Arc Length:

(13.3)

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$L = \int_a^b |r'(t)|\, dt.$$

Find the Length of the arc of the circular helix with vector equation:

$r(t) = \cos t \vec{i} + \sin t \vec{j} + t \vec{k}$ from the Point $(1,0,0)$ to the Point $(1,0,2\pi)$.

$$r(t) = \cos t \vec{i} + \sin t \vec{j} + t \vec{k}$$

$$r'(t) = -\sin t \vec{i} + \cos t \vec{j} + \vec{k}$$

$$|r'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}$$

$$= \sqrt{\sin^2 t + \cos^2 t + 1}$$

$$= \sqrt{1+1} = \sqrt{2}$$

$$0 \leqslant t \leqslant 2\pi$$

$$L = \int_0^{2\pi} |r'(t)|\, dt = \int_0^{2\pi} \sqrt{2}\, dt = \sqrt{2}\,[t]_0^{2\pi}$$

$$= \sqrt{2}\,[2\pi - 0] = 2\sqrt{2}\pi$$

Curvature:

Definition: The curvature of a curve is:

(المعظ)

$$K = \left| \frac{dT}{d\vec{S}} \right|$$

where (T) is the unit tangent vector

$$K(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

(or)

$$K(X) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

(or)

$$K(X) = \frac{|y''|}{[1 + (y')^2]^{3/2}}$$

(13.3)

The curvature of the curve given by the vector function (r) is:

$$K(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

Find the curvature of the twisted cubic $r(t) = \langle t, t^2, t^3 \rangle$ at a general Point and at $(0,0,0)$.

$r(t) = \langle t, t^2, t^3 \rangle$

$r'(t) = \langle 1, 2t, 3t^2 \rangle$

$r''(t) = \langle 0, 2, 6t \rangle$

$$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix}$$

$$= i(12t^2 - 6t^2) - j(6t) + k(2)$$

$$r'(t) \times r''(t) = 6t^2 \vec{i} - 6t\vec{j} + 2\vec{k}$$

$$|r'(t) \times r''(t)| = \sqrt{36t^4+36t^2+4}$$

$$= \sqrt{4(9t^4+9t^2+1)}$$

$$= 2\sqrt{9t^4+9t^2+1}$$

$$K(t) = \frac{2\sqrt{9t^4+9t^2+1}}{\left(\sqrt{1+4t^2+9t^4}\right)^3}$$

$$K(0) = \frac{2}{1} \rightarrow K(0) = 2$$

The curvature of the curve given by the vector function $y = f(x)$:

$$K(x) = \frac{|f''(x)|}{[1+(f'(x))^2]^{3/2}}$$

The curvature of the curve:

$r(t) = \langle t^2, 2t, t^2 \rangle$ at $(t=1)$ is

(A) $\frac{2}{\sqrt{6}}$ (B) $\frac{\sqrt{2}}{\sqrt{3}}$ (C) $\frac{2}{\sqrt{3}}$ (d) $\frac{\sqrt{2}}{6\sqrt{3}}$ (E) $\frac{3}{\sqrt{6}}$

$r(t) = \langle t^2, 2t, t^2 \rangle$

$r'(t) = \langle 2t, 2, 2t \rangle$

$r''(t) = \langle 2, 0, 2 \rangle$

$$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ 2t & 2 & 2t \\ 2 & 0 & 2 \end{vmatrix}$$

$= i(4-0) - j(4t-4t) + k(0-4)$

$= 4\vec{i} - 4\vec{k}$

$|r'(t) \times r''(t)| = \sqrt{16+16} = \sqrt{32} = 4\sqrt{2}$

$$k(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} = \frac{4\sqrt{2}}{\left(\sqrt{4t^2+4+4t^2}\right)^3}$$

$$K(t) = \frac{4\sqrt{2}}{\left(\sqrt{8t^2+4}\right)^3}$$

at $(t=1)$

$$K(1) = \frac{4\sqrt{2}}{\left(\sqrt{8(1)^2+4}\right)^3}$$

$$= \frac{4\sqrt{2}}{\left(\sqrt{8+4}\right)^3}$$

$$= \frac{4\sqrt{2}}{\left(\sqrt{12}\right)^3}$$

$$= \frac{4\sqrt{2}}{(2\sqrt{3})^3} = \frac{4\sqrt{2}}{(8)(3)\sqrt{3}}$$

$$= \frac{4\sqrt{2}}{24\sqrt{3}} = \frac{\sqrt{2}}{6\sqrt{3}}$$

Find the curvature of:

$$r(t) = (\cosh t)\vec{i} + (\sinh t)\vec{j} + t\vec{k}$$

$$r'(t) = (\sinh t)\vec{i} + (\cosh t)\vec{j} + \vec{k}$$

$$r''(t) = (\cosh t)\vec{i} + (\sinh t)\vec{j} + 0\vec{k}$$

$$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ \sinh t & \cosh t & 1 \\ \cosh t & \sinh t & 0 \end{vmatrix}$$

$$= -\sinh t\,\vec{i} + \cosh t\,\vec{j} + (\sinh^2 t - \cosh^2 t)\vec{k}$$

$$= -\sinh t\,\vec{i} + \cosh t\,\vec{j} + (-1)\vec{k}$$

$$= -\sinh t\,\vec{i} + \cosh t\,\vec{j} - \vec{k}$$

$$K(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} = \frac{\sqrt{\sinh^2 t + \cosh^2 t + 1}}{\left(\sqrt{\sinh^2 t + \cosh^2 t + 1}\right)^3}$$

$$= \frac{1}{\left(\sqrt{\sinh^2 t + \cosh^2 t + 1}\right)^2} = \frac{1}{\sinh^2 t + \cosh^2 t + 1} = \frac{1}{2\cosh^2 t}$$

$\cosh^2 t$

Show that the curvature of a circle of radius (a) is $\left(\frac{1}{a}\right)$.

$$\left.\begin{aligned} x &= a\cos t \\ y &= a\sin t \end{aligned}\right\} r(t) = \langle a\cos t, a\sin t\rangle$$

$$r'(t) = \langle -a\sin t, a\cos t\rangle$$

Rule (القاعدة)

$$\boxed{K(t) = \frac{|T'(t)|}{|r'(t)|}}$$

$$|r'(t)| = \sqrt{a^2\sin^2 t + a^2\cos^2 t}$$

$$= \sqrt{a^2(\sin^2 t + \cos^2 t)} = \sqrt{a^2} = |a| = a$$

$$T(t) = \frac{r'(t)}{|r'(t)|} = \frac{\langle -a\sin t, a\cos t\rangle}{a} = \frac{\cancel{a}\,\langle -\sin t, \cos t\rangle}{\cancel{a}}$$

$$\left.\begin{aligned} T(t) &= \langle -\sin t, \cos t\rangle \\ T'(t) &= \langle -\cos t, -\sin t\rangle \end{aligned}\right\} \begin{aligned} &|T'(t)| = \sqrt{(-\cos^2 t) + (-\sin^2 t)} \\ &|T'(t)| = 1 \rightarrow K(t) = \frac{1}{a} \end{aligned}$$

Find the curvature of the ellipse $x = 3\cos t$, $y = 2\sin t$ at the points $(3,0)$ and $(0,2)$

$r(t) = \langle 3\cos t, 2\sin t, 0 \rangle$

$r'(t) = \langle -3\sin t, 2\cos t, 0 \rangle$

$r''(t) = \langle -3\cos t, -2\sin t, 0 \rangle$

$$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ -3\sin t & 2\cos t & 0 \\ -3\cos t & -2\sin t & 0 \end{vmatrix}$$

$$= 0\vec{i} - 0\vec{j} + (6\sin^2 t + 6\cos^2 t)\vec{k}$$

$$= 6(\sin^2 t + \cos^2 t)\vec{k} = 6(1)\vec{k} = 6\vec{k}$$

$$K(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} = \frac{\sqrt{36}}{\left(\sqrt{9\sin^2 t + 4\cos^2 t}\right)^3}$$

at $(0,2)$

$\left.\begin{array}{l}\cos t = 0 \\ \sin t = 1\end{array}\right\} K(t) = \frac{\sqrt{36}}{27} = \frac{6}{27} = \frac{2}{9}$

at $(3,0)$

$\left.\begin{array}{l}\cos t = 1 \\ \sin t = 0\end{array}\right\} K(t) = \frac{\sqrt{36}}{8}$

$K(t) = \frac{6}{8} = \frac{3}{4}$

Find the Curvature of the Parabola $y = x^2$ at the Points $(0,0)$, $(1,1)$, $(2,4)$.

$$\left.\begin{aligned} y &= x^2 \\ y' &= 2x \\ y'' &= 2 \end{aligned}\right\} \quad K(x) = \frac{|y''|}{\left[1+(y')^2\right]^{3/2}}$$

$$K(x) = \frac{2}{(1+4x^2)^{3/2}}$$

$(0,0) \quad \rightarrow K(0) = 2$

$(1,1) \quad \rightarrow K(1) = \dfrac{2}{5^{3/2}}$

$(2,4) \quad \rightarrow K(2) = \dfrac{2}{(17)^{3/2}}$

Find the curvature of the Parabola $y = x^2$ at the Points $(0,0)$, $(1,1)$.

$$y = x^2$$

$$y' = 2x$$

$$y'' = 2$$

$$K(x) = \frac{|y''|}{\left[1+(y')^2\right]^{\frac{3}{2}}}$$

at $(0,0)$:

$$K(0) = \frac{|2|}{\left[1+(0)^2\right]^{\frac{3}{2}}} = 2$$

at $(1,1)$:

$$K(1) = \frac{|2|}{\left[1+(2)^2\right]^{\frac{3}{2}}} = \frac{2}{5^{\frac{3}{2}}}$$

Find the curvature of the curve:

$y = \ln(1 - e^{-x})$ at the Point $(\ln 2, -\ln 2)$

$$K(x) = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}}$$

$$y = \ln(1 - e^{-x})$$

$$y' = \frac{e^{-x}}{1 - e^{-x}}$$

$$y'' = \frac{-e^{-x}(1 - e^{-x}) - e^{-x}(e^{-x})}{(1 - e^{-x})^2}$$

$$y'' = \frac{-e^{-x} + \cancel{e^{-2x}} - \cancel{e^{-2x}}}{(1 - e^{-x})^2}$$

$$y'' = \frac{-e^{-x}}{(1 - e^{-x})^2}$$

The point $(\ln 2, -\ln 2)$

$$y' = \frac{e^{-x}}{1-e^{-x}} = \frac{e^{-\ln 2}}{1-e^{-\ln 2}}$$

$$y' = \frac{\frac{1}{e^{\ln 2}}}{1-\frac{1}{e^{\ln 2}}} = \frac{\frac{1}{2}}{1-\frac{1}{2}}$$

$$y' = \frac{\frac{1}{2}}{\frac{1}{2}} = 1 \to (y' = 1)$$

$$y'' = \frac{-e^{-x}}{(1-e^{-x})^2} = \frac{-e^{-\ln 2}}{(1-e^{-\ln 2})^2}$$

$$= \frac{-\frac{1}{e^{\ln 2}}}{\left(1-\frac{1}{e^{\ln 2}}\right)^2} = \frac{-\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2}$$

$$= \frac{-\frac{1}{2}}{\left(\frac{1}{2}\right)^2} = \frac{-\frac{1}{2}}{\frac{1}{4}} = -\frac{1}{2}\cdot\frac{4}{1} = -2$$

$$\to (y'' = -2)$$

$$K(x) = \frac{|y''|}{\left[1 + (y')^2\right]^{3/2}}$$

$$= \frac{|-2|}{\left[1 + (1)^2\right]^{3/2}}$$

$$= \frac{2}{(2)^{3/2}}$$

$$= \frac{2}{(\sqrt{2})^3} = \frac{2}{2\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}$$

The curvature of the curve :

$y = \ln(\cos x)$ at $x = \frac{\pi}{3}$ is

(A) $\frac{1}{2}$ (B) 2 (C) $\sqrt{3}$ (D) $\frac{2}{\sqrt{3}}$ (E) 1

$$y = \ln(\cos x)$$

$$y' = \frac{-\sin x}{\cos x} = -\tan x.$$

$$y'' = -\sec^2 x.$$

$$K(x) = \frac{|y''|}{\sqrt{[1+(y')^2]^{3/2}}}$$

$$K\left(\tfrac{\pi}{3}\right) = \frac{\left|-\sec^2 \frac{\pi}{3}\right|}{\sqrt{[1+(-\tan \frac{\pi}{3})^2]^{3/2}}}$$

$$K\left(\tfrac{\pi}{3}\right) = \frac{|-(2)^2|}{\sqrt{[1+(-\sqrt{3})^2]^{3/2}}} = \frac{4}{(4)^{3/2}} = \frac{4}{8}$$

$$= \frac{1}{2}$$

Principal unit normal vector:

$$N(t) = \frac{T'(t)}{|T'(t)|}$$

Binormal Vector:

$$B(t) = T(t) \times N(t)$$

- Find the unit normal and binormal vectors for the circular helix:

$$r(t) = \cos t \, \vec{i} + \sin t \, \vec{j} + t \, \vec{k}$$

$$r'(t) = -\sin t \, \vec{i} + \cos t \, \vec{j} + \vec{k}$$

$$|r'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2}$$

$$= \sqrt{\sin^2 t + \cos^2 t + 1}$$

$$= \sqrt{1+1} = \sqrt{2}$$

$$T(t) = \frac{r'(t)}{|r'(t)|} = \frac{-\sin t\,\vec{i} + \cos t\,\vec{j} + \vec{k}}{\sqrt{2}},$$

$$T'(t) = \frac{-\cos t\,\vec{i} - \sin t\,\vec{j}}{\sqrt{2}} = \left\langle \frac{-\cos t}{\sqrt{2}}, \frac{-\sin t}{\sqrt{2}} \right\rangle$$

$$|T'(t)| = \sqrt{\frac{\cos^2 t}{2} + \frac{\sin^2 t}{2}} = \sqrt{\frac{\cos^2 t + \sin^2 t}{2}}$$

$$|T'(t)| = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$$N(t) = \frac{T'(t)}{|T'(t)|} = \frac{(-\cos t\,\vec{i} - \sin t\,\vec{j}) / \cancel{\sqrt{2}}}{\cancel{\frac{1}{\sqrt{2}}}}$$

$$N(t) = -\cos t\,\vec{i} - \sin t\,\vec{j} + 0\,\vec{k}$$

$$= \langle -\cos t, -\sin t, 0 \rangle$$

• $B(t) = T(t) \times N(t)$

$$= \begin{vmatrix} i & j & k \\ -\frac{\sin t}{\sqrt{2}} & \frac{\cos t}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\cos t & -\sin t & 0 \end{vmatrix} = \frac{1}{\sqrt{2}}\sin t\,\vec{i} - \frac{1}{\sqrt{2}}\cos t\,\vec{j} + \left(\frac{\sin^2 t}{\sqrt{2}} + \frac{\cos^2 t}{\sqrt{2}}\right)\vec{k}$$

$$= \frac{\sin t}{\sqrt{2}}\vec{i} - \frac{\cos t}{\sqrt{2}}\vec{j} + \frac{1}{\sqrt{2}}\vec{k}$$

Find an equation for the binormal vector $\vec{B}(t)$ at the point $(1,1,0)$ for the curve:

$$r(t) = \langle e^{t}, e^{t}\cos t, e^{t}\sin t \rangle$$

$$r'(t) = \langle e^{t}, e^{t}\cos t - e^{t}\sin t, e^{t}\sin t + e^{t}\cos t \rangle$$

$$r'(t) = \langle e^{t}, e^{t}(\cos t - \sin t), e^{t}(\sin t + \cos t) \rangle$$

Binormal vector: $\vec{B}(t) = T(t) \times N(t)$

$$T(t) = \frac{r'(t)}{|r'(t)|} \quad (\text{and}) \quad N(t) = \frac{T'(t)}{|T'(t)|}$$

$$|r'(t)| = \sqrt{e^{2t} + e^{2t}(\cos t - \sin t)^2 + e^{2t}(\sin t + \cos t)^2}$$

$$= e^{t}\sqrt{1 + (\cos t - \sin t)^2 + (\sin t + \cos t)^2}$$

$$= e^{t}\sqrt{1 + \cos^2 t \cancel{- 2\cos t \sin t} + \sin^2 t + \sin^2 t \cancel{+ 2\sin t \cos t} + \cos^2 t}$$

$$= e^{t}\sqrt{1 + \underbrace{\cos^2 t + \sin^2 t}_{1} + \underbrace{\sin^2 t + \cos^2 t}_{1}} = e^{t}\sqrt{3}$$

$$T(t) = \frac{r'(t)}{|r'(t)|}$$

$$= \frac{\langle e^t, e^t(\cos t - \sin t), e^t(\sin t + \cos t)\rangle}{e^t\sqrt{3}}$$

$$= \frac{\cancel{e^t}\langle 1, \cos t - \sin t, \sin t + \cos t\rangle}{\cancel{e^t}\sqrt{3}}$$

$$T(t) = \frac{\langle 1, \cos t - \sin t, \sin t + \cos t\rangle}{\sqrt{3}}$$

$$T'(t) = \frac{\langle 0, -\sin t - \cos t, \cos t - \sin t\rangle}{\sqrt{3}}$$

$$|T'(t)| = \frac{\sqrt{0 + (-\sin t - \cos t)^2 + (\cos t - \sin t)^2}}{\sqrt{3}}$$

$$|T'(t)| = \frac{\sqrt{\sin^2 t + \cancel{2\sin t\cos t} + \cos^2 t + \cos^2 t - \cancel{2\cos t\sin t} + \sin^2 t}}{\sqrt{3}}$$

$$|T'(t)| = \frac{\sqrt{\overbrace{\sin^2 t + \cos^2 t}^{1} + \overbrace{\cos^2 t + \sin^2 t}^{1}}}{\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}}$$

$$N(t) = \frac{T'(t)}{|T'(t)|}$$

$$N(t) = \frac{\dfrac{\langle 0, -\sin t - \cos t, \cos t - \sin t \rangle}{\cancel{\sqrt{3}}}}{\dfrac{\sqrt{2}}{\cancel{\sqrt{3}}}}$$

$$N(t) = \frac{\langle 0, -\sin t - \cos t, \cos t - \sin t \rangle}{\sqrt{2}}$$

at the Point $(1, 1, 0)$:

$$r(t) = \langle e^t, e^t \cos t, e^t \sin t \rangle$$

$$e^t = 1 \rightarrow \ln e^t = \ln 1 \rightarrow (t = 0)$$

$$T(t) = \frac{\langle 1, \cos t - \sin t, \sin t + \cos t \rangle}{\sqrt{3}}$$

$$T(t) = \frac{\langle 1, \cos 0 - \sin 0, \sin 0 + \cos 0 \rangle}{\sqrt{3}}$$

$$T(t) = \frac{\langle 1, 1, 1 \rangle}{\sqrt{3}} = \frac{1}{\sqrt{3}} \langle 1, 1, 1 \rangle$$

$$N(t) = \frac{\langle 0, -\sin\cdot - \cos\cdot, \cos\cdot - \sin t \rangle}{\sqrt{2}}$$

$$N(t) = \frac{\langle 0, -\sin 0 - \cos 0, \cos 0 - \sin 0 \rangle}{\sqrt{2}}$$

$$N(0) = \frac{\langle 0, -1, 1 \rangle}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \langle 0, -1, 1 \rangle$$

$$B(t) = T(t) \times N(t)$$

$$B(0) = T(0) \times N(0)$$

$$= \begin{vmatrix} i & j & k \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix} = \frac{1}{\sqrt{3}} \frac{1}{\sqrt{2}} \begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix}$$

$$= \frac{1}{\sqrt{6}} \langle 1+1, -1, -1 \rangle$$

$$= \frac{1}{\sqrt{6}} \langle 2, -1, -1 \rangle$$