**The sum of the first n natural numbers raised to some power a**

**Abstract:**

By finding patterns in the coefficients of this popular sum, we can express it in a different way

## Part 1: General derivation

I will begin with the most well-known proof of this sum:

$$\sum_{r=1}^{n} r^{a+1} - \sum_{r=1}^{n}(r-1)^{a+1} = n^{a+1}$$

By expanding the $(r-1)^{a+1}$ by polynomial expansion we get,

$$\sum_{r=1}^{n}\left(r^{a+1} - \sum_{t=0}^{a+1}\frac{(a+1)!\cdot(-1)^t\cdot r^{a+1-t}}{(a+1-t)!\cdot t!}\right) = n^{a+1}$$

Next I will separate the case where t=0 from the sum

$$\sum_{r=1}^{n}\left(r^{a+1} - r^{a+1} - \sum_{t=1}^{a+1}\frac{(a+1)!\cdot(-1)^t\cdot r^{a+1-t}}{(a+1-t)!\cdot t!}\right) = n^{a+1}$$

$$\sum_{r=1}^{n}\left(-\sum_{t=1}^{a+1}\frac{(a+1)!\cdot(-1)^t\cdot r^{a+1-t}}{(a+1-t)!\cdot t!}\right) = n^{a+1}$$

$$\sum_{r=1}^{n}\left(\sum_{t=1}^{a+1}\frac{(a+1)!\cdot(-1)^t\cdot r^{a+1-t}}{(a+1-t)!\cdot t!}\right) = -n^{a+1}$$

Seeing how this is a sum of a sum I can switch the positions of the sum signs.

$$\sum_{t=1}^{a+1}\left(\sum_{r=1}^{n}\frac{(a+1)!\cdot(-1)^t\cdot r^{a+1-t}}{(a+1-t)!\cdot t!}\right) = -n^{a+1}$$

As the only part of the equation that the sum of r affects is $r^{a+1-t}$, I can move that sum there

$$\sum_{t=1}^{a+1}\left(\frac{(a+1)!\cdot(-1)^t}{(a+1-t)!\cdot t!}\cdot\sum_{r=1}^{n} r^{a+1-t}\right) = -n^{a+1}$$

Now, separating the t=1 case

$$-(a+1)\cdot\sum_{r=1}^{n} r^{a} + \sum_{t=2}^{a+1}\left(\frac{(a+1)!\cdot(-1)^t}{(a+1-t)!\cdot t!}\cdot\sum_{r=1}^{n} r^{a+1-t}\right) = -n^{a+1}$$

Rearranging to isolate the sum of $r^a$ on the left side

$$\sum_{r=1}^{n} r^a = \frac{n^{a+1}}{a+1} + \sum_{t=2}^{a+1}\left(\frac{a!\cdot(-1)^t}{(a+1-t)!\cdot t!}\cdot\sum_{r=1}^{n} r^{a+1-t}\right)$$

Now I will shift the sum to be from t=1 to a by replacing any instance of t to t+1

$$\sum_{r=1}^{n} r^a = \frac{n^{a+1}}{a+1} + \sum_{t=1}^{a}\left(\frac{a!\cdot(-1)^{t+1}}{(a-t)!\cdot(t+1)!}\cdot\sum_{r=1}^{n} r^{a-t}\right)$$

**Part 2: Further simplification**

Here I will substitute the parts of this equation with 2 functions

$$A(a) = \frac{n^{a+1}}{a+1}\ , B(a,t) = \frac{a!}{(a-t)!}\ \cdot\frac{(-1)^{t+1}}{(t+1)!}$$

Hence, I can rewrite the equation as

$$\sum_{r=1}^{n} r^a = A(a) + \sum_{t=1}^{a}\left(B(a,t)\cdot\sum_{r=1}^{n} r^{a-t}\right)$$

Next, I will expand the sum of t to a

$$\sum_{r=1}^{n} r^a = A(a) + B(a,1)\cdot\sum_{r=1}^{n} r^{a-1} + B(a,2)\cdot\sum_{r=1}^{n} r^{a-2} + B(a,3)\cdot\sum_{r=1}^{n} r^{a-3} \ldots\ B(a,a)\cdot\sum_{r=1}^{n} r^0$$

With this equation, I can say that

$$\sum_{r=1}^{n} r^{a-1} = A(a-1) + B(a-1,1)\cdot\sum_{r=1}^{n} r^{a-2} + B(a-1,2)\cdot\sum_{r=1}^{n} r^{a-3} \ldots\ B(a-1,a-1)\cdot\sum_{r=1}^{n} r^0$$

And if I substitute that where the sum of $r^{a-1}$ is and expand, we get

$$\sum_{r=1}^{n} r^a = A(a) + A(a-1)\cdot B(a,1) +$$

$$\left(B(a,2) + B(a,1)\cdot B(a-1,1)\right)\cdot\sum_{r=1}^{n} r^{a-2} +$$

$$\left(B(a,3) + B(a,1) \cdot B(a-1,2)\right) \cdot \sum_{r=1}^{n} r^{a-3} \ldots$$

$$(B(a,a) + B(a,1) \cdot B(a-1,a-1)) \cdot \sum_{r=1}^{n} r^{0}$$

And then substitute sum of $r^{a-2}$ and expand

$$\sum_{r=1}^{n} r^{a} = A(a) + A(a-1) \cdot B(a,1) +$$

$$A(a-2) \cdot \left(B(a,2) + B(a,1) \cdot B(a-1,1)\right) +$$

$$\left(B(a,3) + B(a,1) \cdot B(a-1,2) + B(a,2) \cdot B(a-2,1) + B(a,1) \cdot B(a-1,1) \cdot B(a-2,1)\right) \cdot \sum_{r=1}^{n} r^{a-3} \ldots$$

It is clear that if we continue this pattern, the sum could be expressed as

$$\sum_{r=1}^{n} r^{a} = A(a) + \sum_{t=1}^{a} C_t \cdot A(a-t)$$

Where $C_t$ is some coefficient that is a complicated function of function B. Now to find the value of $C_t$, we should consider the first few values and look for a pattern.

When t=1, $C_t$ = B(a,1)

When t=2, $C_t$ = B(a,2) + B(a,1)*B(a-1,1)

When t=3 $C_t$ = B(a,3) + B(a,1)*B(a-1,2) + B(a,2)*B(a-2,1) + B(a,1)*B(a-1,1)*B(a-2,1)

Here I notice a few patterns. If you consider the second input of the B function, it resembles a partition. Ex: the partition of 2 is 2 and 1,1, and the value of $C_t$ at t=2 is B(a,**2**) + B(a,**1**)*B(a-1,**1**). Another noticeable pattern is that when there is a product of B functions, the first input is a minus the sum of the previous second inputs. Ex: the last term of $C_t$ when t=3 is B(a,**1**)*B(a-**1**,**1**)*B(a-**2**,1).

A good analogy for how to think about $C_t$ is the following. Imagine you are on a number line, standing on point a. The point a distance 1 unit to the right is a-1 then a-2 and so on. To reach the point a-3 from point a there are 4 ways:

1- You either jump 3 units from a all at once (equivalent to B(a,3))
2- You jump 2 units to a-2 then from a-2 1 unit (equivalent to B(a,2)*B(a-2,1))
3- You jump 1 unit to a-1 then from a-1 2 units (equivalent to B(a,1)*B(a-1,2))
4- You make 3 single jumps (equivalent to B(a,1)*B(a-1,1)*B(a-2,1))

Using this analogy, the connection to partitions is clear and this pattern can then be generalized to any value of $C_t$. More specifically, $C_t$ is a sum of every possible path you can take to move t units from point a on a number line to point a-t, each of these paths being a unique order of jumps and is equivalent to the partition of t. We then describe each of these paths as a product of these jumps, and we describe each jump as a function B with the first input being the starting point and the second input, the number of units jumped.

So, can we make a formula to calculate it? Well first, it would be helpful to make some simplifications. Remember that so far, we said that:

$$\sum_{r=1}^{n} r^a = A(a) + \sum_{t=1}^{a} C_t \cdot A(a-t)$$

Where $C_t$ is a function of B and that

$$A(a) = \frac{n^{a+1}}{a+1}\ , B(a,t) = \frac{a!}{(a-t)!} \cdot \frac{(-1)^{t+1}}{(t+1)!}$$

Now we will focus on one part of the function B, specifically a!/(a-t)!. So for some value of t, let's assume that one of the paths is by taking n1 steps from a to a-n1 then n2 steps and then n3 steps and so on. What will the value of function B be in each jump? Well, in the first jump it will be B(a,n1), in the second it will be B(a-n1,n2) then B(a-n1-n1,n3) and so on. If we focus on the a!/(a-t)! part we see that the first will be a!/(a-n1)!, then (a-n1)!/(a-n1-n2)!, then (a-n1-n2)!/(a-n1-n2-n3)! And so on. Since in the calculation we multiply all these values, it is clear that the will cancel out and in the end what will remain is a!/(a- the total number of steps)! So we arrive to the conclusion that in any path the first part of the function B will always simplify to a!/(a-t)! and since $C_t$ is a sum of these paths, we can factor it out. And so we get:

$$\sum_{r=1}^{n} r^a = A(a) + \sum_{t=1}^{a} C_t \cdot A(a-t) = A(a) + \sum_{t=1}^{a} K_t \cdot \frac{a!}{(a-t)!} \cdot A(a-t)$$

Where $K_t$ is a new function. We have basically factored out the first input of the function B. Algebraically this means that after factoring out a!/(a-t)!, what remains of the function is $(-1)^{t+1}/(t+1)!$ Which I will call function N(t).

Now to calculate the new coefficient we will follow the same procedure. We find every single path to reach a-t from a and we fill in function N, but this time the part that cares about where we start from has been factored out and so function N only takes in the number of steps. This links better to partitions because you can fill them in easily. Then you multiply the N functions of consecutive jumps (or of the different groups in one partition), and then add the results of each path, and that is $K_t$.

Now we can expand function A and rearrange

$$\sum_{r=1}^{n} r^a = \frac{n^{a+1}}{a+1} + \sum_{t=1}^{a} K_t \cdot \frac{a!}{(a-t)!} \cdot \frac{n^{a+1-t}}{a+1-t}$$

$$\sum_{r=1}^{n} r^a = \frac{n^{a+1}}{a+1} + a! \cdot \sum_{t=1}^{a} K_t \cdot \frac{n^{a+1-t}}{(a+1-t)!}$$

**Part 3: Finding $K_t$**

The next question would be, "Is there a way we can calculate $K_t$ with an equation?" well, kind of. First, let us take the partition of 3: 3, 2/1, 1/2, 1/1/1. So, calculating $K_3$ would be as so:

$K_3$ = N(3) + N(2)*N(1) + N(1)*N(2) + N(1)*N(1)*N(1)

We can imagine that the partition of 3 could be expressed as taking 3 steps at once or taking 1 step followed by taking all routes that jump 2 points which is the partition of 2! Or by taking 2 steps followed by all ways to jump one point or the partition of 1! Now I can re-express this as:

$$K_3 = N(3) + N(2) \cdot N(1) + N(1) \cdot N(2) + N(1) \cdot N(1) \cdot N(1)$$

$$K_3 = N(3) + N(2) \cdot N(1) + N(1) \cdot (N(2) + N(1) \cdot N(1))$$

$$K_3 = N(3) + N(2) \cdot K_1 + N(1) \cdot K_2$$

Hence I can rewrite $K_t$ as a function of its values from $K_1$ to $K_{t-1}$

$$K_t = N(t) + \sum_{L=1}^{t-1} N(t-L) \cdot K_L$$

I will rename $K_t$ to function F(t). Now, substituting in function N we getn

$$F(t) = \frac{(-1)^{t+1}}{(t+1)!} + \sum_{L=1}^{t-1} \frac{(-1)^{t+1-L}}{(t+1-L)!} \cdot F(L)$$

Finally putting everything together we get

$$\sum_{r=1}^{n} r^a = \frac{n^{a+1}}{a+1} + a! \cdot \sum_{t=1}^{a} F(t) \cdot \frac{n^{a+1-t}}{(a+1-t)!}$$

$$F(t) = \frac{(-1)^{t+1}}{(t+1)!} + \sum_{L=1}^{t-1} \frac{(-1)^{t+1-L}}{(t+1-L)!} \cdot F(L)$$

**Part 4: Simplifying function F(t)**

Here is a bit of a tricky part. I don't want to keep making new functions to substitute old ones every time I wish to manipulate this equation, and since F(t) is a function of itself many interesting manipulations can be made.

I want to redefine function F using the following trick. I will make the new function F(t) = -1* the old function F(t). So, every instance of F(t) in both equations should be substituted for -F(t), so we get

$$\sum_{r=1}^{n} r^a = \frac{n^{a+1}}{a+1} + a! \cdot \sum_{t=1}^{a} -F(t) \cdot \frac{n^{a+1-t}}{(a+1-t)!}$$

$$-F(t) = \frac{(-1)^{t+1}}{(t+1)!} + \sum_{L=1}^{t-1} \frac{(-1)^{t+1-L}}{(t+1-L)!} \cdot -F(L)$$

Simplifying a bit, we get

$$\sum_{r=1}^{n} r^a = \frac{n^{a+1}}{a+1} - a! \cdot \sum_{t=1}^{a} F(t) \cdot \frac{n^{a+1-t}}{(a+1-t)!}$$

$$F(t) = \frac{(-1)^{t}}{(t+1)!} - \sum_{L=1}^{t-1} \frac{(-1)^{t-L}}{(t+1-L)!} \cdot F(L)$$

Next I can factor out (-1) ^t from the definition of function F

$$F(t) = (-1)^t \cdot \left( \frac{1}{(t+1)!} - \sum_{L=1}^{t-1} \frac{(-1)^{-L}}{(t+1-L)!} \cdot F(L) \right)$$

Now, I can replace every instance of F(t) with F(t)*(-1) ^t

$$\sum_{r=1}^{n} r^a = \frac{n^{a+1}}{a+1} - a! \cdot \sum_{t=1}^{a} (-1)^t \cdot F(t) \cdot \frac{n^{a+1-t}}{(a+1-t)!}$$

$$(-1)^t \cdot F(t) = (-1)^t \cdot \left( \frac{1}{(t+1)!} - \sum_{L=1}^{t-1} \frac{(-1)^{-L}}{(t+1-L)!} \cdot (-1)^L \cdot F(L) \right)$$

Rearranging and simplifying we get

$$\sum_{r=1}^{n} r^a = \frac{n^{a+1}}{a+1} - a! \cdot \sum_{t=1}^{a} F(t) \cdot \frac{(-1)^t \cdot n^{a+1-t}}{(a+1-t)!}$$

$$F(t) = \frac{1}{(t+1)!} - \sum_{L=1}^{t-1} \frac{F(L)}{(t+1-L)!}$$

Next, I will make all the sums start from zero,

$$\sum_{r=1}^{n} r^a = \frac{n^{a+1}}{a+1} - a! \cdot \sum_{t=0}^{a-1} F(t+1) \cdot \frac{(-1)^{t+1} \cdot n^{a+1-(t+1)}}{(a+1-(t+1))!}$$

$$F(t) = \frac{1}{(t+1)!} - \sum_{L=0}^{t-2} \frac{F(L+1)}{(t+1-(L+1))!}$$

simplify

$$\sum_{r=1}^{n} r^a = \frac{n^{a+1}}{a+1} + a! \cdot \sum_{t=0}^{a-1} F(t+1) \cdot \frac{(-1)^{t} \cdot n^{a-t}}{(a-t)!}$$

$$F(t) = \frac{1}{(t+1)!} - \sum_{L=0}^{t-2} \frac{F(L+1)}{(t-L)!}$$

Well, since the main equation now has F(t+1), we find F(t+1) from the F(t) equation

$$F(t+1) = \frac{1}{(t+2)!} - \sum_{L=0}^{t-1} \frac{F(L+1)}{(t+1-L)!}$$

Next, I can redefine F(t), by replacing every instance of F(t+1) with F(t), giving us the final equation of

$$\sum_{r=1}^{n} r^a = \frac{n^{a+1}}{a+1} + a! \cdot \sum_{t=0}^{a-1} F(t) \cdot \frac{(-1)^{t} \cdot n^{a-t}}{(a-t)!}$$

$$F(t) = \frac{1}{(t+2)!} - \sum_{L=0}^{t-1} \frac{F(L)}{(t+1-L)!}$$

**Part 5: Calculating F(t)**

Well, we need to calculate the values of F if we want to use this equation or find some pattern if we wish to simplify it further. Let us begin with F(0)

$$F(0) = \frac{1}{(0+2)!} - \sum_{L=0}^{-1} \frac{F(L)}{(1-L)!}$$

The sum is invalid and so F(0) = 1/(2!) = 1/2.

$$F(1)=\frac{1}{(1+2)!}-\sum_{L=0}^{0}\frac{F(L)}{(2-L)!}=\frac{1}{6}-\frac{F(0)}{2}=\frac{1}{6}-\frac{1}{4}=-\frac{1}{12}$$

$$F(2)=\frac{1}{(2+2)!}-\sum_{L=0}^{1}\frac{F(L)}{(3-L)!}=\frac{1}{24}-\frac{F(0)}{6}-\frac{F(1)}{2}=\frac{1}{24}-\frac{1}{12}+\frac{1}{24}=0$$

And so on. Interestingly F(any even number > 0) = 0, but I can't seem to find a reason why. The values of F(t) when t is odd switch signs with every consecutive odd number and are always smaller than the value before it.