Section 2: Integrating factors

Solutions to Exercise level 1

1. (i) $x^2\frac{dy}{dx}+2xy=\cos x$

$$\frac{dy}{dx}(x^2y)=\cos x$$

$$x^2y=\sin x+c$$

$$y=\frac{1}{x^2}\sin x+\frac{c}{x^2}$$

(ii) $x^3e^y\frac{dy}{dx}+3x^2e^y=4$

$$\frac{d}{dx}\left(x^3e^y\right)=4$$

$$x^3e^y=4x+c$$

$$e^y=\frac{4}{x^2}+\frac{c}{x^3}$$

$$y=\ln\left|\frac{4}{x^2}+\frac{c}{x^3}\right|$$

(iii) $\frac{2y}{x}\frac{dy}{dx}-\frac{y^2}{x^2}=e^x$

$$\frac{d}{dx}\left(\frac{y^2}{x}\right)=e^x$$

$$\frac{y^2}{x}=e^x+c$$

$$y=\sqrt{xe^x+cx}$$

(iv) $\sin x\frac{dy}{dx}+y\cos x=x^2$

$$\frac{dy}{dx}(y\sin x)=x^2$$

$$y\sin x=\tfrac{1}{3}x^3+c$$

$$y=\frac{x^3+k}{3\sin x}$$

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2. (i) $\frac{dy}{dx}+\frac{y}{x}=\frac{1}{x}$

$\text{Integrating factor} = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

$x\frac{dy}{dx}+y=1$

$\frac{d}{dx}\left(xy\right)=1$

$xy=x+c$

$y=1+\frac{c}{x}$

(ii) $\frac{dy}{dx}+y\cos x=e^{-\sin x}$

$\text{Integrating factor} = e^{\int \cos x\,dx} = e^{\sin x}$

$e^{\sin x}\frac{dy}{dx}+ye^{\sin x}\cos x=1$

$\frac{d}{dx}\left(ye^{\sin x}\right)=1$

$ye^{\sin x}=x+c$

$y=(x+c)e^{-\sin x}$

(iii) $\frac{dy}{dx}+\frac{y}{x+2}=x-3$

$\text{Integrating factor} = e^{\int \frac{1}{x+2}dx} = e^{\ln(x+2)} = x+2$

$(x+2)\frac{dy}{dx}+y=(x-3)(x+2)$

$\frac{d}{dx}\left(y(x+2)\right)=x^2-x-6$

$y(x+2)=\frac{1}{3}x^3-\frac{1}{2}x^2-6x+c$

$y=\frac{2x^3-3x^2-36x+k}{6(x+2)}$

(iv) $x\frac{dy}{dx}-y=x^3e^{2x}$

$\frac{dy}{dx}-\frac{y}{x}=x^2e^{2x}$

$\text{Integrating factor} = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$

$$\frac{1}{x}\frac{dy}{dx}-\frac{y}{x^2}=xe^{2x}$$

$$\frac{d}{dx}\left(\frac{y}{x}\right)=xe^{2x}$$

$$\frac{y}{x}=\tfrac{1}{2}xe^{2x}-\int \tfrac{1}{2}e^{2x}dx+c$$

$$\frac{y}{x}=\tfrac{1}{2}xe^{2x}-\tfrac{1}{4}e^{2x}+c$$

$$y=\tfrac{1}{2}x^2e^{2x}-\tfrac{1}{4}xe^{2x}+cx$$

(v) $\dfrac{dy}{dx}+y\cot x=x\operatorname{cosec} x$

Integrating factor $=e^{\int \cot x\,dx}=e^{\int \frac{\cos x}{\sin x}dx}=e^{\ln(\sin x)}=\sin x$

$$\sin x\frac{dy}{dx}+y\cos x=x$$

$$\frac{d}{dx}(y\sin x)=x$$

$$y\sin x=\tfrac{1}{2}x^2+c$$

$$y=\frac{x^2+k}{2\sin x}$$

(vi) $x\dfrac{dy}{dx}+2y=x+1$

$$\frac{dy}{dx}+\frac{2y}{x}=1+\frac{1}{x}$$

Integrating factor $=e^{\int \frac{2}{x}dx}=e^{2\ln x}=x^2$

$$x^2\frac{dy}{dx}+2xy=x^2+x$$

$$\frac{d}{dx}(x^2y)=x^2+x$$

$$x^2y=\tfrac{1}{3}x^3+\tfrac{1}{2}x^2+c$$

$$y=\tfrac{1}{3}x+\tfrac{1}{2}+\frac{c}{x^2}$$

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3. (i) $\frac{dy}{dx} + \frac{y}{x} = e^x$

Integrating factor $= e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

$x\frac{dy}{dx} + y = xe^x$

$\frac{d}{dx}(xy) = xe^x$

$xy = xe^x - \int e^x dx$ (using integrating by parts)

$xy = xe^x - e^x + A$

$y = e^x - \frac{e^x}{x} + \frac{A}{x}$

$y = 1$ when $x = 1$, so $1 = e - e + A$

$A = 1$

Particular solution is $y = e^x - \frac{e^x}{x} + \frac{1}{x}$

(ii) $\frac{dy}{dx} + 2xy = e^{-(x-2)^2}$

Integrating factor $= e^{\int 2x\, dx} = e^{x^2}$

$e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y = e^{-(x-2)^2}e^{x^2}$

$\frac{d}{dx}(e^{x^2}y) = e^{-x^2+4x-4+x^2} = e^{4x-4}$

$e^{x^2}y = \frac{1}{4}e^{4x-4} + A$

$y = \frac{1}{4}e^{-x^2+4x-4} + Ae^{-x^2}$

$y = \frac{1}{4}e^{-(x-2)^2} + Ae^{-x^2}$

$y = 0$ when $x = 1$, so $0 = \frac{1}{4}e^{-1} + Ae^{-1}$

$A = -\frac{1}{4}$

$y = \frac{1}{4}e^{-(x-2)^2} - \frac{1}{4}e^{-x^2}$