Circle Arrangement Upper Bound Estimations

Michael Drennan

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Introduction:

When considering only perfect circles on the plane, there are many obvious ways to arrange them. Various arrangements have been used in the past to serve as Venn diagrams or to demonstrate how categories of objects are related, but to the curious, there is a natural question: How many unique ways are there to arrange a given number of circles in the plane? To make the problem a bit more precise, we do not allow for circles to "kiss" or to completely overlap. Nor will we allow more than 2 circles to intersect at the same point. This last condition is simply so that our description of a circle arrangement is unique and there are not more than 1 way to express the same description. This problem has already been solved in the cases for 1, 2, 3, and 4 circles yielding totals of 1, 3, 14, and 173, but there is still a lack of a decent general upper bound for $n = 5$ or greater. The purpose of this paper is to generate different upper bounds for this question as the number of circles considered becomes too large to brute force enumerate.

The Approach:

The answer for 1 circle is obviously, 1. There is only 1 way to arrange 1 circle in the plane. For 2 circles, the answer is 3 as shown below in Fig 1.

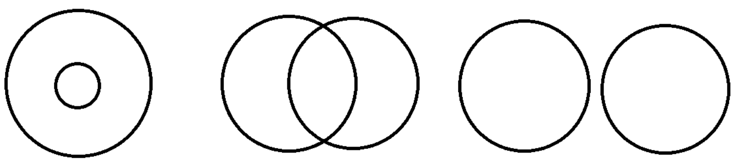


Figure 1

There it is easy to see that we can only have 2 circles not interact, overlap partially, or have one inside the other. Any other arrangement of 2 circles will necessarily be one of these. The question for 3 circles becomes much more complicated. The ultimate answer is already known, as well as for 4 circles, but to illustrate the approach employed here let us start from the ground up.

To make the process of enumerating all possible circle arrangements for n circles, we will construct an mapping between a circle arrangement and a code number which describes it. This relationship will ensure that each code number can only describe 1 circle arrangement and we can then analyze the circle arrangement by analyzing features of the code number alone and develop filters for which code numbers describe valid arrangements and which ones do not.

There are many ways to construct a circle arrangement methodically, but the system employed here is to describe the regions a circle arrangement cuts the plane into by which circles the region is inside of. For example, with 2 circles, we can label them A and B and then ask how many regions of different inclusion are there?

This question turns out to be the same as asking how many ways can you choose either A or B or both from the set of A and B. Naturally, for 2 objects, you can only have the one, the other, or both, which is a total of 3 types of regions. To make it easier to see, we shall enumerate the code numbers in the following array:

A	B	AB
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

Figure 2.

Here we can see that there are a total of 8 encoding numbers listing out every possible way for an arrangement of 2 circles to cut the plane. If we look at the rows we see a present/absent description of each type of region. The first row describes a circle arrangement which has no region inside any circle. That is clearly impossible and so cannot describe a possible circle arrangement. The same is true for the second and third rows as they also only describe arrangements with 1 region which is impossible for a diagram with 2 circles in it. The first code number which describes a possible arrangement is the 4th row stating that the arrangement cuts the plane into 2 regions, one being only inside circle B and the other being inside both A and B. This is describing the following:

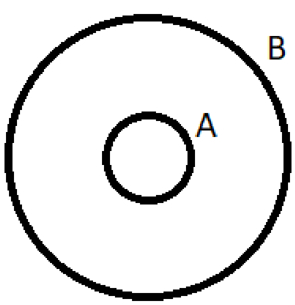


Figure 3.]

The next valid code number is the 6th row describing the following:

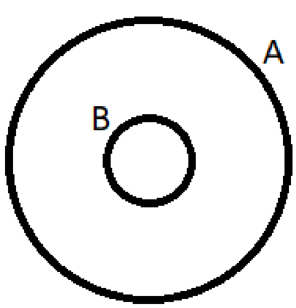


Figure 4.

While the code number describes a unique circle arrangement if the circles are labeled, when they are unlabeled as is the case for the original problem, there will be redundancy in the enumeration. This is something we will come back to as one of our filters to help estimate an upper bound on unique circle arrangements. The final two rows describe the following completing the list:

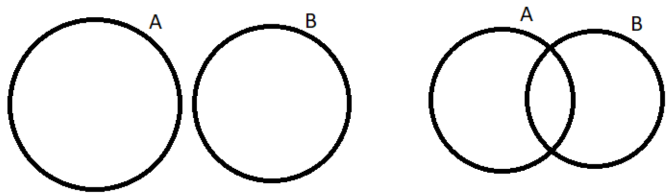


Figure 5.

Formalizing:

Here we can see that for 2 circles, our enumeration yielded 8 possibilities, of which only 4 describe valid arrangements and of those, 3 were unique. The process we employed was somewhat haphazard and while it got the right result we must be much more precise when dealing with higher n values. The first thing we can describe fully is how many region types are there for n circles? We can have regions inside a single circle, 2 circles, 3 circles, 4, 5 and so on up to n circles. The way to calculate this is to use combinitorics and recognize that we are asking for the total number of ways to choose 1 circle from n, then the total number of ways to choose 2 circles from n and so on all the way up to choosing n circles from n circles. This is the same as asking for the sum of all the $\begin{pmatrix} n \\ k \end{pmatrix}$ for $1 \leq k \leq n$ This can be easily written as $\sum_{k=1}^{n} \frac{n!}{k!(n-k)!}$ which would be a perfect formula to tell us how many digits are in our encoding numbers except that we can take we know from Pascal's triangle to see that since each entry in the triangle is the $\begin{pmatrix} n \\ k \end{pmatrix}$ value for the row and position, and that the sum of each row of the triangle is 2 to the power of the row number. These combine to tell us that the sum $\sum_{k=1}^{n} \frac{n!}{k!(n-k)!} = 2^n - 1$ which is much easier to utilize. The effect of this is that for any n value of circles, there will only ever be $2^n - 1$ types of regions a diagram composed of them can display.

Notice how this is also true for 2 circles, $2^2 - 1 = 3$ and we saw exactly 3 types of regions. The next thing to formalize is the total number of encoding numbers generated describing the full possibility space of circle arrangements for n circles. Given that before we stated that each digit of the code number is a present/absent statement about the region encoded by that digit position, we can say that for n circles, there will be 2^{2^n-1} total code numbers.

Right here we have our first (if crude) upper bound on the number of unique circle arrangements for n circles, it will be impossible for any n value to have more than 2^{2^n-1} diagrams in the plane. Plugging this in for $n = 3$ we get a value of $2^{2^3-1} = 128$. This is quite a bit more than the actual number of 14, but it at least puts a cap on the unknown.

The next step is to come up with elimination rules which can act on the code numbers alone, and thus allow us to check if a given code number is a valid description of not. Before we start listing elimination rules, it is important to formalize how we list the regions in the digits of the code numbers. There are many different ways to list them, but for our purposes we will highlight 2 different methods, each with their own benefits. The first way is to basically list them out is sections. Each section corresponds to the $\binom{n}{k}$ selections, so we would start on the left with $\binom{n}{1}$ then the next section would be all the $\binom{n}{2}$ and so on until on the right we would have the section for $\binom{n}{n}$. This is in fact the way we listed the regions when we enumerated it out for 2 circles. For 3 circles we would list it out as follows:

$$
\begin{array}{ccc|ccc|c}
A & B & C & AB & AC & BC & ABC \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\vdots & & & & & & \\
1 & 1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}
$$

Figure 6.

The other method for listing region types is going to be based on a recursive system which ultimately leads to a very nice scale-ability and whose sections now are not for each $\binom{n}{k}$ but for each circle in the diagram. We will start by listing all the region types for 1 circle, then all the ways to have regions with 2 circles provided that some have already been listed in the 1 circle section. This process repeats, always listing all the ways to have regions with the new total of circles, provided that some have already been listed in the previous sections. Below shows how to generate this method iteratively:

$$
\begin{array}{cccccccc|cccc|cc|c}
 & & & & & & & & & & & & & & A \\
 & & & & & & & & & & & & B & BA & A \\
 & & & & & & & & C & CB & CBA & CA & B & BA & A \\
D & DC & DCB & DCBA & DCA & DB & DBA & DA & C & CB & CBA & CA & B & BA & A
\end{array}
$$

For $n = 3$ we would see an enumeration like the following:

$$
\begin{array}{cccc|cc|c}
C & BC & ABC & AC & B & AB & A \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\vdots & & & & & & \\
1 & 1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}
$$

Figure 7.

The sections now act as separations for the different circles present, the section farthest left will always be the only section that the last circle will be mentioned in. Since the labeling of circles is arbitrary, you can always relabel to group the circle you are focusing on into the left section.

This region listing method allows you to see what would happen if you removed a single circle, since that would correspond to the same exact code number, but with the left section taken off. This scale-ability is very useful when looking at sub-arrangements since code numbers generated under this ordering preserve the structure of the other circles when the last one is removed, a feature the other method lacks.

Due to the hassle of rewriting these encoding schemes, from now on, we will talk about code number sections, and if the section is called a $\binom{n}{k}$ then it is referring to the first code number type, and if it is, for example, called the circle D section it is referring to the latter code number type. Now we can list out the 5 elimination rules I have identified:

Rules of Elimination:

In order for a code number to describe a valid circle arrangement for n circles it:

1. Must have at least one 1 in each circle section
2. Must have between n and $n(n-1)+1$ 1's in total
3. Must have no more than n 1's in each $\binom{n}{k}$ section
4. Must have at least one 1 in $\binom{n}{k-1}$ for each non-zero $\binom{n}{k}$
5. Must not be an invalid code number for $n-1$ circles when the n^{th} circle section is removed

Unfortunately, these rules are not mutually exclusive, and so we cannot immediately stack them to achieve better and better upper bounds, but each one independently can give us a different upper bound. But first, where do they come from?

The first one is fairly obvious, in order to be a description of a circle arrangement with n circles, all n circles must be mentioned at least once in the code number. If we had a code number for 6 circles, but it only ever mentioned 5 labeled circles, then that is lacking the 6th circle completely and is obviously an invalid code number. To develop the upper bound from this rule, we must look at the difference between the unaltered upper bound of 2^{2^n-1} and how many code numbers which fail to mention a circle. To do things in the most efficient way, we will only consider code numbers failing to mention 1 circle, since all of the numbers failing to mention any more would necessarily fail to mention 1 and so that group will cover all the rest. To calculate how many code numbers fail to mention a circle, all we need to do is remove the n^{th} circle section and see how many that removes.

The remaining code numbers depends on how many regions there are in the n^{th} circle section. This is actually simple to figure out, if you have n circles, then the n^{th} circle section will have exactly 2^{n-1} regions in it (note that above we listed it out for 3 circles and the third section did in fact have $2^{3-1}=4$ regions in it). This allows us to say that there will be $2^{2^n-1-2^{n-1}}$ code numbers that fail to mention a single circle. Finally, we take the difference to find $2^{2^n-1}-2^{2^n-1-2^{n-1}} = 2^{2^n-1}(1-\frac{1}{2^{2^{n-1}}})$.

This upper bound estimates for $n=3,4,5$ the number to be 96, 28672, and 2.01x10^9.

Rule 2 is also fairly easy to understand, the minimum number of regions an arrangement can have is going to be when all of the circles are isolated on their own, and so the minimum is n. This means that any code number which has fewer than n 1's in it is describing a circle arrangement with fewer regions than is possible and so is then an invalid code number. Likewise, the maximum number of regions for n circles is $n(n-1)+1$ which happens when all the circles overlap like a Venn diagram as seen below for $n = 5$:

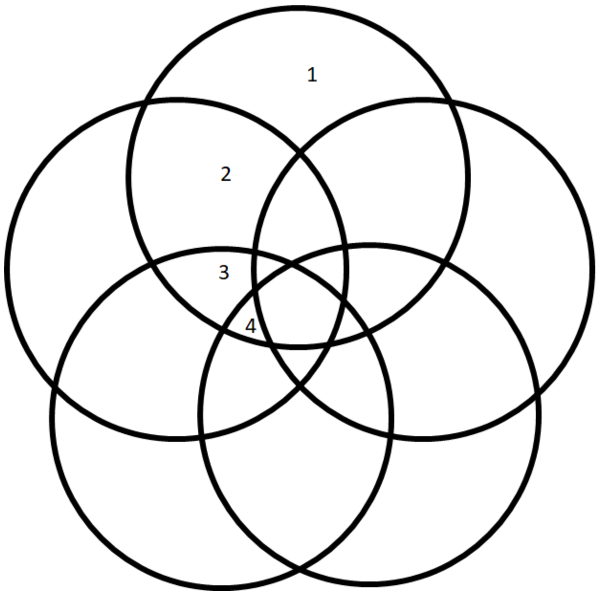


Figure 8.

There are 4 layers of regions, each with 5 copies rotated around the diagram, and then one single region in the middle, thus, $n(n-1)+1$. Here we can figure out how many code numbers have between n and $n(n-1)+1$ 1's them by considering all the ways to choose n 0's out of $2^n - 1$ digits and turn them into 1's. This is of course the same as asking for $\binom{2^n-1}{n}$ and then asking the same for $n+1$ which is of course $\binom{2^n-1}{n+1}$ and so on up to $\binom{2^n-1}{n(n-1)+1}$. We can therefore obtain a formula for the number of code numbers with between n and $n(n-1)+1$ 1's in them as follows: $\sum_{k=n}^{n(n-1)+1} \frac{(2^n-1)!}{k!(2^n-1-k)!}$.

This upper bound estimates that for $n = 3, 4, 5$ the number is 99, 32176, and 2.11x10^9. Which are comparable to the estimates given by the rule 1 formula.

Rule 3 is based on a the same reasoning as rule 2, being that in Fig. 8 you can also notice that there are a maximum of n regions of each type. By type, we mean that for any diagram there can only ever be up to n regions inside 1 circles, or inside 2 circles and so on. This means that in our code number it is impossible to have more than n 1's in each $\binom{n}{k}$. If we were to find the difference between each $\binom{n}{k}$ and n then we could move all the extra digits which must remain 0's to the front of the code number and thus reduce the total by some amount. The total number of digits in the code number is still $2^n - 1$ and so we want to subtract the number of digits left over when each

$\begin{pmatrix} n \\ k \end{pmatrix}$ section is maxed out with n 1's. The equation for this is thus $2^n - 1 - \left(\sum_{k=1}^{n-1} \frac{n!}{k!(n-k)!} - n \right)$.
We set the index from 1 to $n-1$ simply because $\begin{pmatrix} n \\ n \end{pmatrix} = 1 < n$ and so removing n from it would artificially inflate the number of fixed 0 digits. We can however notice 2 things, the first being that for any sum: $\sum_{k=1}^{y} k - x = \left(\sum_{k=1}^{y} k \right) - (xy)$. This allows us to pull the $-n$ out from the sum and simplify the sum along with the second observation: $\sum_{k=1}^{n-1} \frac{n!}{k!(n-k)!} = \left(\sum_{k=1}^{n} \frac{n!}{k!(n-k)!} \right) - 1$.
The two of these observations along with the previous fact that $\sum_{k=1}^{n} \frac{n!}{k!(n-k)!} = 2^n - 1$ allows us to simplify the equation to $2^n - 1 - (2^n - 2 - n(n-1)) = n(n-1) + 1$. Then, to find the total number of code numbers which satisfy this condition, we raise 2 to this new power $2^{n(n-1)+1}$. This gives estimates of 128, 8192, and 2097152 for $n = 3, 4, 5$ respectively. Notice how much smaller this estimate is than the others.

Here we also notice the same $n(n-1) + 1$ term that popped up in Rule 2, which makes sense since they are both describing a code number with a maximum of that many 1's. The only difference is in the distribution of those 1's. For rule 2 they would be in any concentration or any distribution in the code number, but for rule 3 they are grouped into each $\begin{pmatrix} n \\ k \end{pmatrix}$ section of the number. Since they don't describe the exact same types of code numbers this will act as a highly related, but distinct elimination rule.

The fourth rule is also pretty simple but a little less obvious. To translate what the rule says into plain English: you cannot have a diagram where circles overlap or perfectly fit into a circle intersection. The intuition behind this can be demonstrated in the following code numbers:

A	B	C	AB	AC	BC	ABC
0	0	0	1	0	1	1
1	0	1	0	0	0	1
0	0	0	1	1	1	0

Each of these code numbers obey the first 3 rules but fail the 4th rule and if we attempt to draw the diagrams these circle arrangements describe we will quickly run into a problem. Below is the diagram that satisfies the first number:

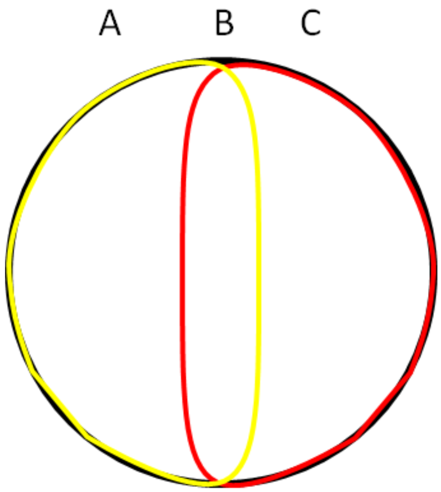


Figure 9.

In order to follow the description, neither circle A or C can exist outside of circle B, and circle B cannot exist apart from the other two. This is in fact the only way to achieve this code number and so it requires not only overlap of edges, but it requires non-circular shapes. The second code number can be seen bellow:

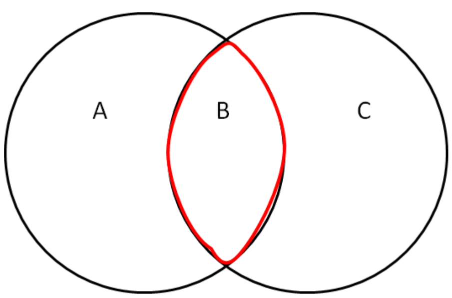


Figure 10.

This one can only be accomplished if the circle labeled B takes on the same shape as the overlap of A and C, something not possible for a circle. It is however (as a slight tangent) possible for squares as seen below:

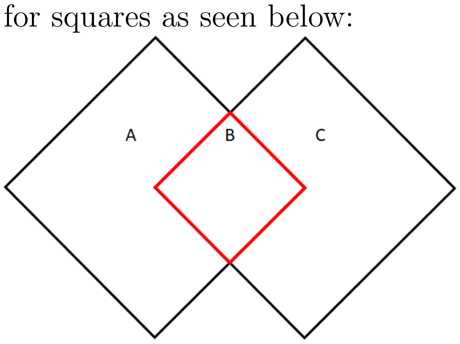


Figure 11.

The third and final invalid code number we will showcase is seen below:

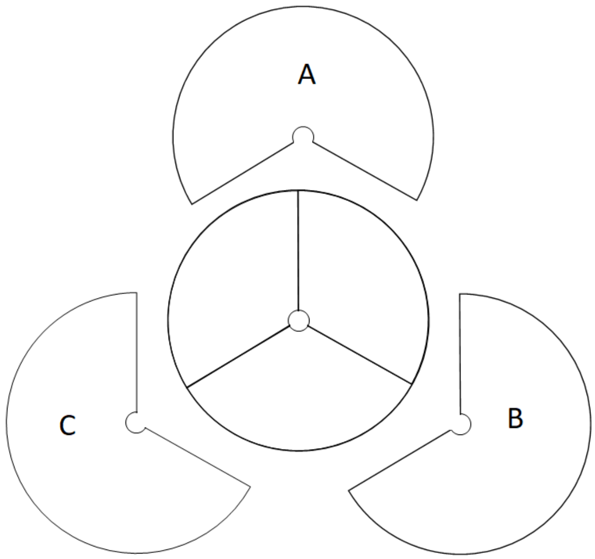


Figure 12.

The last diagram required some very non-circular shapes, both to accomplish the description as well as to not have all 3 intersect at the same point in the middle. Because of this the shapes were also shown on their own to see how they fit together.

With all of these failures, the reason is that they describe situations where you have some region inside of 2 or more circles, but no region inside of 1 fewer. This leads to "circles" which take on strange and definitively non-circular shapes to satisfy the code number description. All of these can be fixed by simply adding a 1 to each section left of a section where there is a 1. Hence the wording of rule 4, it requires that if there is a 1 in the $\binom{n}{k}$ section, then there must be a 1 in the $\binom{n}{k-1}$ section to avoid these issues. We can also observe that this implies any code number with a 1 in it at all must have a 1 in the $\binom{n}{1}$ section since this "add 1 to the left of" fix applies to itself. Therefore, the elimination rule can be written as an equation describing what happens when the $\binom{n}{1}$ section is all 0's.

The total number of digits is once again $2^n - 1$ and the number of digits in the $\binom{n}{1}$ section is just n so we can write the new total as $2^{2^n-1} - 2^{2^n-1-n} = 2^{2^n-1}(1-\frac{1}{2^n})$ which for $n = 3, 4, 5$ gives an estimate of 112, 30720, 2.08x10^9 respectively.

The fifth and final elimination rule i have been able to isolate is one that relies heavily on the second method of listing digits. To recap, for n circles, you can start by listing all ways to have a region with 1 circle, then all the ways to have regions with 2 circles provided that some have already been listed in the 1 circle section. This process repeats, always listing all the ways to have regions with the new total of circles, provided that some have already been listed in the previous sections. Below shows how to generate this method iteratively:

$$
\begin{array}{cccccccccccccccccc}
 & & & & & & & & & & & & & & & & & A \\
 & & & & & & & & & & & & & & B & BA & | & A \\
 & & & & & & & & & C & CB & CBA & CA & | & B & BA & | & A \\
D & DC & DCB & DCBA & DCA & DB & DBA & DA & | & C & CB & CBA & CA & | & B & BA & | & A
\end{array}
$$

Shown here you can easily see how each listing for the next n value is simply the previous listing but with a new section added. As stated before, the length of this new circle section is always 2^{n-1}. This method of listing regions does more than just lets us all the ways a new circle can be added, by thinking about that property in reverse we can also look at all the sub-diagrams for a given code number by removing the last circle section since it then describes the exact same arrangement but for $n-1$. This is the nice scale-ability mentioned earlier. With this digit listing method we preserve the structure of all previous n value arrangements when we add the next circle.

The intuition for this rule is that if a code number describes an arrangement that has as an impossibility as one of its' sub-diagrams, then clearly that code number is also invalid. The really nice feature of this rule is that how we define impossible is based on all the prior rules. The equation for this rule then is going to be how many code numbers for n do not have a previously impossible arrangement as a sub-diagram. This is then going to be $2^{2^n-1} - 2^{2^{n-1}}\mathbb{I}(n-1)$ since there are $2^{2^{n-1}}$ code numbers for each impossible code number in $n-1$ and where $\mathbb{I}(n)$ is the number of invalid code numbers in n circles.

The way to find $\mathbb{I}(n-1)$ is only by solving this problem for $n-1$ and then counting them. This brings us to the fact that even when we have eliminated all the invalid code numbers, many of them will be redundant. The way we estimate the number of unique is to realize that for any valid circle arrangement in a diagram, you can relabel all the circles arbitrarily, and so there will be $n!$ ways to relabel the same exact arrangement. This means that there are $n!$ duplicates of the same code number as well and so we can estimate the unique arrangements by dividing the number of valid code numbers by $n!$. This also allows us to figure out the number of impossible code numbers by taking the unique arrangements (which have to be previously known) in n and multiply by $n!$, then subtract that from the total in n which is 2^{2^n-1} to yield $\mathbb{I}(n) = 2^{2^n-1} - (n)!\mathbb{U}(n)$ where $\mathbb{U}(n)$ is the number of unique circle arrangements in n circles (the thing we are actually looking for).

We know that of the 8 code numbers for 2 circles, 4 of them are impossible, so for $n = 2$, $\mathbb{I}(2) = 4$ and by applying the process above to 3 circles, we find it estimates $2^{2^n-1} - 2^{2^{n-1}}\mathbb{I}(n-1) = 2^{2^3-1} - 2^{2^2}\mathbb{I}(2) = 2^7 - 2^4(4) = 64$ total valid code numbers. If we do the same for 4 circles, we first need to find $\mathbb{I}(3) = 2^{2^3-1} - (3)!\mathbb{U}(3) = 2^7 - 6(14) = 128 - 84 = 44$ which we can then plug into the same process to obtain:

$$2^{2^n-1} - 2^{2^{n-1}}\mathbb{I}(n-1) = 2^{2^4-1} - 2^{2^3}\mathbb{I}(3) = 2^15 - 2^8(44) = 21504 \text{ valid code numbers for 4 circles.}$$

Doing the same for 5 circles we first need to know $\mathbb{I}(4) = 2^{2^4-1} - (4)!\mathbb{U}(4) = 2^15 - 24(173) = 28616$ which we can plug into the formula for 5 circles to find:

$$2^{2^5-1} - 2^{2^4}\mathbb{I}(4) = 2^{31} - 2^{16}(28616) = 272105472.$$

Results:

The very last thing we can do for each of these elimination rules and upper bounds is to apply what we just said about redundancy and divide each of them by $n!$. If we do this, we can see the upper bounds from each rules below:

n	*Real*	$R1$	$R2$	$R3$	$R4$	$R5$
		$\frac{2^{2^n-1}(1-\frac{1}{2^{2^{n-1}}})}{n!}$	$\frac{\sum_{k=n}^{n(n-1)+1}\frac{(2^n-1)!}{k!(2^n-1-k)!}}{n!}$	$\frac{2^{n(n-1)+1}}{n!}$	$\frac{2^{2^n-1}(1-\frac{1}{2^n})}{n!}$	$\frac{2^{2^n-1}-2^{2^{n-1}}\mathbb{I}(n-1)}{n!}$
3	14	20	16.5	21.33	18.66	**10.66**
4	173	$1,360$	$1,340.66$	**341.33**	$1,280$	896
5	?	$17,895,424$	$17,631,884.725$	**17,476.266**	$17,336,456.533$	$2,267,545.6$

Unfortunately, since we don't know the exact value of $\mathbb{U}(6)$ we cannot estimate the values for 6 circles, but even coming this close for 5 circles appears to be a new result. It also appears that as n gets larger and larger, the smallest upper bound will always be given by $\frac{2^{n(n-1)+1}}{n!}$ with a possible upper bound on 5 circles being 17,476.

Additional Topics:

Beyond the problem of finding all unique circle arrangements for n circles, there are other questions one might ask. The following are the two that I worked on a while.

The first is that, once you know all the unique diagrams, is there a single diagram which contains all of them inside it in an optimal way? To be clear, I am referring to something like a super-permutation but with these diagrams, and long story short, yes it is possible. The question comes when you try to figure out how many circles is the minimum you can get away with. Below is the first diagram i discovered which contains all 14 diagrams for 3 circles in it as sub-diagrams as well as the 14:

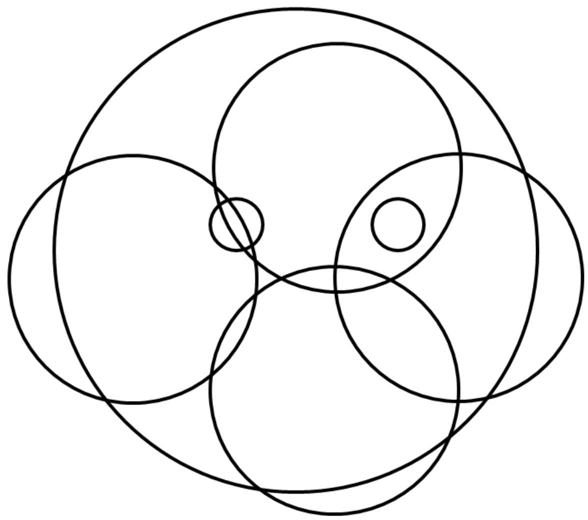


Figure 13.

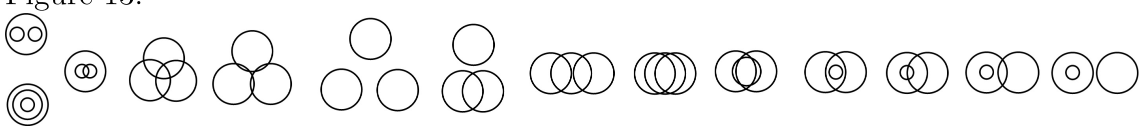


Figure 14.

It might be hard to find all of them in it, but they are most definetely there. This diagram also required 7 circles, which makes sense. You are going to need more than 3 for sure, probably more than 4, but what is the minimum necessary to contain all 14 as sub-diagrams? Well doing the math, we need a number of circles C such that the number of ways to pick 3 circles from it is greater than 14 so we can embed all 14 in it. this is basically asking $\begin{pmatrix} C \\ 3 \end{pmatrix} \geq 14$ and for $C = 5$ we get $\begin{pmatrix} 5 \\ 3 \end{pmatrix} = 10$ which is too few, but for $C = 6$ we see that $\begin{pmatrix} n \\ 1 \end{pmatrix} = 20$ which is larger and so works. This means the fewest circles you could ever do this with is 6, and that even allows for 6 redundancies.

All of this is frustrating then when no matter how hard i try, i still can only do it using 7 circles. The following are all different diagrams i have found that work but all require 7 circles to do it:

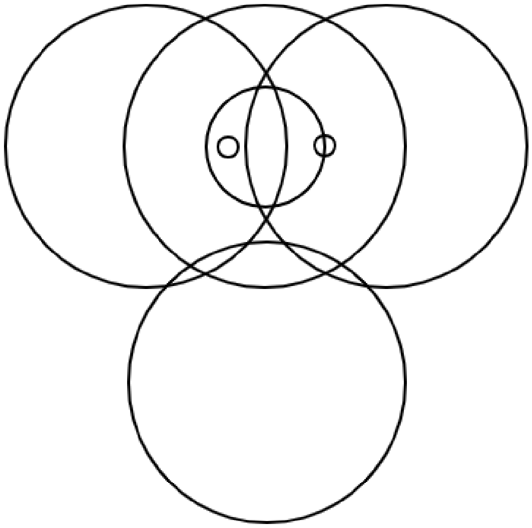
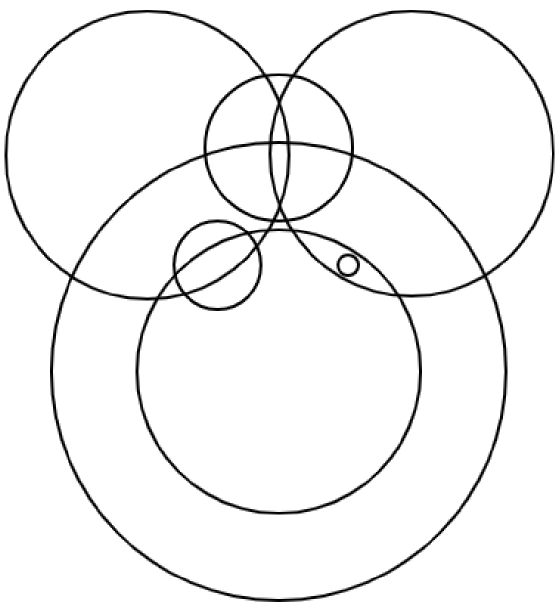


Figure 15.



(1).png
Figure 16.

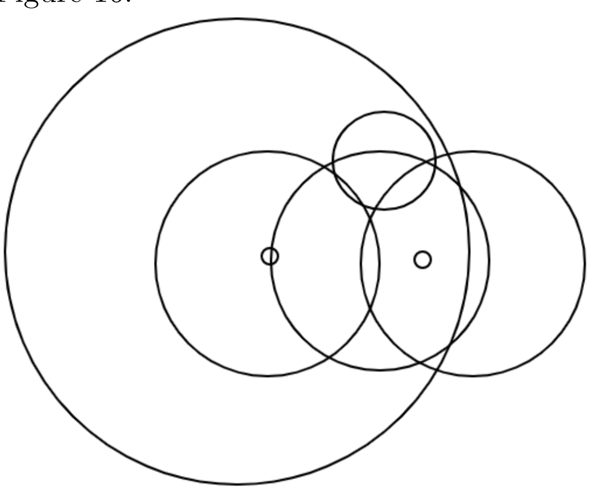


Figure 17.

If there is a way to prove it must take 7 circles and not 6, then i have not found it yet.

The last topic discussed here is that of a polyhedron composed of these arrangements. There is a technique of studying algebraic structures as literal structures where permutations, associations, and other such discrete objects act as the vertices of a solid, the edges act as simple transpositions or rearrangements connecting these objects, and faces act as definitions for families of these objects all connected via simple operations. The most well known of these is probably the permutahedron which takes as it's vertices all the ways to permute a string of 4 symbols, then the edges are all the ways to navigate from one vertex to another by executing a single transposition. It creates a truncated octahedron as seen below:

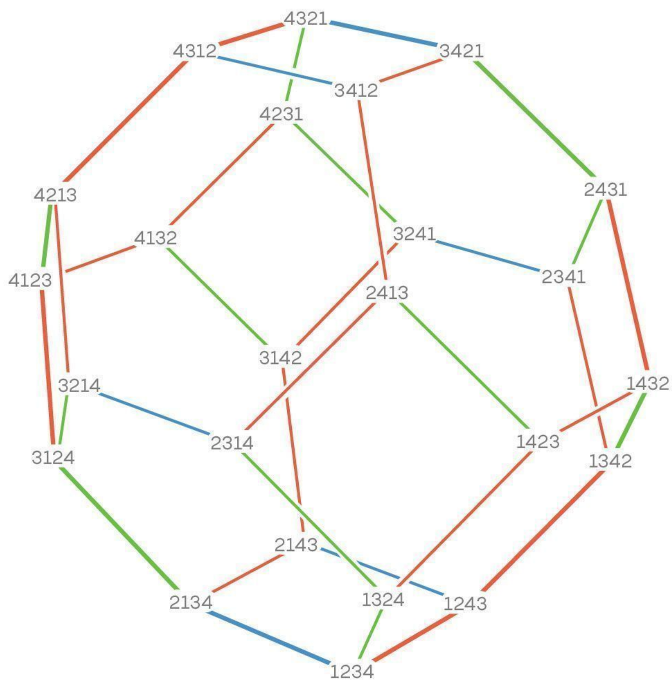


Figure 18.(Credit: Ken Abbott abbottsystems@gmail.com)

Another well known polyhedron like this is the associahedron which is built from all the ways to put parentheses around 4 symbols associatively. But my question is, can we look at all 14 circle arrangements for 3 circles as 2-dimensional permutations? And if so, can we construct a solid in the same manner? We can take the diagrams as the vertices and the edges would be simple motions of a circle to shift from one diagram to another. I have constructed a diagram representing how i see these vertices relating, but it does not seem to be possible to build an actual polyhedron from these relationships:

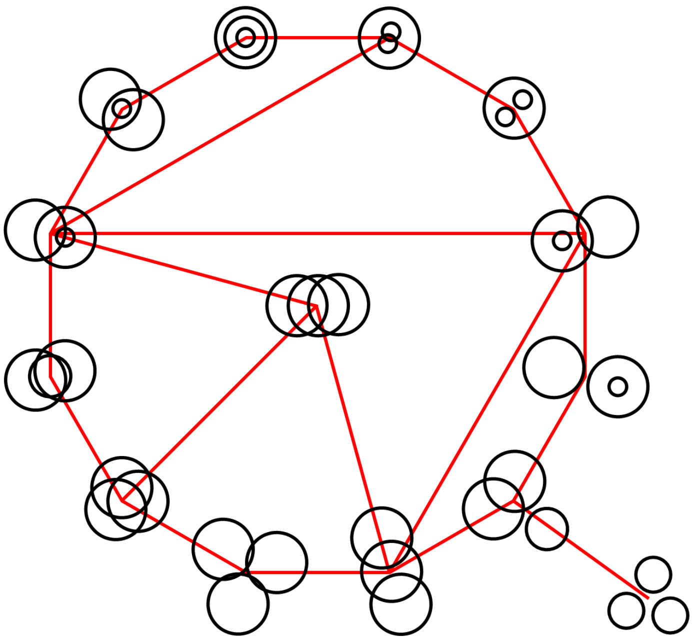


Figure 19.