Introduction:

Given a choice between armor and health runes, what combination would be optimal? HP and armor are different stats, but they both have the same purpose - to make you tankier. A champion with 10 HP could effectively have a pool of 10.000 hp, given enough armor while facing exclusively physical damage, but at the same time still have base 10 facing AP opponents. This corresponds to saying that effective HP is dependent on the type of damage you are facing. So we need to mathematically derive "effective HP", with the generalized formula we can unify resistances with health, and by extension compare runes (and items for that matter) to each other. However actually solving for a pair of optimal runes turns out to be a rather complicated problem (from a mathematical point of view), I will include examples of how to find a pair of optimal runes after the solutions have been derived. First of all we need to derive the formulas for effective health:

The math of effective HP:

EHP is derived from the way league calculates damage:

$$D = \left(\frac{100}{100 + armor}\right) P + \left(\frac{100}{100 + MR}\right) M$$

Where *P* denotes physical damage and *M* magical damage. This core function states that all incoming raw damage will get "cut" by a percentage depending on the resistances of the enemy. Since D is the true corrected damage, to kill a champion you would need to deal $D = H$ worth of damage, where *H* is the health of your enemy, so:

$$D = \left(\frac{100}{100 + armor}\right) P + \left(\frac{100}{100 + MR}\right) M = H$$

The "effective HP" (EHP), is the isolation of the uncorrected raw damages in the above; Divide equation by *M+P:*

$$\left(\frac{100}{100 + armor}\right) \frac{P}{M + P} + \left(\frac{100}{100 + MR}\right) \frac{M}{M + P} = \frac{H}{M + P}$$

Multiply then divide under *H*:

$$\Rightarrow M + P = \frac{H}{\left(\frac{100}{100 + armor}\right) \frac{P}{M + P} + \left(\frac{100}{100 + MR}\right) \frac{M}{M + P}} \equiv EHP$$

Notice the numbers; $\frac{P}{M+P}$ and $\frac{M}{M+P}$, these are the multipliers for effective *hp* when you are facing *mixtures* of damage. We will give these values special names:

$$\left(\frac{P}{M + P}\right) = \rho \quad , \quad \left(\frac{M}{M + P}\right) = 1 - \rho$$

"ρ", is the percentage of physical damage, and "$1 - \rho$" is the percentage of magical damage. Thus:

$$EHP = \frac{H}{\left(\frac{100}{100 + armor}\right) \rho + \left(\frac{100}{100 + MR}\right) (1 - \rho)}$$

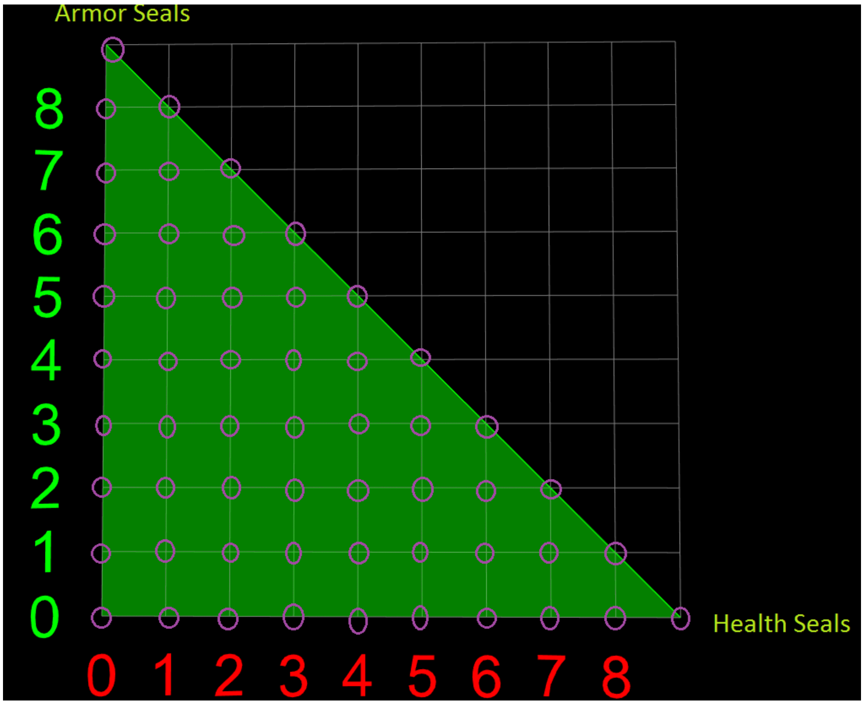
For an extended laning phase we will also need to consider the health regeneration effects:

$$EHP = \frac{H + H_{Regen}}{\left(\frac{100}{100 + armor}\right)\rho + \left(\frac{100}{100 + MR}\right)(1 - \rho)}$$

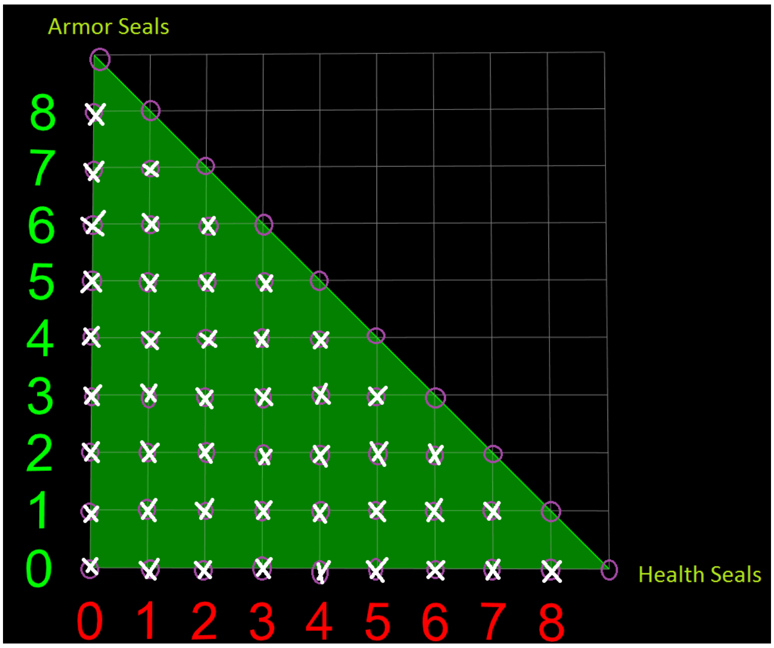
The health you regenerate is also subject to your resistances, and it behaves in the same way as *hp.* This will be important for the bottom lane where massive regen stats are present.

The Seals Problem:

We have 9 rune slots for Seals, which gives us 55 different combinations of armor and health seals:



The vast majority of these vertices will fail to be optimal, for example we would prefer $(H_{seals}, A_{seals}) = (9,0) \succcurlyeq (H_{seals}, A_{seals}) = (8,0) \succcurlyeq \cdots \succcurlyeq (H_{seals}, A_{seals}) = (0,0)$, since *EHP* is rising in health, same statement is valid with regards to armor. This is equivalent of saying the optimal vertices are placed on the outer part of the triangle spanned by the rune limits (also equivalent of saying we have to fill all 9 of our slots..), this reduces the problem to this:



One of the none-crossed vertices is our maximum. The formal problem is to maximize *EHP,* given our rune combinations must be chosen on this line, while being integral points and none-negative. The *EHP* function (object function) we will call:

$$f(H_{seals}, A_{seals}) \ = \frac{H + H_{Regen} + 8H_{seals}}{\left(\frac{100}{100 + armor + A_{seals}}\right)\rho \ + \left(\frac{100}{100 + MR}\right) \ (1-\rho)}$$

The problem would then be:

$$\begin{gathered} \max \ f(H_{seals}, A_{seals}) \ \ st. \\ H_{seals} + A_{seals} = 9 \\ 0 \leq H_{seals} \\ 0 \leq A_{seals} \\ H_{seals}, A_{seals} \in \mathbb{Z} \end{gathered}$$

(scary math begin): f (or EHP) is differentiable at all inner vertices. Also if a combination $(H^*_{seals}, A^*_{seals})$ is an optimal solution, then the following conditions are true by the KKT conditions: $\exists constants: \mu_1, \mu_2, \lambda$ (also called Karush-Kuhn-Tucker multipliers) such that these 4 conditions are satisfied:

(1)**Stationarity**:

$$\nabla f(H^*_{seals}, A^*_{seals}) = \mu_1 \nabla g_1(H^*_{seals}, A^*_{seals}) + \mu_2 \nabla g_2(H^*_{seals}, A^*_{seals}) + \lambda \nabla \mathrm{h}(H^*_{seals}, A^*_{seals})$$

This means no other combination of runes, “nearby” is optimal.

(2)**Primal feasibility**:

$$g_1(H^*_{seals}, A^*_{seals}) \leq 0$$
$$g_2(H^*_{seals}, A^*_{seals}) \leq 0$$
$$\mathrm{h}(H^*_{seals}, A^*_{seals}) = 0$$

These are the constraints; they are called “primal” constraints, in duality theory.

$(\mathbf{3})$**Dual feasibility**:

$\mu_1, \mu_2 \geq 0$

From duality theory; the multipliers μ_1, μ_2 are the variables associated with a shadow/dual problem.

$(\mathbf{4})$**Complementary slackness**:

$$\mu_1 g_1(H^*_{seals}, A^*_{seals}) = 0$$
$$\mu_2 g_2(H^*_{seals}, A^*_{seals}) = 0$$

Duality theory says there is a relationship between a primal and its dual; this is known as complementary slackness. Complementary slackness refers to a relationship between the slackness in a primal constraint, and the slackness associated with the dual variable. In our case the constraints are:

$$g_1(H^*_{seals}, A^*_{seals}) = -H_{seals}$$
$$g_2(H^*_{seals}, A^*_{seals}) = -A_{seals}$$
$$h(H^*_{seals}, A^*_{seals}) = H_{seals} + A_{seals} - 9$$

All of our constraints are affine functions, which is a sufficient condition to secure the maximum found in the KKT is a true maximum. Written out one would need to find a solution to this system:

$$\begin{pmatrix} \dfrac{d\, f(H_{seals}, A_{seals})}{d\, H_{seals}} \\ \dfrac{d\, f(H_{seals}, A_{seals})}{d\, A_{seals}} \end{pmatrix} = \mu_1 \begin{pmatrix} \dfrac{d\, g_1(H_{seals}, A_{seals})}{d\, H_{seals}} \\ \dfrac{d\, g_1(H_{seals}, A_{seals})}{d\, A_{seals}} \end{pmatrix} + \mu_2 \begin{pmatrix} \dfrac{d\, g_2(H_{seals}, A_{seals})}{d\, H_{seals}} \\ \dfrac{d\, g_2(H_{seals}, A_{seals})}{d\, A_{seals}} \end{pmatrix} + \lambda \begin{pmatrix} \dfrac{d\, h(H_{seals}, A_{seals})}{d\, H_{seals}} \\ \dfrac{d\, h(H_{seals}, A_{seals})}{d\, A_{seals}} \end{pmatrix}$$
$$g_1(H_{seals}, A_{seals}) \leq 0$$
$$g_2(H_{seals}, A_{seals}) \leq 0$$
$$h(H_{seals}, A_{seals}) = 0$$
$$\mu_1 \geq 0$$
$$\mu_2 \geq 0$$
$$\mu_1 g_1(H_{seals}, A_{seals}) = 0$$
$$\mu_2 g_2(H_{seals}, A_{seals}) = 0$$
$$H_{seals}, A_{seals} \in \mathbb{Z}$$

We will solve the integer relaxation problem, and in the end show that the KKT predicted solutions are “near” the none-relaxed optimum. We can deduce that there can only be 3 solutions to this system. We can see this from complementary slackness, either our optimum is pure, or both dual variables are 0, and optimum is a mixture of runes.

First KKT solution:

$assume\ \mu_1 = 0, \mu_2 \neq 0 \Longrightarrow (H_{seals}, A_{seals}) = (9{,}0)$ Else complementary slackness will fail and primal conditions will fail. Thus if this solution is optimal we must have:

$$\left.\frac{d\, f(H_{seals}, A_{seals})}{d\, H_{seals}}\right|_{(H_{seals}, A_{seals})=(9,0)} = \lambda$$
$$\left.\frac{d\, f(H_{seals}, A_{seals})}{d\, A_{seals}}\right|_{(H_{seals}, A_{seals})=(9,0)} = \lambda - \mu_2$$
$$\mu_2 > 0$$

From the first equation we get the second KKT multiplier, the third multiplier we get from the second equation:

$$\mu_2 = \left.\frac{d\, f(H_{seals}, A_{seals})}{d\, H_{seals}}\right|_{(H_{seals}, A_{seals})=(9,0)} - \left.\frac{d\, f(H_{seals}, A_{seals})}{d\, A_{seals}}\right|_{(H_{seals}, A_{seals})=(9,0)}$$

From the dual constraint we get that $(H_{seals}, A_{seals}) = (9,0)$ is optimal if and only if:

$$\left.\frac{d\, f(H_{seals}, A_{seals})}{d\, H_{seals}}\right|_{(H_{seals}, A_{seals})=(9,0)} > \left.\frac{d\, f(H_{seals}, A_{seals})}{d\, A_{seals}}\right|_{(H_{seals}, A_{seals})=(9,0)}$$

Using that:

$$armor = [Armor_{base} + Armor_{items}] + [Armor_{scaling}] \cdot Level = A_1 + A_2 level$$

$$HP = [HP_{base} + HP_{items}] + [HP_{scaling}] \cdot Level = H_1 + H_2 level$$

$$MR = [MR_{base} + MR_{items}] + [MR_{scaling}] level = M_1 + M_2 level$$

We will call the base values with notation *1*, and the scaling bases with notation *2,* also these values are the constants that are used to find optimum. We can "reduce" this inequality to:

$$\begin{aligned}
&[8{A_2}^2(1-\rho) + \rho(8A_2 - H_2)M_2] \cdot l^2 \\
&\quad + [-(8A_1(2A_2(\rho-1) - M_2\rho) - 8A_2((M_1 - 100)\rho + 200) \\
&\quad + (H_{regen}M_2 + H_1M_2 + H_2(M_1 + 100) - 728M_2)\rho)] \cdot l \\
&\quad + \left[-\left(8{A_1}^2(\rho - 1) - 8A_1((M_1 - 100)\rho + 200)\right.\right. \\
&\quad \left.\left. + \left(H_{regen}(M_1 + 100) + H_1(M_1 + 100) - 8(91M_1 - 900)\right)\rho - 80000\right)\right] > 0
\end{aligned}$$

This is a second degree polynomial inequality in level: $a \cdot l^2 + b \cdot l + c > 0$, where the big square parenthesis are the constants *a, b and c*, the solution in *level* is:

$$a > 0\,,\ \ c > \frac{b^2}{4a} \Rightarrow l \in \mathbb{R}$$
$$a = 0,\ \ b = 0, c > 0\ \Rightarrow l \in \mathbb{R}$$
$$a = 0,\ \ b > 0\ \Rightarrow l > -\frac{c}{b}$$
$$a = 0, b < 0 \Rightarrow l < -\frac{c}{b}$$
$$a > 0,\ \ c < \frac{b^2}{4a} \Rightarrow \frac{a\sqrt{\frac{b^2 - 4ac}{a^2}} - b}{2a} < l \text{ and } l < \frac{a\left(-\sqrt{\frac{b^2 - 4ac}{a^2}}\right) - b}{2a}$$
$$a < 0, \qquad c > \frac{b^2}{4a}\ \Rightarrow \frac{a\left(-\sqrt{\frac{b^2 - 4ac}{a^2}}\right) - b}{2a} < l < \frac{a\sqrt{\frac{b^2 - 4ac}{a^2}} - b}{2a}$$

Thus if $(H_{seals}, A_{seals}) = (9,0)$ is optimal; your level is must be between the roots, spanned by your base stats.

Second KKT solution:

$assume\ \mu_1 \neq 0, \mu_2 = 0 \Longrightarrow (H_{seals}, A_{seals}) = (0,9)$ Else complementary slackness will fail and primal conditions will fail. Thus

$$\left.\frac{d\, f(H_{seals}, A_{seals})}{d\, H_{seals}}\right|_{(H_{seals},A_{seals})=(0,9)} = \lambda - \mu_1$$

$$\left.\frac{d\, f(H_{seals}, A_{seals})}{d\, A_{seals}}\right|_{(H_{seals},A_{seals})=(0,9)} = \lambda$$

$$\mu_2 > 0$$

From the second equation we get the second KKT multiplier, the third we get from the first equation:

$$\mu_1 = \left.\frac{d\, f(H_{seals}, A_{seals})}{d\, A_{seals}}\right|_{(H_{seals},A_{seals})=(0,9)} - \left.\frac{d\, f(H_{seals}, A_{seals})}{d\, H_{seals}}\right|_{(H_{seals},A_{seals})=(0,9)}$$

From the dual constraint we get that $(H_{seals}, A_{seals}) = (0,9)$ is optimal if and only if:

$$\left.\frac{d\, f(H_{seals}, A_{seals})}{d\, A_{seals}}\right|_{(H_{seals},A_{seals})=(0,9)} > \left.\frac{d\, f(H_{seals}, A_{seals})}{d\, H_{seals}}\right|_{(H_{seals},A_{seals})=(0,9)}$$

This is the reverse of the first KKT, giving the negative parameters:

$$\begin{aligned}
&[M_2\rho(H_2 - 8A_2) - 8{A_2}^2(1-\rho)]l^2 \\
&\quad + [8A_1(2A_2(\rho - 1) - M_2\rho) - 8A_2((M_1 - 118)\rho + 218) \\
&\quad + (H_{regen}M_2 + H_1M_2 + H_2(M_1 + 100) - 872M_2)\rho]l \\
&\quad + \Big[8{A_1}^2(\rho - 1) - 8A_1((M_1 - 118)\rho + 218) + \big(H_{regen}(M_1 + 100) + H_1(M_1 + 100) - 872(M_1 - 9)\big)\rho \\
&\quad - 95048\Big] > 0
\end{aligned}$$

Not surprisingly this is also a second degree polynomial inequality, but it now has different constants:

$$a_1 \cdot l^2 + b_1 \cdot l + c_1 > 0$$

The solution is given on the same form as before.

The last *KKT* solution (mix optimum):

$assume\ \mu_1 = 0, \mu_2 = 0 \Longrightarrow (H_{seals}, A_{seals}) \in (0,9) \times (0,9)$, thus:

$$\frac{d\,f(H_{seals}, A_{seals})}{d\,H_{seals}} = \lambda$$
$$\frac{d\,f(H_{seals}, A_{seals})}{d\,A_{seals}} = \lambda$$
$$g_1(H_{seals}, A_{seals}) \leq 0$$
$$g_2(H_{seals}, A_{seals}) \leq 0$$
$$h(H_{seals}, A_{seals}) = 0$$
$$H_{seals}, A_{seals} \in \mathbb{Z}$$

From the first equation we get:

$$\frac{d\,f(H_{seals}, A_{seals})}{d\,H_{seals}} = \lambda \Longrightarrow A_{seals} = K^1_{(\lambda)}$$

Where $K^1_{(\lambda)}$ is the optimal choice of A_{seals} given λ, insert this into equation 2:

$$\left.\frac{d\,f(H_{seals}, A_{seals})}{d\,A_{seals}}\right|_{\left(H_{seals}, K^1_{(\lambda)}\right)} = \lambda \Longrightarrow H_{seals} = K^2_{(\lambda)}$$

Where $K^2_{(\lambda)}$ is the optimal choice of H_{seals} given λ. Inserting these into the equality *h* gets you:

$$K^1_{(\lambda)} + K^2_{(\lambda)} = 9$$

The sum: $K^1_{(\lambda)} + K^2_{(\lambda)}$ is a second degree polynomial in λ, the roots to this equation is the KKT multiplier that satisfies stationarity, it has two roots; λ_1, λ_2:

$$\lambda_1 = \frac{-2\left((MR+100)\sqrt{\left(H + H_{regen} + 8(armor - MR + 9)\right)\rho - H - H_{regen} - 8(armor + 109)} + 2\sqrt{-2(MR+100)^3}\sqrt{\rho}\right)}{25(\rho - 1)\sqrt{\left(H + H_{regen} + 8(armor - MR + 9)\right)\rho - H - H_{regen} - 8(armor + 109)}}$$

$$\lambda_2 = \frac{-2\left((MR+100)\sqrt{\left(H + H_{regen} + 8(armor - MR + 9)\right)\rho - H - H_{regen} - 8(armor + 109)} - 2\sqrt{-2(MR+100)^3}\sqrt{\rho}\right)}{25(\rho - 1)\sqrt{\left(H + H_{regen} + 8(armor - MR + 9)\right)\rho - H - H_{regen} - 8(armor + 109)}}$$

The roots are complex functions, but given our parameters they will always be rooted in $\mathbb{R}$. The solution given the first root is:

$$A_{seals}|_{\lambda_1} = \frac{-\sqrt{2\rho}}{4(1-\rho)}\sqrt{(H + H_{regen} + 8armor)(1-\rho) + (8MR - 72)\rho + 872}\sqrt{(MR+100)} - \frac{armor(1-\rho) + \rho MR + 100}{1-p}$$

This cannot be our optimum, since $\rho \in (0,1)$ will break our primal constraints. The second root will give us the optimal solutions:

$$A_{seals}|^*_{\lambda_2} = \frac{\sqrt{2\rho}}{4(1-\rho)}\sqrt{\left[(H + H_{regen} + 8armor)(1-\rho) + (8MR - 72)\rho + 872\right]}\sqrt{(MR+100)} - \frac{armor(1-\rho) + \rho MR + 100}{1-p}$$

$$H_{seals}|^*_{\lambda_2} = 9 - A_{seals}|^*_{\lambda_2}$$

Notice however there is a problem when $\rho = 1$; then$\left(\frac{\sqrt{\rho}}{1-\rho}\right) = \infty$, and $-\frac{1}{1-p} = -\infty$. Not to worry, we can find the optimum in this case as:

$$\lim_{\rho\to 1} A_{seals}|_{\lambda_2} = \frac{H + H_{regen} - 8(armor + 91)}{16}$$

And:

$$\lim_{\rho\to 1} H_{seals}|_{\lambda_2} = 9 - \lim_{\rho\to 1} A_{seals}|_{\lambda_2} = \frac{8(armor + 109) - H - H_{regen}}{16}$$

For now let's focus on the case where $\rho \neq 1$, we need the primal conditions to be fulfilled:

$$A_{seals}|^*_{\lambda_2} \geq 0 : \Leftrightarrow$$

$$\begin{aligned}\left[2(1-\rho)\left(8{A_2}^2(\rho-1) - 8M_2A_2\rho + H_2M_2\rho\right)\right]l^2 \\ &+ \left[2(1-\rho)\left(8A_1(2A_2(\rho-1) - M_2\rho\right) - 8A_2\left((M_1-100)\rho + 200\right) \right. \\ &\left. + \left(H_{regen}M_2 + H_1M_2 + H_2(M_1+100) - 728M_2\right)\rho)\right]l \\ &+ \left[2(1-\rho)\left(8{A_1}^2(\rho-1) - 8A_1\left((M_1-100)\rho + 200\right) \right.\right. \\ &\left.\left. + \left(H_{regen}(M_1+100) + H_1(M_1+100) - 8(91M_1 - 900)\right)\rho - 80000\right)\right] \geq 0\end{aligned}$$

This is also a polynomial inequality, with the parameters:

$$a_2l^2 + b_2l + c_2 \geq 0$$

Same conditions apply to health seals: $H_{seals} \geq 0 : \Leftrightarrow H_{seals}|^*_{\lambda_2} = 9 - A_{seals}|^*_{\lambda_2} \geq 0 \Leftrightarrow 9 \geq A_{seals}|^*_{\lambda_2} \Leftrightarrow$:

$$\begin{aligned}\left[2(1-\rho)\left(8{A_2}^2(1-\rho) + 8M_2A_2\rho - H_2M_2\rho\right)\right]l^2 \\ &+ \left[2(\rho-1)\left(8A_1(2A_2(\rho-1) - M_2\rho\right) - 8A_2\left((M_1-118)\rho + 218\right) \right. \\ &\left. + \left(H_{regen}M_2 + H_1M_2 + H_2(M_1+100) - 872M_2\right)\rho)\right]l \\ &+ \left[2(\rho-1)\left(8{A_1}^2(\rho-1) - 8A_1\left((M_1-118)\rho + 218\right) \right.\right. \\ &\left.\left. + \left(H_{regen}(M_1+100) + H_1(M_1+100) - 872(M_1 - 9)\right)\rho - 95048\right)\right] \geq 0\end{aligned}$$

Which we will call:

$$a_3l^2 + b_3l + c_3 \geq 0$$

For the special case of $\rho = 1$ the primal conditions simplify:

$$A^*_{seals} = \frac{H_1 + H_2level + H_{regen} - 8(A_1 + A_2level + 91)}{16} \geq 0$$

$$H^*_{seals} = \frac{8(A_1 + A_2level + 109) - \left(H_1 + H_2level + H_{regen}\right)}{16} \geq 0$$

$$\Rightarrow \frac{8A_1 + 728 - H_1 - H_{regen}}{H_2 - 8A_2} \leq level \leq \frac{8A_1 + 872 - H_1 - H_{regen}}{H_2 - 8A_2}$$

The integer condition is not accounted for by the KKT solutions; we solved the integer relaxation by the KKT. We can still trace optimal integral solutions inside of our KKT with certainty, this will however yield none-integer levels- we cannot have none-integer levels. Best hope is that the true optimum is "near" the KKT optimum, meaning optimum must behave nicely moving along the KKT limits. In our case the optimal pair of runes is the combination whose distance to the nearest positive integer level is chosen minimally, given an integer pair of runes has been found by the KKT solutions. Now we only need to calculate the polynomials and check at what ranges a pair of runes are optimal.

Example RYZE:

Assume we play ryze top lane, versus an enemy that exclusively deals physical damage($\rho = 1$). We will start the laning phase with a mana crystal; also we play with increased HP masteries, here is the format for ryze base stats:

$$A_1 = 15, A_2 = \frac{39}{10}(= 3.9) \; and \; H_1 = \frac{927}{2}(= 463.5), H_2 = \frac{4429}{50} \; (= 88.58) and \; M_1 = 30, M_2 = 0$$

Now we calculate the three characteristic polynomials, to find the optimality in level:

$(\boldsymbol{H_{seals}}, \boldsymbol{A_{seals}}) = (\mathbf{9}, \mathbf{0})$ ***is optimal***:

$\Leftrightarrow level < 6.70094 - 0.017428 H_{regen}$

$(\boldsymbol{H_{seals}}, \boldsymbol{A_{seals}}) = (\mathbf{0}, \mathbf{9})$ ***is optimal***:

$\Leftrightarrow level > 9.21053 - 0.017428 H_{regen}$

$(\boldsymbol{H_{seals}}, \boldsymbol{A_{seals}}) \in (\mathbf{0}, \mathbf{9}) \times (\mathbf{0}, \mathbf{9})$ ***is optimal***:

$\Leftrightarrow 6.70094 - 0.017428 H_{regen} \leq level \leq 9.21053 - 0.017428 H_{regen}$

The regen effect pushes the optimum toward armor combinations, ignore regen effects for now:

$(\boldsymbol{H_{seals}}, \boldsymbol{A_{seals}}) = (\mathbf{9}, \mathbf{0})$ ***is optimal***:

$\Leftrightarrow level \in \{1,2,3,4,5,6\}$

from this we can see that the *hp* seals are stronger in terms of all-out power and gank survivability in the first part of laning phase. Now we will include regen effects, with my 3 potions I will have *at least* 510 regen stats in this part of laning:

$(\boldsymbol{H_{seals}}, \boldsymbol{A_{seals}}) = (\mathbf{9}, \mathbf{0})$ ***is optimal***:

$\Leftrightarrow level < 6.70094 - 0.017428 * 510 = -2.18 \; (never)$

Counting regen effects there will be no optimal solutions in pure health, or a mixture of armor and health, meaning armor runes are stronger in terms of lane presence. After your first recall you will return with flat HP items, further increasing the potency of armor runes (better even without regen effects).

Conclusion: Health Seals are better if you expect an all-out or lethal gank before your first recall, otherwise *definitely* go for full Armor seals, mixed runes will likewise never be optimal with ryze scaling and item choices. Note also this is not true for all top champions, even when facing physical damage enemies, if you for example are playing a high base stat champion at top, and for some reason spec raw armor as a start, most likely you will be better off with a mixture or a full health combination, for a substantial amount of time.

Let's go to the bottom lane, this will most likely be the only lane where you can physically feel the effects of choosing a bad combination of protective seals. Now we allow mixtures of damage, we defined mixtures of damage through the parameters:

$$\left(\frac{P}{M+P}\right) = \rho \qquad \left(\frac{M}{M+P}\right) = 1 - \rho$$

We are interested in the physical damage percent up to some level *n*, that is:

$$\rho = \frac{P_1 + \dots + P_n}{M_1 + P_1 + \dots + M_n + P_n} = \frac{\sum_{i=1}^{n} P_i}{\sum_{i=1}^{n}(M_i + P_i)}$$

In other words we are adding up all the incoming damage received up to some level *n*, both physical and magical damage. This size itself is *stochastic*, and we can estimate it through large data pools, although these I don't have access to. Also if you are facing *k* opponents this number would be:

$$\rho = \frac{(P_{11} + \dots + P_{1k}) + \dots + (P_{n1} + \dots + P_{nk})}{(M_{11} + \dots + M_{1k}) + \dots + (M_{n1} + \dots + M_{nk}) + (P_{11} + \dots + P_{1k}) + \dots + (P_{n1} + \dots + P_{nk})}$$
$$= \frac{\sum_{i=1}^{n}\sum_{m=1}^{k} P_{im}}{\sum_{i=1}^{n}\sum_{m=1}^{k} M_{im} + \sum_{i=1}^{n}\sum_{m=1}^{k} P_{im}}$$

We will assume all incoming damage is from one rotation of abilities, with an added auto attack at each level. However a new complication will appear, each possible combination of {adc,support}, result in different values of ρ, I myself am interested in Annie match ups, so we will assume enemy botlane is {Graves, Annie}. With my idealization of damage distributions we can calculate:

$$\Rightarrow \rho \approx 0.618$$

Which means the estimated amount of physical damage in this lane is around 62%, this is pretty reasonable and we will continue with this. With my base stats as twitch (with increased HP masteries):

$$M_1 = 30\,, M_2 = 0 \text{ and } A_1 = 18, A_2 = 3 \text{ and } H_1 = \frac{1751}{4}(= 437.75), H_2 = \frac{8343}{100}(= 83.43)$$

First calculating the characteristic polynomials, you will find:

$(\boldsymbol{H_{seals}}, \boldsymbol{A_{seals}}) = (\mathbf{9}, \mathbf{0})$ ***is optimal***:

$$\left[\frac{2421}{88}\right] l^2 + \left[-\frac{3674301}{1408}\right] l + \left[\frac{9912961}{128} - \frac{28275}{352} H_{regen}\right] > 0$$

Solution with regen effects:

$$\frac{\sqrt{1131(1721600 H_{regen} - 333530237)} + 1224767}{25824} < level$$

But also:

$$level < \frac{-\sqrt{1131(1721600 H_{regen} - 333530237)} + 1224767}{25824}$$

$(\boldsymbol{H_{seals}}, \boldsymbol{A_{seals}}) = (\mathbf{0}, \mathbf{9})$ ***is optimal***:

$$\left[-\frac{2421}{8}\right] l^2 + \left[\frac{3441885}{1408}\right] l + \left[-\frac{134819291}{1408} + \frac{28275}{352} H_{regen}\right] > 0$$

Solution with regen:

$$\frac{-\sqrt{1131(1721600H_{regen}-888384701)}+1147295}{25824}<level<\frac{\sqrt{1131(1721600H_{regen}-888384701)}+1147295}{25824}$$

$(\boldsymbol{H_{seals}}, \boldsymbol{A_{seals}}) \in (\mathbf{0},\mathbf{9}) \times (\mathbf{0},\mathbf{9})$ $\boldsymbol{is\ optimal}$:

$A_{seals}|^{*}_{\lambda_2} \geq 0 :\Leftrightarrow$

$$\left[-\frac{651249}{30976}\right]l^2+\left[\frac{988386969}{495616}\right]l+\left[-\frac{2666586509}{45056}+\frac{7605975}{123904}H_{regen}\right]\geq 0$$

Solution with regen:

$$\frac{-\sqrt{1131(1721600H_{regen}-333530237)}+1224767}{25824}\leq l\leq\frac{\sqrt{1131(1721600H_{regen}-333530237)}+1224767}{25824}$$

$9 \geq A_{seals}|^{*}_{\lambda_2} :\Leftrightarrow$

$$\left[\frac{651249}{30976}\right]l^2+\left[-\frac{925867065}{495616}\right]l+\left[\frac{36266389279}{495616}-\frac{7605975}{123904}H_{regen}\right]\geq 0$$

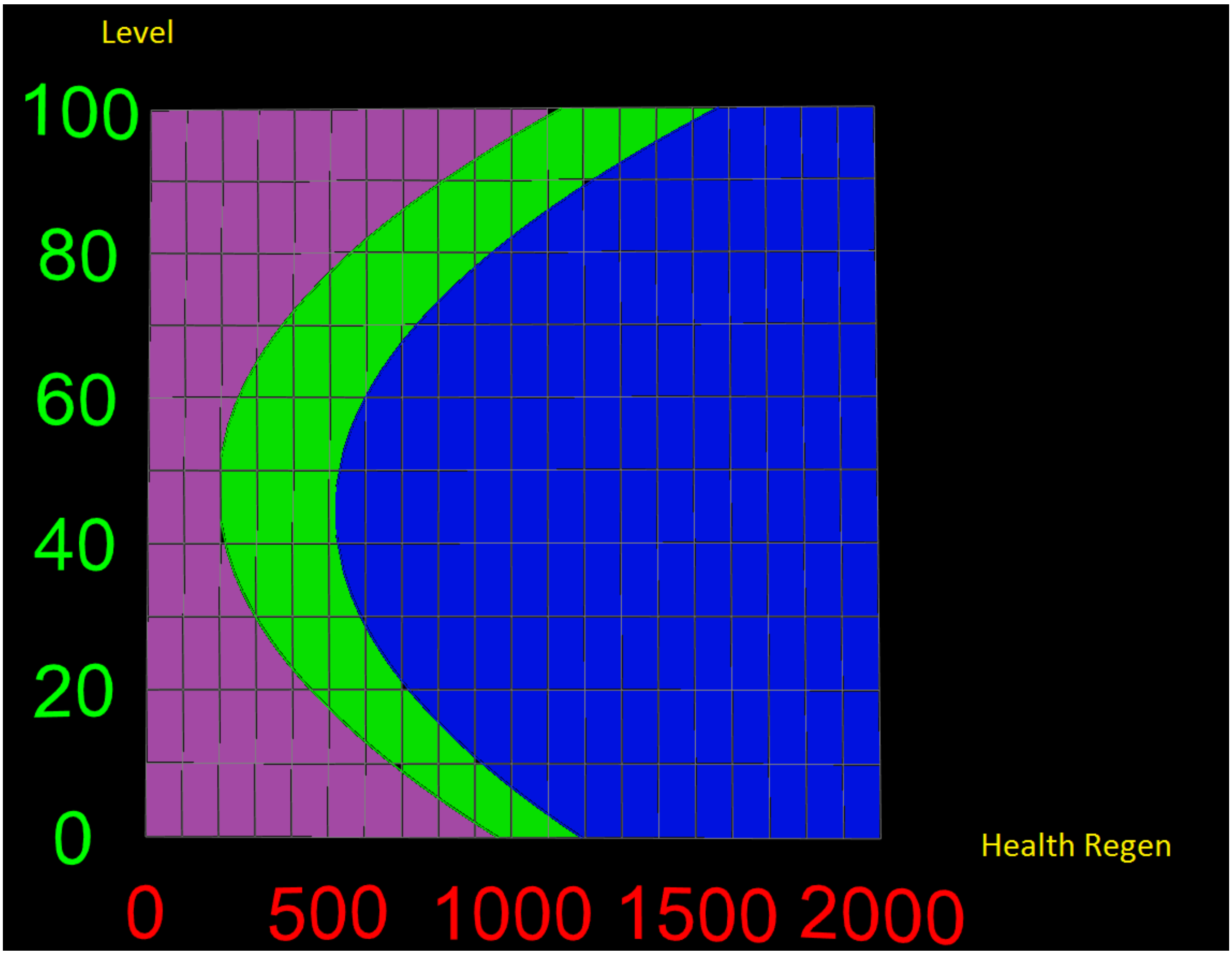
Solution with regen:

$$\frac{\sqrt{1131(1721600H_{regen}-888384701)}+1147295}{25824}<level$$

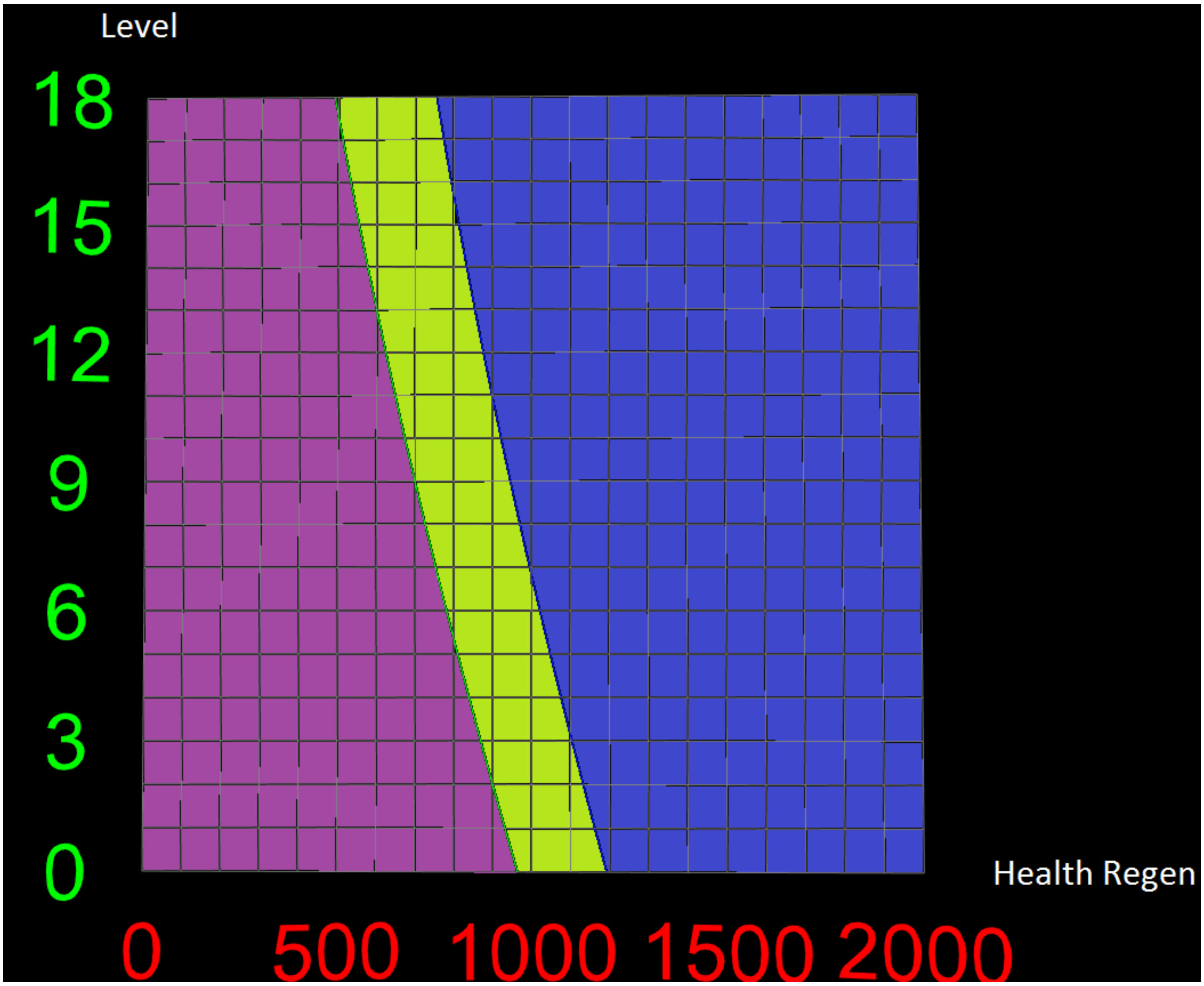
But also:

$$level<\frac{-\sqrt{1131(1721600H_{regen}-888384701)}+1147295}{25824}$$

Help for visualizing these regions:



The purple region is when pure health is optimum. The blue region is when we have pure armor optimum. The smaller green region is when a mixture of seals is optimum. These are the polynomials we found in the beginning in their root forms, let's zoom in on the relevant region of levels:



As for the interpretation of the regions, we must lock onto some level we consider "laning phase", let's say we define the initial laning phase to be up to level 6. Then we can see (roughly) that the pure optimum in health dominates all, until you reach regen values of about 780, then we have a brief moment where a mixture is optimal around the regen interval[780,1020], after this point pure armor runes would flat out outperform all other combinations. The only question left to answer is, how much are we able to regen?

$$g(n) = 99(n-1) + \frac{1}{8}\left(2\left\lfloor\frac{n-1}{3}\right\rfloor^2 + 320\left\lfloor\frac{n-1}{3}\right\rfloor + (-1)^{\left\lfloor\frac{n-1}{3}\right\rfloor} - 1\right) + 57(n-1)$$