Title: Primes In Arithmetic Progressions And Idea For Goldbach's Conjecture

By C.K(Initials)(Student)

Introduction

There are many arithmetic progressions (AP) that contain prime numbers.
Some of which contain infinitely many primes. One of which is 2n+3 for n≥0.
In this paper we would use a rule called the "Exemption rule", $n = Pk + \frac{P-3}{2}$**, for** P **is an odd prime and** k **is a positive integer, to spot which terms are primes in 2n+3.**

The idea for Goldbach's Conjecture is that for every even integer greater than 4 (≥6) we would express them as 2(N+3) for N≥0, so as to use 2n+3 as an expression for primes.

Primes in 2n+3 and "Exemption Rule"

First of all, the starting n value is 0, and we are going to figure out which terms are prime.

In order to do that we would need to employ the "Exemption rule" to sieve out the composite terms in the AP.
For that we let $n = Pk + \frac{P-3}{2}$**,**

$$2(Pk + \frac{P-3}{2})+3=2Pk + P - 3+3=P(2k+1)$$

$P(2k+1)$ **gives all the odd multiples of the prime P, and 2n+3 contains all the odd numbers ≥3. Thus, sieving out these n values, we are left with n values that result in 2n+3 being prime.**

- **Do note that** $Pk + \frac{P-3}{2}$ **is an integer for all odd prime P, which every prime after 2 is an odd prime number and are terms in this AP. And this is a sequence on its own.**

Now to use the "Exemption rule" with 2n+3,

At first let us assume that we do not know any odd prime P,i.e, 3,5,7,....etc.
So in the beginning we do not have any odd prime P to use the "Exemption rule" with. Therefore, we start the AP sequence as per normal from n=0 since there is no exempted value for now,

Let n=0, 2(0)+3=3 (prime), now we generate its "Exemption rule" sequence by letting P=3,

$3k+\frac{3-3}{2}=3k$**: 3,6,9,12,15,18,....,etc for** k **is a positive integer. Thus, the very first exempted n value we have is n=3, so we carry on normally for n=1 and 2, we get,**

2(1)+3=5: $5k+\frac{5-3}{2}=5k+1$**: 6,11,16,21,26,....,etc**

2(2)+3=7: $7k+\frac{7-3}{2}=7k+2$**: 9,16,23,30,....,etc**

But when we get to n=3, the first exempted n value, we get 2(3)+3=9=3×3⇒The first odd multiple of 3 as we have seen from $P(2k+1)$ **earlier.**

In general, at the beginning the sequence starts as per normal, but as we get each term that is prime, we generate its "Exemption rule" sequence and get the values that are exempted.
Also, as known from $P(2k+1)$**, for each** k **value we get all the odd multiples of P which can make the generating of exempted n values easier for larger prime numbers we spot. Example; Take** $5k+1$ **sequence,**

$5k+1$**: 6(1st odd number greater than 1 is 3) , 31(6th odd number is 13)**

⇓ ⇓

2(6)+3=15=3×5 2(31)+3=5×13

Thus, making it easier to tell when a particular n value would be repeated in two different "Exemption rule" sequences. The first unique n value occurs for $2k+1=P$**.**

2(N+3) and Goldbach's Conjecture

Goldbach's Conjecture states that every even integer greater than 2 can be expressed as a sum of two prime numbers.

There is only one even integer that can be expressed as the sum of two even prime numbers, and there is only one even prime number, which is 4, which is expressed as 2+2.

Every even integer greater than 4 (≥6) can only be expressed as the sum of two odd primes since the addition of 2 and another odd prime would result in an odd integer.

Now instead of expressing even integers greater than 4 as 2N we express them as 2(N+3) for N≥0, as we can write this expression as,

2(N+3)=[2a+3]+[2b+3], for N=a+b such that [2a+3] and [2b+3] are odd primes

Before carrying on, let's introduce two sets, A and B,

A={N∈ℤ | 2N+3 is prime}⇒ such as N being 0,1,2,4,5,...,etc that are <u>not exempted</u>

B={N∈ℤ | 2N+3 is not prime}⇒ such as N being 3,6,9,11,12,...,etc that are <u>exempted</u>

So we need a and b from earlier to be elements of Set A so that each even number from 6 onwards can be expressed as the sum of two odd prime numbers. Or to find a N value that cannot be expressed with such a and b values to find a counter-example.

Finally, we would split the problem into two cases;

<u>Case 1: N∈A,</u>

This is the easier case as when N∈A, then 2N+3 is prime, and 0∈A as well so we got our desired a and b values for N values in this case.

2(N+3)=[2N+3]+[2(0)+3]=[2N+3]+3, where [2N+3] is an odd prime.

Take N=4 for example, 2(4+3)=14⇒14=[2(4)+3]+3=11+3.

Thus, the conjecture holds true for such N values, and since there are infinitely many primes, and everyone of them except 2 can be expressed as 2n+3, we get the conjecture to hold true for many even integers even beyond the 4×10^{18} numbers that have been searched for already.

<u>Case 2: N∈B,</u>

When N is an element of Set B, it is no longer as straightforward as Case 1. The idea now is to split N into a and b such that both a and b are in Set A.

For every N, there is a M less than N such that M∈A, and N=M+h. h can be in either set A or B, but when added with M gives N. Ideally we would like h to be in set A to prove the conjecture, but for all M in A, if we cannot find such h value, we have found our counter-example. Also, that (M+h+1)∈A, we can confidently say this last statement because the AP contains infinitely many primes, thus, after a sequence of N∈B, there will be a N'∈A.

I.e, let M=10∈A, we have N∈{11,12}⊆B, such that N'=13∈A.Where (M+h+1)=13, and there two consecutive N values in set B.

Note:{M+1,M+2,.....,M+h} is a finite sequence as the AP 2n+3 contains all the primes so the sequence must terminate at some (h+1)≥2, so that the next prime after 2(M+h+1)+3 can appear.

Also, we need to establish how big this h value is actually (estimation). Luckily we can with Bertrand's Postulate (https://en.wikipedia.org/wiki/Bertrand%27s_postulate), in which the less restrictive formulation states that for every n>1, there is always at least one prime,p, such that n<p<2n.

So for M∈A, which gives 2M+3 to be prime. Letting n be M we know that there is another prime between 2M+3 and 4M+6. We also know that p is odd so the largest possible odd number in that range is 4M+5 that can be prime, so the preceding odd number 4M+3 is the largest odd number in this range that can be composite,

2(M+h)+3≤4M+3
2(M+h)≤4M
M+h≤2M
h≤M

The lower bound for h was 1 so ultimately we have, 1≤h≤M. So the largest gap between two consecutive primes is 2(M) units on the number line.

Now carrying forward, we would split the case into two further sub-cases,

Sub-Case 1: h∈A

For M∈A, let N=M+h, for h∈A. 2(M+h+3)=2M+2h+3+3
=(2M+3)+(2h+3), and since M,h∈A.
=(Prime)+(Prime)

Thus, these even numbers can be written as sum of two primes as well. Also, Case 1 is just an example of this sub-case where h=0∈A, and N=M∈A.

Sub-Case 2: h∈B

One example is N=104∈A, we have {105,106,107,108}∈B, N=107=104+3, 3∈B.

Earlier, we used B.P to find out the maximum range of consecutive N values that can be in set B, which is {1,....,M-1,M}.

In the set stated above, M∈A, so the largest value in it that can be in set B is M-1. We will apply B.P once again to find the maximum range of h values that can be in set B.

Since 2M+3 is prime, and 2(M-1)+3 is the largest composite we can get with the values in the set {1,....,M-1,M}.

By B.P, n<p<2n, where 2n is 2M3+1=2M+4=2(M+2),and p is 2M+3 (largest candidate in the range given by B.P for prime). Hence, n=M+2.

For M is odd, M+2 is likely the prime before 2M+3, so for x∈A,

$$2x+3=M+2$$
$$2x=M-1$$
$$x=\frac{M-1}{2}\in A$$

Hence, for M is odd: {1,...,$\frac{M-1}{2}$,$\frac{M+1}{2}$,...,M-1,M}.

M+h=(M-1)+(h+1)
=(M-2)+(h+2)
.
.
=($\frac{M+1}{2}$)+(h+$\frac{M-1}{2}$), where $\frac{M+1}{2}$∈B
=($\frac{M-1}{2}$)+(h+$\frac{M+1}{2}$)

Where for the smallest h value in set B from the above set {1,...,$\frac{M-1}{2}$,$\frac{M+1}{2}$,...,M-1,M}, is h=$\frac{M+1}{2}$, and every value on the right-hand side is in set B except for h+$\frac{M-1}{2}$=M. But when h=$\frac{M+1}{2}$, the value on the left-hand side is $\frac{M+1}{2}$which is in set B.

For the largest value of h, where h=M-1, we have to take a look at the set after {1,...,$\frac{M-1}{2}$,$\frac{M+1}{2}$,...,M-1,M} which by B.P is {1,...,$\frac{M-1}{2}$,$\frac{M+1}{2}$,....4M}, and we will arrive at the same conclusion as for the smallest value of h earlier.

Thus, theoretically for extremely large numbers, where primes become sparse, we can find a N=M+h, where it is impossible to write them as elements of set A together.

Annex

1) **We can make small changes to the "Exemption rule" so that we can use the AP an+b for a is even and b is odd,$n = Pk + \frac{P-1}{2}$, and follow the steps above for the idea of Goldbach's Conjecture.**

2) **We can have the exemption rule work with an±b for a is even and b is odd to work for prime gaps of 2b.**

Thank you for reading.