Use the Well Ordering Principle to prove that

$$n \leq 3^{n/3} \qquad (1)$$

for every nonnegative integer, n.

Hint: Verify (1) for $n \leq 4$ by explicit calculation.

Proof:

$$P(n) ::= \forall n \in \mathbb{N}.(n \leq 3^{n/3} \;\equiv\; n^3 \leq 3^n) \qquad (2)$$

$$C(n) ::= \textit{nonempty set}.\{n \in \mathbb{N} \,|\, n \neq 3^{n/3}\} \qquad (3)$$

$$P(0) ::= 0 \leq 1 \qquad (4)$$
$$P(1) ::= 1 \leq 3 \qquad (5)$$
$$P(2) ::= 8 \leq 9 \qquad (6)$$
$$P(3) ::= 27 \leq 27 \qquad (7)$$
$$P(4) ::= 64 \leq 81 \qquad (8)$$

By the well-ordering principle, let c be the lesser element $\in C$ (9)
$\therefore c - 1$ must be an integer ≥ 4 and P(c-1) must be true (10)

$$P(c-1) ::= (c-1)^3 \leq 3^{c-1} \qquad (11)$$
$$\equiv 3(c-1)^3 \leq 3^c \qquad (12)$$
$$P(c) ::= c^3 \leq 3^c \qquad (13)$$
$$\equiv c^3 \leq 3(c-1)^3$$
$$\equiv c \leq \sqrt[3]{3} \times (c-1)$$

which is always true for $n \geq 4$

$$\therefore C(n) \textit{ is an empty set and } P(n) \textit{ holds}$$

QED