Wave packet

Consider a solution of the time-dependent Schrödinger-Equation in position space $\psi_t = \psi(x,t)$ for a free particle of mass m in one dimension. Following relation holds:

$$\psi_t(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} \varphi_t(k) e^{ikx}$$

with $\varphi_t(k) = \phi(k) e^{-iE(k)t/\hbar}$ and $\langle \phi | \phi \rangle = 1$.

(a) What is the relation between $E(k)$ and k? (No complicated math required)
(b) Show that the expectation value $X(t) = \langle \psi_t | \hat{X} | \psi_t \rangle$ is:

$$X(t) = X(0) + tW$$

with

$$W = \frac{\hbar}{m} \int_{-\infty}^{\infty} \mathrm{d}k \phi^\star(k) k \phi(k).$$

Hint: $\int_{-\infty}^{\infty} \mathrm{d}y e^{iby} = 2\pi\delta(b)$.
(c) Show that for real $\phi(k)$: $X(0) = 0$.

Solution attempts

(a) Since $p = \hbar k$ and $E = p^2/2m$ we can write $E(k)$ as:

$$E(k) = \frac{\hbar^2 k^2}{2m}.$$

(b) Using the definition of the expectation value, we obtain for $X(t)$:

$$X(t) = \langle \psi_t | \hat{X} | \psi_t \rangle = \int_{-\infty}^{\infty} \mathrm{d}x \Big(\int_{-\infty}^{\infty} \frac{\mathrm{d}k}{\sqrt{2\pi}} \phi^\star(k) e^{iE(k)t/\hbar} e^{-ikx} \Big) x \Big(\int_{-\infty}^{\infty} \frac{\mathrm{d}k}{\sqrt{2\pi}} \phi(k) e^{-iE(k)t/\hbar} e^{ikx} \Big).$$