 t, 0, 0) dt = 0 \ ,$$

$$W(C_2) = -\int\limits_{C_2} \mathbf{F}[\mathbf{r}(t)] \cdot \dot{\mathbf{r}}(t) dt = -y_0 \int\limits_0^1 F_y(x_0, y_0 t, 0) dt = 0 \ ,$$

$$W(C_3) = -\int\limits_{C_3} \mathbf{F}[\mathbf{r}(t)] \cdot \dot{\mathbf{r}}(t) dt = -z_0 \int\limits_0^1 F_z(x_0, y_0, z_0 t) dt =$$

$$= -z_0 \int\limits_0^1 \left(3\alpha_1 x_0 y_0^2 z_0^2 t^2 - 6\alpha_2 x_0^2 z_0 t\right) dt = -\alpha_1 x_0 y_0^2 z_0^3 + 3\alpha_2 x_0^2 z_0^2$$

$$\Longrightarrow W = 3\alpha_2 x_0^2 z_0^2 - \alpha_1 x_0 y_0^2 z_0^3 \ .$$

3. **F** is conservative and therefore possesses a potential:

$$V(\mathbf{r}) = -\alpha_1 x y^2 z^3 + 3\alpha_2 x^2 z^2 + V_0 \ .$$

Solution 2.4.2 Path C_1: parameter representation:

$$C_{11} : \qquad \mathbf{r}(t) = (1-t)\mathbf{r}_1 \ ; \quad \dot{\mathbf{r}}(t) = -\mathbf{r}_1 \ ; \quad (0 \le t \le 1) \ ,$$

$$C_{12} : \qquad \mathbf{r}(t) = t \cdot \mathbf{r}_2 \ ; \quad \dot{\mathbf{r}}(t) = \mathbf{r}_2 \ ; \quad (0 \le t \le 1) \ .$$

work:

$$W_{C_1} = -\alpha \int\limits_{C_{11}} \mathbf{r} \cdot d\mathbf{r} - \alpha \int\limits_{C_{12}} \mathbf{r} \cdot d\mathbf{r} = \alpha \, r_1^2 \int\limits_0^1 (1-t) \, dt - \alpha \, r_2^2 \int\limits_0^1 t \, dt$$

$$= \frac{1}{2} \alpha \left(r_1^2 - r_2^2\right) \ .$$

Path C_2: parameter representation:

$$
\begin{aligned}
C_{21}: \qquad & \mathbf{r}(t) = r_1[\cos(\varphi \cdot t),\ \sin(\varphi \cdot t)]\ ; \quad (0 \le t \le t)\ , \\
& \dot{\mathbf{r}}(t) = r_1\varphi[-\sin(\varphi \cdot t),\ \cos(\varphi \cdot t)] \\
\Longrightarrow\ & \mathbf{r}(t) \cdot \dot{\mathbf{r}}(t) = 0\ . \\
C_{22}: \qquad & \mathbf{r}(t) = (\mathbf{r}_2 - \mathbf{r}_A)\, t + \mathbf{r}_A; \quad (0 \le t \le 1)\ , \\
& \dot{\mathbf{r}}(t) = \mathbf{r}_2 - \mathbf{r}_A \\
\Longrightarrow\ & \mathbf{r}(t) \cdot \dot{\mathbf{r}}(t) = (\mathbf{r}_2 - \mathbf{r}_A)^2 t + \mathbf{r}_A \cdot\ (\mathbf{r}_2 - \mathbf{r}_A)\ .
\end{aligned}
$$

work:

$$
\begin{aligned}
W_{C_2} &= 0 - \alpha(\mathbf{r}_2 - \mathbf{r}_A)^2 \int\limits_0^1 t\, dt - \alpha\, \mathbf{r}_A \cdot\ (\mathbf{r}_2 - \mathbf{r}_A) \int\limits_0^1 dt \\
&= \frac{1}{2}\alpha \left(r_1^2 - r_2^2\right)\ ; \quad \left(r_A^2 = r_1^2\right)\ .
\end{aligned}
$$

Path C_3: parameter representation:

$$
\begin{aligned}
C_{31}: \qquad & \mathbf{r}(t) = (\mathbf{r}_A - \mathbf{r}_1)t + \mathbf{r}_1\ ; \quad (0 \le t \le 1)\ , \\
& \dot{\mathbf{r}}(t) = \mathbf{r}_A - \mathbf{r}_1 \\
\Longrightarrow\ & \mathbf{r}(t) \cdot \dot{\mathbf{r}}(t) = (\mathbf{r}_A - \mathbf{r}_1)^2\, t + \mathbf{r}_1 \cdot\ (\mathbf{r}_A - \mathbf{r}_1)\ . \\
C_{32}: \qquad & \text{as } C_{22}\ .
\end{aligned}
$$

work:

$$
\begin{aligned}
W_{C_3} &= W_{C_{31}} + W_{C_{32}}\ , \\
W_{C_{31}} &= -\frac{1}{2}\alpha(\mathbf{r}_A - \mathbf{r}_1)^2 - \alpha\, \mathbf{r}_1 \cdot\ (\mathbf{r}_A - \mathbf{r}_1) = 0 \\
\Longrightarrow W_{C_3} &= W_{C_{32}} = W_{C_{22}} = W_{C_2} = \frac{1}{2}\alpha \left(r_1^2 - r_2^2\right)\ .
\end{aligned}
$$

The carried out work is obviously the same on each of the three paths. However, that is not at all astonishing since

$$
\nabla \times \mathbf{F}(\mathbf{r}) = \nabla \times (\alpha \mathbf{r}) = 0
$$

(see Exercise 1.3.5). $\mathbf{F}(\mathbf{r})$ is therefore a conservative force and

$$\int_A^B \mathbf{F} \cdot d\mathbf{r}$$

is path-independent! Thus there exists a potential $V = V(\mathbf{r})$,

$$\mathbf{F}(\mathbf{r}) = -\left(\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}, \frac{\partial V}{\partial x_3}\right) ,$$

which can be determined as follows:

$$-\frac{\partial V}{\partial x_1} = \alpha\, x_1 \Longrightarrow V(x_1, x_2, x_3) = -\frac{\alpha}{2}x_1^2 + g(x_2, x_3) ,$$
$$-\frac{\partial V}{\partial x_2} = \alpha\, x_2 = -\frac{\partial g}{\partial x_2} \Longrightarrow V(x_1, x_2, x_3) = -\frac{\alpha}{2}\left(x_1^2 + x_2^2\right) + f(x_3) ,$$
$$-\frac{\partial V}{\partial x_3} = \alpha\, x_3 = -\frac{df}{dx_3} \Longrightarrow V(x_1, x_2, x_3) = -\frac{\alpha}{2}\left(x_1^2 + x_2^2 + x_3^2\right) + C .$$

Hence, the potential of the force $\mathbf{F}$ reads:

$$V(\mathbf{r}) = -\frac{\alpha}{2}r^2 + C .$$

The work

$$W_{P_1 \to P_2} = V(P_2) - V(P_1) = \frac{1}{2}\alpha\left(r_1^2 - r_2^2\right)$$

is path-independent!

Solution 2.4.3

1. No, because:

$$\nabla \times \mathbf{F} = f\left(0 + \frac{9}{\alpha^2}y, 0 - \frac{8}{\alpha^3}z^2, 0 - \frac{2}{\alpha}\right)$$
$$= f\left(\frac{9}{\alpha^2}y, -\frac{8}{\alpha^3}z^2, -\frac{2}{\alpha}\right) \not\equiv (0, 0, 0)$$

   Line integrals will be path-dependent!

2. Parametrization of the path:

$$\begin{aligned}
\mathbf{r}(t) &= (\alpha t, \alpha t, \alpha t)\ ; \quad 0 \le t \le 1 \\
\Longrightarrow\ \frac{\partial \mathbf{r}}{\partial t} &= (\alpha, \alpha, \alpha) \\
\mathbf{F} &= f\left(3t^2 + 2t, -9t^2, 8t^3\right) \\
\Longrightarrow\ \mathbf{F}\cdot\frac{\partial \mathbf{r}}{\partial t} &= f\alpha\left(3t^2 + 2t - 9t^2 + 8t^3\right) \\
&= f\alpha\left(8t^3 - 6t^2 + 2t\right) \\
\Longrightarrow\ W_1 &= -\int_0^1 \mathbf{F}\cdot\frac{\partial \mathbf{r}}{\partial t}dt \\
&= -f\alpha\left(\frac{8}{4} - \frac{6}{3} + \frac{2}{2}\right) \\
&= -f\alpha
\end{aligned}$$

3. Parametrization:

$$\mathbf{r}(t) = \begin{cases} (\alpha t, 0, 0) \\ (\alpha, \alpha t, 0) \\ (\alpha, \alpha, \alpha t) \end{cases} \quad 0 \le t \le 1$$

$$W_2 = W_{2x} + W_{2y} + W_{2z}$$

$$W_{2x}: \qquad \mathbf{F} = f(3t^2, 0, 0)\ ; \quad \frac{\partial \mathbf{r}}{\partial t} = (\alpha, 0, 0)$$

$$\Longrightarrow\ W_{2x} = -\int_0^1 f\alpha 3t^2 dt = -f\alpha$$

$$W_{2y}: \qquad \mathbf{F} = f(3 + 2t, 0, 0)\ ; \quad \frac{\partial \mathbf{r}}{\partial t} = (0, \alpha, 0)$$

$$\Longrightarrow\ W_{2y} = -\int_0^1 f\alpha(1\cdot 0)dt = 0$$

$$W_{2z}: \qquad \mathbf{F} = f(5, -9t, 8t^2)\ ; \quad \frac{\partial \mathbf{r}}{\partial t} = (0, 0, \alpha)$$

$$\Longrightarrow\ W_{2z} = -\int_0^1 8t^2\alpha f dt = -\frac{8}{3}f\alpha$$

It follows:

$$W_2 = -\frac{11}{3} f\alpha$$

4.

$$\begin{aligned}
\mathbf{r}(t) &= \alpha(t, t^2, t^4) \quad 0 \le t \le 1 \\
\Longrightarrow \quad \mathbf{F} &= f(3t^2 + 2t^2, -9t^6, 8t^9) \\
\frac{\partial \mathbf{r}}{\partial t} &= \alpha(1, 2t, 4t^3) \\
\Longrightarrow \quad \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial t} &= f\alpha(5t^2 - 18t^7 + 32t^{12}) \\
\Longrightarrow \quad W_3 &= -f\alpha \int_0^1 (5t^2 - 18t^7 + 32t^{12}) dt \\
&= -f\alpha \left(\frac{5}{3} - \frac{18}{8} + \frac{32}{13} \right) \\
&= -f\alpha \frac{293}{156}
\end{aligned}$$

5.

$$\begin{aligned}
\mathbf{r}(\varphi) &= \alpha(\cos\varphi, \sin\varphi, 0) \quad 0 \le \varphi \le 2\pi \\
\mathbf{F} &= f\left(3\cos^2\varphi + 2\sin\varphi, 0, 0\right) \\
\frac{\partial \mathbf{r}}{\partial \varphi} &= \alpha(-\sin\varphi, \cos\varphi, 0) \\
\Longrightarrow \quad \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial \varphi} &= f\alpha(-3\cos^2\varphi \sin\varphi - 2\sin^2\varphi) \\
&= f\alpha \left(\frac{d}{d\varphi} \cos^3\varphi - 2\sin^2\varphi \right) \\
\Longrightarrow \quad W &= -\oint \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial \varphi} d\varphi \\
&= 2f\alpha \int_0^{2\pi} \sin^2\varphi d\varphi \\
&= 2f\alpha \left(-\frac{1}{2} \sin\varphi \cos\varphi + \frac{\varphi}{2} \right)_0^{2\pi} \\
&= 2f\alpha\pi \qquad \text{(path-dependent!)}
\end{aligned}$$

Solution 2.4.4

1.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \partial_x & \partial_y & \partial_z \\ \alpha xy & -\alpha z & 0 \end{vmatrix}$$
$$= \mathbf{e}_x(0 + 2\alpha z) + \mathbf{e}_y(0 - 0) + \mathbf{e}_z(0 - \alpha x) \equiv \alpha(2z, 0, -x)$$
$$\not\equiv 0 \, .$$

$\curvearrowright$ The force is **not** conservative!

2. *'direct path'*: $(0,0,0) \longrightarrow (1,1,3)$
We choose as parameter the (dimensionless) *'time'* t.

$$\mathbf{r}(t) = (t, t, 3t)$$
$$\dot{\mathbf{r}} = (1, 1, 3)$$
$$\mathbf{F}(\mathbf{r}(t)) = \alpha(t^2, -9t^2, 0)$$
$$\mathbf{F}(\mathbf{r}(t)) \cdot \dot{\mathbf{r}}(t) = \alpha(t^2 - 9t^2) = -8\alpha t^2$$
$$\curvearrowright \quad W_{(b)} = +\alpha \int_0^1 8t^2 dt = \frac{8}{3}\alpha \, .$$

3. Parametrization of the *'curved'* path $[(0,0,0) \longrightarrow (1,1,3)]$:

$$0 \leq t \leq 1 \quad : \; y = t \; , \; x = t^2 \; , \; z = 3\sqrt{t} \, .$$

Therewith holds:

$$\mathbf{r}(t) = (t^2, t, 3\sqrt{t})$$
$$\dot{\mathbf{r}} = (2t, 1, \frac{3}{2\sqrt{t}})$$
$$\mathbf{F}(\mathbf{r}(t)) = \alpha(t^3, -9t, 0)$$
$$\mathbf{F}(\mathbf{r}(t)) \cdot \dot{\mathbf{r}}(t) = \alpha(2t^4 - 9t)$$
$$\curvearrowright \quad W_{(c)} = +\alpha \int_0^1 (2t^4 - 9t) dt = \alpha\left(\frac{2}{5} - \frac{9}{2}\right) = \frac{41}{10}\alpha$$
$$\neq W_{(b)} \, .$$

Solution 2.4.5

$$\nabla \times \mathbf{F}(\mathbf{r}) = \nabla \times (\mathbf{a} \times \mathbf{r}) = 2\mathbf{a}$$

[see Exercise 1.5.7]. This force is not conservative. Consequently the line integral will be path-dependent. We use for the various paths the same parameter-representations as in Exercise 2.4.2.

Path C_1:

$$\int_{C_{11}} \mathbf{F}\cdot d\mathbf{r} = +(\mathbf{a}\times\mathbf{r}_1)\cdot\mathbf{r}_1 \int_0^1 dt(1-t) = 0\ ,$$

$$\int_{C_{12}} \mathbf{F}\cdot d\mathbf{r} = -(\mathbf{a}\times\mathbf{r}_2)\cdot\mathbf{r}_2 \int_0^1 dt\ t = 0$$

$$\Longrightarrow W_{C_1} = 0\ .$$

Path C_2:

$$C_{21}: \qquad (\mathbf{a}\times\mathbf{r}) = r_1\left(-a_3\sin(\varphi t), a_3\cos(\varphi t), a_1\sin(\varphi t) - a_2\cos(\varphi t)\right)\ ,$$

$$(\mathbf{a}\times\mathbf{r})\cdot\dot{\mathbf{r}} = r_1^2\varphi[a_3\sin^2(\varphi t) + a_3\cos^2(\varphi t)] = a_3 r_1^2\varphi$$

$$\Longrightarrow W_{C_{21}} = -a_3 r_1^2\varphi \int_0^1 dt = -a_3 r_1^2\varphi\ .$$

$$C_{22}: \qquad (\mathbf{a}\times\mathbf{r})\cdot\dot{\mathbf{r}} = t[\mathbf{a}\times(\mathbf{r}_2-\mathbf{r}_A)]\cdot(\mathbf{r}_2-\mathbf{r}_A) + (\mathbf{a}\times\mathbf{r}_A)\cdot(\mathbf{r}_2-\mathbf{r}_A) =$$

$$= (\mathbf{a}\times\mathbf{r}_A)\cdot\mathbf{r}_2 = 0\ , \quad \text{since } \mathbf{r}_A \uparrow\uparrow \mathbf{r}_2$$

$$\Longrightarrow W_{C_{22}} = 0\ .$$

Altogether it holds for the path C_2:

$$W_{C_2} = -a_3 r_1^2\varphi\ .$$

Path C_3:

$$C_{31}: \qquad (\mathbf{a}\times\mathbf{r})\cdot\dot{\mathbf{r}} = (\mathbf{a}\times\mathbf{r}_1)\cdot(\mathbf{r}_A-\mathbf{r}_1) = (\mathbf{a}\times\mathbf{r}_1)\cdot\mathbf{r}_A$$

$$\Longrightarrow W_{C_{31}} = -(\mathbf{a}\times\mathbf{r}_1)\cdot\mathbf{r}_A\ ;$$

C_{32} as C_{22}, therefore $W_{C_{32}} = 0$

$$\Longrightarrow W_{C_3} = -(\mathbf{a}\times\mathbf{r}_1)\cdot\mathbf{r}_A\ .$$

The works to be carried out on the various paths are obviously rather different!

Solution 2.4.6

1.

$$
\begin{aligned}
\nabla \times \mathbf{F} &= \left(\frac{\partial b}{\partial y} - \frac{\partial ax}{\partial z}, \frac{\partial ay}{\partial z} - \frac{\partial b}{\partial x}, \frac{\partial ax}{\partial x} - \frac{\partial ay}{\partial y} \right) = \\
&= (0, 0, a - a) = 0 \\
\Longrightarrow \quad & \mathbf{F} \text{ is conservative!}
\end{aligned}
$$

2. Parameter-representation of the path:

$$
\begin{aligned}
\mathbf{r}(\alpha) &= (\alpha x, \alpha y, \alpha z) \; ; \quad (0 \le \alpha \le 1) \\
\Longrightarrow \frac{d\mathbf{r}}{d\alpha} &= (x, y, z) \; ; \quad \mathbf{F}[\mathbf{r}(\alpha)] = (a\alpha y, a\alpha x, b) \\
\Longrightarrow \mathbf{F} \cdot \frac{d\mathbf{r}}{d\alpha} &= 2a\alpha xy + bz \; .
\end{aligned}
$$

Therewith we get the needed work for moving the mass point:

$$
W_{P_0 \to P} = \int_0^1 \mathbf{F} \cdot \frac{d\mathbf{r}}{d\alpha} \, d\alpha = axy + bz \; .
$$

3.

$$
\begin{aligned}
-\frac{\partial V}{\partial x} &= ay \Longrightarrow V = -axy + g(yz) \; , \\
-\frac{\partial V}{\partial y} &= ax = ax + \frac{\partial g}{\partial y} \Longrightarrow \frac{\partial g}{\partial y} = 0 \Longrightarrow V = -axy + g(z) \; , \\
-\frac{\partial V}{\partial z} &= b \Longrightarrow g(z) = -bz + c \\
\Longrightarrow V(\mathbf{r}) &= -axy - bz + c \; .
\end{aligned}
$$

4. The work is the same as in part 2. because $\mathbf{F}$ is conservative.

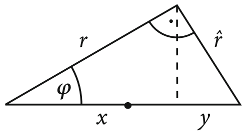
Solution 2.4.7

1.

$$
\mathbf{F} = -\nabla V = -(kx, ky, kz) = -k\mathbf{r} \; .
$$

It is the potential of the harmonic oscillator. $\mathbf{F}(\mathbf{r})$ is a central force.

Fig. A.30



2.

$$\frac{\partial}{\partial x} V(\mathbf{r}) = \frac{m}{2}[2\omega_x(\boldsymbol{\omega} \cdot \mathbf{r}) - 2\omega^2 x] \; .$$

Analogous expressions hold for the other two components:

$$\mathbf{F}(\mathbf{r}) = -\nabla V(\mathbf{r}) = -m[\boldsymbol{\omega}(\boldsymbol{\omega} \cdot \mathbf{r}) - \omega^2 \mathbf{r}] = -m[\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})] \; .$$

It is about the potential of the centrifugal force (2.79). In this case $\mathbf{F}$ is not a central force!

Solution 2.4.8

1. The angle in the semicircle is a right angle (90°) (Thales theorem, see Exercise 1.3.5) (Fig. A.30).

$$\begin{aligned} x &= r\cos\varphi \\ y &= \hat{r}\cos\left(\frac{\pi}{2} - \varphi\right) = \hat{r}\sin\varphi \\ \hat{r} &= \sqrt{4R^2 - r^2} \\ 2R &= x + y \\ &= r\cos\varphi + \sqrt{4R^2 - r^2}\,\sin\varphi \end{aligned}$$

$$\begin{aligned} \Longrightarrow \quad \left(4R^2 - r^2\right)\sin^2\varphi &= 4R^2 + r^2\cos^2\varphi - 4Rr\cos\varphi \\ 4Rr\cos\varphi &= 4R^2\left(1 - \sin^2\varphi\right) + r^2 \\ 0 &= (2R\cos\varphi - r)^2 \\ \Longrightarrow \quad r &= r(\varphi) = 2R\cos\varphi \; . \end{aligned}$$

2. Conservative central force $\mathbf{F}$
   $\Rightarrow \exists$ potential with $V(\mathbf{r}) = V(r)$
   $\Rightarrow$ angular momentum $\mathbf{L} = \text{const}$, $|\mathbf{L}| = mr^2\dot{\varphi}$
   $\Rightarrow \mathbf{F} = f(r)\mathbf{e}_r$; $f(r) = -dV/dr$.

Energy theorem

$$
\begin{aligned}
E &= \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) + V(r) = \frac{m}{2}\dot{r}^2 + \frac{L^2}{2mr^2} + V(r) \\
\dot{r} &= \frac{d}{dt}r = \frac{dr}{d\varphi}\frac{d\varphi}{dt} = \frac{L}{mr^2}\frac{dr}{d\varphi} \\
\Longrightarrow \quad \frac{m}{2}\dot{r}^2 &= \frac{L^2}{2mr^4}\left(\frac{dr}{d\varphi}\right)^2 \\
\Longrightarrow \quad E &= \frac{L^2}{2mr^4}\left[\left(\frac{dr}{d\varphi}\right)^2 + r^2\right] + V(r)\ .
\end{aligned}
$$

3.

$$
f(r) = -\frac{dV}{dr}
$$

Differentiate the energy theorem with respect to r:

$$
0 = -\frac{2L^2}{mr^5}\left[\left(\frac{dr}{d\varphi}\right)^2 + r^2\right] + \frac{L^2}{2mr^4}\left[\frac{d}{dr}\left(\frac{dr}{d\varphi}\right)^2 + 2r\right] + \frac{dV}{dr}
$$

$$
\begin{aligned}
\frac{dr}{d\varphi} = -2R\sin\varphi \;\Longrightarrow\; \left(\frac{dr}{d\varphi}\right)^2 &= 4R^2\sin^2\varphi = 4R^2(1-\cos^2\varphi) \\
&= 4R^2\left(1 - \frac{r^2}{4R^2}\right) = 4R^2 - r^2
\end{aligned}
$$

$$
\Longrightarrow \quad \frac{d}{dr}\left(\frac{dr}{d\varphi}\right)^2 = -2r\ .
$$

Insertion:

$$
-\frac{dV}{dr} = -\frac{8R^2L^2}{mr^5}\ .
$$

Central force:

$$
\mathbf{F}(\mathbf{r}) = -\frac{8R^2L^2}{mr^5}\mathbf{e}_r\ .
$$

Solution 2.4.9

1.

$$\mathbf{F} = m\ddot{\mathbf{r}} = m\frac{d}{dt}\dot{\mathbf{r}}$$

$$\Longrightarrow \dot{\mathbf{r}}(t) - \dot{\mathbf{r}}(t=0) = \frac{1}{m}\int_0^t \mathbf{F}(t')dt' =$$

$$= \int_0^t (15\,t'^2, 2t' - 1, -6t')\,dt'\,\text{cm}\,\text{s}^{-1} =$$

$$= (5t^3, t^2 - t, -3t^2)\,\text{cm}\,\text{s}^{-1}$$

$$\Longrightarrow \dot{\mathbf{r}}(t=1) = (5,0,-3) + (0,0,6) = (5,0,3)\,\text{cm}\,\text{s}^{-1}\ .$$

2.

$$\dot{\mathbf{r}}^2(t=1) = 34\,\text{cm}^2\text{s}^{-2}$$

$$\Longrightarrow T_1 = \frac{3}{2}\cdot 34\,\text{cm}^2\,\text{g}\,\text{s}^{-2} = 51\,\text{cm}^2\,\text{g}\,\text{s}^{-2}\ .$$

3. $W_{10} = T_0 - T_1$.

$$T_0 = \frac{3}{2}\,36\,\text{cm}^2\,\text{g}\,\text{s}^{-2} = 54\,\text{cm}^2\,\text{g}\,\text{s}^{-2}$$

$$\Longrightarrow W_{10} = 3\,\text{cm}^2\,\text{g}\,\text{s}^{-2}\ .$$

Solution 2.4.10

1. The force $F(x) = -kx$ is conservative thus possessing a potential:

$$V(x) = \frac{k}{2}x^2 + C\ .$$

No other forces are present so that according to Eq. (2.231) the energy conservation law holds:

$$E = \frac{m}{2}\dot{x}^2 + \frac{k}{2}x^2 = \text{const}$$

This one easily sees as follows:

$$0 = m\ddot{x} + kx = (m\ddot{x} + kx)\dot{x} = \frac{dE}{dt} \; .$$

2. The energy conservation law leads to:

$$\dot{x}^2 = \frac{2E}{m} - \omega_0^2 x^2 \; ; \quad \omega_0^2 = \frac{k}{m} \; .$$

This we exploit for a separation of variables:

$$dt = \frac{dx}{\sqrt{\frac{2E}{m} - \omega_0^2 x^2}} \; .$$

After Exercise 2.3.15 the velocity $\dot{x}$ is zero for $x = x_{\max}$. That means:

$$\frac{2E}{m\,\omega_0^2} = x_{\max}^2 \; .$$

Therewith follows:

$$\begin{aligned} t - t_1 &= \frac{1}{\omega_0} \int\limits_{x_{\max}}^{x} \frac{dx'}{\sqrt{x_{\max}^2 - x'^2}} = \frac{1}{\omega_0} \int\limits_{1}^{x/x_{\max}} \frac{dy}{\sqrt{1 - y^2}} = \\ &= \frac{1}{\omega_0} \left[\arcsin\left(\frac{x}{x_{\max}} \right) - \arcsin 1 \right] \\ \Longrightarrow \arcsin\left(\frac{x}{x_{\max}} \right) &= \omega_0 (t - t_1) + \frac{\pi}{2} \; . \end{aligned}$$

The quantity $x_{\max}$ is fixed by t_1 so that there is no additional free parameter:

$$x(t) = \frac{1}{\omega_0} \sqrt{\frac{2E}{m}} \cos\left(\omega_0 (t - t_1) \right) \; .$$

3. After Exercise 2.3.15 the maximal velocity is reached at the zero crossing. Hence it follows from $x(t_2) = 0$

$$\begin{aligned} t - t_2 &= \frac{1}{\omega_0} \int\limits_{0}^{x/x_{\max}} \frac{dy}{\sqrt{1 - y^2}} \\ \Longrightarrow x(t) &= \frac{1}{\omega_0} \sqrt{\frac{2E}{m}} \sin\left(\omega_0 (t - t_2) \right) \; . \end{aligned}$$

Solution 2.4.11

1.

$$
\begin{aligned}
& m\ddot{x} = -\frac{dV}{dx} = F(x) \qquad \text{(conservative!)} \\
\iff\ & m\ddot{x}\dot{x} = -\frac{dV}{dx}\dot{x} \\
\iff\ & \frac{d}{dt}\left(\frac{m}{2}\dot{x}^2\right) = -\frac{d}{dt}V(x) \\
\iff\ & \frac{d}{dt}\left(\frac{m}{2}\dot{x}^2 + V(x)\right) = 0
\end{aligned}
$$

$$\frac{m}{2}\dot{x}^2 + V(x) = E = \text{const} .$$

The constant of integration E corresponds to the total energy.

$$\implies \dot{x} = \sqrt{\frac{2}{m}(E - V(x))}$$

Separation of variables:

$$dt = \frac{dx}{\sqrt{\frac{2}{m}(E - V(x))}}$$

$$\implies t - t_0 = \int_{x_0}^{x} dy \frac{1}{\sqrt{\frac{2}{m}(E - V(y))}}$$

free parameters: t_0, E.

2. From $\dot{x} = 0$ for the oscillation amplitude one can conclude:

$$E = V(-a) = V(b) .$$

Oscillation period:

$$\frac{\tau}{2} = \int_{-a}^{b} dy \frac{1}{\sqrt{\frac{2}{m}(E - V(y))}}$$

Symmetry:

$$
\begin{aligned}
E = \text{const} \quad &\iff \quad \text{parallels to the } x \text{ axis} \\
\forall E \quad a = b \quad &\iff \quad V(x) = V(-x) \quad \forall\, x
\end{aligned}
$$

3.

$$V(x) = \frac{k}{2}x^2 = \frac{1}{2}m\omega_0^2 x^2$$

Turning points ($a = b$):

$$E = V(-a) = V(a) = \frac{1}{2}m\omega_0^2 a^2$$

$$\begin{aligned}
\frac{\tau}{2} &= \frac{1}{\omega_0}\int_{-a}^{a}\frac{dy}{\sqrt{a^2 - y^2}} = \frac{1}{\omega_0}\arcsin\frac{y}{|a|}\Big|_{-a}^{a} \\
&= \frac{1}{\omega_0}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \frac{\pi}{\omega_0} \\
&\Longrightarrow\ \tau = \frac{2\pi}{\omega_0}
\end{aligned}$$

It is typical for **harmonic** oscillations that τ is independent of the amplitude.

4. E chosen so that $V(x)$ maximal at $x = b$ ($E = V(b)$). Because of

$$\frac{dV}{dx}\Big|_{x=b} = 0 = F(x = b)$$

the restoring force is zero at the turning point.
$\Longrightarrow$ particle does not come back $\Longrightarrow \tau \to \infty$.

5. Potential energy:

$$\begin{aligned}
V(x) &= \frac{1}{2}m\omega_0^2 x^2 + \frac{1}{4}m\varepsilon x^4 \\
V(x) &= V(-x) \ \Longrightarrow\ a = b \\
E = V(a) &= \frac{1}{2}m\omega_0^2 a^2 + \frac{1}{4}m\varepsilon a^4
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow\ E - V(x) &= \frac{1}{2}m\omega_0^2\left(a^2 - x^2\right) + \frac{1}{4}m\varepsilon\left(a^4 - x^4\right) \\
&= \frac{1}{2}m\omega_0^2\left(a^2 - x^2\right)\left[1 + \frac{\varepsilon}{2\omega_0^2}\left(a^2 + x^2\right)\right]
\end{aligned}$$

$$\Longrightarrow\ \frac{1}{\sqrt{\frac{2}{m}(E - V(x))}} = \frac{1}{\omega_0}\frac{1}{\sqrt{a^2 - x^2}}\frac{1}{\sqrt{1 + \frac{\varepsilon}{2\omega_0^2}\left(a^2 + x^2\right)}}$$

Series expansion:

$$(1+x)^{\frac{n}{m}} = 1 + \frac{n}{m}x - \frac{n(m-n)}{2!m^2}x^2 + \dots$$

$$\Longrightarrow \quad \frac{1}{\sqrt{\frac{2}{m}(E-V(x))}} \approx \frac{1}{\omega_0}\frac{1}{\sqrt{a^2-x^2}}\left[1 - \frac{\varepsilon}{4\omega_0^2}\left(a^2+x^2\right)\right]$$
$$= \frac{1-\frac{\varepsilon a^2}{4\omega_0^2}}{\omega_0\sqrt{a^2-x^2}} - \frac{\varepsilon}{4\omega_0^3}\frac{x^2}{\sqrt{a^2-x^2}}$$

With the standard integrals

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{|a|} + c_1$$
$$\int \frac{x^2 dx}{\sqrt{a^2-x^2}} = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{|a|} + c_2$$

the oscillation period can be estimated:

$$\frac{\tau}{2} = \frac{1-\frac{\varepsilon a^2}{4\omega_0^2}}{\omega_0}(\arcsin(1) - \arcsin(-1)) - \frac{\varepsilon}{4\omega_0^3}\frac{a^2}{2}\Big(\underbrace{\arcsin(1)}_{=\pi/2} - \underbrace{\arcsin(-1)}_{=-\pi/2}\Big)$$

$$\Longrightarrow \quad \tau = \frac{2\pi}{\omega_0}\left(1 - \frac{3}{8}\frac{a^2}{\omega_0^2}\varepsilon\right)$$

Now τ does explicitly depend on the amplitude $a \Longrightarrow$ '*anharmonicity*'.

6.

$$t_0 = 0\,; \quad x(0) = 0$$

$$\Longrightarrow \quad t = \int\limits_0^x \frac{dy}{\sqrt{\frac{2}{m}(E-V(y))}}$$
$$\approx \frac{1}{\omega_0}\left(1 - \frac{\varepsilon a^2}{4\omega_0^2}\right)\int\limits_0^x \frac{dy}{\sqrt{a^2-y^2}} - \frac{\varepsilon}{4\omega_0^3}\int\limits_0^x \frac{dy\, y^2}{\sqrt{a^2-y^2}}$$
$$= \frac{1}{\omega_0}\left(1 - \frac{\varepsilon a^2}{4\omega_0^2}\right)\arcsin\frac{x}{a} + \frac{\varepsilon}{4\omega_0^3}\frac{x}{2}\sqrt{a^2-x^2} - \frac{\varepsilon a^2}{8\omega_0^3}\arcsin\frac{x}{a}$$

$$\implies \left(\omega_0 t - \frac{\varepsilon x}{8\omega_0^2}\sqrt{a^2-x^2}\right) = \arcsin\frac{x}{a}\left(1+\frac{3\varepsilon a^2}{8\omega_0^2}\right)$$

$$\implies \arcsin\frac{x}{a} \approx \left(\omega_0 t - \frac{\varepsilon x}{8\omega_0^2}\sqrt{a^2-x^2}\right)\left(1-\frac{3\varepsilon a^2}{8\omega_0^2}\right)$$

$$\approx \omega_0\left(1+\frac{3a^2}{8\omega_0^2}\varepsilon\right)t - \frac{\varepsilon x}{8\omega_0^2}\sqrt{a^2-x^2} + \mathcal{O}(\varepsilon^2)\ .$$

For abbreviation:

$$\omega \equiv \omega_0\left(1+\frac{3a^2}{8\omega_0^2}\varepsilon\right)\ .$$

$$\frac{x}{a} = \sin\left(\omega t - \underbrace{\frac{\varepsilon x}{8\omega_0^2}\sqrt{a^2-x^2}}_{=\varepsilon\cdot\hat{x}}\right)$$

$$= \sin\omega t \underbrace{\cos\varepsilon\hat{x}}_{=1+\mathcal{O}(\varepsilon^2)} - \cos\omega t \underbrace{\sin\varepsilon\hat{x}}_{=\varepsilon\hat{x}+\mathcal{O}(\varepsilon^3)}$$

$$= \sin\omega t - \varepsilon\hat{x}\cos\omega t$$

$$= \sin\omega t + \mathcal{O}(\varepsilon)$$

$$\hat{x} = \frac{x}{8\omega_0^2}a\sqrt{1-\frac{x^2}{a^2}} = \frac{a^2}{8\omega_0^2}\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}$$

$$= \frac{a^2}{8\omega_0^2}\sin\omega t\cos\omega t + \mathcal{O}(\varepsilon)$$

$$\implies x(t) \approx a\sin\omega t\left(1-\frac{\varepsilon a^2}{8\omega_0^2}\cos^2\omega t\right)$$

Initial velocity:

(a)

$$\dot{x}(t) = a\omega\cos\omega t\left(1-\frac{\varepsilon a^2}{8\omega_0^2}\cos^2\omega t\right) + a\sin\omega t\,\frac{d}{dt}\left(1-\frac{\varepsilon a^2}{8\omega_0^2}\cos^2\omega t\right)$$

$$\implies \dot{x}(0) = a\omega\left(1-\frac{\varepsilon a^2}{8\omega_0^2}\right)$$

(b) from the energy theorem

$$E = \frac{m}{2}\left(\omega_0^2 a^2 + \frac{1}{2}\varepsilon a^4\right) = \frac{m}{2}\dot{x}^2 + \frac{m}{2}\left(\omega_0^2 x^2 + \frac{1}{2}\varepsilon x^4\right)$$

$$x(0) = 0$$

$$\Longrightarrow \; \dot{x}(0) = \sqrt{\omega_0^2 a^2 + \frac{1}{2}\varepsilon a^4} = a\omega_0\sqrt{1 + \frac{\varepsilon a^2}{2\omega_0^2}}$$

$$\approx a\omega_0\left(1 + \frac{\varepsilon a^2}{4\omega_0^2}\right) = a\omega\frac{1 + \frac{\varepsilon a^2}{4\omega_0^2}}{1 + \frac{3\varepsilon a^2}{8\omega_0^2}} \approx a\omega\left(1 + \frac{\varepsilon a^2}{4\omega_0^2}\right)\left(1 - \frac{3\varepsilon a^2}{8\omega_0^2}\right)$$

$$\Longrightarrow \; \dot{x}(0) = a\omega\left(1 - \frac{3\varepsilon a^2}{8\omega_0^2} + \mathcal{O}(\varepsilon^2)\right) \qquad \text{(as above!)}$$

Solution 2.4.12

1. Possible starting point: *'area conservation principle'*

$$\frac{dS}{dt} = \frac{1}{2}|\mathbf{r} \times \dot{\mathbf{r}}| = \frac{L}{2m} = \text{const}$$

$$\Longrightarrow \; t_{a,b} = \frac{2m\Delta S_{a,b}}{L}$$

$\Delta S_{a,b}$ is the area swept by the position vector during the time $t_{a,b}$.
central force $\Longrightarrow$ $\mathbf{L}$ =const: motion takes place in a fixed plane perpendicular to $\mathbf{L}$
spherical coordinates: $r, \vartheta = \frac{\pi}{2}, \varphi$
After (2.267):

$$r = \frac{k}{1 + \varepsilon\cos\varphi}\,; \quad k = \frac{L^2}{\alpha m} \qquad \text{conic section !}$$

(a) Circle (Fig. A.31) $\Longrightarrow \varepsilon = 0$

$$r = R = k = \frac{L^2}{\alpha m}$$

$$\Longrightarrow L = \sqrt{\alpha m R}$$

$$\Delta S_a = \frac{1}{2}\pi R^2$$

$$\Longrightarrow t_a = \pi R\sqrt{R\frac{m}{\alpha}}$$

Fig. A.31

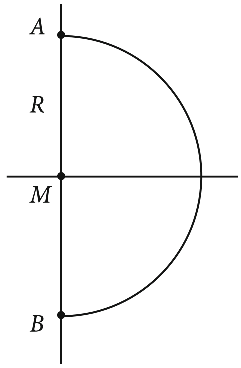


Fig. A.32

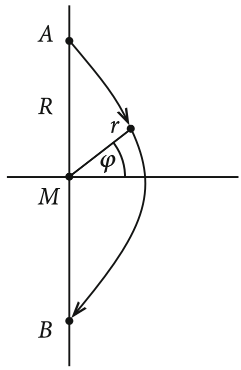
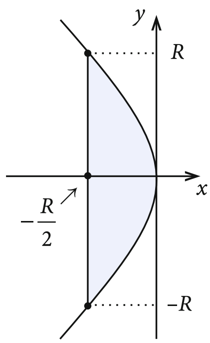


Fig. A.33



(b) Parabola (Fig. A.32) $\Longrightarrow \varepsilon = 1$

$$r = \frac{k}{1+\cos\varphi}\ ;\ r\left(\varphi = \frac{\pi}{2}\right) = R$$

$$\Longrightarrow k = R \Longrightarrow L = \sqrt{\alpha m R}$$

$$r(\varphi = 0) = \frac{1}{2}k = \frac{1}{2}R \qquad \text{(Fig. A.33)}$$

$$
\begin{aligned}
y = c\sqrt{-x} : \quad R &= c\sqrt{\frac{R}{2}} \implies c = \sqrt{2R} \\
\frac{1}{2}\Delta S_b &= \int_{-\frac{R}{2}}^{0} \sqrt{-2Rx}dx \\
&= -\sqrt{-2R}\,\frac{2}{3}\left(-\frac{R}{2}\right)\sqrt{-\frac{R}{2}} \\
&= \frac{1}{3}R^2 \\
\implies t_b &= 2\sqrt{\frac{m}{\alpha R}} \cdot \frac{2}{3}R^2 \\
t_b &= \frac{4}{3}R\sqrt{R\frac{m}{\alpha}} \qquad (t_b < t_a)
\end{aligned}
$$

2. Total energy:

$$
\begin{aligned}
E &= \frac{m}{2}\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{\alpha}{r} \\
&= \frac{m}{2}\dot{r}^2 + \alpha\left(\frac{k}{2r^2} - \frac{1}{r}\right)
\end{aligned}
$$

(a) Circle:

$$
\begin{aligned}
r = R &= \text{const} ; \quad k = R \\
\implies E &= -\frac{\alpha}{2R}
\end{aligned}
$$

It holds energy conservation:

$$
T = E - V = -\frac{\alpha}{2R} + \frac{\alpha}{R} = \frac{\alpha}{2R}
$$

velocity:

$$
\begin{aligned}
\dot{\mathbf{r}} &= R\dot{\varphi}\mathbf{e}_\varphi \\
v_a = |\dot{\mathbf{r}}| &= \sqrt{\frac{\alpha}{mR}}
\end{aligned}
$$

direct way from A to B (uniformly straight-line):

$$\hat{t}_a = \frac{2R}{v_a} = 2R\sqrt{\frac{mR}{\alpha}} < t_a$$

(b) Parabola: *'point closest to the sun'*: $\varphi = 0$

$$\dot{r}(\varphi = 0) = 0\,; \quad r(\varphi = 0) = \frac{k}{2}$$
$$\Longrightarrow\; E = 0 \;\Longrightarrow\; T = -V$$
$$\Longrightarrow\; T(A) = +\frac{\alpha}{R}$$
$$v_b(A) = \sqrt{\frac{2\alpha}{mR}}$$

direct way:

$$\hat{t}_b = \frac{2R}{v_b} = \sqrt{2}R\sqrt{R\frac{m}{\alpha}} > t_b \quad (!)$$

Altogether:

$$t_b < \hat{t}_b < \hat{t}_a < t_a$$

It obviously brings a gain of time by flying by close to the planet!

3. $\vartheta = \frac{\pi}{2} \Longrightarrow \dot{\vartheta} = 0$: Motion in the xy plane

$$\mathbf{r} = r\mathbf{e}_r \;\Longrightarrow\; \dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\varphi}\mathbf{e}_\varphi$$
$$\ddot{\mathbf{r}} = \left(\ddot{r} - r\dot{\varphi}^2\right)\mathbf{e}_r + (r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\,\mathbf{e}_\varphi$$

(a) Equations of motion:
Components of the force of friction

$$\mathbf{F}_R = -m\hat{\alpha}\left(\dot{r}\mathbf{e}_r + r\dot{\varphi}\mathbf{e}_\varphi\right)$$

radial motion:

$$m\left(\ddot{r} - r\dot{\varphi}^2\right) = -\frac{\alpha}{r^2} - m\hat{\alpha}\dot{r}$$
$$\ddot{r} - r\dot{\varphi}^2 = -\frac{\alpha}{mr^2} - \hat{\alpha}\dot{r} \tag{1}$$

azimuthal motion:
force of friction $\Longrightarrow$ torque $\Longrightarrow$ time-dependent change of angular momentum
angular momentum:

$$
\begin{aligned}
\mathbf{L} = m(\mathbf{r} \times \dot{\mathbf{r}}) &= -mr^2\dot{\varphi}\mathbf{e}_\vartheta \\
&\overset{\vartheta = \frac{\pi}{2}}{=} mr^2\dot{\varphi}\mathbf{e}_z \\
L &= mr^2\dot{\varphi} \\
\mathbf{M} = \mathbf{r} \times \mathbf{F}_R &= \frac{d}{dt}\mathbf{L} \\
\Longleftrightarrow \quad -m\hat{\alpha}r^2\dot{\varphi}\,(-\mathbf{e}_\vartheta) &= m\left(2r\dot{r}\dot{\varphi} + r^2\ddot{\varphi}\right)\mathbf{e}_z \\
\Longrightarrow \quad r\ddot{\varphi} + 2\dot{r}\dot{\varphi} &= -\hat{\alpha}r\dot{\varphi} \qquad (2)
\end{aligned}
$$

because friction is **not** a central-force problem!

(b) Circular path

$$
\begin{aligned}
r &= R(1 + c_1 t) : \; \dot{r} = c_1 R \,; \quad \ddot{r} = 0 \\
\dot{\varphi} &= \omega_0(1 + c_2 t) : \; \ddot{\varphi} = c_2\omega_0
\end{aligned}
$$

Insertion into (1):

$$
-R(1 + c_1 t)\omega_0^2(1 + c_2 t)^2 = -\frac{\alpha}{mR^2(1 + c_1 t)^2} - \hat{\alpha}c_1 R
$$

All terms quadratic in $(c_1, c_2, \hat{\alpha})$ can be neglected (for not too large times t!):

$$
\begin{gathered}
-R\omega_0^2(1 + (c_1 + 2c_2)t) \approx -\frac{\alpha}{mR^2}(1 - 2c_1 t) \\
t = 0 \;\Longrightarrow\; \omega_0^2 = +\frac{\alpha}{mR^3} \\
\left(\text{test:} \quad \omega_0 \overset{1.a)}{=} \frac{2\pi}{2t_a} = \frac{1}{R}\sqrt{\frac{\alpha}{mR}}\right) \\
\Longrightarrow \; 1 + (c_1 + 2c_2)t \approx 1 - 2c_1 t
\end{gathered}
$$

Equation is satisfied if

$$
3c_1 + 2c_2 = 0 \qquad (*)
$$

Insertion into (2):

$$R(1+c_1t)c_2\omega_0 + 2c_1R\omega_0(1+c_2t) = -\hat{\alpha}R(1+c_1t)\omega_0(1+c_2t)$$
$$\Longrightarrow \quad R\omega_0(c_2+2c_1) \approx -\hat{\alpha}R\omega_0$$
$$\Longrightarrow \quad 2c_1 + c_2 = -\hat{\alpha} \qquad (**)$$

Combination of ($*$) and ($**$):

$$c_1 = -2\hat{\alpha}\ ; \quad c_2 = 3\hat{\alpha}$$

However, the approximation is valid only for times for which $\hat{\alpha}t \ll 1$ holds! Then:

$$r \approx R(1-2\hat{\alpha}t)$$
$$\dot{\varphi} \approx \omega_0(1+3\hat{\alpha}t)$$

(c) Radius:

$$\frac{d}{dt}r = -2\hat{\alpha}R \qquad \text{becomes } \textit{smaller}!$$

Angular velocity:

$$\frac{d}{dt}\dot{\varphi} = 3\hat{\alpha}\omega_0 \qquad \text{becomes } \textit{larger}!$$

Path velocity:

$$|\dot{\mathbf{r}}| = \sqrt{\dot{r}^2 + r^2\dot{\varphi}^2}$$
$$\approx \sqrt{(-2\hat{\alpha}R)^2 + R^2(1-2\hat{\alpha}t)^2\omega_0^2(1+3\hat{\alpha}t)^2}$$
$$\approx \sqrt{R^2\omega_0^2(1+2\hat{\alpha}t)}$$
$$\approx R\omega_0(1+\hat{\alpha}t)$$
$$\Longrightarrow \quad \frac{d}{dt}|\dot{\mathbf{r}}| \approx R\omega_0\hat{\alpha} \qquad \text{becomes } \textit{larger}!$$

Kinetic energy:

$$\dot{\mathbf{r}}^2 \approx R^2\omega_0^2(1+2\hat{\alpha}t)$$
$$\Longrightarrow \quad \frac{d}{dt}T \approx \underbrace{mR^2\omega_0^2\hat{\alpha}}_{\hat{\alpha}\,\frac{\alpha}{R}} \qquad \text{becomes } \textit{larger}!$$

Potential energy:

$$V = -\frac{\alpha}{r} \approx -\frac{\alpha}{R(1-2\hat{\alpha}t)} \approx -\frac{\alpha}{R}(1+2\hat{\alpha}t)$$

$$\Longrightarrow \quad \frac{d}{dt}V \approx -2\hat{\alpha}\frac{\alpha}{R} \qquad \text{becomes } \textit{smaller}!$$

Decreases twice as strongly as T increases!
Total energy:

$$\frac{d}{dt}E = \frac{d}{dt}(T+V) \approx -\hat{\alpha}\frac{\alpha}{R} \qquad \text{becomes } \textit{smaller}!$$

(d) Friction energy:

$$\begin{aligned}\frac{d}{dt}E_R &= -\mathbf{F}_R \cdot \dot{\mathbf{r}} = m\hat{\alpha}\dot{\mathbf{r}}^2\\ &\approx m\hat{\alpha}R^2\omega_0^2(1+2\hat{\alpha}t)\\ &\approx m\hat{\alpha}R^2\omega_0^2\\ &= \hat{\alpha}\frac{\alpha}{R}\end{aligned}$$

Friction energy is taken from the potential energy!

Solution 2.4.13

1.

$$x(t) = a\cos(\omega t) \Longrightarrow \frac{y^2(t)}{b^2} = 1 - \cos^2(\omega t) = \sin^2(\omega t)\ .$$

Thus it is:

$$y(t) = b\sin(\omega t)\ .$$

The angular frequency ω is given by

$$\omega \cdot 2 = 6\pi \quad \Longrightarrow \quad \omega = 3\pi\ s^{-1}\ .$$

The trajectory therewith reads:

$$\mathbf{r}(t) = (a\cos(3\pi\, t), b\sin(3\pi\, t), 0)\ .$$

2. It obviously holds:

$$\ddot{\mathbf{r}}(t) = -\omega^2\mathbf{r}(t) = -9\pi^2\mathbf{r}(t)\ .$$

Therefore the force acting on the mass point is:

$$\mathbf{F}(\mathbf{r}, t) = -m\,\omega^2 \mathbf{r}(t) \; .$$

3. Angular momentum:

$$\mathbf{L} = m(\mathbf{r} \times \dot{\mathbf{r}}) = m \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ x & y & 0 \\ \dot{x} & \dot{y} & 0 \end{vmatrix} = m(x\dot{y} - y\dot{x})\mathbf{e}_z =$$

$$= m[a\cos\omega t(b\omega\cos\omega t) + b\sin\omega t(a\omega\sin\omega t)]\mathbf{e}_z = m\,a\,b\,\omega\,\mathbf{e}_z \; .$$

$\mathbf{L}$ is constant with respect to direction as well as magnitude since $\mathbf{F}$ is about a central force.

4.

$$\frac{dS}{dt} = \frac{1}{2}|(\mathbf{r} \times \dot{\mathbf{r}})| = \frac{L}{2m} = \frac{1}{2} a\,b\,\omega = \text{const}$$

$$\Longrightarrow \Delta S = \frac{dS}{dt}\Delta t = \frac{3}{2}\pi\,a\,b \; .$$

Section 2.5

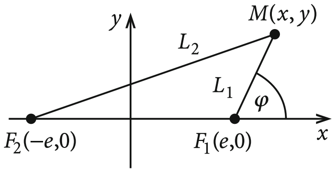
Solution 2.5.1

$$L_1 + L_2 = 2a$$

1. We choose $M = M(0, b)$. Then it is $L_1 = L_2 = a$ (Fig. A.34). Hence, we can use Pythagoras' theorem to get:

$$b^2 = a^2 - e^2 \; .$$

Fig. A.34



2.

$$L_1^2 = y^2 + (x-e)^2 \; ; \quad L_2^2 = y^2 + (x+e)^2 \; .$$

By inserting this into the defining equation of the ellipse

$$L_1 + L_2 = 2a \Longleftrightarrow L_1^2 + L_2^2 + 2L_1L_2 = 4a^2$$

one gets after simple rearrangings and with the result from part 1. the so-called midpoint equation of the ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \; .$$

3.

$$L_2^2 - L_1^2 = (L_2 + L_1)(L_2 - L_1) = 2a(L_2 - L_1) \; .$$

After part 2. it also holds:

$$L_2^2 - L_1^2 = 4ex = 2a\frac{2ex}{a} = 2a\,2\varepsilon x \; .$$

The comparison leads to:

$$L_2 - L_1 = 2\varepsilon x \; .$$

Combination with $L_1 + L_2 = 2a$ yields:

$$\begin{aligned} L_1 &= a - \varepsilon x = a - \varepsilon(e + L_1\cos\varphi) \\ \Longrightarrow L_1(1 + \varepsilon\cos\varphi) &= a - \varepsilon e = a - \frac{e^2}{a} = \frac{b^2}{a} = k \; . \end{aligned}$$

Putting still $L_1 = r$ we have found the equation of the ellipse in polar coordinates:

$$r = \frac{k}{1 + \varepsilon\cos\varphi} \; .$$

4. The parameter-representation

$$\left.\begin{aligned} x &= a\cos t \; , \\ y &= b\sin t \end{aligned}\right\} \quad 0 \leq t \leq 2\pi$$

obviously fulfills the midpoint equation (2):

$$\mathbf{r}(t) = \begin{pmatrix} a\cos t \\ b\sin t \end{pmatrix} .$$

Solution 2.5.2

1. It belongs to the potential

$$V(\mathbf{r}) = V(r) = \frac{\alpha}{r^2}$$

the conservative central force

$$\mathbf{F}(\mathbf{r}) = -\frac{2\alpha}{r^3}\mathbf{e}_r .$$

The angular momentum **L** is therefore a conserved quantity

$$\mathbf{L} = \text{const}$$

The motion occurs in a fixed orbital plane. That shall be the xy plane ($\vartheta = \pi/2$). Then it holds after (2.252):

$$\mathbf{L} = m\,r^2\dot{\varphi}\,\mathbf{e}_z .$$

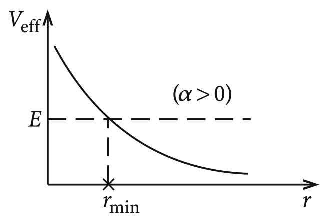
The energy E is likewise a conserved quantity:

$$E = \frac{m}{2}\dot{r}^2 + \frac{L^2}{2mr^2} + \frac{\alpha}{r^2} \quad \text{(see (2.260))} .$$

One defines as effective potential (Fig. A.35):

$$V_{\text{eff}}(r) = \frac{L^2}{2mr^2} + \frac{\alpha}{r^2} .$$

Fig. A.35



2. At $r(t=0) = r_{\rm min}$ it must be $\dot{r}(t=0) = 0$. Then it follows from the energy conservation law:

$$r_{\rm min} = \sqrt{\frac{L^2 + 2m\alpha}{2mE}} \; .$$

Because of $\alpha > 0$ only for $E > 0$ an actual motion is possible.

3.

$$E = \frac{m}{2}\dot{r}^2 + E\frac{r_{\rm min}^2}{r^2}$$

$$\Longrightarrow \; \dot{r} = \frac{1}{r}\sqrt{\frac{2E}{m}}\sqrt{r^2 - r_{\rm min}^2} \; .$$

Separation of the variables:

$$dt = \sqrt{\frac{m}{2E}}\frac{rdr}{\sqrt{r^2 - r_{\rm min}^2}} = \sqrt{\frac{m}{2E}}\frac{d}{dr}\sqrt{r^2 - r_{\rm min}^2}\, dr \; .$$

That allows with $r_{\rm min} = r(t=0)$ a simple integration:

$$t = \sqrt{\frac{m}{2E}}\sqrt{r^2 - r_{\rm min}^2}$$

$$\Longrightarrow r(t) = \sqrt{r_{\rm min}^2 + \frac{2E}{m}t^2} \; .$$

To determine the path $r = r(\varphi)$ we start with the angular-momentum conservation law:

$$\dot{\varphi} = \frac{L}{mr^2} \Longrightarrow d\varphi = \frac{L}{mr^2} \cdot \frac{dr}{\dot{r}} =$$

$$= \frac{L}{\sqrt{2mE}} \cdot \frac{1}{r^2} \cdot \frac{dr}{\sqrt{1 - (r_{\rm min}/r)^2}} \; .$$

With $\varphi(r_{\rm min}) = 0$ the formal solution reads:

$$\varphi = \int\limits_{r_{\rm min}}^{r} \frac{L}{\sqrt{2mE}}\frac{dr'}{r'^2\sqrt{1 - (r_{\rm min}/r')^2}} \; .$$

Proper substitution:

$$y = \frac{r_{\min}}{r'} \Longrightarrow dy = -\frac{r_{\min}}{r'^2}dr'$$

$$\Longrightarrow \varphi = \frac{-L}{r_{\min}\sqrt{2mE}} \int_1^{r_{\min}/r} \frac{dy}{\sqrt{1-y^2}} =$$

$$= \frac{-L}{r_{\min}\sqrt{2mE}}\left[\arcsin\left(\frac{r_{\min}}{r}\right) - \frac{\pi}{2}\right] .$$

The reversal yields:

$$\frac{r_{\min}}{r} = \cos\left(\frac{r_{\min}\sqrt{2mE}}{L} \cdot \varphi\right) .$$

We still insert into the cosine function $r_{\min}$ from 2.:

$$r(\varphi) = \frac{r_{\min}}{\cos\left(\sqrt{\frac{L^2+2m\alpha}{L^2}} \cdot \varphi\right)} .$$

The special case $\alpha = 0$ leads to

$$r_{\min} = r\cos\varphi .$$

The path is then a straight-line (Fig. A.36):

$$-\frac{\pi}{2} \le \varphi \le +\frac{\pi}{2} .$$

4. It is now (Fig. A.37)

$$V_{\text{eff}}(r) = \frac{1}{r^2}\left(\frac{L^2}{2m} - |\alpha|\right) .$$

Fig. A.36

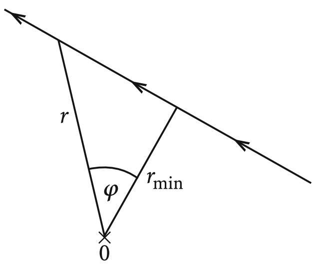
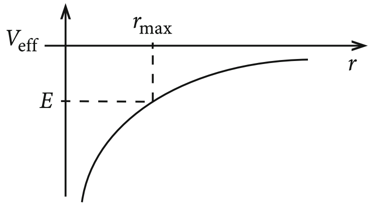


Fig. A.37



The *bound* motion in Fig. A.37 requires:

$$|\alpha| > \frac{L^2}{2m} \; ; \quad E < 0 \; .$$

Using the same way of computation as in part 2. one arrives at:

$$r_{\max} = \sqrt{\frac{2m|\alpha| - L^2}{2m|E|}} \; .$$

5. We can express the constant total energy E by $r_{\max}$:

$$E = \frac{m}{2}\dot{r}^2 + E\frac{r_{\max}^2}{r^2} \; .$$

The same considerations as in part 3. then lead to:

$$r(t) = \sqrt{r_{\max}^2 - \frac{2|E|}{m}t^2} \; .$$

So the particle is landing after the time

$$t_0 = r_{\max} \cdot \sqrt{\frac{m}{2|E|}}$$

in the center $r = 0$.

6. The calculation of the path line $r(\varphi)$ runs analogous to that in part 3.:

$$d\varphi = \frac{L}{mr^2}\frac{dr}{\dot{r}} = \frac{L}{\sqrt{2m|E|}}\frac{1}{r^2}\frac{dr}{\sqrt{\left(\frac{r_{\max}}{r}\right)^2 - 1}} \; .$$

With $\varphi(r_{\max}) = 0$ and the substitution $y = \frac{r_{\max}}{r}$,

$$dy = -\frac{r_{\max}}{r^2}dr \; ,$$

it follows:

$$\varphi = \frac{L}{r_{\max}\sqrt{2m|E|}} \int\limits_{r_{\max}/r}^{1} \frac{dy}{\sqrt{y^2-1}} = \frac{-L}{r_{\max}\sqrt{2m|E|}} \operatorname{arccosh}\left(\frac{r_{\max}}{r}\right) .$$

The energy theorem still delivers:

$$r_{\max} = \sqrt{\frac{2m|\alpha| - L^2}{2m|E|}} .$$

Inserted into the above expression it eventually yields the required path line:

$$r(\varphi) = \sqrt{\frac{2m|\alpha| - L^2}{2m|E|}} \cdot \frac{1}{\cosh\left(\sqrt{\frac{2m|\alpha|}{L^2} - 1} \cdot \varphi\right)} .$$

This is the equation of a helical line. The particle touches down in the center $r = 0$ after infinitely many circulations ($\varphi \to \infty$), but after a finite time t_0.

Solution 2.5.3

1.

$$\dot{\mathbf{A}} = (\ddot{\mathbf{r}} \times \mathbf{L}) + (\dot{\mathbf{r}} \times \dot{\mathbf{L}}) + (\nabla V \cdot \dot{r})\mathbf{r} + V(r)\dot{\mathbf{r}} .$$

In a central potential it is $\dot{\mathbf{L}} = 0$ and furthermore:

$$\ddot{\mathbf{r}} = -\frac{1}{m}\frac{dV}{dr}\mathbf{e}_r .$$

This has the consequence:

$$\dot{\mathbf{A}} = -\frac{1}{m}\frac{dV}{dr}\frac{m}{r}[\mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}})] + \frac{dV}{dr}\frac{1}{r}(\mathbf{r} \cdot \dot{\mathbf{r}})\mathbf{r} + V(r)\dot{\mathbf{r}} =$$
$$= \dot{\mathbf{r}}\left[r\frac{dV}{dr} + V(r)\right] = 0 \quad \text{for } V(r) = -\frac{\alpha}{r} .$$

2. We obtain the magnitude of the Lenz vector from:

$$\mathbf{A} \cdot \mathbf{A} = [(\dot{\mathbf{r}} \times \mathbf{L}) + V(r)\mathbf{r}] \cdot [(\dot{\mathbf{r}} \times \mathbf{L}) + V(r)\mathbf{r}] .$$

For the central potential it is $\dot{\mathbf{r}} \perp \mathbf{L}$:

$$\mathbf{A}^2 = \dot{\mathbf{r}}^2\mathbf{L}^2 + V(r)\left[\mathbf{r} \cdot (\dot{\mathbf{r}} \times \mathbf{L}) + (\dot{\mathbf{r}} \times \mathbf{L}) \cdot \mathbf{r}\right] + V^2(r)\mathbf{r}^2 .$$

With

$$(\dot{\mathbf{r}} \times \mathbf{L}) \cdot \mathbf{r} = \mathbf{L} \cdot (\mathbf{r} \times \dot{\mathbf{r}}) = \frac{L^2}{m}$$

it furthermore follows:

$$\mathbf{A}^2 = \frac{2L^2}{m} \left[V(r) + \frac{m}{2}\dot{\mathbf{r}}^2 \right] + V^2(r)\mathbf{r}^2$$

$$\Longrightarrow |\mathbf{A}| = \sqrt{\alpha^2 + \frac{2L^2}{m} E} \, .$$

3.

$$\mathbf{A} \cdot \mathbf{r} = (\dot{\mathbf{r}} \times \mathbf{L}) \cdot \mathbf{r} + V(r) r^2 = \frac{L^2}{m} + V(r) r^2 = |A| r \cos\varphi \, .$$

If one abbreviates

$$\varepsilon = \frac{|\mathbf{A}|}{\alpha} = \sqrt{1 + \frac{2L^2}{m\alpha^2} E} \, ; \quad k = \frac{L^2}{m\alpha} \, ,$$

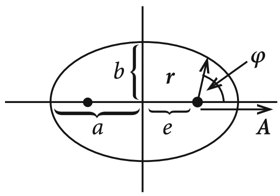
then one gets the compact result:

$$r = \frac{k}{1 + \varepsilon \cos\varphi} \, .$$

for $E < 0$ follows: $\varepsilon < 1 \Longrightarrow$ ellipse,
for $E > 0$ follows: $\varepsilon > 1 \Longrightarrow$ hyperbola.
A lies in the orbital plane pointing from the center to the perihelion (Fig. A.38) and has the magnitude

$$\alpha\varepsilon = \alpha \frac{e}{a} \, .$$

Fig. A.38



Solution 2.5.4

1. According to Sect. 2.4.5 it holds for the conservative central field:

$$E = \frac{m}{2}\dot{r}^2 + V_{\text{eff}}(r)$$
$$V_{\text{eff}}(r) = V(r) + \frac{L^2}{2mr^2} = -\frac{\alpha}{r} + \frac{L^2}{2mr^2}$$
$$\alpha = \gamma m M \ .$$

γ is the gravitational constant, m the mass of the earth satellite and M the mass of the earth.
Force of gravity:

$$mg \stackrel{!}{=} \frac{\gamma m M}{R^2} \quad \curvearrowright \quad \gamma M = gR^2$$
$$R = 6370\,\text{km} \quad (\text{earth radius})$$
$$g = 9,81\,\frac{\text{m}}{\text{s}^2} \ .$$

The satellite is on a circular path ($r = \text{const} = R_0$) if and only if r corresponds to the minimum of V_{eff} (see Fig. 2.54).

$$0 \stackrel{!}{=} \left.\frac{dV_{\text{eff}}}{dr}\right|_{r=R_0} = \frac{\alpha}{R_0^2} - \frac{L^2}{mR_0^3}$$
$$\curvearrowright \quad R_0 = \frac{L^2}{m\alpha} = k \ .$$

Comparison with (2.268) (conic section)

$$r = \frac{k}{1+\varepsilon\cos\phi} \quad \curvearrowright \quad r = R_0 \ \text{ for } \ \varepsilon = 0 \ \ (\text{circle}) \ .$$

in addition, because of the central force

$$L = mr^2\dot{\varphi} \longrightarrow mR_0^2\omega$$
$$\curvearrowright \quad R_0 = \frac{m^2R_0^4\omega^2}{m\gamma m M} = \frac{R_0^4\omega^2}{\gamma M} = \frac{R_0^4\omega^2}{gR^2}$$
$$\curvearrowright \quad R_0 = \left(\frac{gR^2}{\omega^2}\right)^{\frac{1}{3}}$$

geostationary orbit:

satellite remains always above the same point of the earth's surface!

$$\curvearrowright \quad \omega = \frac{2\pi}{24h} \approx 7,29 \cdot 10^{-5}\,\frac{1}{\mathrm{s}}\ .$$

That means

$$R_0 \approx 4,22 \cdot 10^7 m \approx 6,6 \cdot R\ .$$

2. Minimum requirement:

$$R_0 \overset{!}{=} R\ .$$

angular momentum for $r = R$:

$$L = m\,|\mathbf{r} \times \dot{\mathbf{r}}| \longrightarrow mRv_1 \quad (\dot{\mathbf{r}} \perp \mathbf{r})$$
$$R_0 = R = \frac{L^2}{m\alpha} = \frac{m^2R^2v_1^2}{m\gamma mM} = \frac{R^2v_1^2}{\gamma M} = \frac{R^2v_1^2}{gR^2}\ .$$

First cosmic velocity:

$$v_1 = \sqrt{gR} = 7,9\,\frac{\mathrm{km}}{\mathrm{s}}\ .$$

3. To leave the attracting region of the earth the satellite needs at least the energy $E = 0$ (see Fig. 2.54). That means:

$$0 \overset{!}{=} \frac{m}{2}v_2^2 - \frac{\alpha}{R} = \frac{m}{2}v_2^2 - \frac{mgR^2}{R}\ .$$

Second cosmic velocity:

$$v_2 = \sqrt{2gR} = 11,2\,\frac{\mathrm{km}}{\mathrm{s}}\ .$$

Solution 2.5.5

1. Because of the central field:

$$\mathbf{L} = \mathrm{const}\,.$$

The motion takes place within a plane perpendicular to $\mathbf{L}$, which contains the origin. If $\mathbf{L} \propto \mathbf{e}_z$ then the motion takes place in the xy plane with (equation after (2.252)):

$$\mathbf{L} = mr^2\dot{\varphi}\,\mathbf{e}_z \; .$$

r, φ: plane polar coordinates. With these the acceleration reads according to (2.13):

$$\ddot{\mathbf{r}} = \left(\ddot{r} - r\dot{\varphi}^2\right)\mathbf{e}_r + \left(r\ddot{\varphi} + 2\dot{r}\dot{\varphi}\right)\mathbf{e}_\varphi \; .$$

Therewith, out of

$$f(r)\mathbf{e}_r = m\ddot{\mathbf{r}}$$

the following two conditional equations arise:

$$f(r) = m\ddot{r} - mr\dot{\varphi}^2 \tag{1}$$

$$0 = mr\ddot{\varphi} + 2m\dot{r}\dot{\varphi} \; . \tag{2}$$

In addition it holds:

$$\ddot{r} = \frac{d}{dt}\left(\frac{dr}{d\varphi}\dot{\varphi}\right) = \frac{d^2r}{d\varphi^2}\,\dot{\varphi}^2 + \frac{dr}{d\varphi}\,\ddot{\varphi} \; .$$

Equation (1) now becomes:

$$f(r) = m\left(\frac{d^2r}{d\varphi^2} - r\right)\dot{\varphi}^2 + m\frac{dr}{d\varphi}\,\ddot{\varphi} \; .$$

Equation (2) can be rewritten as:

$$\ddot{\varphi} = -\frac{2\dot{r}\dot{\varphi}}{r} = -\frac{2}{r}\left(\frac{dr}{d\varphi}\right)\dot{\varphi}^2 \; .$$

Inserting into the previous equation:

$$f(r) = m\dot{\varphi}^2\left(\frac{d^2r}{d\varphi^2} - \frac{2}{r}\left(\frac{dr}{d\varphi}\right)^2 - r\right) \; .$$

Now we still exploit the angular-momentum conservation $\dot{\varphi}^2 = \frac{L^2}{m^2 r^4}$:

$$f(r) = \frac{L^2}{mr^4}\left(\frac{d^2 r}{d\varphi^2} - \frac{2}{r}\left(\frac{dr}{d\varphi}\right)^2 - r\right) .$$

That was to be shown.

2. Mass point on an ellipse (conic section) in a central field, the center of which lies at one of the two focal points (2.268):

$$r = r(\varphi) = \frac{k}{1 + \varepsilon\cos(\varphi)} \qquad (\varepsilon < 1) .$$

k is here undetermined. It follows

$$\frac{dr}{d\varphi} = \frac{k\varepsilon\sin(\varphi)}{(1 + \varepsilon\cos(\varphi))^2}$$

$$\frac{d^2 r}{d\varphi^2} = \frac{2k\varepsilon^2\sin^2(\varphi)}{(1 + \varepsilon\cos(\varphi))^3} + \frac{k\varepsilon\cos(\varphi)}{(1 + \varepsilon\cos(\varphi))^2} .$$

Insertion into $f(r)$:

$$\begin{aligned} f(r) &= \frac{L^2}{mr^4}\left(\frac{2k\varepsilon^2\sin^2(\varphi)}{(1 + \varepsilon\cos(\varphi))^3} + \frac{k\varepsilon\cos(\varphi)}{(1 + \varepsilon\cos(\varphi))^2}\right.\\ &\qquad \left. -2\,\frac{1 + \varepsilon\cos(\varphi)}{k}\,\frac{k^2\varepsilon^2\sin^2(\varphi)}{(1 + \varepsilon\cos(\varphi))^4} - \frac{k}{1 + \varepsilon\cos(\varphi)}\right)\\ &= \frac{L^2}{mr^3}\left(\frac{\varepsilon\cos(\varphi)}{1 + \varepsilon\cos(\varphi)} - 1\right)\\ &= \frac{L^2}{mr^3}\,\frac{-1}{1 + \varepsilon\cos(\varphi)}\\ &= -\frac{L^2}{mk}\,\frac{1}{r^2}\\ &\propto -\frac{1}{r^2} . \end{aligned}$$

That is related to a potential of the form:

$$V(\mathbf{r}) = V(r) \propto -\frac{1}{r} \qquad \text{(gravitation, Coulomb)} .$$

In case of a *conservative central-force field* one can *uniquely* propose the explicit force $\mathbf{F}(\mathbf{r})$ from the path line $r = r(\varphi)$!

3. It holds

$$r(\varphi) = r_0\, e^{-\varphi} \quad \curvearrowright \quad \frac{dr}{d\varphi} = -r \;\; ; \;\; \frac{d^2 r}{d\varphi^2} = r$$

$$\curvearrowright \quad f(r) = \frac{L^2}{mr^4}\left(r - \frac{2}{r}\, r^2 - r\right) = -\frac{2L^2}{mr^3}$$

$$\curvearrowright \quad f(r) \propto -\frac{1}{r^3} \;\; ; \;\; V(r) = -\frac{1}{r^2}$$

Solution 2.5.6

1.

$$\mathbf{F}(\mathbf{r}) = f(r)\,\mathbf{e}_r \;\; ; \;\; f(r) = -\frac{\alpha}{r^n}\ .$$

We use (1.289):

$$\begin{aligned}
\nabla \times \mathbf{F} &= \nabla \times (f(r)\mathbf{e}_r) = \nabla \times \left(\frac{f(r)}{r}\mathbf{r}\right) \\
&= \frac{f(r)}{r}\underbrace{\nabla \times \mathbf{r}}_{=0} + \nabla\left(\frac{f(r)}{r}\right) \times \mathbf{r} \\
&= \frac{d}{dr}\left(\frac{f(r)}{r}\right)\mathbf{e}_r \times \mathbf{e}_r r \\
&= 0
\end{aligned}$$

Hence, $\mathbf{F}$ is conservative!

2.

$$\begin{aligned}
&\mathbf{L} = m\,(\mathbf{r} \times \dot{\mathbf{r}}) \\
\curvearrowright \quad &\frac{d}{dt}\mathbf{L} = m\,\underbrace{(\dot{\mathbf{r}} \times \dot{\mathbf{r}})}_{=0} + m(\mathbf{r} \times \ddot{\mathbf{r}}) = \mathbf{r} \times \mathbf{F} = \frac{f(r)}{r}\mathbf{r} \times \mathbf{r} \\
&\phantom{\frac{d}{dt}\mathbf{L}} = 0 \\
\curvearrowright \quad &\mathbf{L} = \text{const}\ .
\end{aligned}$$

The motion thus takes place in a fixed plane perpendicular to $\mathbf{L}$, which contains the origin.

3. The fixed plane may be chosen as the xy-plane. For spherical coordinates (r, ϑ, φ) that means $\vartheta = \frac{\pi}{2}$ and therewith according to (2.21):

$$\dot{\mathbf{r}} \to \dot{r}\mathbf{e}_r + r\dot{\varphi}\mathbf{e}_\varphi$$

conservative central field $\curvearrowright$ energy conservation with $V(\mathbf{r}) = V(r)$:

$$E = \frac{m}{2}\,\dot{\mathbf{r}}^2 + V(\mathbf{r}) = \frac{m}{2}\,\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) + V(r)\ .$$

$\dot{\varphi}$ from angular-momentum conservation:

$$\begin{aligned} \mathbf{L} &= m\,(\mathbf{r} \times \dot{\mathbf{r}}) = mr^2\dot{\varphi}(\mathbf{e}_r \times \mathbf{e}_\varphi) \\ &= mr^2\dot{\varphi}(-\mathbf{e}_\vartheta) = mr^2\dot{\varphi}\mathbf{e}_z = \text{const} \\ \curvearrowright\quad \dot{\varphi}^2 &= \frac{L^2}{m^2r^4}\ . \end{aligned}$$

Thus it is left:

$$\begin{aligned} E &= \frac{m}{2}\,\dot{r}^2 + \frac{L^2}{2mr^2} + V(r) = \frac{m}{2}\,\dot{r}^2 + V_{\text{eff}}(r) \\ V_{\text{eff}}(r) &= V(r) + \frac{L^2}{2mr^2}\ . \end{aligned}$$

With

$$\mathbf{F}(\mathbf{r}) \stackrel{!}{=} -\nabla V(r) = -\frac{dV}{dr}\mathbf{e}_r \stackrel{!}{=} -\frac{\alpha}{r^n}\mathbf{e}_r$$

it holds except for an unimportant additive constant:

$$V(r) = -\frac{\alpha}{n-1}\cdot\frac{1}{r^{n-1}}\ .$$

4. Condition for a circular path:

$$V_{\text{eff}}(r) = \frac{-\alpha}{(n-1)r^{n-1}} + \frac{L^2}{2mr^2}$$

must possess a minimum!

- Necessary condition:

$$\frac{d}{dr}\,V_{\text{eff}}(r)\big|_{r=r_0} = \frac{\alpha}{r_0^n} - \frac{L^2}{mr_0^3} \stackrel{!}{=} 0 \quad\curvearrowright\quad r_0^{3-n} = \frac{L^2}{m\alpha}\ .$$

- Sufficient condition:

$$\frac{d^2}{dr^2}\,V_{\text{eff}}(r)|_{r=r_0} = -\frac{n\alpha}{r_0^{n+1}} + \frac{3L^2}{mr_0^4} \overset{!}{>} 0$$

$$\curvearrowright \quad -n\alpha + \frac{3L^2}{mr_0^{3-n}} = -n\alpha + 3\alpha \overset{!}{>} 0 \quad \curvearrowright \quad \alpha(3-n) \overset{!}{>} 0 \;\; (\alpha > 0)\;.$$

Conclusion:

$$\textit{condition for a circular path:} \quad n < 3$$

$$\textit{radius:} \quad r_0 = \left(\frac{L^2}{m\alpha}\right)^{\frac{1}{3-n}}\;.$$

Section 3.3

Solution 3.3.1

1. The forces which act on m_1 are:

$$F_1 = -k_1(x_1 - x_{01})\;,$$
$$F_{12} = -k_{12}\left[(x_1 - x_{01}) - (x_2 - x_{02})\right]\;.$$

   The forces which act on m_2 are:

$$F_2 = -k_2(x_2 - x_{02})\;,$$
$$F_{21} = -F_{12}\;.$$

2. With the abbreviations

$$\bar{x}_i = x_i - x_{0i}\;; \quad i = 1, 2$$

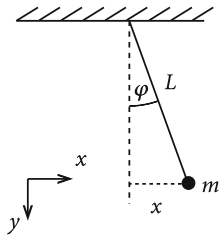
   the equations of motion read:

$$m_1\ddot{\bar{x}}_1 = -k_1\bar{x}_1 - k_{12}(\bar{x}_1 - \bar{x}_2)\;,$$
$$m_2\ddot{\bar{x}}_2 = -k_2\bar{x}_2 + k_{12}(\bar{x}_1 - \bar{x}_2)\;.$$

3. Using the ansatz

$$\bar{x}_i = \alpha_i \cos\omega t$$

Fig. A.39



one gets the following homogeneous system of equations:

$$\begin{pmatrix} k_1 + k_{12} - m_1\omega^2 & -k_{12} \\ -k_{12} & k_2 + k_{12} - m_2\omega^2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

For a non-trivial solution the secular-determinant must vanish:

$$\begin{aligned} 0 &\stackrel{!}{=} \left(k_1 + k_{12} - m_1\omega^2\right)\left(k_2 + k_{12} - m_2\omega^2\right) - k_{12}^2 = \\ &= (3k - m\omega^2)(6k - 2m\omega^2) - k^2 = 2(3k - m\omega^2)^2 - k^2 \\ \Longrightarrow \omega_\pm^2 &= \left(3 \pm \frac{1}{\sqrt{2}}\right)\frac{k}{m} . \end{aligned}$$

Solution 3.3.2 Oscillation equation of the simple pendulum:

$$\ddot{\varphi} + \frac{g}{L}\sin\varphi = 0 .$$

For **small** pendulum oscillations (Fig. A.39):

$$\frac{x}{L} = \sin\varphi \approx \varphi$$

$\Longrightarrow$ approximate equation of motion for x

$$\ddot{x} + \frac{g}{L}x = 0 .$$

That corresponds to the *'external'* force on the mass m:

$$\mathbf{F}^{(\mathrm{ext})} \approx -m\frac{g}{L}x .$$

Additionally there is an *'internal'* force due to the coupling of the pendulums:

$$F_{12} = -k(\hat{x}_1 - \hat{x}_2) \qquad \text{with} \qquad \hat{x}_i = x_i - x_{i0}$$

x_{i0}: rest position.

$\Longrightarrow$ Coupled equations of motion:

$$\ddot{\hat{x}}_1 + \frac{g}{L}\hat{x}_1 + \frac{k}{m}(\hat{x}_1 - \hat{x}_2) = 0$$
$$\ddot{\hat{x}}_2 + \frac{g}{L}\hat{x}_2 - \frac{k}{m}(\hat{x}_1 - \hat{x}_2) = 0 \; .$$

Subtraction and addition, respectively, of the two equations lead to:

$$\frac{d^2}{dt^2}(\hat{x}_1 - \hat{x}_2) + \left(\frac{g}{L} + \frac{2k}{m}\right)(\hat{x}_1 - \hat{x}_2) = 0$$
$$\frac{d^2}{dt^2}(\hat{x}_1 + \hat{x}_2) + \frac{g}{L}(\hat{x}_1 + \hat{x}_2) = 0 \; .$$

Change of variables

$$u = \hat{x}_1 - \hat{x}_2 \; ; \quad v = \hat{x}_1 + \hat{x}_2$$

and

$$\omega^2 \equiv \frac{g}{L} + \frac{2k}{m} \; ; \quad \omega_0^2 = \frac{g}{L}$$

where ω_0 is the eigenfrequency of the simple pendulum yields

$$\ddot{u} + \omega^2 u = 0$$
$$\ddot{v} + \omega_0^2 v = 0 \; .$$

General solutions of the decoupled(!) equations of motion:

$$u = a\sin\omega t + b\cos\omega t \iff \text{inversely phased oscillation}$$
$$v = A\sin\omega_0 t + B\cos\omega_0 t \iff \text{in phase oscillation} \; .$$

Initial conditions:

$$u(0) = -x_0 \; , \quad v(0) = x_0$$
$$\dot{u}(0) = \dot{v}(0) = 0$$
$$\Longrightarrow \; b = -x_0 \; ; \quad B = x_0 \; ; \quad a = A = 0 \; .$$

Solution:

$$\hat{x}_1(t) = \frac{x_0}{2}(-\cos\omega t + \cos\omega_0 t)$$
$$\hat{x}_2(t) = \frac{x_0}{2}(\cos\omega t + \cos\omega_0 t)\ .$$

For a simpler interpretation we still reformulate the solution a bit:

$$\omega = \frac{\omega-\omega_0}{2} + \frac{\omega+\omega_0}{2}\ ; \quad \omega_0 = \frac{\omega+\omega_0}{2} - \frac{\omega-\omega_0}{2}$$
$$\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\Longrightarrow\ \hat{x}_1(t) = x_0\sin\left(\frac{\omega-\omega_0}{2}t\right)\sin\left(\frac{\omega+\omega_0}{2}t\right)$$
$$\hat{x}_2(t) = x_0\cos\left(\frac{\omega-\omega_0}{2}t\right)\cos\left(\frac{\omega+\omega_0}{2}t\right)\ .$$

For *'weak'* coupling

$$\omega_0 = \sqrt{\frac{g}{L}} \approx \omega = \sqrt{\frac{g}{L}+\frac{2k}{m}}$$

$\hat{x}_{1,2}(t)$ represent oscillations with the frequency $\frac{1}{2}(\omega+\omega_0)\approx\omega_0$, which are via the amplitude functions

$$x_0\sin\left(\frac{\omega-\omega_0}{2}t\right)\ ; \quad x_0\cos\left(\frac{\omega-\omega_0}{2}t\right)$$

'weakly', i.e. with small frequency $\frac{1}{2}(\omega-\omega_0)$, temporally modulated $\Longrightarrow$ *'beat of oscillation'*.

Solution 3.3.3 Equations of motion

$$m\ddot{x}_1 = -kx_1 + k(x_2-x_1) = -2kx_1 + kx_2$$
$$m\ddot{x}_2 = -k(x_2-x_1)$$

$x_1,\ x_2$: deviations from the rest positions. Ansatz for the solution:

$$x_i = A_i\cos\omega t\ ; \quad i = 1,2\ .$$

Insertion yields:

$$\left(2k - m\omega^2\right) A_1 - kA_2 = 0$$
$$-kA_1 + \left(k - m\omega^2\right) A_2 = 0$$

$\Leftrightarrow$ linear, homogeneous system of equations for $A_1,\ A_2$. Solvability condition:

$$0 \overset{!}{=} \begin{vmatrix} 2k - m\omega^2 & -k \\ -k & k - m\omega^2 \end{vmatrix} = m^2\omega^4 - 3km\omega^2 + k^2$$

$$\Longleftrightarrow \quad \omega^4 - 3\omega_0^2\omega^2 + \omega_0^4 = 0 \qquad \text{with} \qquad \omega_0^2 \equiv \frac{k}{m}$$

$$\Longleftrightarrow \quad \left(\omega^2 - \frac{3}{2}\omega_0^2\right)^2 = \frac{5}{4}\omega_0^4$$

$$\Longrightarrow \quad \omega_+^2 = \left(\frac{3}{2} + \frac{\sqrt{5}}{2}\right)\omega_0^2 \ ; \quad \omega_-^2 = \left(\frac{3}{2} - \frac{\sqrt{5}}{2}\right)\omega_0^2 \ .$$

Eigenfrequencies must be positive:

$$\omega_+ = \frac{1}{\sqrt{2}}\sqrt{3 + \sqrt{5}}\,\omega_0 \qquad \text{inversely phased oscillation}$$
$$\omega_- = \frac{1}{\sqrt{2}}\sqrt{3 - \sqrt{5}}\,\omega_0 \qquad \text{in phase oscillation}\,.$$

Ratio of the amplitudes

$$\frac{A_1^{(\pm)}}{A_2^{(\pm)}} = \frac{k}{2k - m\omega_\pm^2} = \frac{1}{2 - \frac{1}{2}\left(3 \pm \sqrt{5}\right)} = \frac{2}{1 \mp \sqrt{5}}$$
$$A_2^\pm = \frac{1}{2}\left(1 \mp \sqrt{5}\right) A_1^\pm$$

General solution

$$x_1(t) = \alpha_+ \cos(\omega_+ t + \varphi_+) + \alpha_- \cos(\omega_- t + \varphi_-)$$
$$x_2(t) = \frac{1}{2}\left(1 - \sqrt{5}\right)\alpha_+ \cos(\omega_+ t + \varphi_+) + \frac{1}{2}\left(1 + \sqrt{5}\right)\alpha_- \cos(\omega_- t + \varphi_-)$$

The four constants $\alpha_\pm,\ \varphi_\pm$ are fixed by initial conditions.

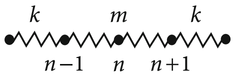


Fig. A.40

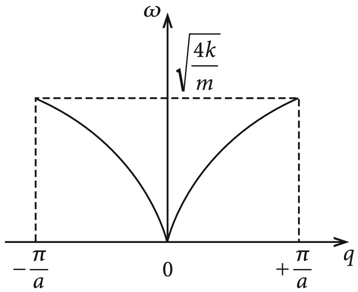


Fig. A.41

Solution 3.3.4

1. Equation of motion of the n-th atom (Fig. A.40):

$$\begin{aligned} m\ddot{u}_n &= k(u_{n+1} - u_n) + k(u_{n-1} - u_n) \\ &= k(u_{n+1} - 2u_n + u_{n-1}) \end{aligned}$$

Ansatz:

$$\begin{aligned} -m\omega^2 A e^{i(qR_n - \omega t)} &= kAe^{-i\omega t}\left(e^{iqR_{n+1}} - 2e^{iqR_n} + e^{iqR_{n-1}}\right) \\ \iff \quad -m\omega^2 e^{iqna} &= k\left(e^{iq(n+1)a} - 2e^{iqna} + e^{iq(n-1)a}\right) \\ \iff \quad \omega^2 &= -\frac{k}{m}\left(e^{iqa} - 2 + e^{-iqa}\right) \\ &= \frac{2k}{m}(1 - \cos qa) \\ \implies \quad \omega = \omega(q) &= \sqrt{\frac{2k}{m}(1 - \cos qa)} \end{aligned}$$

The resulting eigenfrequency ω is periodic with the period $\frac{2\pi}{a}$. Therefore we can restrict our considerations to the interval $-\frac{\pi}{a} \le q \le +\frac{\pi}{a}$ (1. Brillouin zone) (Fig. A.41).

2. The sites R_n and $R_{n\pm 1}$ are no longer equivalent:

$$\begin{aligned} u_{2n}(t) &= Ae^{i(q2na - \omega t)} \\ u_{2n+1}(t) &= Be^{i(q(2n+1)a - \omega t)} \end{aligned}$$

Equations of motion:

$$\begin{aligned} m\ddot{u}_{2n} &= f_1\,(u_{2n+1} - u_{2n}) + f_2\,(u_{2n-1} - u_{2n}) \\ m\ddot{u}_{2n-1} &= f_2\,(u_{2n} - u_{2n-1}) + f_1\,(u_{2n-2} - u_{2n-1}) \end{aligned}$$

$$\begin{aligned} \Longrightarrow \quad -m\omega^2 A &= f_1\left(Be^{iqa} - A\right) + f_2\left(Be^{-iqa} - A\right) \\ -m\omega^2 B &= f_2\left(Ae^{iqa} - B\right) + f_1\left(Ae^{-iqa} - B\right) \end{aligned}$$

homogeneous system of equations:

$$\begin{aligned} A\left(-m\omega^2 + f_1 + f_2\right) + B\left(-f_1 e^{iqa} - f_2 e^{-iqa}\right) &= 0 \\ A\left(-f_2 e^{iqa} - f_1 e^{-iqa}\right) + B\left(-m\omega^2 + f_2 + f_1\right) &= 0\,. \end{aligned}$$

Non-trivial solution $\Longleftrightarrow$ $\det(\ldots) = 0$:

$$\begin{aligned} 0 &= \left(-m\omega^2 + f_1 + f_2\right)^2 - \left(f_1 e^{iqa} + f_2 e^{-iqa}\right)\left(f_2 e^{iqa} + f_1 e^{-iqa}\right) \\ &= \left(-m\omega^2 + f_1 + f_2\right)^2 - \left(f_1^2 + f_2^2 + 2f_1 f_2 \cos(2qa)\right)\,. \end{aligned}$$

As solution we get the dispersion relation:

$$\omega_\pm^2(q) = \frac{1}{m}\left(f_1 + f_2 \pm \sqrt{f_1^2 + f_2^2 + 2f_1 f_2 \cos(2qa)}\right)\,.$$

The eigenfrequency is now periodic with the period $\frac{\pi}{a} \Longrightarrow q$ can be restricted to the region $-\frac{\pi}{2a} \le q \le +\frac{\pi}{2a}$, $\Longrightarrow$ compared to the situation in part 1. the Brillouin zone has halved.
Discussion (Fig. A.42):

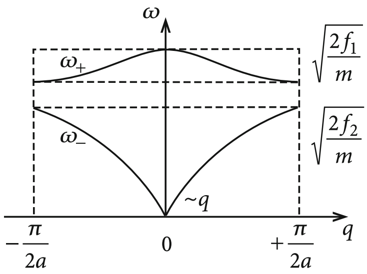
- $q = 0$:

$$\begin{aligned} \omega_+(q=0) &= \sqrt{\tfrac{2}{m}(f_1 + f_2)} && \text{(optical branch)} \\ \omega_-(q=0) &= 0 && \text{(acoustical branch)} \end{aligned}$$

- $q \ll \frac{\pi}{a}$:

$$\begin{aligned} \omega_-^2(q) &\approx \frac{1}{m}\left(f_1 + f_2 - \sqrt{f_1^2 + f_2^2 + 2f_1 f_2\left(1 - \frac{1}{2}(2qa)^2\right)}\right) \\ &= \frac{1}{m}\left(f_1 + f_2 - \sqrt{(f_1 + f_2)^2 - 4f_1 f_2 a^2 q^2}\right) \end{aligned}$$

Fig. A.42



$$
\begin{aligned}
&= \frac{1}{m}(f_1 + f_2)\left(1 - \sqrt{1 - \frac{4f_1f_2}{(f_1+f_2)^2}\, a^2q^2}\right) \\
&\approx \frac{1}{m}(f_1 + f_2)\left(\frac{2f_1f_2}{(f_1+f_2)^2}\, a^2q^2\right) \\
&= \frac{2f_1f_2}{m(f_1+f_2)}\, a^2q^2
\end{aligned}
$$

$$\Longrightarrow \quad \omega_-(q) = v_s \cdot q \; .$$

This behavior is typical for the acoustical branch. v_s is the sound velocity:

$$v_s = a\sqrt{\frac{2f_1f_2}{m(f_1+f_2)}} \; .$$

- $q = \pm\frac{\pi}{2a}$:

$$\omega_\pm^2\left(\pm\frac{\pi}{2a}\right) \underset{\substack{\uparrow \\ f_1 \geq f_2}}{=} \frac{1}{m}\left(f_1 + f_2 \pm (f_1 - f_2)\right)$$

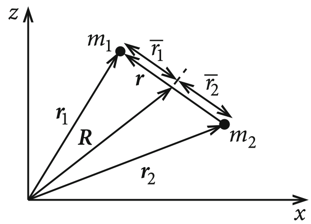
$$
\begin{aligned}
\Longrightarrow \quad \omega_+\left(\pm\frac{\pi}{2a}\right) &= \sqrt{\frac{2f_1}{m}} \\
\omega_-\left(\pm\frac{\pi}{2a}\right) &= \sqrt{\frac{2f_2}{m}}
\end{aligned}
$$

Solution 3.3.5

1.

$$\mathbf{g} = (0, 0, -g) \; .$$

Fig. A.43



Equations of motion:

$$m_1\ddot{\mathbf{r}}_1 = \mathbf{F}_1^{(\mathrm{ex})} + \mathbf{F}_{12} \;,$$
$$m_2\ddot{\mathbf{r}}_2 = \mathbf{F}_2^{(\mathrm{ex})} + \mathbf{F}_{21} \;.$$

It holds for the involved forces:

$$\mathbf{F}_1^{(\mathrm{ex})} = m_1\mathbf{g}\,; \quad \mathbf{F}_2^{(\mathrm{ex})} = m_2\mathbf{g}\,; \quad \mathbf{F}_{12} = -\mathbf{F}_{21} \;.$$

The total external force

$$\mathbf{F}^{(\mathrm{ex})} = \sum_i \mathbf{F}_i^{(\mathrm{ex})} = M\,\mathbf{g}\,; \quad M = m_1 + m_2$$

moves the center of gravity (Fig. A.43)

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$$

fulfilling the center of mass theorem:

$$M\ddot{\mathbf{R}} = \mathbf{F}^{(\mathrm{ex})} = M\,\mathbf{g} \;.$$

2. With the initial conditions

$$\mathbf{R}(t=0) = 0 \;; \quad \dot{\mathbf{R}}(t=0) = \mathbf{v}_0$$

the center of gravity follows the path:

$$\mathbf{R}(t) = \frac{1}{2}\mathbf{g}\,t^2 + \mathbf{v}_0 \cdot t \;.$$

3. The total angular momentum $\mathbf{L}$ can be decomposed into a relative part $\mathbf{L}_r$ and a center-of-gravity part $\mathbf{L}_s$:

$$\mathbf{L} = \sum_{i=1}^{2} m_i(\mathbf{r}_i \times \dot{\mathbf{r}}_i) = \mathbf{L}_r + \mathbf{L}_s \;,$$

for which we have found in (3.53) and (3.54):

$$\mathbf{L}_s = M(\mathbf{R} \times \dot{\mathbf{R}})$$
$$\mathbf{L}_r = \mu(\mathbf{r} \times \dot{\mathbf{r}}) \;, \qquad \mu = \frac{m_1 \cdot m_2}{m_1 + m_2}$$

$\mathbf{L}_s$ can explicitly be calculated:

$$\mathbf{L}_s = M\left(\frac{1}{2}\mathbf{g}\,t^2 + \mathbf{v}_0 t\right) \times (\mathbf{g}\,t + \mathbf{v}_0) =$$
$$= \frac{1}{2}M(\mathbf{v}_0 \times \mathbf{g})t^2 \;.$$

4.

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = \frac{\mathbf{F}_1^{(\mathrm{ex})}}{m_1} - \frac{\mathbf{F}_2^{(\mathrm{ex})}}{m_2} + \frac{\mathbf{F}_{12}}{m_1} - \frac{\mathbf{F}_{21}}{m_2} = \frac{1}{\mu}\mathbf{F}_{12}$$
$$\Longrightarrow \mathbf{F}_{12} = \mu\ddot{\mathbf{r}} \;; \quad \mathbf{F}_{12} \propto \mathbf{r} \;.$$

It is about an effective one-particle-central-field problem. Hence it must be:

$$\mathbf{L}_r = \text{const}$$

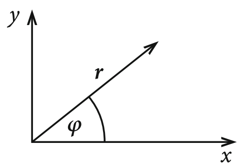
5. Because of $\mathbf{L}_r = \text{const}$ the relative motion takes place within a fixed orbital plane and therefore can be conveniently described by use of plane polar coordinates (Fig. A.44).

$$r = |\mathbf{r}_1 - \mathbf{r}_2| = l = \text{const}$$

Equations (2.8) till (2.13) provide in our case because of $\dot{r} = 0$:

$$\mathbf{r}(t) = l\,\mathbf{e}_r(t) \;,$$
$$\dot{\mathbf{r}}(t) = l\,\dot{\varphi}\,\mathbf{e}_\varphi \;,$$
$$\ddot{\mathbf{r}}(t) = -l\,\dot{\varphi}^2\,\mathbf{e}_r + l\,\ddot{\varphi}\,\mathbf{e}_\varphi \;.$$

Fig. A.44



Since a central force is present it must be

$$\ddot{\mathbf{r}} \propto \mathbf{e}_r$$

That means

$$\ddot{\varphi} = 0 \Longrightarrow \dot{\varphi} = \omega = \text{const}$$

and therewith

$$\ddot{\mathbf{r}} = -l\,\omega^2\,\mathbf{e}_r = -\omega^2\mathbf{r}\ .$$

So the solutions are of the type

$$\mathbf{r}(t) = l(\cos\omega t\,\mathbf{e}_x + \sin\omega t\,\mathbf{e}_y)\ .$$

The motions of the masses m_1, m_2 relative to the center of gravity are described by ((3.43), (3.44)) (Fig. A.43):

$$\bar{\mathbf{r}}_1 = \frac{m_2}{M}\mathbf{r}\ ; \quad \bar{\mathbf{r}}_2 = -\frac{m_1}{M}\mathbf{r}$$

$$\bar{\mathbf{r}}_1(t) = l\frac{m_2}{M}(\cos\omega t\,\mathbf{e}_x + \sin\omega t\,\mathbf{e}_y)\ ,$$

$$\bar{\mathbf{r}}_2(t) = -l\frac{m_1}{M}(\cos\omega t\,\mathbf{e}_x + \sin\omega t\,\mathbf{e}_y)\ .$$

These are obviously circular paths with radiuses

$$\rho_1 = l\frac{m_2}{M}\ ; \quad \rho_2 = l\frac{m_1}{M}\ ; \quad \frac{\rho_1}{\rho_2} = \frac{m_2}{m_1}\ ,$$

which are passed through with constant angular velocity ω.

Solution 3.3.6 Particle 2 is at rest before the collision (Fig. A.45) $\Longrightarrow$ the motion happens in a fixed plane.

1. momentum conservation law

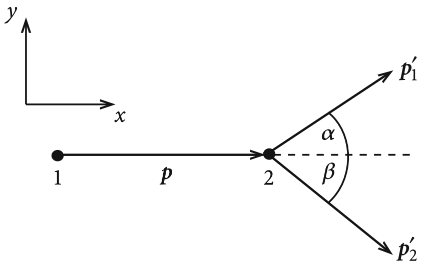
$$\mathbf{p} = \mathbf{p}_1' + \mathbf{p}_2'$$

component by component:

$$p = p_1'\cos\alpha + p_2'\cos\beta$$

$$0 = p_1'\sin\alpha - p_2'\sin\beta\ .$$

Fig. A.45



This can be gathered as follows:

$$\begin{aligned}
p\sin\alpha &= p_2'(\cos\beta\sin\alpha + \sin\beta\cos\alpha)\\
&= p_2'\sin(\alpha+\beta)\\
p\sin\beta &= p_1'(\cos\alpha\sin\beta + \sin\alpha\cos\beta)\\
&= p_1'\sin(\alpha+\beta)\ .
\end{aligned}$$

energy theorem:

$$\begin{aligned}
&\frac{\mathbf{p}^2}{2m} = \frac{\mathbf{p}_1'^{\,2}}{2m} + \frac{\mathbf{p}_2'^{\,2}}{2m} + Q\\
\Longleftrightarrow\ & p^2 = \mathbf{p}_1'^{\,2} + \mathbf{p}_2'^{\,2} + 2mQ\\
&\quad = p^2\frac{\sin^2\beta}{\sin^2(\alpha+\beta)} + p^2\frac{\sin^2\alpha}{\sin^2(\alpha+\beta)} + 2mQ\\
&\Longrightarrow\ \frac{\sin^2(\alpha+\beta)}{\sin^2\alpha+\sin^2\beta} = \frac{p^2}{p^2-2mQ}
\end{aligned}$$

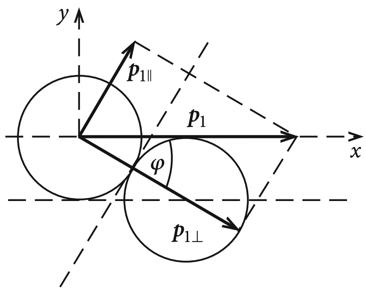
2. Special case $\alpha = \beta$:

$$\begin{aligned}
&\frac{\sin^2(2\alpha)}{2\sin^2\alpha} = \frac{4\sin^2\alpha\cos^2\alpha}{2\sin^2\alpha} = 2\cos^2\alpha\\
\Longrightarrow\ & \cos^2\alpha = \frac{1}{2}\frac{p^2}{p^2-2mQ}
\end{aligned}$$

elastic collision ($Q = 0$):

$$\Longrightarrow\ \cos^2\alpha = \frac{1}{2}\ \Longrightarrow\ \alpha = 45^\circ\ .$$

Fig. A.46



inelastic collision ($Q > 0$):

$$\Longrightarrow \cos^2\alpha \quad \text{becomes bigger} \quad \Longrightarrow \quad \alpha \quad \text{smaller} .$$

$$Q = \frac{p^2}{2m}\left(1 - \frac{1}{2\cos^2\alpha}\right) > 0 \Longrightarrow \; 2\cos^2\alpha > 1 \;\Longrightarrow\; 0 \leq \alpha < 45°$$

$\alpha = 0$:

$$Q = \frac{1}{2}\frac{p^2}{2m} = \frac{1}{2}T$$

That is the maximum energy which can be detracted from the kinetic energy of the collision partners.

Solution 3.3.7

1. The initial momentum $\mathbf{p}_1$ is decomposed in its components parallel and perpendicular to the contact plane ($\mathbf{p}_{1\parallel}, \mathbf{p}_{1\perp}$) (Fig. A.46). Since according to the precondition friction effects do not appear there is no force transfer within the contact plane. The parallel component of the momentum thus does not change:

$$\mathbf{p}_{1\parallel} = \mathbf{p}'_{1\parallel} \; ; \quad \mathbf{p}_{2\parallel} = \mathbf{p}'_{2\parallel} = 0 \; .$$

momentum conservation law:

$$\begin{aligned} \mathbf{p}_1 + \mathbf{p}_2 &= \mathbf{p}'_1 + \mathbf{p}'_2 \\ \Longrightarrow \mathbf{p}_{1\perp} + \mathbf{p}_{2\perp} &= \mathbf{p}'_{1\perp} + \mathbf{p}'_{2\perp} = \mathbf{p}_{1\perp} \; . \end{aligned}$$

Furthermore we exploit the energy conservation law:

$$\frac{1}{2m_1}\mathbf{p}^2_{1\perp} = \frac{1}{2m_1}\mathbf{p}'^2_{1\perp} + \frac{1}{2m_2}\mathbf{p}'^2_{2\perp} \; .$$

The two conservation laws lead to the following conditional equations:

$$p_{1\perp}^2 = p_{1\perp}'^2 + p_{2\perp}'^2 + 2p_{1\perp}' p_{2\perp}' \ ,$$
$$p_{1\perp}^2 = p_{1\perp}'^2 + \frac{m_1}{m_2} p_{2\perp}'^2 \ .$$

These are solved by:

$$p_{1\perp}' = \frac{m_1 - m_2}{m_1 + m_2} p_{1\perp} \ ,$$
$$p_{2\perp}' = \frac{2m_2}{m_1 + m_2} p_{1\perp} \ .$$

That can be evaluated a bit further:

$$p_{1\perp} = p_1 \cos\varphi \ ; \quad p_{1\parallel} = p_1 \sin\varphi \ ,$$
$$\sin\varphi = \frac{A}{2A} = \frac{1}{2} \Longrightarrow \varphi = 30^\circ \Longrightarrow \cos\varphi = \frac{1}{2}\sqrt{3} \ .$$

Therewith it follows:

$$\mathbf{p}_1' = \frac{1}{2} p_1 \left(\sqrt{3} \frac{m_1 - m_2}{m_1 + m_2} \mathbf{e}_\perp + \mathbf{e}_\parallel \right) \ ,$$
$$\mathbf{p}_2' = \sqrt{3}\, p_1 \frac{m_2}{m_1 + m_2} \mathbf{e}_\perp \ .$$

With

$$\mathbf{e}_\perp = \cos\varphi\, \mathbf{e}_x - \sin\varphi\, \mathbf{e}_y = \frac{1}{2} \left(\sqrt{3}, -1 \right) \ ,$$
$$\mathbf{e}_\parallel = \cos\left(\frac{\pi}{2} - \varphi\right) \mathbf{e}_x + \sin\left(\frac{\pi}{2} - \varphi\right) \mathbf{e}_y = \frac{1}{2} \left(1, \sqrt{3}\right)$$

the momenta after the collision read:

$$\mathbf{p}_1' = \frac{1}{2} \frac{p_1}{m_1 + m_2} \left(2m_1 - m_2, \sqrt{3}\, m_2 \right) \ ,$$
$$\mathbf{p}_2' = \frac{1}{2} \frac{p_1}{m_1 + m_2} \left(3m_2, -\sqrt{3}\, m_2 \right) \ .$$

An interesting special case appears for equal masses $m_1 = m_2 = m$:

$$p'_{2\perp} = p_{1\perp} \; , \quad p'_{1\perp} = 0 \; .$$

That means for the final momenta:

$$\mathbf{p}'_1 \cdot \mathbf{p}'_2 = 0 \iff \mathbf{p}'_1 \perp \mathbf{p}'_2 \; .$$

2. In the center-of-gravity system it holds for the momenta:

$$\bar{\mathbf{p}}_i = \mathbf{p}_i - \frac{m_i}{M}\mathbf{P} \; ,$$
$$\bar{\mathbf{p}}'_i = p'_i - \frac{m_i}{M}\mathbf{P} \; .$$

Here it is

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}'_1 + \mathbf{p}'_2 = \mathbf{p}_1 \; .$$

Before the collision it therefore holds:

$$\bar{\mathbf{p}}_1 = \mathbf{p}_1 - \frac{m_1}{M}\mathbf{p}_1 = \frac{m_2}{M}\mathbf{p}_1 \; ,$$
$$\bar{\mathbf{p}}_2 = 0 - \frac{m_2}{M}\mathbf{p}_1 = -\bar{\mathbf{p}}_1 \; .$$

After the collision the two balls have the following momenta:

$$\bar{\mathbf{p}}'_1 = \mathbf{p}'_1 - \frac{m_1}{M}\mathbf{p}_1 = \frac{1}{2}\frac{m_2 p_1}{m_1 + m_2}(-1, \sqrt{3}) \; ,$$
$$\bar{\mathbf{p}}'_2 = \mathbf{p}'_2 - \frac{m_2}{M}\mathbf{p}_1 = \frac{1}{2}\frac{m_2 p_1}{m_1 + m_2}(1, -\sqrt{3}) \; .$$

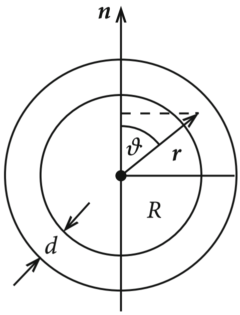
Section 4.5

Solution 4.5.1

1. Mass density (Fig. A.47):

$$\rho(\mathbf{r}) = \begin{cases} \rho_0 \; , \text{ for} \quad R - d \le r \le R \\ 0 \quad \text{otherwise} \end{cases}$$

Fig. A.47



Moment of inertia:

$$J = \int d^3r\, \rho(\mathbf{r})(\mathbf{n} \times \mathbf{r})^2 =$$

$$= \rho_0 \int\limits_{R-d}^{R} r^4 dr \int\limits_0^{2\pi} d\varphi \int\limits_{-1}^{+1} d\cos\vartheta \,(1 - \cos^2\vartheta) =$$

$$= \frac{8\pi}{15}\rho_0 \left[R^5 - (R-d)^5\right] \; .$$

Since $d \ll R$ it holds approximately:

$$(R-d)^5 = R^5 \left(1 - \frac{d}{R}\right)^5 \approx R^5 - 5d\,R^4 \; .$$

That means:

$$J \approx \frac{8\pi}{3}\rho_0\, d\, R^4 \; .$$

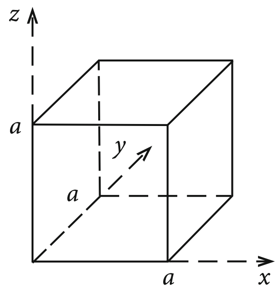
For the mass M of the spherical shell one calculates

$$M = \frac{4\pi}{3}\rho_0 \left[R^3 - (R-d)^3\right] \approx 4\pi\rho_0 R^2 d$$

and therewith for the moment of inertia:

$$J \approx \frac{2}{3} M\, R^2 \; .$$

Fig. A.48



2. The z axis may be the rotation axis (Fig. A.48):

$$J = \int d^3r\, \rho(\mathbf{r})(x^2 + y^2) =$$

$$= \rho_0 \int\int\int_0^a dx\, dy\, dz\, (x^2 + y^2) =$$

$$= \rho_0 a^2 \frac{a^3}{3} 2 \,.$$

For the mass M holds:

$$M = \rho_0 \cdot a^3 \,.$$

That means:

$$J = \frac{2}{3} M\, a^2 \,.$$

3. Mass density (Fig. A.49) (cylindrical coordinates: $\overline{\rho}$, φ , z):

$$\rho(\mathbf{r}) = \begin{cases} \alpha \cdot \overline{\rho} \,, \text{ if } 0 \le \overline{\rho} \le R \quad \text{and } -\frac{L}{2} \le z \le +\frac{L}{2} \\ 0 \qquad \text{otherwise.} \end{cases}$$

For the mass M it holds in this case:

$$M = \int \rho(\mathbf{r}) d^3r = \alpha \int_0^R \overline{\rho}^2 d\overline{\rho} \int_0^{2\pi} d\varphi \int_{-\frac{L}{2}}^{+\frac{L}{2}} dz = \alpha \frac{R^3}{3} 2\pi\, L \,.$$

That determines the constant α:

$$\alpha = \frac{3M}{2\pi\, L \cdot R^3} \,.$$

Fig. A.49

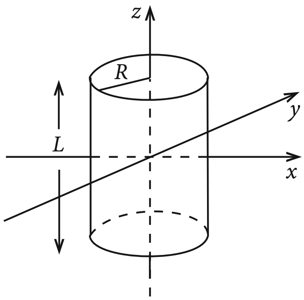
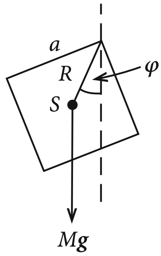


Fig. A.50



Now we calculate the moment of inertia:

$$J = \int d^3r\,\rho(\mathbf{r})\overline{\rho}^2 = 2\pi\,L\alpha\int_0^R \overline{\rho}^4 d\overline{\rho} = \frac{2\pi\,L\alpha}{5}R^5 = \frac{3}{5}M\,R^2\ .$$

Solution 4.5.2 It is about a realization of the physical pendulum treated in Sect. 4.2.3 (Fig. A.50). The equation of motion is derived in (4.22):

$$J\,\ddot{\varphi} + M\,g\,R\sin\varphi = 0\ .$$

R is the vertical distance of the center of gravity to the rotation axis

$$R = \frac{a}{\sqrt{2}}\ .$$

The moment of inertia J we have calculated in part 2. of Exercise 4.5.1:

$$J = \frac{2}{3}M\,a^2\ .$$

For small oscillations ($\sin\varphi \simeq \varphi$) the equation of motion then reads:

$$\ddot{\varphi} + \frac{3g}{2\sqrt{2}\,a}\varphi = 0\ .$$

The oscillation period and the angular frequency one can take directly from this expression:

$$\omega = 2^{-\frac{3}{4}}\sqrt{\frac{3g}{a}}\ ; \qquad \tau = \frac{2\pi}{\omega}\ .$$

According to Eq. (4.23) the thread length of the equivalent mathematical pendulum would be:

$$l = \frac{J}{M R} \Longrightarrow l = \frac{2\sqrt{2}}{3}a\ .$$

Solution 4.5.3

1. We use the same notation as in Fig. 4.11. For the potential energy we can directly adopt Eq. (4.34):

$$V = M\,g(l - s)\sin\alpha\ .$$

For the kinetic energy it holds:

$$T = \frac{1}{2}J\,\omega^2 + \frac{1}{2}M\,\dot{s}^2\ .$$

J is the moment of inertia with respect to the symmetry axis of the infinitely thin-walled hollow cylinder:

$$J = M\,R^2\ .$$

From the rolling off condition (4.31)

$$\Delta s = R\Delta\varphi \Longleftrightarrow \dot{s} = R\dot{\varphi}$$

it follows with $\dot{\varphi} = \omega$ and $\dot{s} = v$:

$$\omega = \frac{v(t)}{R}\ .$$

The total kinetic energy T is then:

$$T = Mv^2(t)$$

At $t = 0$ the potential energy amounts to

$$V(s = 0) = M\,g\,l\sin\alpha \equiv V_0\ .$$

The kinetic energy is zero at $t = 0$. Therewith the energy theorem reads:

$$\begin{aligned} V(s) + T(\dot{s}) &= \text{const} = V_0 \\ \Longrightarrow M\,g(l-s)\sin\alpha + M\,v^2(t) &= M\,g\,l\sin\alpha \\ \Longrightarrow v^2(t) &= g\,s\sin\alpha = \dot{s}^2(t)\ . \end{aligned}$$

2. The last relation can be written as:

$$\frac{ds}{dt} = \sqrt{g\sin\alpha}\cdot s^{1/2}\ .$$

We separate the variables and integrate:

$$\begin{aligned} \int_0^s \frac{ds'}{\sqrt{s'}} &= \sqrt{g\sin\alpha}\int_0^t dt' \\ \Longrightarrow 2\sqrt{s(t)} &= t\cdot\sqrt{g\sin\alpha}\ . \end{aligned}$$

That gives the solution:

$$s(t) = \frac{1}{4}t^2\,g\sin\alpha\ .$$

The sought-after velocity $v(t) = \dot{s}(t)$ therewith reads:

$$v(t) = \frac{1}{2}t\,g\sin\alpha\ .$$

Solution 4.5.4 Starting point shall be two body-fixed Cartesian systems of coordinates with parallel axes as sketched in Fig. 4.22. The origins of the coordinates are at the middle of the respective cylinder axis. Directions of the rotation axes:

$$\mathbf{n}_1 = \mathbf{n}_2 = -\mathbf{e}_z\ .$$

Let $\mathbf{r}_1$, $\mathbf{r}_2$ be the support points and therewith the points where the thread tensions are acting:

$$\mathbf{r}_1 = (0, R_1, z_1) \quad ; \quad \mathbf{r}_2 = (0, -R_2, z_2)\ .$$

Thread tensions:

$$\mathbf{F}_1 = (F, 0, 0) = -\mathbf{F}_2\ .$$

Torques:

$$\mathbf{M}_{\mathrm{ex}}^{(1)} = (0, R_1, z_1) \times (F, 0, 0) = (0, z_1 F, -R_1 F)$$
$$\mathbf{M}_{\mathrm{ex}}^{(2)} = (0, -R_2, z_2) \times (-F, 0, 0) = (0, -z_2 F, -R_2 F) \ .$$

Paraxial components:

$$\mathbf{M}_{\mathrm{ex}}^{(1)} \cdot \mathbf{n}_1 = R_1 F \quad ; \quad \mathbf{M}_{\mathrm{ex}}^{(2)} \cdot \mathbf{n}_2 = R_2 F \ .$$

Angular momentum law (4.17):

$$J_1 \ddot{\varphi}_1 = R_1 F \quad ; \quad J_2 \ddot{\varphi}_2 = R_2 F \ .$$

Moments of inertia of the cylinders with their homogeneous mass density are given in (4.13):

$$J_1 = \frac{1}{2} M_1 R_1^2 \quad ; \quad J_2 = \frac{1}{2} M_2 R_2^2 \ .$$

Rolling off condition:

$$x_2 = \mathrm{const} + R_1 \varphi_1 + R_2 \varphi_2$$
$$\curvearrowright \ \ddot{x}_2 = R_1 \ddot{\varphi}_1 + R_2 \ddot{\varphi}_2 \ .$$

Translation of cylinder 2 according to the center of mass theorem:

$$M_2 \ddot{x}_2 = M_2 g - F \ .$$

It follows then by inserting:

$$M_2 R_1 \ddot{\varphi}_1 + M_2 R_2 \ddot{\varphi}_2 = M_2 g - F$$
$$\curvearrowright \ M_2 R_1 \frac{R_1 F}{J_1} + M_2 R_2 \frac{R_2 F}{J_2} = M_2 g - F$$
$$\curvearrowright \ F \left(1 + \frac{R_1^2 M_2}{J_1} + \frac{R_2^2 M_2}{J_2} \right) = M_2 g$$
$$\curvearrowright \ F \left(1 + 2 \frac{M_2}{M_1} + 2 \right) = M_2 g \ .$$

Therewith it holds for the thread tension:

$$F = \frac{M_1 M_2}{3 M_1 + 2 M_2} \, g \ .$$

Solution 4.5.5

1. Angular-momentum law:

$$\frac{d}{dt}\mathbf{L} = \frac{dL}{dt}\,\mathbf{l} + L\,\frac{d\mathbf{l}}{dt} = \mathbf{M}_{\text{ex}} = M(\mathbf{n}\times\mathbf{l})\ .$$

Scalar multiplication of this equation by $\mathbf{n}$, where $\mathbf{n}$ is the unit vector of the fixed (!) axis, i.e. $\frac{d}{dt}\mathbf{n} = 0$:

$$\mathbf{n}\cdot\frac{d}{dt}\mathbf{L} = \frac{d}{dt}(\mathbf{n}\cdot\mathbf{L}) = \mathbf{n}\cdot\mathbf{M}_{\text{ex}} = 0\ .$$

That means:

$$\mathbf{n}\cdot\mathbf{L} = \text{const}\ .$$

Now scalar multiplication of the angular momentum law by $\mathbf{l}$:

$$\mathbf{l}\cdot\frac{d}{dt}\mathbf{L} = \frac{dL}{dt}\,\mathbf{l}^2 + L\mathbf{l}\cdot\frac{d\mathbf{l}}{dt} = \frac{dL}{dt} + \frac{1}{2}L\underbrace{\frac{d}{dt}\mathbf{l}^2}_{=0} = \frac{dL}{dt} \overset{!}{=} \mathbf{l}\cdot\mathbf{M}_{\text{ex}} = 0\ .$$

That means:

$$|\mathbf{L}| = \text{const}\ .$$

2. Because of $\frac{dL}{dt} = 0$ it follows from the angular momentum law:

$$L\,\frac{d\mathbf{l}}{dt} = \mathbf{M}_{\text{ex}} = M(\mathbf{n}\times\mathbf{l})\ .$$

That means:

$$\frac{d}{dt}\mathbf{L} = \frac{M}{L}(\mathbf{n}\times\mathbf{L})\ .$$

The angular momentum $\mathbf{L}$ is precessing around the axis $\mathbf{n}$ with the angular velocity:

$$\boldsymbol{\omega}_p = \frac{M}{L}\,\mathbf{n}\ .$$

Solution 4.5.6

1. Components of the inertial tensor in the particle picture (4.45):

$$J_{mn} = \sum_{i=1}^{N} m_i \left(\mathbf{r}_i^2 \delta_{mn} - x_{in} x_{im}\right) \qquad \mathbf{r}_i\,(x_{i1}, x_{i2}, x_{i3})$$

$$\Sigma \to \Sigma' \qquad \curvearrowright \qquad \mathbf{r}_i \to \mathbf{r}_i' = \mathbf{r}_i + \mathbf{a}\ .$$

Therewith it follows:

$$\begin{aligned} J'_{mn} &= \sum_i m_i \left(\mathbf{r}_i^{2\prime} \delta_{mn} - x'_{in} x'_{im}\right) \\ &= \sum_i m_i \left((\mathbf{r}_i + \mathbf{a})^2 \delta_{mn} - (x_{im} + a_m)(x_{in} + a_n)\right) \\ &= \sum_i \left(\mathbf{r}_i^2 \delta_{mn} - x_{im} x_{in}\right) + \sum_i m_i \left(\mathbf{a}^2 \delta_{mn} - a_m a_n\right) \\ &\quad + 2\mathbf{a} \cdot \sum_i m_i \mathbf{r}_i \delta_{mn} - \sum_i \left(x_{im} a_n + a_m x_{in}\right)\ . \end{aligned}$$

The origin in Σ coincides with the center of gravity. That means:

$$\sum_i m_i \mathbf{r}_i = M\mathbf{R} = 0$$

$$\sum_i m_i x_{im} = M R_m = 0$$

$$\sum_i m_i x_{in} = M R_n = 0\ .$$

That leads us to the generalized Steiner's theorem:

$$J'_{mn} = J_{mn} + M\left(\mathbf{a}^2 \delta_{mn} - a_m a_n\right)\ .$$

2. Rotation $\Sigma \to \Sigma'$ means:

$$x'_i = \sum_j d_{ij} x_j \ ; \ d_{ij} = \cos\varphi_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j\ .$$

Inertial tensor in Σ' (particle picture):

$$J'_{ij} = \sum_\alpha m_\alpha \left(\mathbf{r}'^2_\alpha\, \delta_{ij} - x'_{i\alpha} x'_{i\alpha}\right) \qquad \alpha = 1, 2, \ldots, N\ .$$

We calculate step by step the various terms in this expression:

$$\begin{aligned}
\mathbf{r}'^2_\alpha = \sum_{k=1}^{3} x'^2_{\alpha k} &= \sum_k \sum_{st} d_{ks} d_{kt} x_{\alpha s} x_{\alpha t} \\
&= \sum_{st} \delta_{st} x_{\alpha s} x_{\alpha t} = \sum_s x^2_{\alpha s} \\
&= \mathbf{r}^2_\alpha \ .
\end{aligned}$$

Here we have exploited the orthogonality of the columns of the rotation matrix. It is clear that the length of a vector cannot change with the rotation.

$$\begin{aligned}
\delta_{ij} &= \sum_m d_{im} d_{jm} = \sum_{mn} d_{im} d_{jn} \delta_{mn} \\
x'_{\alpha i} x'_{\alpha j} &= \sum_{mn} d_{im} d_{jn} x_{\alpha m} x_{\alpha n} \ .
\end{aligned}$$

That eventually yields:

$$\begin{aligned}
J'_{ij} &= \sum_{mn} d_{im} d_{jn} \left\{ \sum_\alpha m_\alpha \left(\mathbf{r}^2_\alpha \delta_{mn} - x_{\alpha m} x_{\alpha n} \right) \right\} \\
&= \sum_{mn} d_{im} d_{jn} J_{mn} \ .
\end{aligned}$$

Thus the inertial tensor transforms as it is expected for a second-rank tensor.

Solution 4.5.7

1. $\widehat{\Sigma}$: body-fixed Cartesian system of coordinates with its origin in the lower-left edge of the cuboid and axes along the edges of the cuboid. Inertial tensor:

$$J_{mn} = \int_V d^3\hat{r} \rho(\hat{r}) \left(\hat{\mathbf{r}}^2 \delta_{mn} - \hat{x}_m \hat{x}_n \right) \qquad \hat{\mathbf{r}} = (\hat{x}_1, \hat{x}_2, \hat{x}_3) \ .$$

Homogeneous mass density ρ_0: Mass: $M = \rho_0 abc$. Therewith it holds:

$$\begin{aligned}
J_{11} &= \rho_0 \int_0^a d\hat{x} \int_0^b d\hat{y} \int_0^c d\hat{z} \left(\hat{y}^2 + \hat{z}^2 \right) \\
&= \rho a \int_0^b d\hat{y} \int_0^c d\hat{z} \left(\hat{y}^2 + \hat{z}^2 \right) \\
&= \rho_0 a \int_0^b d\hat{y} \left(c\hat{y}^2 + \frac{1}{3} c^3 \right)
\end{aligned}$$

$$
\begin{aligned}
&= \rho_0 a \left(\frac{1}{3} c b^3 + \frac{1}{3} c^3 b \right) \\
&= \rho_0 abc \frac{1}{3} \left(b^2 + c^2 \right) \\
&= \frac{1}{3} M \left(b^2 + c^2 \right) \; .
\end{aligned}
$$

Symmetry:

$$
J_{22} = \frac{1}{3} M \left(a^2 + c^2 \right) \quad ; \quad J_{33} = \frac{1}{3} M \left(a^2 + b^2 \right)
$$

$$
\begin{aligned}
J_{12} &= \rho_0 \int_0^a d\hat{x} \int_0^b d\hat{y} \int_0^c d\hat{z} \; (-\hat{x}\hat{y}) \\
&= -\rho_0 c \int_0^a d\hat{x} \, \hat{x} \frac{b^2}{2} = -\rho_0 c \frac{a^2}{2} \frac{b^2}{2} \\
&= -\frac{1}{4} M \, ab = J_{21} \; .
\end{aligned}
$$

Symmetry:

$$
J_{13} = J_{31} = -\frac{1}{4} M \, ac \quad ; \quad J_{23} = J_{32} = -\frac{1}{4} M \, bc \; .
$$

Inertial tensor:

$$
\underline{\widehat{\mathbf{J}}} = M \begin{pmatrix} \frac{1}{3}\left(b^2 + c^2\right) & -\frac{1}{4} ab & -\frac{1}{4} ac \\ -\frac{1}{4} ba & \frac{1}{3}\left(a^2 + c^2\right) & -\frac{1}{4} bc \\ -\frac{1}{4} ca & -\frac{1}{4} cb & \frac{1}{3}\left(a^2 + b^2\right) \end{pmatrix} \; .
$$

2. Rotation around the space diagonal of the cuboid:

$$
\boldsymbol{\omega} = \omega \, \mathbf{n} \qquad \mathbf{n} = \frac{1}{\sqrt{a^2 + b^2 + c^2}} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \; .
$$

Moment of inertia related to $\mathbf{n}$ (4.50):

$$
J_n = \sum_{i,j} J_{ij} \, n_i n_j \; .
$$

That means:

$$\begin{aligned}
J_n = &\frac{M}{a^2+b^2+c^2}\left(\frac{1}{3}a^2\left(b^2+c^2\right)-\frac{1}{4}a^2b^2-\frac{1}{4}a^2c^2-\frac{1}{4}b^2a^2\right.\\
&\left.+\frac{1}{3}b^2\left(a^2+c^2\right)-\frac{1}{4}b^2c^2-\frac{1}{4}c^2a^2-\frac{1}{4}c^2b^2+\frac{1}{3}c^2\left(a^2+b^2\right)\right)\\
= &\frac{M}{a^2+b^2+c^2}\left(\frac{2}{3}-\frac{2}{4}\right)\left(a^2b^2+a^2c^2+b^2c^2\right) .
\end{aligned}$$

Therewith we have found an expression likewise valid for all the four space diagonals:

$$J_n = \frac{1}{6}M\,\frac{a^2b^2+a^2c^2+b^2c^2}{a^2+b^2+c^2} .$$

3. $\overline{\Sigma}$: System of coordinates with axes parallel to the edges of the cuboid as in part 1., but now with its origin at the center of gravity of the cuboid. Seen from $\widehat{\Sigma}$ the latter lies because of the homogeneous mass density at $\mathbf{R} = \frac{1}{2}(a,b,c)$. (Verify that explicitly!)

$$\begin{aligned}
\bar{J}_{11} &= \rho_0 \int_{-\frac{a}{2}}^{+\frac{a}{2}} d\bar{x} \int_{-\frac{b}{2}}^{+\frac{b}{2}} d\bar{y} \int_{-\frac{c}{2}}^{+\frac{c}{2}} d\bar{z}\left(\bar{y}^2+\bar{z}^2\right)\\
&= \rho_0 a \int_{-\frac{b}{2}}^{+\frac{b}{2}} d\bar{y}\left(\bar{y}^2 c+\frac{1}{12}c^3\right)\\
&= \rho_0 a\left(\frac{1}{12}b^3c+\frac{1}{12}c^3\right)\\
&= \frac{1}{12}M\left(b^2+c^2\right) .
\end{aligned}$$

Symmetry:

$$\bar{J}_{22} = \frac{1}{12}M\left(a^2+c^2\right) \qquad \bar{J}_{33} = \frac{1}{12}M\left(a^2+b^2\right) .$$

Non-diagonal elements:

$$\begin{aligned}
\bar{J}_{12} &= \rho_0 \int_{-\frac{a}{2}}^{+\frac{a}{2}} d\bar{x} \int_{-\frac{b}{2}}^{+\frac{b}{2}} d\bar{y} \int_{-\frac{c}{2}}^{+\frac{c}{2}} d\bar{z}\,(-\bar{x}\bar{y})\\
&= \rho_0 c \int_{-\frac{a}{2}}^{+\frac{a}{2}} d\bar{x}(-\bar{x})\left(\frac{c^2}{4}-\frac{c^2}{4}\right)\\
&= 0 .
\end{aligned}$$

Analogously we get the other non-diagonal elements of the inertial tensor! The Cartesian axes of $\overline{\Sigma}$ thus represent the principal axes of inertia of the cuboid:

$$\underline{\mathbf{J}} = \frac{1}{12} M \begin{pmatrix} b^2+c^2 & 0 & 0 \\ 0 & a^2+c^2 & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix} .$$

Moment of inertia with respect to the space diagonal $\mathbf{n}$ as in part 2.:

$$\begin{aligned} J_n &= \sum_{i,j} J_{ij} n_i n_j \\ &= \frac{1}{a^2+b^2+c^2} \frac{1}{12} M \left(a^2\left(b^2+c^2\right) + b^2\left(a^2+c^2\right) + c^2\left(a^2+b^2\right)\right) \\ &= \frac{1}{6} M \, \frac{a^2b^2 + a^2c^2 + b^2c^2}{a^2+b^2+c^2} . \end{aligned}$$

That is the same result as in part 2. Clear, because for $\widehat{\Sigma}$ as well as for $\overline{\Sigma}$ the origin lies on the rotation axis.

4. Rotation axis now coincides with the cuboid-edge in y direction. Then $\widehat{\Sigma}$ from part 1. has its origin on the rotation axis, however $\overline{\Sigma}$ does not. Therefore the inertial tensor from part 1. has to be used. Rotation axis:

$$\mathbf{n} = \mathbf{e}_{\hat{y}} = (0, 1, 0) .$$

It follows:

$$J_y = \widehat{J}_{22} = \frac{1}{3} M \left(a^2 + c^2\right) .$$

Now the rotation axis shall be again in y direction but passing through the center of gravity of the cuboid. Now $\overline{\Sigma}$ has the origin on the axis, but $\widehat{\Sigma}$ does not. Hence, the inertial tensor from part 3. has to be applied. The direction of the rotation axis in $\overline{\Sigma}$, however, is analog.

$$\mathbf{n} = \mathbf{e}_{\bar{y}} = (0, 1, 0) .$$

That means

$$J = \overline{J}_{22} = \frac{1}{12} M \left(a^2 + c^2\right) .$$

Steiner's theorem:

$$J_y = J + M s^2 = \frac{1}{12} M \left(a^2 + c^2\right) + M s^2 .$$

Thereby s is the vertical distance of the origin of $\overline{\Sigma}$ (cuboid corner) from the parallel axis through the center of gravity of the cuboid:

$$s = \frac{1}{2}\sqrt{a^2 + c^2} \; .$$

Therewith:

$$J_y = J + M\,s^2 = \frac{1}{12}\,M\left(a^2 + c^2\right) + \frac{1}{4}M\left(a^2 + c^2\right) = \frac{1}{3}M\left(a^2 + c^2\right) \; .$$

That was to be shown!

Solution 4.5.8 The principal moments of inertia are found by solving the eigenvalue equation:

$$\underline{\mathbf{J}}\,\boldsymbol{\omega} = j\,\boldsymbol{\omega} \; .$$

The angular velocity $\boldsymbol{\omega}$ has thereby the direction of one of the principal axes of inertia. After Exercise 4.5.6 it holds here:

$$\underline{\mathbf{J}} = \frac{1}{4}Ma^2 \begin{pmatrix} \frac{8}{3} & -1 & -1 \\ -1 & \frac{8}{3} & -1 \\ -1 & -1 & \frac{8}{3} \end{pmatrix} \; .$$

Condition for a non-trivial solution of the homogeneous system of equations which results from the eigenvalue equation:

$$\det\left(\underline{\mathbf{J}} - j\,\mathbb{1}\right) \overset{!}{=} 0$$

or

$$\det\left(\underline{\mathbf{J}}' - j'\,\mathbb{1}\right) = 0 \quad \text{with} \quad j' = \frac{4}{Ma^2}\,j$$

and

$$\underline{\mathbf{J}}' = \begin{pmatrix} \frac{8}{3} & -1 & -1 \\ -1 & \frac{8}{3} & -1 \\ -1 & -1 & \frac{8}{3} \end{pmatrix} \; .$$

1. *Eigenvalues (principal moments of inertia)*

$$\begin{vmatrix} \frac{8}{3}-j' & -1 & -1 \\ -1 & \frac{8}{3}-j' & -1 \\ -1 & -1 & \frac{8}{3}-j' \end{vmatrix} \stackrel{!}{=} 0 \,.$$

With $x = \frac{8}{3} - j'$ one has to solve:

$$x^3 - 3x - 2 = 0 \quad \curvearrowright \quad x_1 = 2 \,;\; x_2 = x_3 = -1$$

That means:

$$j_1' = \frac{8}{3} - x_1 = \frac{2}{3} \,;\; j_{2,3}' = \frac{8}{3} - x_{2,3} = \frac{11}{3} \,.$$

Principal moments of inertia:

$$A = \frac{1}{4}Ma^2 j_1' = \frac{1}{6}Ma^2$$
$$B = C = \frac{1}{4}Ma^2 j_{2,3}' = \frac{11}{12}Ma^2 \,.$$

2. *Eigenvectors (principal axes of inertia)*
Eigenvectors of $\underline{\mathbf{J}}$ are also eigenvectors of $\underline{\mathbf{J}}'$!

(a) $A = \frac{1}{6}Ma^2$

$$\left(\underline{\mathbf{J}}' - j_1' \mathbb{1}\right) \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0 \,.$$

This is equivalent to:

$$2a_1 = a_2 + a_3$$
$$2a_2 = a_1 + a_3$$
$$\curvearrowright \quad a_1 = a_2 = a_3 \,.$$

(normalized) unit vector:

$$\mathbf{e}_\xi = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \,.$$

One of the principal axes of inertia thus is the space diagonal of the cube. The two others must therefore lie within the plane perpendicular to the space diagonal being orthogonal to each other. Apart from that, however, they should be arbitrarily rotatable in this plane.

(b) $B = \frac{11}{12}Ma^2$

$$\left(\underline{\mathbf{J}}' - j_2'\mathbb{1}\right)\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = 0 \; .$$

It follows

$$b_1 + b_2 + b_3 = 0 \; .$$

i.e. only one conditional equation. If one chooses $b_1 = b_2 = 1$ it arises as (normalized) unit vector:

$$\mathbf{e}_\eta = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \; .$$

Orthogonality:

$$\mathbf{e}_\eta \cdot \mathbf{e}_\xi = 0 \; .$$

(c) $C = \frac{11}{12}Ma^2$
From

$$\left(\underline{\mathbf{J}}' - j_3'\mathbb{1}\right)\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0$$

it follows now analogously:

$$c_1 + c_2 + c_3 = 0 \; .$$

That leads to the ansatz

$$\mathbf{e}_\zeta \propto \begin{pmatrix} c_1 \\ c_2 \\ -c_1 - c_2 \end{pmatrix} \; ,$$

where

$$\mathbf{e}_\zeta \cdot \mathbf{e}_\xi = 0$$

is already guaranteed. Furthermore it should hold:

$$0 \stackrel{!}{=} \mathbf{e}_\zeta \cdot \mathbf{e}_\eta = \frac{1}{\sqrt{6}} (c_1 + c_2 + 2c_1 + 2c_2) \curvearrowright c_1 = -c_2 \ .$$

That yields the (normalized) unit vector:

$$\mathbf{e}_\zeta = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \ .$$

The arbitrariness in the last step concerning the sign is removed by the requirement that the unit vectors build a right-handed system:

$$\mathbf{e}_\zeta \cdot \left(\mathbf{e}_\eta \times \mathbf{e}_\xi\right) \stackrel{!}{=} 1 \ .$$

The unit vectors $\mathbf{e}_\xi$, $\mathbf{e}_\eta$, $\mathbf{e}_\zeta$ define the directions of the principal axes of inertia!

Index

W. Nolting, *Theoretical Physics 1*, DOI 10.1007/978-3-319-40108-9