Physics/Engineering Equation Sheet

1 Fluid Dynamics Conservation Equations

1.1 Integral Form

Mass:

$$\frac{\partial}{\partial t}\oiiint \rho \mathrm{d}V + \oiint \rho \mathbf{u}\cdot \mathrm{d}\mathbf{S} = 0$$

Momentum:

$$\frac{\partial}{\partial t}\oiiint \rho \mathbf{u} \mathrm{d}V + \oiint \rho\left(\mathbf{u}\cdot\mathrm{d}\mathbf{S}\right)\mathbf{u} = \oiiint \rho \mathbf{F}\mathrm{d}V - \oiint p\mathrm{d}\mathbf{S} + \mathbf{F}_{viscous}$$

Energy:

$$\frac{\partial}{\partial t}\oiiint \rho\left(e+\tfrac{1}{2}\mathbf{u}\cdot\mathbf{u}\right)\mathrm{d}V + \oiint \rho\left(e+\tfrac{1}{2}\mathbf{u}\cdot\mathbf{u}\right)\mathbf{u}\cdot\mathrm{d}\mathbf{S} = \oiiint \dot{q}\rho\mathrm{d}V + \dot{Q}_{viscous} - \oiint p\mathbf{u}\cdot\mathrm{d}\mathbf{S} + \oiiint \rho\left(\mathbf{f}\cdot\mathbf{u}\right)\mathrm{d}V + \dot{W}_{viscous}$$

1.2 Differential Form

Mass:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0$$

Momentum:

$$\begin{aligned}
\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) &= \rho\mathbf{F} - \nabla p + \nabla\cdot\boldsymbol{\tau} \\
&= \rho\mathbf{F} - \nabla p + \mu\nabla^2\mathbf{u} + \tfrac{1}{3}\mu\nabla(\nabla\cdot\mathbf{u}) \\
&\quad + \kappa\nabla(\nabla\cdot\mathbf{u}) + 2(\nabla\mu)\cdot\nabla\mathbf{u} + (\nabla\mu)\times(\nabla\times\mathbf{u}) \\
&\quad - \tfrac{2}{3}(\nabla\mu)(\nabla\cdot\mathbf{u}) + (\nabla\kappa)(\nabla\cdot\mathbf{u})
\end{aligned}$$

$$\kappa = \lambda + \tfrac{2}{3}\mu \qquad \text{(bulk viscosity)}$$

$$\boldsymbol{\tau} = \mu\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^{\intercal}\right] + \left(\kappa - \tfrac{2}{3}\mu\right)(\nabla\cdot\mathbf{u})\mathbf{I}$$

$$\mathbf{I} = \sum_{i=1}^{3}\sum_{j=1}^{3}\delta_{ij}$$

Vorticity:

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} + \nabla\times(\boldsymbol{\Omega}\times\mathbf{u}) = \nabla\times\mathbf{F} + \frac{\nabla\rho\times\nabla p}{\rho^2} + \nu\nabla^2\boldsymbol{\Omega} + (\nabla\nu)\times\left[\tfrac{3}{2}\nabla(\nabla\cdot\mathbf{u}) - \nabla^2\boldsymbol{\Omega}\right]$$

$$\nabla\times(\boldsymbol{\Omega}\times\mathbf{u}) = \boldsymbol{\Omega}(\nabla\cdot\mathbf{u}) + (\mathbf{u}\cdot\nabla)\boldsymbol{\Omega} - (\boldsymbol{\Omega}\cdot\nabla)\mathbf{u}$$

$$\boldsymbol{\Omega} = \nabla\times\mathbf{u} \qquad \text{and} \qquad \nu = \mu/\rho$$

Energy:

$$\frac{\partial(\rho e_t)}{\partial t} = -\nabla\cdot\left[(p+\rho e_t) - \boldsymbol{\tau}\cdot\mathbf{u} + \mathbf{q}\right] \qquad \text{(total energy)}$$

$$\rho\frac{De}{Dt} = -\nabla\cdot\mathbf{q} - p\nabla\cdot\mathbf{u} + \boldsymbol{\tau}:\nabla\mathbf{u} \qquad \text{(internal energy)}$$

$$\rho\frac{Dh}{Dt} = -\nabla\cdot\mathbf{q} + \frac{Dp}{Dt} + \boldsymbol{\tau}:\nabla\mathbf{u} \qquad \text{(specific enthalpy)}$$

$$\rho c_p\frac{DT}{Dt} = -\nabla\cdot\mathbf{q} + \beta T\frac{Dp}{Dt} + \boldsymbol{\tau}:\nabla\mathbf{u} \qquad \text{(temperature)}$$

$$\rho\frac{Ds}{Dt} = -\nabla\cdot\left(\frac{\mathbf{q}}{T}\right) - \frac{\mathbf{q}}{T^2}\cdot\nabla T + \frac{\boldsymbol{\tau}:\nabla\mathbf{u}}{T} \qquad \text{(specific entropy)}$$

$$\rho c_p\frac{DT}{Dt} = \nabla(\lambda\nabla T) + \beta T\frac{Dp}{Dt} + \boldsymbol{\tau}:\nabla\mathbf{u} \qquad \text{(Fourier conduction)}$$

$$\begin{aligned}
\boldsymbol{\tau}:\nabla\mathbf{u} = \mu\Big\{&2(\nabla\mathbf{u})_{11}^2 + 2(\nabla\mathbf{u})_{22}^2 + 2(\nabla\mathbf{u})_{33}^2 \\
&+ \left[(\nabla\mathbf{u})_{12} + (\nabla\mathbf{u})_{21}\right]^2 \\
&+ \left[(\nabla\mathbf{u})_{13} + (\nabla\mathbf{u})_{31}\right]^2 \\
&+ \left[(\nabla\mathbf{u})_{23} + (\nabla\mathbf{u})_{32}\right]^2\Big\} \\
&+ \left(\kappa - \tfrac{2}{3}\mu\right)(\nabla\cdot\mathbf{u})^2
\end{aligned}$$

Thermodynamic Relations:

$$e_t = e + \tfrac{1}{2}\mathbf{u}\cdot\mathbf{u} + \psi \qquad \text{(total energy)}$$

Constant Specific Heat:

$$\hat{u} = \int c_v\mathrm{d}T \approx c_v T + \text{const}$$

2 Vector Identities

2.1 Algebraic Identities

$$\sin\theta = \frac{\|\mathbf{A}\times\mathbf{B}\|}{\|\mathbf{A}\|\,\|\mathbf{B}\|}$$
$$\cos\theta = \frac{\|\mathbf{A}\times\mathbf{B}\|}{\|\mathbf{A}\|\,\|\mathbf{B}\|}$$
$$\|\mathbf{A}\times\mathbf{B}\|^2 + (\mathbf{A}\cdot\mathbf{B})^2 = \|\mathbf{A}\|^2\|\mathbf{B}\|^2$$
$$\mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A}$$
$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$$
$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$$
$$= \mathbf{e}_1(A_2B_3 - A_3B_2) + \mathbf{e}_2(A_3B_1 - A_1B_3) + \mathbf{e}_3(A_1B_2 - A_2B_1)$$
$$c(\mathbf{A}+\mathbf{B}) = c\mathbf{A} + c\mathbf{B}$$
$$(\mathbf{A}+\mathbf{B})\cdot\mathbf{C} = \mathbf{A}\cdot\mathbf{C} + \mathbf{B}\cdot\mathbf{C}$$
$$(\mathbf{A}+\mathbf{B})\times\mathbf{C} = \mathbf{A}\times\mathbf{C} + \mathbf{B}\times\mathbf{C}$$
$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$
$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\,\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\,\mathbf{C}$$
$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) + \mathbf{C}\times(\mathbf{A}\times\mathbf{B}) + \mathbf{B}\times(\mathbf{C}\times\mathbf{A}) = 0$$
$$(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{B}\cdot\mathbf{C})(\mathbf{A}\cdot\mathbf{D})$$
$$|\mathbf{A}\times\mathbf{B}|^2 = (\mathbf{A}\cdot\mathbf{A})(\mathbf{B}\cdot\mathbf{B}) - (\mathbf{A}\cdot\mathbf{B})^2$$

2.2 Differentiation of vectors

$$\frac{d\mathbf{A}}{dt} = \frac{dA_1}{dt}\mathbf{i} + \frac{dA_2}{dt}\mathbf{j} + \frac{dA_3}{dt}\mathbf{k}$$
$$\frac{d}{dt}(\mathbf{A}+\mathbf{B}) = \frac{d\mathbf{A}}{dt} + \frac{d\mathbf{B}}{dt}$$
$$\frac{d}{dt}\left(\phi\mathbf{A}\right) = \frac{d\phi}{dt}\mathbf{A} + \phi\frac{d\mathbf{A}}{dt}$$
$$\frac{d}{dt}(\mathbf{A}\cdot\mathbf{B}) = \frac{d\mathbf{A}}{dt}\cdot\mathbf{B} + \mathbf{A}\cdot\frac{d\mathbf{B}}{dt}$$
$$\frac{d}{dt}(\mathbf{A}\times\mathbf{B}) = \frac{d\mathbf{A}}{dt}\times\mathbf{B} + \mathbf{A}\times\frac{d\mathbf{B}}{dt}$$
$$\frac{d}{dt}(\mathbf{A}\times\mathbf{B}\cdot\mathbf{C}) = \frac{d\mathbf{A}}{dt}\times\mathbf{B}\cdot\mathbf{C} + \mathbf{A}\times\frac{d\mathbf{B}}{dt}\cdot\mathbf{C} + \mathbf{A}\times\mathbf{B}\cdot\frac{d\mathbf{C}}{dt}$$

2.3 First Derivative Identities

2.3.1 Distributive Properties

$$\nabla\left(\psi+\phi\right) = \nabla\psi + \nabla\phi$$
$$\nabla(\mathbf{A}+\mathbf{B}) = \nabla\mathbf{A} + \nabla\mathbf{B}$$
$$\nabla\cdot(\mathbf{A}+\mathbf{B}) = \nabla\cdot\mathbf{A} + \nabla\cdot\mathbf{B}$$
$$\nabla\times(\mathbf{A}+\mathbf{B}) = \nabla\times\mathbf{A} + \nabla\times\mathbf{B}$$

2.3.2 Product rule for multiplication by a Scalar

$$\nabla\left(\psi\phi\right) = \phi\nabla\psi + \psi\nabla\phi$$
$$\nabla(\psi\mathbf{A}) = (\nabla\psi)\,\mathbf{A}^\intercal + \psi\nabla\mathbf{A} = \nabla\psi\otimes\mathbf{A} + \psi\nabla\mathbf{A}$$
$$\nabla\cdot(\psi\mathbf{A}) = \psi\nabla\cdot\mathbf{A} + (\nabla\psi)\cdot\mathbf{A}$$
$$\nabla\times(\psi\mathbf{A}) = \psi\nabla\times\mathbf{A} + (\nabla\psi)\times\mathbf{A}$$
$$\nabla^2(fg) = f\nabla^2 g + 2\nabla f\cdot\nabla g + g\nabla^2 f$$

2.3.3 Quotient rule for division by a scalar

$$\nabla\left(\frac{\psi}{\phi}\right) = \frac{\phi\nabla\psi - \psi\nabla\phi}{\phi^2}$$
$$\nabla\left(\frac{\mathbf{A}}{\phi}\right) = \frac{\phi\nabla\mathbf{A} - \nabla\phi\otimes\mathbf{A}}{\phi^2}$$
$$\nabla\cdot\left(\frac{\mathbf{A}}{\phi}\right) = \frac{\phi\nabla\cdot\mathbf{A} - \nabla\phi\cdot\mathbf{A}}{\phi^2}$$
$$\nabla\times\left(\frac{\mathbf{A}}{\phi}\right) = \frac{\phi\nabla\times\mathbf{A} - \nabla\phi\times\mathbf{A}}{\phi^2}$$

2.3.4 Dot Product Rule

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\,\mathbf{B} + (\mathbf{B}\cdot\nabla)\,\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A})$$
$$\tfrac{1}{2}\nabla(\mathbf{A}\cdot\mathbf{A}) = (\mathbf{A}\cdot\nabla)\,\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{A})$$

2.3.5 Cross product rule

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = (\nabla\times\mathbf{A})\cdot\mathbf{B} - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$
$$\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\,\mathbf{A} - (\mathbf{A}\cdot\nabla)\,\mathbf{B}$$
$$= \nabla\cdot(\mathbf{B}\mathbf{A}^\intercal - \mathbf{A}\mathbf{B}^\intercal)$$
$$\mathbf{A}\times(\nabla\times\mathbf{B}) = (\nabla\mathbf{B})\cdot\mathbf{A} - (\mathbf{A}\cdot\nabla)\,\mathbf{B}$$
$$(\mathbf{A}\times\nabla)\times\mathbf{B} = (\nabla\mathbf{B})\cdot\mathbf{A} - \mathbf{A}(\nabla\cdot\mathbf{B})$$
$$= \mathbf{A}\times(\nabla\times\mathbf{B}) + (\mathbf{A}\cdot\nabla)\,\mathbf{B} - \mathbf{A}(\nabla\cdot\mathbf{B})$$

2.4 Second Derivative Identities

$$\nabla\cdot(\nabla\times\mathbf{A}) = 0$$
$$\Delta\psi = \nabla^2\psi = \nabla\cdot(\nabla\psi)$$
$$\nabla\cdot(\nabla\cdot\mathbf{A})\ \text{ is undefined}$$
$$\nabla\times\left(\nabla\phi\right) = 0$$
$$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$$
$$\nabla\times(\nabla\cdot\mathbf{A})\ \text{ is undefined}$$

In the 2nd to last identity, ∇^2 is the vector Laplacian operating on the vector field $\mathbf{A}$.

2.5 Third Derivative Identities

$$\nabla^2(\nabla\psi) = \nabla[\nabla\cdot(\nabla\psi)] = \nabla\left(\nabla^2\psi\right)$$
$$\nabla^2(\nabla\cdot\mathbf{A}) = \nabla\cdot[\nabla(\nabla\cdot\mathbf{A})] = \nabla\cdot\left(\nabla^2\mathbf{A}\right)$$
$$\nabla^2(\nabla\times\mathbf{A}) = -\nabla\times[\nabla\times(\nabla\times\mathbf{A})] = \nabla\times\left(\nabla^2\mathbf{A}\right)$$

3 Vector Differential Invariants

$$h_i = \sqrt{\sum_{k=1}^{n} \left(\frac{\partial x_k}{\partial q_i}\right)^2} \qquad \text{(scale factors for orthogonal coordinate systems)}$$

$$\nabla\psi \equiv \text{grad } \psi = \frac{\mathbf{i}_i}{h_i}\frac{\partial \psi}{\partial q_i}$$

$$\nabla \cdot \mathbf{u} \equiv \text{div } \mathbf{u} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial (u_1 h_2 h_3)}{\partial q_1} + \frac{\partial (u_2 h_3 h_1)}{\partial q_2} + \frac{\partial (u_3 h_1 h_2)}{\partial q_3}\right]$$

$$\nabla \times \mathbf{u} \equiv \text{curl } \mathbf{u} = \frac{\mathbf{i}_1}{h_2 h_3}\left[\frac{\partial (u_3 h_3)}{\partial q_2} - \frac{\partial (u_2 h_2)}{\partial q_3}\right] + \cdots$$

$$\nabla \cdot \nabla\psi = \nabla^2\psi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \psi}{\partial q_1}\right) + \cdots\right]$$

$$\nabla(\nabla \cdot \mathbf{u}) = \frac{\mathbf{i}_1}{h_1}\frac{\partial}{\partial q_1}\left\{\frac{1}{h_1 h_2 h_3}\left[\frac{\partial (u_1 h_2 h_3)}{\partial q_1} + \frac{\partial (u_2 h_3 h_1)}{\partial q_2} + \frac{\partial (u_3 h_1 h_2)}{\partial q_3}\right]\right\} + \cdots$$

$$\nabla \times (\nabla \times \mathbf{u}) = \frac{\mathbf{i}_1}{h_2 h_3}\left\langle \frac{\partial}{\partial q_2}\left\{\frac{h_3}{h_1 h_2}\left[\frac{\partial (u_2 h_2)}{\partial q_1} - \frac{\partial (u_1 h_1)}{\partial q_2}\right]\right\} - \frac{\partial}{\partial q_3}\left\{\frac{h_2}{h_3 h_1}\left[\frac{\partial (u_1 h_1)}{\partial q_3} - \frac{\partial (u_3 h_3)}{\partial q_1}\right]\right\}\right\rangle + \cdots$$

$$\begin{aligned}
\nabla^2\mathbf{u} = \mathbf{i}_1 \Bigg[& \nabla^2 u_1 - u_1 h_1 \nabla^2 \frac{1}{h_1} \\
& + \frac{u_1 h_1}{h_1}\frac{\partial}{\partial q_1}\left(\nabla^2 q_1\right) + \frac{u_2 h_2}{h_1}\frac{\partial}{\partial q_1}\left(\nabla^2 q_2\right) + \frac{u_3 h_3}{h_1}\frac{\partial}{\partial q_1}\left(\nabla^2 q_3\right) \\
& - \frac{2}{h_1^2}\frac{\partial (1/h_1)}{\partial q_1}\frac{\partial}{\partial q_1}(u_1 h_1) - \frac{2}{h_2^2}\frac{\partial (1/h_1)}{\partial q_2}\frac{\partial}{\partial q_1}(u_2 h_2) - \frac{2}{h_3^2}\frac{\partial (1/h_1)}{\partial q_3}\frac{\partial}{\partial q_1}(u_3 h_3) \\
& + \frac{1}{h_1}\frac{\partial\left(1/h_1^2\right)}{\partial q_1}\frac{\partial}{\partial q_1}(u_1 h_1) + \frac{1}{h_1}\frac{\partial\left(1/h_2^2\right)}{\partial q_1}\frac{\partial}{\partial q_2}(u_2 h_2) + \frac{1}{h_1}\frac{\partial\left(1/h_3^2\right)}{\partial q_1}\frac{\partial}{\partial q_3}(u_3 h_3) \Bigg] + \cdots
\end{aligned}$$

$$\nabla\mathbf{u} = \sum_{j=1}^{3}\sum_{k=1}^{3}\frac{\mathbf{i}_j}{h_j}\mathbf{i}_j\left(\mathbf{i}_k\frac{\partial u_k}{\partial q_j} + u_k\frac{\partial \mathbf{i}_k}{\partial q_j}\right); \qquad \frac{\partial \mathbf{i}_k}{\partial q_j} = \frac{\mathbf{i}_j}{h_k}\frac{\partial h_j}{\partial q_k} \quad (j \neq k), \qquad \frac{\partial \mathbf{i}_j}{\partial q_j} = -\frac{\mathbf{i}_k}{h_k}\frac{\partial h_j}{\partial q_k} - \frac{\mathbf{i}_l}{h_l}\frac{\partial h_j}{\partial q_l}$$

$$\begin{aligned}
= & \frac{\mathbf{i}_1\mathbf{i}_1}{h_1}\left[\frac{\partial u_1}{\partial q_1} + \frac{u_2}{h_2}\frac{\partial h_1}{\partial q_2} + \frac{u_3}{h_3}\frac{\partial h_1}{\partial q_3}\right] + \frac{\mathbf{i}_1\mathbf{i}_2}{h_1}\left[\frac{\partial u_2}{\partial q_1} - \frac{u_1}{h_2}\frac{\partial h_1}{\partial q_2}\right] + \frac{\mathbf{i}_1\mathbf{i}_3}{h_1}\left[\frac{\partial u_3}{\partial q_1} - \frac{u_1}{h_3}\frac{\partial h_1}{\partial q_3}\right] \\
& + \frac{\mathbf{i}_2\mathbf{i}_1}{h_2}\left[\frac{\partial u_1}{\partial q_2} - \frac{u_2}{h_1}\frac{\partial h_2}{\partial q_1}\right] + \frac{\mathbf{i}_2\mathbf{i}_2}{h_2}\left[\frac{\partial u_2}{\partial q_2} + \frac{u_3}{h_3}\frac{\partial h_2}{\partial q_3} + \frac{u_1}{h_1}\frac{\partial h_2}{\partial q_1}\right] + \frac{\mathbf{i}_2\mathbf{i}_3}{h_2}\left[\frac{\partial u_3}{\partial q_2} - \frac{u_2}{h_3}\frac{\partial h_2}{\partial q_3}\right] \\
& + \frac{\mathbf{i}_3\mathbf{i}_1}{h_3}\left[\frac{\partial u_1}{\partial q_3} - \frac{u_3}{h_1}\frac{\partial h_3}{\partial q_1}\right] + \frac{\mathbf{i}_3\mathbf{i}_2}{h_3}\left[\frac{\partial u_2}{\partial q_3} - \frac{u_3}{h_2}\frac{\partial h_3}{\partial q_2}\right] + \frac{\mathbf{i}_3\mathbf{i}_3}{h_3}\left[\frac{\partial u_3}{\partial q_3} + \frac{u_1}{h_1}\frac{\partial h_3}{\partial q_1} + \frac{u_2}{h_2}\frac{\partial h_3}{\partial q_2}\right]
\end{aligned}$$

$$\boldsymbol{\phi} = \sum_{j=1}^{3}\sum_{k=1}^{3}\mathbf{i}_j\mathbf{i}_k\phi_{jk} \qquad \text{(Dyadic)}$$

$$\boldsymbol{\phi}^{\mathsf{T}} = \sum_{j=1}^{3}\sum_{k=1}^{3}\mathbf{i}_j\mathbf{i}_k\phi_{kj} \qquad \text{(Dyadic transpose)}$$

4 Cylindrical/Spherical Vector Identities

4.1 Cylindrical

$$\nabla\psi = \mathbf{i}_r\frac{\partial\psi}{\partial r} + \mathbf{i}_\theta\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{i}_z\frac{\partial\psi}{\partial z}$$

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2}$$

$$\nabla\cdot\mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_z}{\partial z}$$

$$\nabla\times\mathbf{u} = \mathbf{i}_r\left(\frac{1}{r}\frac{\partial u_z}{\partial\theta} - \frac{\partial u_\theta}{\partial z}\right) + \mathbf{i}_\theta\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) + \mathbf{i}_z\left[\frac{1}{r}\frac{\partial}{\partial r}(ru_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right]$$

$$\nabla^2\mathbf{u} = \mathbf{i}_r\left(\nabla^2 u_r - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta} - \frac{u_r}{r^2}\right) + \mathbf{i}_\theta\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2}\right) + \mathbf{i}_z\nabla^2 u_z$$

$$\begin{aligned}\nabla\mathbf{u} &= \mathbf{i}_r\mathbf{i}_r\frac{\partial u_r}{\partial r} + \mathbf{i}_r\mathbf{i}_\theta\frac{\partial u_\theta}{\partial r} + \mathbf{i}_r\mathbf{i}_z\frac{\partial u_z}{\partial r}\\ &+ \mathbf{i}_\theta\mathbf{i}_r\frac{1}{r}\left(\frac{\partial u_r}{\partial\theta} - u_\theta\right) + \mathbf{i}_\theta\mathbf{i}_\theta\frac{1}{r}\left(\frac{\partial u_\theta}{\partial\theta} + u_r\right) + \mathbf{i}_\theta\mathbf{i}_z\frac{1}{r}\frac{\partial u_z}{\partial\theta}\\ &+ \mathbf{i}_z\mathbf{i}_r\frac{\partial u_r}{\partial z} + \mathbf{i}_z\mathbf{i}_\theta\frac{\partial u_\theta}{\partial z} + \mathbf{i}_z\mathbf{i}_z\frac{\partial u_z}{\partial z}\end{aligned}$$

4.2 Spherical

$$\nabla\psi = \mathbf{i}_R\frac{\partial\psi}{\partial R} + \mathbf{i}_\phi\frac{1}{R}\frac{\partial\psi}{\partial\phi} + \mathbf{i}_\theta\frac{1}{R\sin\phi}\frac{\partial\psi}{\partial\theta}$$

$$\nabla^2\psi = \frac{1}{R^2}\left[\frac{\partial}{\partial R}\left(R^2\frac{\partial\psi}{\partial R}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial\psi}{\partial\phi}\right) + \frac{1}{\sin^2\phi}\frac{\partial^2\psi}{\partial^2\theta}\right]$$

$$\nabla\cdot\mathbf{u} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 u_R\right) + \frac{1}{R\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi u_\phi\right) + \frac{1}{R\sin\phi}\frac{\partial u_\theta}{\partial\theta}$$

$$\nabla\times\mathbf{u} = \mathbf{i}_R\frac{1}{R\sin\phi}\left[\frac{\partial}{\partial\phi}\left(\sin\phi u_\theta\right) - \frac{\partial u_\phi}{\partial\theta}\right] + \mathbf{i}_\phi\frac{1}{R}\left[\frac{1}{\sin\phi}\frac{\partial u_R}{\partial\theta} - \frac{\partial}{\partial R}(Ru_\theta)\right] + \mathbf{i}_\theta\frac{1}{R}\left[\frac{\partial}{\partial R}\left(Ru_\phi\right) - \frac{\partial u_R}{\partial\phi}\right]$$

$$\begin{aligned}\nabla^2\mathbf{u} &= \mathbf{i}_R\left[\nabla^2 u_R - \frac{2}{R^2}\left(u_R + \frac{\partial u_\phi}{\partial\phi} + u_\phi\cot\phi - \frac{1}{\sin\phi}\frac{\partial u_\theta}{\partial\theta}\right)\right]\\ &+ \mathbf{i}_\phi\left[\nabla^2 u_\phi + \frac{1}{R^2}\left(2\frac{\partial u_R}{\partial\phi} - \frac{u_\phi}{\sin^2\phi} - 2\frac{\cos\phi}{\sin^2\phi}\frac{\partial u_\theta}{\partial\theta}\right)\right]\\ &+ \mathbf{i}_\theta\left[\nabla^2 u_\theta + \frac{1}{R^2}\left(\frac{2}{\sin\phi}\frac{\partial u_R}{\partial\theta} + 2\frac{\cos\phi}{\sin^2\phi}\frac{\partial u_\phi}{\partial\theta} - \frac{u_\theta}{\sin^2\phi}\right)\right]\end{aligned}$$

$$\begin{aligned}\nabla\mathbf{u} &= \mathbf{i}_R\mathbf{i}_R\frac{\partial u_R}{\partial R} + \mathbf{i}_R\mathbf{i}_\phi\frac{\partial u_\phi}{\partial R} + \mathbf{i}_R\mathbf{i}_\theta\frac{\partial u_\theta}{\partial R}\\ &+ \mathbf{i}_\phi\mathbf{i}_R\frac{1}{R}\left(\frac{\partial u_R}{\partial\phi} - u_\phi\right) + \mathbf{i}_\phi\mathbf{i}_\phi\frac{1}{R}\left(\frac{\partial u_\phi}{\partial\phi} + u_R\right) + \mathbf{i}_\phi\mathbf{i}_\theta\frac{1}{R}\frac{\partial u_\theta}{\partial\phi}\\ &+ \mathbf{i}_\theta\mathbf{i}_R\frac{1}{R}\left(\frac{1}{\sin\phi}\frac{\partial u_R}{\partial\theta} - u_\theta\right) + \mathbf{i}_\theta\mathbf{i}_\phi\frac{1}{R}\left(\frac{1}{\sin\phi}\frac{\partial u_\phi}{\partial\theta} - \cot\phi u_\theta\right) + \mathbf{i}_\theta\mathbf{i}_\theta\frac{1}{R}\left(\frac{1}{\sin\phi}\frac{\partial u_\theta}{\partial\theta} + u_R + \cot\phi u_\phi\right)\end{aligned}$$

5 Vector Integral Theorems

5.1 Gauss's Divergence Theorem

$$\oiint \mathbf{f}\cdot \mathrm{d}\mathbf{S} = \oiiint \boldsymbol{\nabla}\cdot \mathbf{f}\mathrm{d}V$$

5.2 Green's Theorems

5.2.1 Green's First Theorem

$$\oiint \phi\frac{\partial\psi}{\partial n}\mathrm{d}S = \oiiint \left(\phi\boldsymbol{\nabla}^2\psi + \boldsymbol{\nabla}\phi\cdot\boldsymbol{\nabla}\psi\right)\mathrm{d}V$$

5.2.2 Green's Second Theorem

$$\oiint \left(\phi\frac{\partial\phi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)\mathrm{d}S = \oiiint \left(\phi\boldsymbol{\nabla}^2\psi - \psi\boldsymbol{\nabla}^2\phi\right)\mathrm{d}V$$

5.2.3 Special Cases

$$\oiint \left(\phi\boldsymbol{\nabla}\phi\right)\cdot \mathrm{d}\mathbf{S} = \oiiint \left[\phi\boldsymbol{\nabla}^2\phi + \left(\boldsymbol{\nabla}\phi\right)^2\right]\mathrm{d}V$$

$$\oiint \frac{\partial\phi}{\partial n}\mathrm{d}S = \oiiint \boldsymbol{\nabla}^2\phi\mathrm{d}V$$

5.3 Stokes' Theorem

Let a simple closed curve C be spanned by a surface S. Define the positive normal $\mathbf{n}$ to S, and the positive sense of description of the curve C with line element $\mathrm{d}\mathbf{r}$, such that the positive sense of the contour C is clockwise when we look through the surface S in the direction of the normal. Then, if $\mathbf{f}$ is continuously differentiable vector field defined on S and C with vector element $\mathbf{S} = \mathbf{n}\mathrm{d}S$

$$\oint \mathbf{f}\cdot \mathrm{d}\mathbf{r} = \oiint \boldsymbol{\nabla}\times\mathbf{f}\cdot \mathrm{d}\mathbf{S}$$

where the line integral around C is taken in the positive sense.

5.4 Integral rate of change theorems

5.4.1 Rate of change of volume integral bounded by a moving closed surface.

Let f be a continuous scalar function of position and time t defined throughout the volume $V(t)$, which is itself bounded by a simple closed surface $S(t)$ moving with velocity $\mathbf{v}$. Then the rate of change of the volume integral of f is given by

$$\frac{D}{Dt}\int_{V(t)} f\mathrm{d}V = \int_{V(t)}\frac{\partial f}{\partial t}\mathrm{d}V + \int_{S(t)} f\mathbf{v}\cdot\mathrm{d}\mathbf{S}$$

where $\mathrm{d}\mathbf{S}$ is the outward drawn vector element of area, and

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})$$

5.4.2 Rate of change of flux through a surface.

Let $\mathbf{q}$ be a vector function that may also depend on the time t, and $\mathbf{n}$ be the unit outward drawn normal to the surface S that moves with velocity $\mathbf{v}$. Defining the flux of $\mathbf{q}$ through S as

$$m = \oint_S \mathbf{q}\cdot\mathbf{n}\mathrm{d}S$$

then

$$\frac{Dm}{Dt} = \oint\left[\frac{\partial\mathbf{q}}{\partial t} + \mathbf{v}\left(\boldsymbol{\nabla}\cdot\mathbf{q}\right) + \boldsymbol{\nabla}\times\left(\mathbf{q}\times\mathbf{v}\right)\right]\cdot\mathbf{n}\mathrm{d}S$$

5.4.3 Rate of change of the circulation around a given moving curve.

Let C be a closed curve, moving with velocity $\mathbf{v}$, on which is defined a vector field $\mathbf{q}$. Defining the circulation ζ of $\mathbf{q}$ around C by

$$\zeta = \oint \mathbf{q}\cdot\mathrm{d}\mathbf{r}$$

then

$$\frac{D\zeta}{Dt} = \oint\left[\frac{\partial\mathbf{q}}{\partial t} + \left(\boldsymbol{\nabla}\times\mathbf{q}\right)\times\mathbf{v}\right]\cdot\mathrm{d}\mathbf{r}$$

6 Ideal Flow (irrotational/inviscid/incompressible)

6.1 Governing equations

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{(continuity)}$$

$$\rho \left[\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right] = -\nabla p \qquad \text{(momentum)}$$

6.2 Fundamentals

6.2.1 Streamfunction

May be defined when continuity eq. reduces to two terms. For cartesian coordinates:

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$$

The streamfunction may be defined as:

$$u \equiv \frac{\partial \psi}{\partial y} \qquad \text{and} \qquad v \equiv -\frac{\partial \psi}{\partial x}$$

while vorticity may be defined as

$$\omega_z = -\nabla^2 \psi$$

6.2.2 Velocity potential

$$u \equiv \frac{\partial \phi}{\partial x} \qquad \text{and} \qquad v \equiv \frac{\partial \phi}{\partial y}$$

6.2.3 Bernoulli's equation

If flow is irrotational throughout:

$$p + \tfrac{1}{2}\rho V^2 = \text{const.}$$

else, only valid along streamlines.

6.2.4 Incompressible pressure coefficient

$$C_p = 1 - \left(\frac{V}{V_\infty} \right)^2$$

7 Viscous Flow

7.1 Governing Equatinos

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{(continuity)}$$
$$\rho\left(\mathbf{u} \cdot \nabla \mathbf{u}\right) = \rho \mathbf{F} - \nabla p + \mu \nabla^2 \mathbf{u} \qquad \text{(momentum)}$$

7.2 Boundary layer terms

7.2.1 Reynolds Numbers

$$\mathrm{Re} = \frac{\rho U l}{\mu}$$

7.2.2 Skin criction coefficient

$$C_f(x) = \frac{\tau_w(x)}{\frac{1}{2}\rho U^2} = \frac{2}{\sqrt{\mathrm{Re}}}\left(\frac{\partial u^*}{\partial \bar{y}}\right)_w$$

7.2.3 Drag coefficient

$$\begin{aligned} C_D &= \frac{F_D}{\frac{1}{2}\rho U^2 A} \\ &= b\int_{\mathrm{LE}}^{\mathrm{TE}} \left(-p_u \sin\theta + \tau_u \cos\theta\right) \mathrm{d}S_u \\ &+ b\int_{\mathrm{LE}}^{\mathrm{TE}} \left(p_l \sin\theta + \tau_l \cos\theta\right) \mathrm{d}S_l \end{aligned}$$

where "u", "l", and "b"
denote "upper", "lower", and "width"

7.2.4 Separation point

$$\left(\frac{\partial u}{\partial y}\right)_{-} y = 0$$

7.2.5 Displacement thickness

$$\delta^* = \int_{y=0}^{\infty} \left(1 - \frac{u}{U_e}\right) dy$$

7.2.6 Momentum thickness

$$\theta = \int_{y=0}^{\infty} \frac{u}{U_e}\left(1 - \frac{u}{U_e}\right)$$

7.2.7 Shape factor

$$H = \frac{\delta^*}{\theta}$$

7.3 Blasius equation

Assumptions:

- Incompressible
- Laminar
- Steady
- $\frac{\partial p}{\partial x} = 0$

7.3.1 Equation

$$f'' + \tfrac{1}{2} f f'' = 0 \qquad \text{where} \qquad \eta = \sqrt{\nu x / U}$$

7.3.2 Shear stress

$$\tau_w = \mu\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)_{y=0} = 0.332 \rho U^2 / \sqrt{\mathrm{Re}_x}$$

7.3.3 Skin friction coefficient

$$C_f \equiv \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.664}{\sqrt{\mathrm{Re}_x}}$$

7.3.4 Drag on plate

$$F_D = \int_0^L \tau_w \mathrm{d}x = \frac{0.664 \rho U^2 L}{\sqrt{\mathrm{re}_L}}$$

(drag force per unit width)

7.3.5 Drag coefficient

$$C_D = \frac{F_D}{\frac{1}{2}\rho U^2 L} = \frac{1.33}{\sqrt{\mathrm{Re}_L}}$$

7.4 Pohlhausen's Polynomial

$$\frac{u(\eta)}{U_\infty} = \left(2 + \tfrac{1}{6}\Lambda\right)\eta - \tfrac{1}{2}\Lambda\eta^2 + \left(\tfrac{1}{2}\Lambda - 2\right)\eta^3 + \left(1 - \tfrac{1}{6}\Lambda\right)\eta^4$$

$$\text{where:} \qquad \eta = y/\delta \qquad \Lambda = \frac{\rho\delta^2}{\mu}\frac{\mathrm{d}U}{\mathrm{d}x}$$

7.5 Von Karman momentum integral equation

$$\tau_w = \rho\left[\frac{d}{dx}\left(U_e^2\theta\right) + U_e\delta^*\frac{dU_e}{dx}\right]$$

$$\tau_w = \mu\left.\frac{\partial u}{\partial y}\right|_{y\to 0}$$

solve for $\delta(x)$ (boundary layer thickness)

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2}$$ (skin friction coefficient)

$$\mathrm{Re} = \frac{\rho U\ell}{\mu} \quad \text{and} \quad \mathrm{Re}_x = \frac{\rho U x}{\mu}$$

(Reynolds number and x-Reynolds number)

7.6 Thwaites' method

For solving Von Karmar mom. int. eq.:

$$\theta^2 = \frac{0.45\nu}{U_e^2(x)}\int_0^x U_e^5(x')\,\mathrm{d}x' + \frac{\theta_0^2 U_0^6}{U_e^6(x)}$$

$$\lambda \equiv \frac{\theta^2}{\nu}\frac{\mathrm{d}U_e}{\mathrm{d}x}$$ (solve for θ)

$$\tau \equiv \mu\frac{U_e}{\theta}l(\lambda)$$

$$H(\lambda) = \frac{\delta^*}{\theta}$$

(use table to find surface shear stress and displacement thickness)

8 Compressible Flow

8.1 Governing equations

$$\frac{\partial}{\partial t}\oiiint_V \rho \mathrm{d}V + \oiint_S \rho \mathbf{u}\cdot \mathrm{d}\mathbf{S} = 0 \qquad \text{(continuity)}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0$$

$$\frac{\partial}{\partial t}\oiiint \rho \mathbf{u}\mathrm{d}V + \oiint \rho\left(\mathbf{u}\cdot\mathrm{d}\mathbf{S}\right)\mathbf{u} = \oiiint \rho \mathbf{F}\mathrm{d}V - \oiint p\mathrm{d}\mathbf{S} \qquad \text{(momentum)}$$

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = \rho\mathbf{F} - \nabla p$$

8.2 General Definitions

8.2.1 Mach number

$$M \equiv U/c$$

$$\text{where} \qquad c^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_s$$

8.2.2 Compressibility

$$\tau = -\frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}p} = \frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}p}$$

8.2.3 Isothermal compressibility

$$\tau_T = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T$$

8.2.4 Isentropic compressibility

$$\tau_s = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_s$$

8.3 Perfect Gas Thermodynamic Relations

8.3.1 Internal Energy/Enthalpy/Thermal eq. of state

$e = c_\mathrm{v}T$ (internal energy)

$h = c_\mathrm{p}T$ (enthalpy)

$p = \rho RT$ (thermal equation of state)

8.3.2 Specific heats

$$c_\mathrm{v} = \frac{R}{\gamma-1} \qquad c_\mathrm{p} = \frac{\gamma R}{\gamma - 1} \qquad \gamma = c_\mathrm{p}/c_\mathrm{v}$$

8.3.3 Speed of sound

$$c = \sqrt{\gamma RT} = \sqrt{\gamma p/\rho}$$

8.3.4 Entropy change

$$s_2 - s_1 = c_\mathrm{p}\ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right) = c_\mathrm{v}\ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{\rho_2}{\rho_1}\right)$$

8.3.5 Isentropic relations

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma}$$

$$\frac{T_2}{T_2} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma-1} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}$$

8.3.6 Important air properties

$$R = 287\ \mathrm{m}^2/\left(\mathrm{s}^2\mathrm{K}\right)$$

$$c_\mathrm{v} = 717\mathrm{m}^2/\left(\mathrm{s}^2\mathrm{K}\right)$$

$$c_\mathrm{p} = 1004\mathrm{m}^2/\left(\mathrm{s}^2\mathrm{K}\right)$$

$$\gamma = 1.40$$

8.4 One-dimensional isentropic flow

8.4.1 Continuity

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2$$

8.4.2 Stagnation enthalpy

$$h_0 = h + \tfrac{1}{2}u^2 = \text{const.}$$

8.4.3 Stagnation property relations

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2$$
$$\frac{p_0}{p} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma - 1}}$$
$$\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{1}{\gamma - 1}}$$

8.4.4 Nozzle area relation

$$\frac{A}{A*} = \frac{1}{M}\left[\frac{2}{\gamma + 1}\left(1 + \frac{\gamma - 1}{2} M^2\right)\right]^{\frac{\gamma + 1}{2(\gamma - 1)}}$$

8.4.5 Velocity-area differential relationship

$$\frac{\mathrm{d}u}{u} = -\frac{1}{1 - M^2}\frac{\mathrm{d}A}{A}$$

8.5 Normal shock waves

8.5.1 Mach number, pressure, temperature, and density relations across a shock

$$M_2^2 = \frac{(\gamma - 1) M_1^2 + 2}{2\gamma M_1^2 + 1 - \gamma}$$
$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}\left(M_1^2 - 1\right)$$
$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1) M_1^2}{(\gamma - 1) M_1^2 + 2}$$
$$\frac{T_2}{T_1} = 1 + \frac{2(\gamma - 1)}{(\gamma + 1)^2}\frac{\gamma M_1^2 + 1}{M_1^2}\left(M_1^2 - 1\right)$$

8.5.2 Entropy change

$$\frac{s_2 - s_1}{c_\mathrm{v}} = \ln\left\{\frac{p_2}{p_1}\left(\frac{\rho_1}{\rho_2}\right)^{\gamma}\right\}$$
$$= \ln\left\{\left[1 + \frac{2\gamma}{\gamma - 1}\left(M_1^2 - 1\right)\right]\left[\frac{(\gamma - 1) M_2^1 + 2}{(\gamma + 1) M_1^2}\right]^{\gamma}\right\}$$

8.6 Wind Tunnel equations

8.6.1 Mass flow rate

$$\dot{m} = \frac{\rho_0 A^*}{T_0}\sqrt{\frac{\gamma}{R}\left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma + 1}{\gamma - 1}}}$$

8.6.2 Nozzle/Diffuser area ratio

$$\frac{A_2^*}{A_1^*} = \frac{p_0}{p_0'}$$
$$= \left[1 + \frac{2\gamma}{\gamma + 1}\left(M_1^2 - 1\right)\right]^{\frac{1}{\gamma - 1}}\left[\frac{(\gamma - 1) M_1^2 + 2}{(\gamma + 1) M_1^2}\right]^{\frac{\gamma}{\gamma - 1}}$$

where A_2^*, A_1^*, p_0, p_0', and M_1 are the diffuser throat area, inflow nozzle throat area, stagnation pressure behind the shock, stagnation pressure after the shock, and Mach number upstream of the shock

8.7 Two-dimensional compressible flow

8.7.1 Definitions

$$\delta = \text{deflection angle}$$
$$\sigma = \text{wave angle}$$
$$M_1 = \text{upstream Mach number}$$
$$M_2 = \text{downstream Mach number}$$

8.7.2 Mach wave angle

$$\sin\mu = 1/M$$

8.7.3 Mach number relation

$$M_2^2 \sin^2(\sigma - \delta) = \frac{(\gamma - 1) M_1^2 \sin^2\sigma + 2}{2\gamma M_1^2 \sin^2\sigma + 1 - \gamma}$$

8.7.4 Oblique shockwave angle

$$\tan\delta = 2\cot\sigma\frac{M_1^2 \sin^2\sigma - 1}{M_1^2(\gamma - \cos 2\sigma) + 2}$$

8.7.5 Prandtl-Meyer function (expansion fans)

$$\delta_1 + \nu(M_1) = \delta_2 + \nu(M_2) = \text{const.}$$
$$\nu(M) = \sqrt{\frac{\gamma + 1}{\gamma - 1}}\arctan\sqrt{\frac{\gamma - 1}{\gamma + 1}\left(M^2 - 1\right)} - \arctan\sqrt{M^2 - 1}$$

8.8 Thin-airfoil theory

8.8.1 Lift and Drag coefficient

$$\begin{aligned} C_L &= \frac{L}{\frac{1}{2}\rho_\infty U_\infty^2 b} \\ &\approx \frac{4\alpha}{\sqrt{M_\infty^2 - 1}} \\ C_D &= \frac{D}{\frac{1}{2}\rho_\infty U_\infty^2 b} \\ &\approx \frac{4\alpha^2}{\sqrt{M_\infty^2 - 1}} \end{aligned}$$

8.9 Velocity Potential

$$\nabla^2\phi = \tfrac{1}{2}M_0^2\left[(\gamma - 1)\,\nabla^2\phi\left(\nabla\phi\cdot\nabla\phi\right) + \nabla\phi\cdot\nabla\left(\nabla\phi\cdot\nabla\phi\right)\right]$$

where

$$a^2 = a_0^2 - \frac{\gamma - 1}{2}V^2 = a_0^2 - \frac{\gamma - 1}{2}\left(u^2 + v^2 + w^2\right)$$

9 Rocket Propulsion Equations

9.1 Definitions and Fundamentals

9.1.1 Total impulse

$$I_t = \int_0^t F \mathrm{d}t$$
$$= Ft \quad \text{(for constant thrust)}$$

9.1.2 Specific impulse

$$\begin{aligned} I_s &= I_t / \left(m_p g_0\right) \\ &= F / \left(\dot{m} g_0\right) \\ &= F / \dot{w} \\ &= I_t / w \end{aligned}$$

9.1.3 Effective exhaust velocity

$$c = I_s g_0 = F / \dot{m}$$

9.1.4 Mass ratio MR

$$m_f / m_0$$

9.1.5 Propellant mass fraction ζ

$$\begin{aligned} \zeta &= m_p / m_0 \\ &= \left(m_0 - m_f\right) = m_p / \left(m_p + m_f\right) \end{aligned}$$

where $m_0 = m_p + m_f$

9.1.6 Impulse-to-weight Ratio

$$\frac{I_t}{w_0} = \frac{I_t}{\left(m_f + m_p\right) g_0} = \frac{I_s}{m_f / m_p + 1}$$

9.1.7 Thrust F

$$\begin{aligned} F &= \frac{\mathrm{d}\left(m v_2\right)}{\mathrm{d}t} = \dot{m} v_2 \quad \text{at sea level} \quad = \frac{\dot{w}}{g_0} v_2 \\ &= \dot{m} v_2 + \left(p_2 - p_3\right) A_2 \\ &= \dot{m} c \\ &= \dot{m} v_2 + p_2 A_2 \quad \text{(in vacuum of space)} \\ &= F_{\text{opt}} + p_1 A_t \left(\frac{p_2}{p_1} - \frac{p_3}{p_1}\right) \epsilon \\ &= \frac{A_t v_t v_2}{V_t} + \left(p_2 - p_3\right) A_2 \\ &= A_t p_1 \sqrt{\frac{2\gamma^2}{\gamma - 1} \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{\gamma-1}} \left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right]} + \left(p_2 - p_3\right) A_2 \\ &= C_F A_t p_1 \\ &= \dot{m} c^* C_F \end{aligned}$$

9.1.8 Thrust Coefficient

$$\begin{aligned} C_F &= \frac{F}{p_1 A_t} \\ &= \sqrt{\frac{2\gamma^2}{\gamma - 1} \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{\gamma-1}} \left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right]} + \frac{p_2 - p_3}{p_1} \frac{A_2}{A_t} \end{aligned}$$

9.1.9 Exhaust velocity c

$$c = v_2 + \left(p_2 - p_3\right) A_2 / \dot{m} = I_s g_0$$

9.1.10 Characteristic velocity c^*

$$\begin{aligned} c^* &= p_1 A_t / \dot{m} \\ &= \frac{I_s g_0}{C_f} \\ &= \frac{c}{C_F} \\ &= \frac{1}{\gamma} \sqrt{\frac{\gamma R T_1}{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}} \end{aligned}$$

9.1.11 Power of the jet P_{jet}

$$P_{\text{jet}} = \tfrac{1}{2} \dot{m} v_2^2 = \tfrac{1}{2} \dot{w} g_0 I_s^2 = \tfrac{1}{2} F g_0 I_s = \tfrac{1}{2} F v_2$$

9.1.12 Power input to a chemical engine

$$P_{\text{chem}} = \dot{m} Q_R$$

where Q_R is the heat of combustion per unit of propellant mass.

9.1.13 Power transmitted to vehicle

$$P_{\text{vehicle}} = F u \qquad \text{(where } u \text{ is vehicle velocity)}$$

9.1.14 Internal efficiency η_{p}

$$\eta_{\text{int}} = \frac{\text{kinetic power in jet}}{\text{available chemical power}} = \frac{\tfrac{1}{2} \dot{m} v^2}{\eta_{\text{comp}} P_{\text{chem}}}$$

9.1.15 Propulsive efficiency

$$\begin{aligned} \eta_p &= \frac{\text{vehicle power}}{\text{vehicle power + residual kinetic jet power}} \\ &= \frac{F u}{F u + \tfrac{1}{2} \dot{m} \left(c + u\right)^2} = \frac{2 u / c}{1 + \left(u / c\right)^2} \end{aligned}$$

9.1.16 Multiple propulsion systems

$$
\begin{aligned}
F_{\text{oa}} &= \sum F = F_1 + F_2 + F_3 + \cdots \\
\dot{m}_{\text{oa}} &= \sum \dot{m} = \dot{m}_1 + \dot{m}_2 + \dot{m}_3 + \cdots \\
(I_s)_{\text{oa}} &= F_{\text{oa}} / \left(g_0 \dot{m}_{\text{oa}}\right)
\end{aligned}
$$

9.2 Nozzle theory and thermodynamic relations

9.2.1 Stagnation enthalpy

$$h_0 = h + \frac{1}{2} v^2 = \text{const.}$$

9.2.2 Perfect gas law

$$p_x V_x = R T_x$$

9.2.3 Specific heats c and their γ

$$
\begin{aligned}
k &= c_p / c_v \\
c_p - c_v &= R \\
c_p &= \gamma R / (\gamma - 1)
\end{aligned}
$$

9.2.4 Isentropic flow between x and y in nozzle

$$T_x / T_y = \left(p_x / p_y\right)^{\frac{\gamma-1}{\gamma}} = \left(V_y / V_x\right)^{\gamma - 1}$$

9.2.5 Stagnation temperature

$$T_0 = T + \tfrac{1}{2} v^2 / c_p$$

9.2.6 Speed of sound & Mach number

$$
\begin{aligned}
a &= \sqrt{\gamma R T} \\
M &= v / a = v / \sqrt{\gamma R T}
\end{aligned}
$$

9.2.7 Isentropic relations (temperature, Mach number, pressure)

$$
\begin{aligned}
T_0 &= T \left[1 + \tfrac{1}{2} (\gamma - 1) M^2\right] \\
M &= \sqrt{\frac{2}{\gamma - 1} \left(\frac{T_0}{T} - 1\right)} \\
p_0 &= p \left[1 + \tfrac{1}{2} (\gamma - 1) M^2\right]^{\frac{\gamma}{\gamma - 1}}
\end{aligned}
$$

9.2.8 General area ratio

$$\frac{A_y}{A_x} = \frac{M_x}{M_y} \sqrt{\left[\frac{1 + M_y^2 (\gamma - 1)/2}{1 + M_x^2 (\gamma - 1)/2}\right]^{\frac{\gamma+1}{\gamma-1}}}$$

9.2.9 Exit velocity

$$
\begin{aligned}
v_2 &= \sqrt{2 (h_1 - h_2) + v_1^2} \\
&= \sqrt{\frac{2\gamma}{\gamma - 1} R T_1 \left[1 - \left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma}\right] + v_1^2} \\
&= \sqrt{\frac{2\gamma}{\gamma - 1} R T_1 \left[1 - \left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma}\right]} \quad (\text{when } v_1 \approx 0 (\text{chamber})) \\
&= \sqrt{\frac{2\gamma}{\gamma - 1} \frac{R' T_1}{\mathbb{M}} \left[1 - \left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma}\right]}
\end{aligned}
$$

where $\mathbb{M}$ is molecular mass.

9.2.10 Critical/Throat velocity

$$v_t = \sqrt{\frac{2\gamma}{\gamma + 1} R T_1} = \sqrt{\gamma R T_t} = a_t$$

9.2.11 Mass flow rate

$$\dot{m} = \frac{A_t v_t}{V_t} = A_t p_1 \gamma \sqrt{\frac{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}{\gamma R T_1}}$$

9.2.12 Ratio between throat and downstream area with pressure p_y

$$\frac{A_t}{A_y} = \frac{V_t v_y}{V_y v_t} = \left(\frac{\gamma + 1}{2}\right)^{\frac{1}{\gamma-1}} \left(\frac{p_y}{p_1}\right)^{1/\gamma} \sqrt{\frac{\gamma + 1}{\gamma - 1} \left[1 - \left(\frac{p_y}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right]}$$

9.2.13 Velocity of downstream point from throat

$$\frac{v_y}{v_t} = \sqrt{\frac{\gamma + 1}{\gamma - 1}}$$

9.2.14 Cone-shaped nozzle correction factor for exhaust velocity

$$\gamma = \tfrac{1}{2} (1 + \cos \alpha) \qquad \alpha = 15^o \text{ cone is accepted standard}$$

where α is the cone angle.

9.2.15 Rocket Equation

$$e^{\Delta u / c} = 1 / \mathbf{MR} = m_0 / m_f$$

9.3 Chemical Rocket Performance Analysis

9.3.1 Mol fraction

$$X_j = \frac{n_j}{n} \quad \text{and} \quad n = \sum_{j=1}^{m} n_j$$

9.3.2 Effective average molecular mass

$$\mathbb{M} = \frac{\sum_{j=1}^{m} n_j \mathbb{M}_j}{\sum_{j=1}^{m} n_j}$$

9.3.3 Molar specific heat and specific heat ratio

$$\left(C_p\right)_{\text{mix}} = \frac{\sum_{j=1}^{m} n_j \left(C_p\right)_j}{\sum_{\text{j=1}}^{m} n_j}$$

$$\gamma_{\text{mix}} = \frac{\left(C_p\right)_{\text{mix}}}{\left(C_p\right)_{\text{mix}} - R'}$$

9.3.4 Heat of reaction

$$\Delta_r H^0 = \sum \left[n_j \left(\Delta_f H^0\right)\right]_{\text{products}} - \sum \left[n_j \left(\Delta_f H^0\right)_j\right]_{\text{reactants}}$$

9.3.5 Gibbs free energy

$$G = \sum_{j=1}^{m} n_j G_j$$

$$G_j = U_j + p_j V_j - T_j S_j = h_j - T_j S_j$$

$$\Delta_r G^0 = \sum_{j=1}^{m} \left[n_j \left(\Delta_f G^0\right)\right]_{\text{products}} - \sum_{j=1}^{r} \left[n_j \left(\Delta_f G^0\right)_j\right]_{\text{reactants}}$$

9.3.6 Heat of reaction

$$\Delta_r H = \sum_{1}^{m} n_j \int_{T_{\text{ref}}}^{T_1} C_{pj} \mathrm{d}T = \sum_{1}^{m} n_J \Delta h_j \Big|_{T_{\text{ref}}}^{T_1}$$

9.4 Solid Propellant Rocket Motor Fundamentals

9.4.1 Mass flow rate

$$\begin{aligned} \dot{m} &= A_b r \rho_b \\ &= \frac{\mathrm{d}\left(\rho_1 V_1\right)}{\mathrm{d}t} + \frac{A_t p_1}{c^*} \\ &= A_b \rho_b a p_1^n \end{aligned}$$

where A_b, r, and ρ_b are the burn area, burn rate, and density of solid propellant before burning

9.4.2 Pressure in steady burning

$$p_1 = K \rho_b r c^* \quad \text{where} \quad K \equiv A_b / A_t$$

9.4.3 Burning rate approximation

$$r = a p_1^n$$

10 Aeroacoustics

10.1 Governing Equations

$$\frac{\partial p'}{\partial t} + \rho_0 \nabla \cdot \mathbf{u}' = 0 \qquad \text{(continuity)}$$
$$\rho_0 \frac{\partial \mathbf{u}'}{\partial t} = -\nabla p' \qquad \text{(momentum)}$$

10.2 General definitions

$$k = \frac{2\pi}{\lambda} = \frac{\omega}{a_0} = \frac{2\pi f}{a_0} = \frac{2\pi}{a_0 T} \qquad \text{(waves/distance)}$$
$$\lambda = \frac{2\pi}{k} = \frac{\omega}{2\pi a_0} = \frac{a_0}{f} = a_0 T \qquad \text{(distance/wave)}$$
$$\omega = a_0 k = \frac{2\pi a_0}{\lambda} = 2\pi f = \frac{2\pi}{T} \qquad \text{(radians/second)}$$
$$a_0 = \frac{\omega}{k} = \lambda f = \frac{2\pi f}{k} = \frac{2\pi}{kT} \qquad \text{(distance/time)}$$
$$f = \frac{a_0 k}{2\pi} = \frac{a_0}{\lambda} = \frac{\omega}{2\pi} = \frac{1}{T} \qquad \text{(wavelength/time)}$$
$$T = \frac{a_0 k}{2\pi} = \frac{\lambda}{a_0} = \frac{2\pi}{\omega} = \frac{1}{f} \qquad \text{(time/wavelength)}$$

10.2.1 D'Alembert Solution (plane wave)

$$f(x,t) = F(x - a_0 t) + G(x + a_0 t)$$

10.2.2 Fourier Series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi n t}{T}\right) + b_n \sin\left(\frac{2\pi n t}{T}\right)\right]$$
$$a_0 = \frac{2}{T}\int_0^T f(t)\,\mathrm{d}t$$
$$a_n = \frac{2}{T}\int_0^T f(t)\cos\left(\frac{2\pi n t}{T}\right)\mathrm{d}t$$
$$b_n = \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2\pi n t}{T}\right)\mathrm{d}t$$
$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi n t}{T}}$$
$$c_n = \frac{1}{T}\int_0^T f(t) e^{i\frac{2\pi n t}{T}}\,\mathrm{d}t$$
$$c_n = \begin{cases} \dfrac{a_{-n} + i b_{-n}}{2}; & n \leq -1 \\ \dfrac{a_0}{2} & n = 0 \\ \dfrac{a_n - i b_n}{2} & n \geq 1 \end{cases}$$

10.2.3 Fourier Transform

$$\bar{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty} f(t) e^{-i\omega t} \qquad \text{(fourier transform)}$$
$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty} \bar{f}(\omega) e^{i\omega t}\mathrm{d}\omega \qquad \text{(inverse fourier transform)}$$

10.2.4 Acoustic Intensity

$$\mathbf{I} = p'\mathbf{u}'$$
$$\langle I \rangle = \frac{A^2}{2\rho_0 a_0} \qquad \text{Units: } \left[\frac{\mathrm{W}}{\mathrm{m}^2}\right]$$

10.2.5 Acoustic Energy

$$e = \frac{p'^2}{2\rho_0 a_0^2} + \frac{1}{2}\rho_0 \mathbf{u}'^2$$
$$\langle e \rangle = \frac{A^2}{2\rho_0 a_0^2} \qquad \text{Units: } \left[\frac{\mathrm{J}}{\mathrm{m}^2}\right] \qquad \text{(energy density)}$$

10.2.6 Root of Square Average (RMS)

$$p'_{\mathrm{rms}} = \sqrt{\frac{1}{T}\int_0^T p(t)'^2 \mathrm{d}t}$$
$$= \frac{A}{\sqrt{2}} \qquad (\text{for } p(t)' = A\cos t)$$

10.2.7 Decibel

$$10\log_{10}\left(\frac{\text{Power}}{\text{Ref power}}\right)$$

10.2.8 Acoustic Power Level

$$\mathrm{APL} = 10\log_{10}\left(\frac{\mathbb{P}_{\mathrm{acoustic}}}{\mathbb{P}_{\mathrm{ref}}}\right) \quad [\mathrm{dB}]$$
$$\mathbb{P} = \mathbb{P}_{\mathrm{ref}} 10^{\mathrm{APL}/10}$$

where $\mathbb{P}_{\mathrm{ref}} = 10^{-12}$ [W] and $\mathbb{P} = \langle I \rangle A$

where A is the area the sound is going through

10.2.9 Sound Pressure Level

$$\mathrm{SPL} = 10\log_{10}\left(\frac{p'^2_{\mathrm{rms}}}{p^2_{\mathrm{ref}}}\right)$$
$$= 20\log_{10}\left(\frac{p_{\mathrm{rms}}}{p_{\mathrm{ref}}}\right)$$
$$p_{\mathrm{ref}} = 2 \times 10^{-5}\ [\mathrm{Pa}]$$

10.2.10 RMS pressure summation

$$(p_{\mathrm{rms}})_{1+2} = \sqrt{\left(p^2_{\mathrm{rms}}\right)_1 + \left(p^2_{\mathrm{rms}}\right)_2} \qquad (1)$$

10.2.11 SPL summation

$$\mathrm{SPL}_{1+2+\cdots+n} = 10\log_{10}\left(10^{\mathrm{SPL}_1/10} + 10^{\mathrm{SPL}_2/10} + \cdots + 10^{\mathrm{SPL}_n/10}\right) \qquad (2)$$

10.2.12 Impedence

$$
\begin{aligned}
Z &= i\sigma\omega \\
&= \rho_0 a_0 \\
&= \frac{p'}{u'} \\
m/A &= \sigma \quad \text{(mass/area)}
\end{aligned}
$$

10.2.13 Planar wave transmission & reflection

$$
\begin{aligned}
p_t &= \frac{2\rho_0 a_0}{2\rho_0 a_0 + i\sigma\omega} p_i && \text{(transmission)} \\
p_r &= \frac{i\sigma\omega}{2\rho_0 a_0 + i\sigma\omega} p_i && \text{(reflection)} \\
&\text{where} \quad \sigma = m/A
\end{aligned}
$$

10.2.14 Reflection/Transmission coefficient

$$
\begin{aligned}
\alpha_\mathrm{r} &\equiv \frac{\langle I \rangle_\mathrm{r}}{\langle I \rangle_\mathrm{i}} \\
\alpha_\mathrm{t} &\equiv \frac{\langle I \rangle_\mathrm{t}}{\langle I \rangle_\mathrm{i}}
\end{aligned}
$$

10.2.15 Composite surface transmission coefficient

$$
\alpha_{\mathrm{t}_\mathrm{eff}} = \frac{\sum_{j=1}^{n} \alpha_{\mathrm{t}_j} A_j}{A_\mathrm{T}} \quad \text{where} \quad A \text{ is surface area}
$$

10.2.16 Transmission loss

$$
\begin{aligned}
\text{T.L.} &= 10\log\left(\frac{1}{\alpha_\mathrm{t}}\right) \\
&= 10\log\left[\frac{(\sigma\omega)^2}{(2\rho_0 a_0)^2}\right] + 10\log\left[1 + \frac{(2\rho a_0)^2}{(\sigma\omega)^2}\right]
\end{aligned}
$$

10.2.17 Spherical wave equation

$$
p = \frac{A}{r} e^{i(\omega t - kr)}
$$

10.2.18 Spherical Sound Pressure Level

$$
\begin{aligned}
SPL &= 20\log\left(\frac{p_\mathrm{rms}}{p_\mathrm{ref}}\right) \\
&= 20\log\left(\frac{A}{\sqrt{2} p_\mathrm{ref}}\right) - 20\log(r) \\
\Delta_\mathrm{SPL} = \mathrm{SPL}_1 - \mathrm{SPL}_2 &= 20\log\left(\frac{r_2}{r_1}\right)
\end{aligned}
$$

10.2.19 Spherical acoustic velocity

$$
\begin{aligned}
u_r' &= \frac{p'(t)}{\rho_0 a_0}\left(1 + \frac{1}{ikr}\right) \\
&= \frac{A}{\rho_0 a_0 r}\left(1 + \frac{1}{ikr}\right) e^{i(\omega t - kr)}
\end{aligned}
$$

10.2.20 Reverberant Field energy density

$$
\begin{aligned}
\langle e \rangle_\mathrm{RF} &= \frac{4\pi p_\mathrm{rms}^2}{\rho_0 a_0^2} \\
&= \frac{2\pi A^2}{\rho_0 a_0^2}
\end{aligned}
$$

10.2.21 Reverberant field intensity

$$
\begin{aligned}
\langle I \rangle_\mathrm{RF} &= \frac{\pi A^2}{2\rho_0 a_0} \\
&= \frac{\langle e \rangle_\mathrm{RF}\, a_0}{4}
\end{aligned}
$$

10.2.22 Absorption

$$
\begin{aligned}
\langle I \rangle_\mathrm{RF} &= \frac{\mathbb{P}(1 - \alpha_\mathrm{A})}{\alpha_\mathrm{A} A_\mathrm{wall}} = \frac{\mathbb{P}}{R} \\
\alpha_\mathrm{A} &= \frac{\langle I \rangle_\mathrm{a}}{\langle I \rangle_\mathrm{i}} = \frac{\langle I \rangle_\mathrm{i} - \langle I \rangle_\mathrm{r}}{\langle I \rangle_i}
\end{aligned}
$$

10.2.23 Reverberant Field SPL

$$
\begin{aligned}
\mathrm{SPL}_\mathrm{RF} &= 10\log_{10}\left[\frac{\left(p_\mathrm{rms}^2\right)_\mathrm{RF}}{p_\mathrm{ref}^2}\right] \\
&= 10\log\left(\frac{4\rho_0 a_0 \mathbb{P}}{R p_\mathrm{ref}^2}\right) \\
R &= \frac{A_\mathrm{wall}\alpha_\mathrm{A}}{(1 - \alpha_\mathrm{A})}
\end{aligned}
$$

10.2.24 Direct field and Total SPL

$$
\mathrm{SPL}_\mathrm{total} = 10\log_{10}\left(\frac{\rho_0 a_0 \mathbb{P}}{p_\mathrm{ref}^2}\right) + 10\log_{10}\left(\frac{1}{4\pi r^2} + \frac{4}{R}\right)
$$

10.2.25 Average absorption coefficient

$$
\alpha_\mathrm{A} = \frac{\sum \alpha_{\mathrm{A}_i} A_i}{\sum A_i}
$$

10.2.26 Average SPL or "equivalent level"

$$
\mathrm{SPL}_\mathrm{AVG} = L_\mathrm{EQ} = 10\log_{10}\left(\frac{1}{T}\sum_{i=1}^{N} 10^{\mathrm{SPL}_i/10}\Delta t_i\right)
$$

10.2.27 Day/Night level

$$
L_\mathrm{DN} = 10\log_{10}\left[\frac{1}{T}\left(\sum_{0700}^{2200} 10^{\mathrm{SPL(t)}/10}\Delta t_i + \sum_{2200}^{0700} 10^{\frac{\mathrm{SPL(t)}+10}{10}}\Delta t_i\right)\right]
$$

10.2.28 Permissible time & dose

$$T = \frac{8}{2^{\frac{L-90}{5}}}$$

$$D = \sum_{i=1}^{N} \frac{A_i}{T} = \frac{1}{8} \sum_{i=1}^{N} \Delta t_i 2^{\frac{L_i-90}{5}} \leq 1$$

A_i is the actual time

T is the permissible time (8 hours at 90 dB)

11 Flight Dynamics

11.1 Introduction and Notation

Particle kinematics in rotating frame

$$\begin{aligned}
\mathbf{A} &= \mathbf{A}(t) \\
\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} &= \frac{\delta\mathbf{A}}{\delta t} + \boldsymbol{\omega}\times\mathbf{A} \\
\frac{\mathrm{d}^2\mathbf{A}}{\mathrm{d}t^2} &= \frac{\delta^2\mathbf{A}}{\delta t^2} + 2\boldsymbol{\omega}\times\frac{\delta\mathbf{A}}{\delta t} + \frac{\delta\boldsymbol{\omega}}{\delta t}\times\mathbf{A} + \boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{A} \\
\frac{\mathrm{d}^3\mathbf{A}}{\mathrm{d}t^3} &= \frac{\delta^3\mathbf{A}}{\delta t^3} + 3\left(\frac{\delta\boldsymbol{\omega}}{\delta t}\times\frac{\delta\mathbf{A}}{\delta t} + \boldsymbol{\omega}\times\frac{\delta^2\mathbf{A}}{\delta t^2} + \boldsymbol{\omega}\times\boldsymbol{\omega}\times\frac{\delta\mathbf{A}}{\delta t}\right) \\
&\quad + \frac{\delta^2\boldsymbol{\omega}}{\delta t^2}\times\mathbf{A} + \boldsymbol{\omega}\times\frac{\delta\boldsymbol{\omega}}{\delta t}\times\mathbf{A} + \boldsymbol{\omega}\times\boldsymbol{\omega}\times\boldsymbol{\omega}\times\mathbf{A}
\end{aligned}$$

where $\mathbf{A}$ and $\boldsymbol{\omega}$ are the vector and angular velocity vector in the rotating frame.

Rigid body equations of motion (body-fixed frame)

$$\begin{aligned}
\dot{u} &= F_x/m - g\sin\theta + rv - qw \\
\dot{v} &= F_y/m + g\sin\phi\cos\theta - ru + pw \\
\dot{w} &= F_Z/m + g\cos\phi\cos\theta + qu - pv \\
\dot{p} &= \frac{I_{zz}M_x + I_{xz}M_z + I_{xz}\left(I_{xx} - I_{yy} + I_{zz}\right)pq - \left(I_{xz}^2 - I_{yy}I_{zz} + I_{zz}^2\right)qr}{I_{xx}I_{zz} - I_{xz}^2} \\
\dot{q} &= \frac{M_y - I_{xz}p^2 + (I_{zz} - I_{xx})\,pr + I_{xz}r^2}{I_{yy}} \\
\dot{r} &= \frac{I_{xz}M_x + I_{xx}M_z + \left(I_{xx}^2 + I_{xz}^2 - I_{xx}I_{yy}\right)pq - I_{xz}\left(I_{xx} - I_{yy} + I_{zz}\right)qr}{I_{xx}I_{zz} - I_{xz}^2}
\end{aligned}$$

Where u, v, w, p, q, r, ϕ, θ, and ψ are the velocities in the x, y, and z directions and the angular accelerations in x, y, and z directions.

Aerodynamic Forces and Moments:

$$\begin{aligned}
C_D &\equiv \text{drag coefficient} \equiv \frac{D}{\frac{1}{2}\rho_\infty V_\infty^2 S} \\
C_L &\equiv \text{lift coefficient} \equiv \frac{L}{\frac{1}{2}\rho_\infty V_\infty^2 S} \\
C_A &\equiv \text{axial force coefficient} \equiv \frac{A}{\frac{1}{2}\rho_\infty V_\infty^2 S} \\
C_N &\equiv \text{normal force coefficient} \equiv \frac{N}{\frac{1}{2}\rho_\infty V_\infty^2 S} \\
C_m &\equiv \text{pitching moment coefficient} \equiv \frac{m}{\frac{1}{2}\rho_\infty V_\infty^2 Sc}
\end{aligned}$$

Aerodynamic Forces and Moments per unit span (2-D):

$$\begin{aligned}
\tilde{C}_D &\equiv \text{drag coefficient} \equiv \frac{\tilde{D}}{\frac{1}{2}\rho_\infty V_\infty^2 c} \\
\tilde{C}_L &\equiv \text{lift coefficient} \equiv \frac{\tilde{L}}{\frac{1}{2}\rho_\infty V_\infty^2 c} \\
\tilde{C}_A &\equiv \text{axial force coefficient} \equiv \frac{\tilde{A}}{\frac{1}{2}\rho_\infty V_\infty^2 c} \\
\tilde{C}_N &\equiv \text{normal force coefficient} \equiv \frac{\tilde{N}}{\frac{1}{2}\rho_\infty V_\infty^2 c} \\
\tilde{C}_m &\equiv \text{pitching moment coefficient} \equiv \frac{\tilde{m}}{\frac{1}{2}\rho_\infty V_\infty^2 c^2}
\end{aligned}$$

Angle of attack relations:

$\alpha \equiv$ the angle from $\mathbf{V}_\infty$ to the chord line (positive nose up)

$$\begin{aligned}
\tilde{C}_L &= \tilde{C}_N\cos\alpha - \tilde{C}_A\sin\alpha \\
\tilde{C}_D &= \tilde{C}_A\cos\alpha + \tilde{C}_N\sin\alpha \\
\tilde{C}_N &= \tilde{C}_L\cos\alpha + \tilde{C}_D\sin\alpha \\
\tilde{C}_A &= \tilde{C}_D\cos\alpha - \tilde{C}_L\sin\alpha
\end{aligned}$$

11.2 Aircraft Performance

11.2.1 Thrust required

$$\begin{aligned}
T_R &= \frac{W}{(L/D)\cos\alpha_T + \sin\alpha_T} && \text{(Thrust)}\\
D &= T_R\cos\alpha_T && \text{(Total Drag)}\\
&= \tfrac{1}{2}\rho V^2 S_W C_D\\
&= \frac{1}{2}\rho V^2 S_W\left(C_{D_0} + C_{D_0,L}C_L + \frac{C_L^2}{\pi e R_A}\right)\\
C_{D_0} &= \frac{C_L^2}{\pi e R_A} && \text{(Induced Drag)}\\
L &= W - T_R\sin\alpha_T && \text{(Lift)}\\
C_L &= \frac{L}{\frac{1}{2}\rho V^2 S_W} = \frac{W - T_R\sin\alpha_T}{\frac{1}{2}\rho V^2 S_W} && \text{(Lift Coefficient)}\\
&= \frac{W}{\frac{1}{2}\rho V^2 S_W}\left[\frac{1}{1+(D/L)\tan\alpha_T}\right]\\
\frac{L}{D} &= \frac{W - T_R\sin\alpha_T}{T_R\cos\alpha_T} && \text{(Lift-Drag ratio)}\\
&= \frac{C_L}{C_D} = \frac{C_L}{C_{D_0} + C_{D_0,L}CL + \frac{C_L^2}{\pi e R_A}}\\
(L/D)_{\max} &= \frac{\sqrt{\pi e R_A}}{2\sqrt{C_{D_0}} + C_{D_0,L}\sqrt{\pi e R_A}} && \text{(Max Lift/Drag)}
\end{aligned}$$

$$\begin{aligned}
V_{MD} = \frac{\sqrt{2}}{(\pi e R_A C_{D_0})^{1/4}}\sqrt{\frac{W/S_W}{\rho}}\\
\times\sqrt{\left[1 + \left(2\sqrt{\frac{C_{D_0}}{\pi e R_A}} + C_{D_0,L}\right)^{-1}\tan\alpha_T\right]}\\
\text{(Minimum drag airspeed)}
\end{aligned}$$

Small-angle approximation aircraft performance

$$\begin{aligned}
T_R &= D && \text{(Thrust)}\\
&= \frac{W}{L/D}\\
&= \frac{C_D}{C_L}W\\
&= \left(\frac{C_{D_0}}{C_L} + C_{D_0,L} + \frac{C_L}{\pi e R_A}\right)W\\
&= \left(\frac{\frac{1}{2}\rho V^2 C_{D_0}}{W/S_W} + C_{D_0,L} + \frac{W/S_W}{\frac{1}{2}\pi e R_A\rho V^2}\right)W\\
L &= W && \text{(Lift)}\\
C_L &= \frac{W}{\frac{1}{2}\rho V^2 S_W} && \text{(Lift Coefficient)}\\
V_{MD} &= \left(\frac{4}{\pi e R_A C_{D_0}}\right)^{1/4}\sqrt{\frac{W/S_W}{\rho}} && \text{(Minimum drag airspeed)}
\end{aligned}$$

11.3 Longitudinal Static Stability and Trim

11.3.1 Pitch Stability of a Cambered Wing

$$\begin{aligned}
T &= D && \text{(Thrust)}\\
L &= W && \text{(Lift)}
\end{aligned}$$

Pitching moment about center of gravity

$$\begin{aligned}
m &= m_{ac} - l_w L = 0\\
C_m &= C_{m_{ac}} - \frac{l_w}{\bar{c}}C_L = 0
\end{aligned}$$

Pitch Stability

$$\frac{\partial C_m}{\partial\alpha} \equiv C_{m,\alpha} < 0$$

11.3.2 Simplified Pitch Stability Analysis for a Wing-Tail Combination

12 Classical Mechanics

12.1 Newton's Laws

12.1.1 First Law

In the absence of forces, a particle moves with constant velocity $\mathbf{V}$.

12.1.2 Second Law

$$\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} \equiv \dot{\mathbf{p}} = \frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v})$$

12.1.3 Third Law

If object 1 exerts a force $\mathbf{F}_{12}$ on object 2, then object 2 always exerts a reaction force $\mathbf{F}_{12}$ on object 1 given by

$$\mathbf{F}_{12} = -\mathbf{F}_{21}$$

12.2 Angular Momentum for a single particle

$$\mathbf{l} = \mathbf{r} \times \mathbf{p}$$

$$\dot{\mathbf{l}} = (\dot{\mathbf{r}} \times \mathbf{p}) + (\mathbf{r} \times \dot{\mathbf{p}})$$

12.3 Kinetic Energy and Work

12.3.1 Kinetic Energy

$$T = \tfrac{1}{2}mv^2$$

$$\frac{\mathrm{d}T}{\mathrm{d}t} = m\dot{\mathbf{v}} \cdot \mathbf{v} = \mathbf{F} \cdot \mathbf{v}$$

$$\mathrm{d}T = \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

12.3.2 Work

$$W_{1\to 2} = \int_1^2 \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

12.4 Rotational Motion of rigid bodies

12.4.1 Center of Mass

$$\mathbf{R} = \frac{1}{M}\oint \rho\mathbf{r}\mathrm{d}V$$
$$\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

12.4.2 Momentum

$$\mathbf{P} = M\dot{\mathbf{R}}$$

12.4.3 Angular Momentum

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$$
$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$
$$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$
$$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

12.4.4 Inertia tensor

$$I_{ik} = \oint \rho\left[(\mathbf{r}\cdot\mathbf{r})\,\delta_{ik} - x_i x_k\right]\mathrm{d}V$$
$$= \begin{bmatrix} \int \rho\left(y^2+z^2\right)\mathrm{d}V & -\int \rho xy\mathrm{d}V & -\int \rho xz\mathrm{d}V \\ -\int \rho yx\mathrm{d}V & \int \rho\left(x^2+z^2\right)\mathrm{d}V & -\int \rho yz\mathrm{d}V \\ -\int \rho zx\mathrm{d}V & -\int \rho zy\mathrm{d}V & \int \rho\left(x^2+y^2\right)\mathrm{d}V \end{bmatrix}$$
$$\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

12.5 Special Relativity

$$\gamma(v) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

Relativistic values

$$m = \gamma m_0$$
$$\mathbf{p} = \gamma m_0\mathbf{v}$$
$$E = \gamma m_0 c^2$$
$$K = m_0c^2(\gamma - 1)$$

Lorentz-Einstein transformations

$$x' = \gamma(x - vt) \qquad x = \gamma(x' + vt')$$
$$y' = y \qquad y = y'$$
$$z' = z \qquad z = z'$$
$$t' = \gamma\left(t - vx/c^2\right) \qquad t = \gamma\left(t' + vx'/c^2\right)$$

Lorentz contraction

$$l = l_0/\gamma$$

Velocity transformation

$$u_x = \frac{u'_x + v}{1 + vu'_x/c^2} \qquad u'_x = \frac{u_x - v}{1 - vu_x/c^2}$$
$$u_y = \frac{u'_y + v}{1 + vu'_y/c^2} \qquad u'_y = \frac{u_y - v}{1 - vu_y/c^2}$$

Relativistic doppler shift

$$\beta = v/c$$
$$\lambda' = \frac{\lambda}{\gamma(1 + \beta\cos\theta)}$$
$$\lambda = \lambda\gamma(1 - \beta\cos\theta')$$

13 Thermodynamics & Statistical Mechanics

13.1 Quantum Effects

Electrical Charge:

$$e = 1.602 \times 1-^{-19}\ \text{C}$$

Wavelength of particle waves:

$$\lambda = \frac{h}{p}$$

where h is Plank's constant and p is the particle's momentum.

$$p = \frac{h}{\lambda} = \frac{h}{2\pi}\frac{2\pi}{\lambda} = \hbar k$$

Uncertainty Principle:

$$\Delta x \Delta p_x = h$$
$$\Delta y \Delta p_y = h$$
$$\Delta z \Delta p_z = h$$
$$\Delta \theta \Delta L = h$$

Area of accessible states:

$$\text{number of accessible states} = \frac{\text{total area}}{\text{area of one state}} = \frac{[x][p_x]}{h}$$

In three dimensions:

$$\text{number of accessible states} = \frac{V_r V_p}{h^3} = \frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}p_x\mathrm{d}p_y\mathrm{d}p_x}{h^3}$$

Density of states for system of noninteracting particles:

$$g(\epsilon) = \frac{2\pi V(2m)^{3/2}}{h^3}\sqrt{\epsilon} \qquad \text{nonrelativistic gas}$$
$$g(\epsilon) = \frac{4\pi V}{h^3c^3}\epsilon^2 \qquad \text{massless or relativistic gas}$$

Angular momentum:

$$\mathbf{J} = \mathbf{L} + \mathbf{S}$$

where **J**, **L**, and **S** is the total angular momentum, orbital spin, and that due to intrinsic spin. **Orbital angular momentum**

$$|\mathbf{L}| = \sqrt{l(l+1)}\hbar$$

where l is the "angular momentum quantum number."
z-angular momentum

$$L_z = l_z\hbar, \qquad l_z = 0, \pm1, \pm2, ..., \pm l$$

Intrinsic spin angular momentum S:

$$|\mathbf{S}| = \sqrt{s(s+1)}\hbar$$
$$S_z = s_z\hbar \qquad s_z = -s, -1+1, ..., +s$$

Those with integer spin are "bosons." Those with half-integer spins are fermions

Magnetic moment μ

$$\boldsymbol{\mu} = \left(\frac{q}{2m}\right)\mathbf{L}$$
$$\mu_z = \left(\frac{q}{2m}\right)L_z, \qquad L_z = (0, \pm1, \pm2, ..., \pm l)\,\hbar$$

where q is the charge of the particle and m is its mass.
Spin angular momentum S:

$$\boldsymbol{\mu} = g\left(\frac{e}{2m}\right)\mathbf{S} \qquad \mu_z = g\left(\frac{e}{2m}\right)S_z$$

where e is the fundamental charge and g is the "gyromagnetic ratio."
Gyromagnetic moment g values

$$g = -2.00 \quad \text{electron}$$
$$g = +5.58 \quad \text{proton}$$
$$g = -3.82 \quad \text{neutron}$$

Interaction of $\boldsymbol{\mu}$ with B:

$$U = -\boldsymbol{\mu}\cdot\mathbf{B}$$

Harmonic oscillator potential:

$$E = \left(n + \tfrac{1}{2}\right)\hbar\omega, \qquad n = 0, 1, 2, ...$$
$$\omega = \sqrt{\frac{\kappa}{m}} \qquad \text{(3-D harmonic oscillator)}$$

13.2 Probabilities for various configurations

13.2.1 One criterion

Probability of n of N particles satisfying 1 criterion:

$$P_N(n) = \frac{N!}{n!\,(N-n)!}p^n q^{N-n}$$

Number of such configurations:

$$\text{number of such configurations} = \frac{N!}{n!\,(N-n)!}$$

13.2.2 Handling factorials

Stirling's approximation

$$m! \approx \sqrt{2\pi m}\left(\frac{m}{e}\right)^m$$
$$\ln m! \approx m\ln m - m + \tfrac{1}{2}\ln(2\pi m)$$

13.2.3 Many criteria

Propability n_1, n_2,..., n_m particles will satisfy m criteria from N particles:

$$P_N(n_1, n_2, ..., n_m) = \frac{N}{n_1!n_2!...n_m!}p_1^{n_1}p_2^{n_2}...p_m^{n_m}$$

13.3 Systems with many elements

13.3.1 Mean value and standard deviation

Average number of particle $\bar{n}$ satisfying 1 criterion

$$\bar{n} = pN$$

where p is the probability for any given element to satisfy the criterion.
Standard deviation

$$\sigma = \sqrt{Npq}$$
$$\frac{\sigma}{\bar{n}} = \sqrt{\frac{q}{Np}} \approx \frac{1}{\sqrt{N}}$$

13.3.2 The random walk

Total distance travelled:

$$S_N = \sum_{i=1}^{N}$$

Standard deviation after N steps:

$$\sigma_N = \sqrt{N}\sigma$$

where σ is the standard deviation of a single step.

13.4 Internal energy

Potential well u_0:

$$u_0 = u_0(T, p, N)$$

Energy in solids:

$$\begin{aligned}\epsilon &= \epsilon_{\text{potential}} + \epsilon_{\text{kinetic}} \\ &= u_0 + \tfrac{1}{2}\kappa x^2 + \tfrac{1}{2}\kappa y^2 + \tfrac{1}{2}\kappa z^2 + \frac{1}{2m}p_x^2 + \frac{1}{2m}p_y^2 + \frac{1}{2m}p_x^2\end{aligned}$$

Energy in liquids:

$$\begin{aligned}\epsilon &= \epsilon_{\text{potential}} + \epsilon_{\text{kinetic}} \\ &= u_0 + \frac{1}{2m}p_x^2 + \frac{1}{2m}p_y^2 + \frac{1}{2m}p_x^2\end{aligned}$$

Energy in gases:

$$\begin{aligned}\epsilon &= \epsilon_{\text{kinetic}} \\ &= \frac{1}{2m}p_x^2 + \frac{1}{2m}p_y^2 + \frac{1}{2m}p_x^2\end{aligned}$$

Quantum effects:

$$\epsilon = \epsilon_{\text{pot}} + \epsilon_{\text{rot}} + \epsilon_{\text{vib}}$$

Average energy of a particle in any system:

$$\bar{\epsilon} = u_0 + \tfrac{1}{2}\nu kT$$

13.5 Interactions between systems

13.5.1 Work - the mechanical interaction

Change in internal energy

$$\Delta E = \Delta Q - \Delta W \qquad \text{(thermal and mechanical interactions)}$$

13.5.2 Particle transfer

Chemical potential

$$\Delta E = \Delta Q = \mu \Delta N$$

where μ is the chemical potential, and N is the number of particles transferring

13.5.3 First law of thermodynamics

$$\mathrm{d}E = \mathrm{d}Q - \mathrm{d}W + \mu \mathrm{d}N$$

13.6 Internal energy and the number of accessible states

Probability of being in any 1 state

$$P_{\text{any one state}} = 1/\Omega$$

Number of states available for N particles:

$$\Omega = \omega \times \omega \times \omega \times ... = \omega^N$$

Number of states available for N *indistinguishable* particles:

$$\Omega = \frac{\omega^N}{N!} \approx \left(\frac{\mathrm{e}\omega}{N}\right)^N$$

***Corrected* number of states available for particles:**

$$\begin{aligned}\omega_c &= \omega \qquad \text{(distinguishable particles)} \\ \omega_c &= \frac{e\omega}{N} \qquad \text{(indistinguishable particles)}\end{aligned}$$

States available to each independent particle

$$\omega_c = C\left(\frac{V}{N}\right)\left(\frac{\frac{1}{2}kT}{N}\right)^{\nu/2} \qquad \text{(gas)}$$
$$\omega_c = C\left(\frac{\frac{1}{2}kT}{N}\right)^{\nu/2} \qquad \text{(solid)}$$

Density of states

$$g(E) \approx C\left(\frac{V}{N}\right)^N\left(\frac{\frac{1}{2}kT}{N}\right)^{N\nu/2} \qquad \text{(gas)}$$
$$g(E) \approx C\left(\frac{\frac{1}{2}kT}{N}\right)^{N\nu/2} \qquad \text{(solid)}$$

13.7 Entropy and the second law

Number of states for combined system

$$\Omega_0 = \Omega_1\Omega_2$$

Second law of thermodynamics

$$\Delta\Omega_0 > 0$$

Entropy

$$S = k \ln \Omega$$

13.8 Entropy and thermal interactions

Temperature:

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{V,N}$$

where S is entropy, E is internal energy, V is volume, and N is the number of particles
First law rewritten:

$$\begin{aligned} \mathrm{d}E &= T\mathrm{d}S - p\mathrm{d}V + \mu\mathrm{d}N \\ \mathrm{d}S &= \frac{1}{T}\mathrm{d}E + \frac{p}{T}\mathrm{d}V - \frac{\mu}{T}\mathrm{d}N \quad \left(= \frac{1}{T}\mathrm{d}Q\right) \end{aligned}$$

Intrinsic/Extrinsic properties

$$\begin{aligned} &\text{intrinsic:} \quad && T = T_1 = T_1, \quad && p = p_1 = p_2 \\ &\text{extrinsic:} \quad && S = S_1 + S_1, \quad && V = V_1 + V_2 \end{aligned}$$

Definitions in terms of entropy change

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{V,N} \qquad \frac{p}{T} = \left(\frac{\partial S}{\partial V}\right)_{E,N} \qquad \frac{\mu}{T} = \left(\frac{\partial S}{\partial N}\right)_{E,N}$$

Energy per degree of freedom:

$$E_{\text{therm}} = \tfrac{1}{2}N\nu kT$$

Change in states from entropy:

$$\begin{aligned} \frac{\Omega_f}{\Omega_i} &= \mathrm{e}^{\Delta S/k} \\ &= \mathrm{e}^{\left(\Delta E + p\Delta V - \mu\Delta N\right)/kT} \end{aligned}$$

Entropy at finite temperatures:

$$S(T, y) = \int_0^T \frac{C_y \mathrm{d}T'}{T'}$$

Heat capacities:

$$C_y \to 0 \qquad T \to 0$$

13.9 Constraints

Thermodynamic potentials

$$\begin{aligned} &\text{Helmholtz free energy} \quad && F \equiv E - TS; \\ &\text{Enthalpy,} \quad && H = TS + \mu N; \\ &\text{Gibbs free energy} \quad && G \equiv E - TS + pV \end{aligned}$$

Thermodynamic potentials

$$\begin{aligned} \mathrm{d}E &= T\mathrm{d}S - p\mathrm{d}V + \mu\mathrm{d}N \\ \mathrm{d}F &= -S\mathrm{d}T - p\mathrm{d}V + \mu\mathrm{d}N \\ \mathrm{d}H &= T\mathrm{d}SV\mathrm{d}p + \mu\mathrm{d}N \\ \mathrm{d}G &= -S\mathrm{d}T + V\mathrm{d}p + \mu\mathrm{d}N \end{aligned}$$

13.10 Models

13.10.1 Equations of state

Ideal gases

$$\begin{aligned} E &= \tfrac{1}{2}N\nu kT \\ pV &= NkT \\ N\mu &= NkT\left(\frac{\nu+2}{2} - \ln\omega_c\right) \\ &= E + pV - NkT\ln\omega_c \end{aligned}$$

Solids

$$\begin{aligned} E &= Nu_0 + \tfrac{1}{2}N\nu kT \\ p &= -N\left(\frac{\partial u_0}{\partial V}\right)_{E,N} \\ N\mu &= Nu_0 + NkT\left(\tfrac{1}{2}\nu - \ln\omega_c\right) \\ &= E - NkT\ln\omega_c \end{aligned}$$

Van der Waals

$$\left(p + \frac{a}{v^2}\right)(v - b) = RT$$

Differential equations of state

$$\begin{aligned} \mathrm{d}v &= -\frac{v}{p}\mathrm{d}p + \frac{R}{p}\mathrm{d}T \qquad \text{ideal gas} \\ \mathrm{d}v &= -\frac{A}{B}\frac{v}{p}\mathrm{d}p + \frac{1}{B}\frac{R}{p}\mathrm{d}T \qquad \text{van der Waals gas} \\ \text{with} \quad A &= \left(1 - \frac{b}{v}\right), \qquad B = \left(1 - \frac{a}{pv^2} + \frac{2ab}{pv^3}\right) \end{aligned}$$

Molar Heat capacities

$$C_p = \frac{1}{n}\left(\frac{\partial Q}{\partial T}\right)_p, \qquad C_V = \frac{1}{n}\left(\frac{\partial Q}{\partial T}\right)_V$$

Isothermal compressibility

$$\begin{aligned} \kappa &= \frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T \\ &= \frac{1}{p} \qquad \text{ideal gas} \\ &= \frac{A}{B}\frac{1}{p} \qquad \text{van der Waals gas} \end{aligned}$$

Coefficient of volume expansion

$$\begin{aligned} \beta &= \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p \\ &= \frac{1}{T} \qquad \text{ideal gas} \\ &= \frac{1}{B}\frac{R}{pv} \qquad \text{van der Waals gas} \end{aligned}$$

Difference of heat capacities

$$\begin{aligned} C_p - C_V &\approx pv\beta \\ &= R \qquad \text{ideal gas} \\ &= \frac{1}{B}R \qquad \text{van der Waals gas} \end{aligned}$$

Change in internal energy

$$\Delta E = \left(C_p - pV\beta\right)\Delta T + \left(p\kappa - T\beta\right)V\Delta p$$

Change in entropy

$$\begin{aligned}\Delta S &= \frac{C_V\kappa}{\beta T}\Delta p + \frac{C_p}{TV\beta}\Delta V \\ &= \frac{C_p}{T}\Delta T - V\beta\Delta p\end{aligned}$$

13.11 Special processes

Isobaric

$$\mathrm{d}E = \left(C_p - pV\beta\right)\mathrm{d}T$$

Isothermal

$$\mathrm{d}E = \left(\frac{T\beta}{\kappa} - p\right)\mathrm{d}V$$

Adiabatic

$$\begin{aligned}\mathrm{d}E &= -p\mathrm{d}V \\ p\mathrm{d}V + V\mathrm{d}p &= Nk\mathrm{d}T \\ \mathrm{d}E &= \tfrac{1}{2}N\nu\kappa\mathrm{d}T \\ C_V &= \frac{\nu}{2}R \\ C_p &= C_V + R \\ \beta &= 1/T \\ \kappa &= 1/p\end{aligned}$$

Adiabatic ideal gas

$$\begin{aligned}TV^{\gamma-1} &= \text{constant} \\ Tp^{\gamma/(\gamma-1)} &= \text{constant} \\ pV^{\gamma} &= \text{constant}\end{aligned}$$

13.11.1 Nonequilibrium processes

Throttling

$$\Delta T = \frac{\left(T\beta - 1\right)V}{C_p}\Delta p$$

Free expansion

$$\Delta T = \left(\frac{p\kappa - T\beta}{\kappa C_V}\right)\Delta V$$

13.12 Engines

Efficiency

$$e = \frac{Q_h - Q_c}{Q_h}$$

Work done or heat added

$$\begin{aligned}H_2 - H_1 &= Q_{\text{ext}} - W_{\text{ext}} \quad \text{any fluid} \\ H_2 - H_1 &= nC_p\left(T_2 - T_1\right) = Q_{\text{ext}} - W_{\text{ext}}\end{aligned}$$

Carnot efficiency

$$e_{\text{carnot}} = 1 - \frac{T_c}{T_h}$$

13.13 Diffusive interactions

Gibbs free energy

$$\Delta G = \sum_i \mu_i \Delta N_i = 0 \quad \text{at equilibrium}$$

13.14 Classical statistics

Probability system is in state s

$$P_s = C\exp\left(-\frac{\Delta E + p\Delta V - \mu\Delta N}{kT}\right)$$

Excitation temperature

$$T_e = \frac{\epsilon_1 - \epsilon_0}{k}$$

RMS speed

$$v_{\text{rms}} = \sqrt{\frac{3kT}{m}}$$

13.15 Kinetic Theory

Velocity distribution

$$\begin{aligned}P(\mathbf{v})\mathrm{d}^3v &= \left(\frac{\beta m}{2\pi}\right)^{3/2} \mathrm{e}^{-\beta mv^2/2}v^2\mathrm{d}^3v \\ P(v)\mathrm{d}v &= 4\pi\left(\frac{\beta m}{2\pi}\right)^{3/2} \mathrm{e}^{-\beta mv^2/2}v^2\mathrm{d}v\end{aligned}$$

Particle flux

$$J = \rho\sqrt{\frac{kT}{2\pi m}}$$

Collision frequency

$$v_c = \sqrt{2}\rho\sigma\bar{v}$$

where $\sigma = 4\pi R^2$ and R is the effective molecular radius
Mean free path

$$v_c = \sqrt{2}\rho\sigma\bar{v}$$

Average relative speed of colliding particles

$$\bar{u} = \sqrt{2}\bar{v}$$

Diffusion constant

$$D = \frac{n\bar{l}\bar{v}}{6}$$

Thermal conductivity

$$K = \frac{n\bar{l}\bar{v}}{6}\rho\frac{nu}{2}k$$

Coefficient of viscosity

$$\eta = \frac{n\bar{l}\bar{v}}{6}\rho m$$

14 Numerical Analysis

14.1 Second Order PDEs classification

14.1.1 General Form

$$A\frac{\partial^2\phi}{\partial x^2}+B\frac{\partial^2\phi}{\partial x\partial y}+C\frac{\partial^2\phi}{\partial y^2}+D\frac{\partial\phi}{\partial x}+E\frac{\partial\phi}{\partial y}+f\phi+G=0$$

$$A\frac{\partial^2\phi}{\partial x^2}+B\frac{\partial^2\phi}{\partial x\partial y}+C\frac{\partial^2\phi}{\partial y^2}=H$$

$$H=-\left(D\frac{\partial\phi}{\partial x}+E\frac{\partial\phi}{\partial y}+f\phi+G\right)$$

14.1.2 Characteristics in physical space

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{\alpha,\beta}=\frac{B\pm\sqrt{B^2-4AC}}{2A}$$

14.1.3 Equation characteristics

elliptic if	$B^2-4AC<0$
parabolic if	$B^2-4AC=0$
hyperbolic if	$B^2-4AC>0$

14.2 Finite Difference Schemes

14.2.1 First derivative

$$\begin{aligned}u_i'&=\frac{1}{\Delta x}\left(-u_{i-1}+u_i\right)+\tfrac{1}{2}\Delta x u''\\&=\frac{1}{\Delta x}\left(-u_i+u_{i+1}\right)-\tfrac{1}{2}\Delta x u_i''\\&=\frac{1}{\Delta x}\left(-u_{i-1}+u_{i+1}\right)-\tfrac{1}{6}\Delta x^2 u_i^{(3)}\\&=\frac{1}{\Delta x}\left(\tfrac{1}{2}u_{i-2}-2u_{i-1}+\tfrac{3}{2}u_i\right)+\tfrac{1}{3}\Delta x^2 u_i^{(3)}\\&=\frac{1}{\Delta x}\left(-\tfrac{3}{2}u_i+2u_{i+1}-\tfrac{1}{2}u_{i+2}\right)+\tfrac{1}{3}\Delta x^2 u_i^{(3)}\end{aligned}$$

14.3 Stability Analysis

14.3.1 Discrete Perturbation Stability analysis

Consider the parabolic model equation

$$\frac{\partial u}{\partial t}=\alpha\frac{\partial^2 u}{\partial x^2}.$$

Using a first order time and second order spatial derivative, this equation may be written as:

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}=\alpha\frac{u_{i+1}^n-2u_i^n+u_{i-1}^n}{(\Delta x)^2}.$$

If a disturbance ϵ at node i and time level n is introduced and we search for solution at time level $n+1$ for all i nodes, the finite difference eq. becomes:

$$\frac{u_i^{n+1}-\left(u_i^n+\epsilon\right)}{\Delta t}=\alpha\frac{u_{i+1}^n-2\left(u_i^n+\epsilon\right)+u_{i-1}^n}{(\Delta x)^2}.$$

If $u^n=0$ for all i the equation reduces to:

$$\frac{u_i^{n+1}-\epsilon}{\Delta t}=\alpha\frac{-2\epsilon}{(\Delta x)^2}$$

or equivalently

$$\begin{aligned}u_i^{n+1}&=\epsilon\left\{1+2\alpha\left[\frac{\Delta t}{(\Delta x)^2}\right]\right\}\\&=\epsilon\left(1-2d\right)\\\rightarrow\frac{u_i^{n+1}}{\epsilon}&=\left(1-2d\right)\end{aligned}$$

where $d=\alpha\Delta t/\left(\Delta x\right)^2$ is known as the *diffusion number*. It's required that,

$$\left|\frac{u_i^{n+1}}{\epsilon}\right|\le 1$$

or

$$1-2d\le 1\qquad\text{and}\qquad 1-2d\ge -1$$

14.3.2 Von Neumann Stability Analysis

Solution of finite difference equation is expanded in a Fourier series. The decay or growth of the *amplification factor* determines the stability.

Assume a Fourier component for u_i^n as

$$u_i^n=U^n e^{IP(\Delta x)i}$$

where $I=\sqrt{-1}$, U^n is the amplitude at time n, and P is the wave number in the x-direction, i.e., $\lambda_x=2\pi/P$, where λ_x is the wavelength. Similarly,

$$u_i^{n+1}=U^{n+1}e^{IP(\Delta x)i}\quad\text{and}\quad u_{i\pm1}^n=U^n e^{IP(\Delta x)(i\pm1)}$$

If phase angle $\theta=P\Delta x$ is defined, then

$$\begin{aligned}u_i^n&=U^n e^{I\theta i}\\u_i^{n+1}&=U^{n+1}e^{I\theta i}\\u_{i\pm1}^n&=U^n e^{I\theta(i\pm1)}\end{aligned}$$

Consider again

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}=\alpha\frac{u_{i+1}^n-2u_i^n+u_{i-1}^n}{(\Delta x)^2}$$

or in terms of the diffusion number

$$u_i^(n+1)=u_i^n+d\left(u_{i+1}^n-2u_i^n+u_{i-1}^n\right)$$

Substituting the Fourier component and cancelling out terms of $e^{I\theta i}$, we get

$$U^{n+1}=U^n\left[1+d\left(e^{I\theta}-e^{-I\theta}-2\right)\right]$$

Using the relation $\cos\theta=\left(e^{I\theta}+e^{-I\theta}\right)/2$, we get

$$U^{n+1}=U^n\left[1+2d\left(\cos\theta-1\right)\right]$$

Introducing the amplification factor $U^{n+1}=GU^n$, we get

$$G=1-2d\left(1-\cos\theta\right).$$

Stability requires,

$$|G| \leq 1 \qquad \text{and} \qquad |1 - 2d\,(1 - \cos\theta)| \leq 1$$

so that

$$1 - 2d\,(1 - \cos\theta) \leq 1$$
$$\text{and}$$
$$1 - 2d\,(1 - \cos\theta) \geq -1$$

must be valid for all values of θ. With a maximum value of $(1 - \cos\theta) = 2$, the LHS becomes $1 - 4d$. Solving the new equation gives us a final condition

$$d \leq \frac{1}{2} \tag{3}$$

14.3.3 Scheme Requirements

- *Consistency*: A finite difference approximation of a PDE is consistent if the finite difference equation approaches the PDE as the grid size approaches zero.
- *Stability*: A numerical scheme is said to be stable if any error introduced in the finite difference equation does not grow with the solution of the finite difference equation.
- *Convergence*: A finite difference scheme is convergent if the solution of the finite difference scheme approaches that of the PDE as the grid size approaches zero.
- *Lax's equivalence theorem*: For a FDE which approximates a well-posed, linear initial value problem, the necessary and sufficient condition for convergence is that the FDE must be stable and consistent.
- *The conservative (divergent) form of a PDE*: In this formulation of a physical law, the coefficients of the derivatives are either constant or, if variable, their derivatives do not appear anywhere in the equation.
- *Conservative property of a FDE*: If the finite difference approximation of a PDE maintains the integral property of the conservation law over an arbitrary region containing any number of grid points, it is said to possess a conservative property.

14.4 Classification of equations

Elliptic: $$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

Parabolic: $$\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2}$$

Hyperbolic: $$\frac{\partial^2\phi}{\partial t^2} = a^2\frac{\partial^2\phi}{\partial x^2}$$

15 General Mathematics

15.1 Arithmetic and Elementary Algebra

15.1.1 Powers and Logarithms:

Powers and roots:
For real a, b, q, and p where $a, b > 0$

$$a^{-p} = \frac{1}{a^p}, \qquad a^p a^q = a^{p+q}, \qquad \frac{a^p}{a^q} = a^{p-q},$$
$$(ab)^p = a^p b^p, \qquad \left(\frac{a}{b}\right)^q = \frac{a^q}{b^q}, \qquad (a^p)^q = a^{pq}.$$

Logarithms:
Definition:

$$\log_a b = c \longleftrightarrow a^c = b$$
$$a^{\log_a b} = b$$

where $a > 0$, $a \neq 1$, and $b > 0$.
Properties of logarithms:

$$\log_a (bc) = \log_a b + \log_a c \qquad \log_a \left(\frac{b}{c}\right) = \log_a b - \log_a c$$
$$\log_a \left(b^k\right) = k \log_a b \qquad \log_{a^k} b = \frac{1}{k} \log_a b \quad (k \neq 0)$$
$$\log_a b = \frac{a}{\log_b a} \quad (b \neq 1) \qquad \log_a b = \frac{\log_c b}{\log_c a} \quad (c \neq 1)$$

15.1.2 Binomial Theorem and Related Formulas

Binomial coefficients:

$$C_n^k = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Binomial theorem:

$$(a+b)^n = \binom{n}{k} = \sum_{k=0}^{n} C_n^k a^{n-k} b^k$$

15.2 Elementary Functions

Exponential function e:

$$e \equiv \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n \qquad a^x = e^{x \ln a}$$

15.2.1 Trigonometric Functions

Definition of trigonometric functions:

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}, \qquad \cot\alpha = \frac{\cos\alpha}{\sin\alpha},$$
$$\sec\alpha = \frac{1}{\cos\alpha}, \qquad \csc\alpha = \frac{1}{\sin\alpha}$$

Properties of Trigonometric Functions
Simplest relations

$$\sin^2 x + \cos^2 x = 1, \qquad \tan x \cot x = 1,$$
$$\sin(-x) = -\sin x, \qquad \cos(-x) = \cos x,$$
$$\tan x = \frac{\sin x}{\cos x}, \qquad \cot x = \frac{\cos x}{\sin x},$$
$$\tan(-x) = -\tan x, \qquad \cot(x) = -\cot x$$
$$1 + \tan^2 x = \frac{1}{\cos^2 x}, \qquad 1 = \cot^2 x = \frac{1}{\sin^2 x}.$$

Reduction formulas

$$\sin(x \pm 2n\pi) = \sin x \qquad \cos(x \pm 2n\pi) = \cos x$$
$$\sin(x \pm n\pi) = \pm(-1)^n \sin x \qquad \cos(x \pm n\pi) = \mp(-1)^n \cos x$$
$$\sin\left(x + \frac{2n+1}{2}\pi\right) = \pm(-1)^n \cos x \qquad \cos\left(x \pm \frac{2n+1}{2}\pi\right) = \mp(-1)^n \sin x$$
$$\sin\left(x \pm \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(\sin x \pm \cos x) \qquad \cos\left(x \pm \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(\cos x \mp \sin x)$$
$$\tan(x \pm n\pi) = \tan x \qquad \cot(x \pm n\pi) = \cot x$$
$$\tan\left(x \pm \frac{2n+1}{2}\pi\right) = -\cot x \qquad \cot\left(x \pm \frac{2n+1}{2}\pi\right) = -\tan x$$
$$\tan\left(x \pm \frac{\pi}{4}\right) = \frac{\tan x \pm 1}{1 \mp \tan x} \qquad \cot\left(\pm\frac{\pi}{4}\right) \frac{\cot x \mp 1}{1 \pm \cot x}$$

Relations between trigonometric functions of a single argument

$$\sin x = \pm\sqrt{1-\cos^2 x} = \pm\frac{\tan x}{\sqrt{1+\tan^2 x}} = \pm\frac{1}{\sqrt{1+\cot^2 x}}$$
$$\cos x = \pm\sqrt{1-\sin^2 x} = \pm\frac{1}{\sqrt{1+\tan^2 x}} = \pm\frac{\cot x}{\sqrt{1+\cot^2 x}}$$
$$\tan x = \pm\frac{\sin x}{\sqrt{1-\sin^2 x}} = \pm\frac{\sqrt{1-\cos^2 x}}{\cos x} = \frac{1}{\cot x}$$
$$\cot x = \pm\frac{\sqrt{1-\sin^2 x}}{\sin x} = \pm\frac{\cos x}{\sqrt{1-\cos^2 x}} = \frac{1}{\tan x}$$

Addition and subtraction of trigonometric functions

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$
$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$
$$\sin^2 x - \sin^2 y = \cos^2 y - \cos^2 x = \sin(x+y)\sin(x-y)$$
$$\sin^2 x - \cos^2 y = \cos^2 y - \cos^2 x = \sin(x+y)\sin(x-y)$$
$$\tan x \pm \tan y = \frac{\sin(x \pm y)}{\cos x \cos y} \qquad \cot x \pm \cot y = \frac{\sin(y \pm x)}{\sin x \sin y}$$
$$a\cos x - b\sin x = r\sin\left(x + \phi\right) = r\cos(x - \psi)$$

Here, $r = \sqrt{a^2+b^2}$, $\sin\phi = a/r$, $\cos\phi = b/r$, $\sin\psi = b/r$, and $\cos\psi = a/r$.

Products of trigonometric functions

$$\sin x \sin y = \tfrac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$$
$$\cos x \cos y = \tfrac{1}{2}\left[\cos(x-y) + \cos(x+y)\right]$$
$$\sin x \cos y = \tfrac{1}{2}\left[\sin(x-y) + \sin(x+y)\right]$$

Powers of trigonometric functions

$$\cos^2 x = \tfrac{1}{2}\cos 2x + \tfrac{1}{2}$$
$$\cos^3 x = \tfrac{1}{4}\cos 3x + \tfrac{3}{4}\cos x$$
$$\cos^4 x = \tfrac{1}{8}\cos 4x + \tfrac{1}{2}\cos 2x + \tfrac{3}{8}$$
$$\cos^5 x = \tfrac{1}{16}\cos 5x + \tfrac{5}{16}\cos 3x + \tfrac{5}{8}\cos x$$
$$\sin^2 x = -\tfrac{1}{2}\cos 2x + \tfrac{1}{2}$$
$$\sin^3 x = -\tfrac{1}{4}\sin 3x + \tfrac{3}{4}\sin x$$
$$\sin^4 x = \tfrac{1}{8}\cos 4x - \tfrac{1}{2}\cos 2x + \tfrac{3}{8}$$
$$\sin^5 x = \tfrac{1}{16}\sin 5x - \tfrac{5}{16}\sin 3x + \tfrac{5}{8}\sin x$$
$$\cos^{2n} x = \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1} C_{2n}^{k}\cos\left[2(n-k)x\right] + \frac{1}{2^{2n]}}C_{2n}^{n}$$
$$\cos^{2n+1} x = \frac{1}{2^{2n}}\sum_{k=0}^{n} C_{2n+1}^{k}\cos\left[(2n-2k+1)x\right]$$
$$\sin^{2n} x = \frac{1}{2n-1}\sum_{k=0}^{n-1}(-1)^{n-k} C_{2n}^{k}\cos\left[2(n-k)x\right] + \frac{1}{2^{2n}}C_{2n}^{n}$$
$$\sin^{2n-1} = \frac{1}{2^{2n}}\sum_{k=0}^{n}(-1)^{n-k} C_{2n+1}^{k}\sin\left[(2n-2k+1)x\right]$$

Here, $n = 1, 2, \ldots$ and $C_m^k = \frac{m!}{k!(m-k)!}$ are the binomial coefficients $(0! = 1)$.

Addition formulas

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$
$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$
$$\cot(x \pm y) = \frac{1 \pm \tan x \tan y}{\tan x \pm \tan y}$$

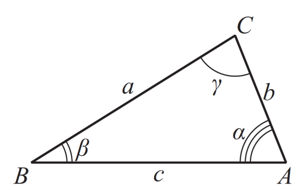
Trigonometric functions of multiple arguments

$$\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$
$$\cos 3x = -3\cos x + 4\cos^3 x$$
$$\cos 4x = 1 - 8\cos^2 x + 8\cos^4 x$$
$$\cos 5x = 5\cos x - 2 - \cos^2 x + 16\cos^4 x$$
$$\sin 2x = 2\sin x \cos x$$
$$\sin 3x = 3\sin x - 4\sin^3 x$$
$$\sin 4x = 4\cos x\left(\sin x - 2\sin^3 x\right)$$
$$\sin 5x = 5\sin x = 5\sin x - 20\sin^3 x + 16\sin^5 x$$
$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$
$$\tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$$
$$\tan 4x = \frac{4\tan x - 4\tan^3 x}{1 - 6\tan^2 x + \tan^4 x}$$

15.3 Elementary Geometry

15.3.1 Plane Geometry

Triangles



Plane triangle relations:

$$\alpha + \beta + \gamma = 180^\circ \qquad \text{(Sum of angles of a triangle)}$$

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R \qquad \text{(Law of sines)}$$

$$c^2 = a^2 + b^2 - 2ab\cos\gamma \qquad \text{(Law of cosines)}$$

$$\frac{a+b}{a-b} = \frac{\tan\left[\frac{1}{2}(\alpha+\beta)\right]}{\tan\left[\frac{1}{2}(\alpha-\beta)\right]} = \frac{\cot\left(\frac{1}{2}\gamma\right)}{\tan\left[\frac{1}{2}(\alpha-\beta)\right]} \qquad \text{(Law of tangents)}$$

$$c = a\cos\beta + b\cos\alpha \qquad \text{(Theorem on projections)}$$

$$p = \tfrac{1}{2}(a+b+c) \qquad \text{(Semi-perimeter)}$$

$$\sin\frac{\gamma}{2} = \sqrt{\frac{(p-a)(p-b)}{ab}} \qquad \text{(Trigonometric angle formulas)}$$

$$\cos\frac{\gamma}{2} = \sqrt{\frac{p(p-c)}{ab}}$$

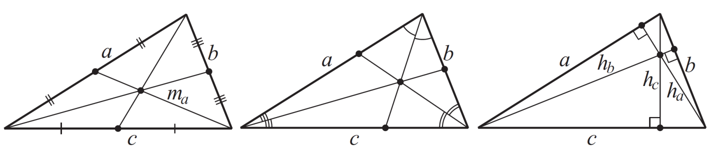
$$\tan\frac{\gamma}{2} = \sqrt{\frac{p(p-c)}{p(p-c)}}$$

$$\sin\gamma = \frac{2}{ab}\sqrt{p(p-a)(p-b)(p-a)}$$

$$\tan\gamma = \frac{c\sin\alpha}{b-c\cos\alpha} = \frac{c\sin\beta}{a-c\cos\beta} \qquad \text{(Law of tangents)}$$

$$\frac{a+b}{c} = \frac{\cos\left[\frac{1}{2}(\alpha-\beta)\right]}{\sin\left(\frac{1}{2}\gamma\right)} = \frac{\cos\left[\frac{1}{2}(\alpha-\beta)\right]}{\cos\left[\frac{1}{2}(\alpha+\beta)\right]} \qquad \text{(Mollweide's)}$$

$$\frac{a-b}{c} = \frac{\sin\left[\frac{1}{2}(\alpha-\beta)\right]}{\cos\left(\frac{1}{2}\gamma\right)} = \frac{\sin\left[\frac{1}{2}(\alpha-\beta)\right]}{\sin\left[\frac{1}{2}(\alpha+\beta)\right]}$$



Medians, angle bisectors, and altitudes of triangle

$$m_a = \tfrac{1}{2}\sqrt{2(b^2+c^2)-a^2} = \tfrac{1}{2}\sqrt{a^2+4b^2-4ab\cos\gamma} \qquad \text{(Median)}$$

$$l_a = \frac{\sqrt{bc\left[(b+c)^2-a^2\right]}}{b+c} = \frac{\sqrt{4p(p-a)bc}}{b+c} \qquad \text{(Angle bisector)}$$

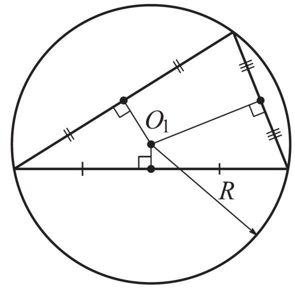
$$= \frac{2cb\cos\left(\frac{1}{2}\alpha\right)}{b+c} = 2R\frac{\sin\beta\sin\gamma}{\cos\left[\frac{1}{2}(\beta-\gamma)\right]}$$

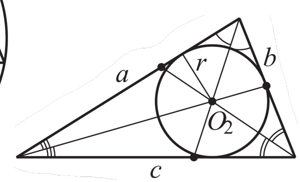
$$= 2p\frac{\sin\left(\frac{1}{2}\beta\right)\sin\left(\frac{1}{2}\gamma\right)}{\sin\beta+\sin\gamma}$$

$$h_a = b\sin\gamma = c\sin\beta = \frac{bc}{2R} \qquad \text{(Altitude)}$$

$$= 2(p-a)\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$$

$$= 2(p-b)\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2}$$





Circumcircle and incircle

$$r = \sqrt{\frac{(p-a)(p-b)(p-c)}{p}}$$

$$= p\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2} = (p-c)\tan\frac{\gamma}{2}$$

$$R = \frac{a}{2\sin\alpha} = \frac{b}{2\sin\beta} = \frac{c}{2\sin\gamma}$$

$$= \frac{p}{4\cos\left(\frac{1}{2}\alpha\right)\cos\left(\frac{1}{2}\beta\right)\cos\left(\frac{1}{2}\gamma\right)}$$

$$d = \sqrt{R^2 - 2Rr}$$

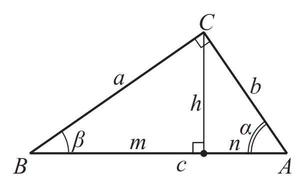
Area of a triangle, S

$$S = \frac{a}{h_a} = \tfrac{1}{2}ab\sin\gamma = rp$$

$$= \sqrt{p(p-a)(p-b)(p-c)}$$

$$= \frac{abc}{4R} = 2R^2\sin\alpha\sin\beta\sin\gamma$$

$$= c^2\frac{\sin\alpha\sin\beta}{2\sin\gamma} = c^2\frac{\sin\alpha\sin\beta}{2\sin(\alpha+\beta)}$$



Right (right-angled) triangles.

$$\alpha + \beta = 90^\circ$$

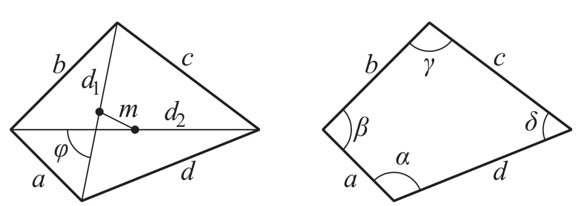
$$\sin\alpha = \cos\beta = \frac{a}{c} \qquad \sin\beta = \cos\alpha = \frac{b}{c}$$

$$\tan\alpha = \cot\beta = \frac{a}{b} \qquad \tan\beta = \cot\alpha = \frac{b}{a}$$

$$a^2 + b^2 = c^2$$

$$h^2 = mn, \qquad a^2 = mc \qquad b^2 = nc$$

Polygons



Areas of quadrilaterals

$$p = \tfrac{1}{2}(a+b+c+d)$$
$$S = \tfrac{1}{2} d_1 d_2 \sin\phi$$
$$= \sqrt{p(p-a)(p-b)(p-c)(p-d) - abcd\cos^2\left(\frac{\beta+\delta}{2}\right)}$$

15.4 Algebra

15.4.1 Polynomials and Algebraic Equations

Linear and Quadratic Equations
Linear equations

$$ax + b = 0$$
$$x = -\frac{b}{a}$$

Quadratic equations

$$ax^2 + bx + c = 0 \tag{4}$$
$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{5}$$

Cubic Equations
Incomplete cubic equation

$$y^3 + py + q = 0$$
$$y_1 = A + B, \qquad y_{2,3} = -\tfrac{1}{2}(A+B) \pm i\frac{\sqrt{3}}{2}(A-B)$$
$$A = \left(-\frac{q}{2} + \sqrt{D}\right)^{1/3}, \qquad B = \left(-\frac{q}{2} - \sqrt{D}\right)^{1/3}$$
$$D = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2, \qquad i^2 = -1$$

Complete cubic equation

$$ax^3 + bx^2 + cx + d = 0$$
$$x_k = y_k - \frac{b}{3a}, \qquad y_k^3 + py_k + q = 0, \qquad k = 1,2,3$$
$$p = -\frac{1}{3}\left(\frac{b}{a}\right)^2 + \frac{c}{a}, \qquad q = \frac{2}{27}\left(\frac{b}{a}\right)^3 - \frac{bc}{3a^2} + \frac{d}{a}.$$

Quartic Equations

$$ax^4 + bx^3 + cx^3 + dx + e = 0 \tag{6}$$
$$x = y - \frac{b}{4a} \tag{7}$$
$$y^4 + py^3 + qy + r = 0 \tag{8}$$
$$z^3 + 2pz^2 + \left(p^2 - 4r\right)z - q^2 = 0 \tag{9}$$
$$y_1 = \tfrac{1}{2}\left(\sqrt{z_1} + \sqrt{z_2} + \sqrt{z_3}\right) \tag{10}$$
$$y_2 = \tfrac{1}{2}\left(\sqrt{z_1} - \sqrt{z_2} - \sqrt{z_3}\right) \tag{11}$$
$$y_3 = \tfrac{1}{2}\left(-\sqrt{z_1} + \sqrt{z_2} - \sqrt{z_3}\right) \tag{12}$$
$$y_4 = \tfrac{1}{2}\left(-\sqrt{z_1} - \sqrt{z_2} + \sqrt{z_3}\right) \tag{13}$$

Algebraic Equations of Arbitrary Degree and Their Properties
Bounds for the roots of algebraic equations with real coefficients All roots of polynomials in absolute value do not exceed

$$N = 1 + \frac{A}{|a_n|} \tag{14}$$

where A is the largest of the coefficients $|a_0|$, $|a_1|$, ..., $|a_{n-1}|$

16 Useful Identities

16.1 Series

16.1.1 Expansion of asymptotic series to a power

$$
\begin{aligned}
\left(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots\right)^n = {} & x_0^n \\
& + \epsilon n x_0^{n-1} x_1 \\
& + \epsilon^2 \left[\frac{n\,(n-1)}{2!} x_0^{n-2} x_1^2 + n x_0^{n-1} x_2\right] \\
& + \epsilon^3 \left[\frac{n\,(n-1)\,(n-2)}{3!} x_0^{n-3} x_1^3 + n\,(n-1)\, x_0^{n-2} x_1 x_2 + n x_0^{n-1} x_3\right] \\
& + \epsilon^4 \left[\frac{n\,(n-1)\,(n-2)\,(n-3)}{4!} x_0^{n-4} x_1^4 + \frac{n\,(n-1)\,(n-2)}{2!} x_0^{n-3} x_1^2 x_2 + \frac{n\,(n-1)}{2!} x_0^{n-2} \left(x_2^2 + 2 x_1 x_3\right) + n x_0^{n-1} x_4\right]
\end{aligned}
$$

Important Series Expansions

Taylor: $f(x)=f(a)+f'(a)(x-a)+\dfrac{f''(a)}{2!}(x-a)^2+\dfrac{f'''(a)}{3!}(x-a)^3+\ldots+\dfrac{f^n(a)}{n!}(x-a)^n+\ldots$

$(1+x)^\alpha=1+ax+\dfrac{\alpha(\alpha-1)}{2!}x^2+\dfrac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3+\ldots$

$(1+x)^{-1}=1-x+x^2-x^3+\ldots;\quad -1<x<1$

$(1+x)^{1/2}=1+\dfrac{x}{2}-\dfrac{x^2}{8}+\dfrac{x^3}{16}-\dfrac{5x^4}{128}+\dfrac{7x^5}{256}+\ldots;\quad -1<x<1$

$(1+x)^{-1/2}=1-\dfrac{x}{2}+\dfrac{3}{8}x^2-\dfrac{5x^3}{16}+\dfrac{35x^4}{128}-\dfrac{63x^5}{256}+\ldots;\quad -1<x<1$

$\ln x=\dfrac{x-1}{x}+\dfrac{1}{2}\left(\dfrac{x-1}{x}\right)^2+\dfrac{1}{3}\left(\dfrac{x-1}{x}\right)^3+\ldots;\quad x\geq\tfrac{1}{2};$

$\ln x=(x-1)-\dfrac{1}{2}(x-1)^2+\dfrac{1}{3}(x-1)^3-\ldots;\quad |x-1|\leq 1,\quad x\neq 0$

$\ln(1+x)=x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+\ldots+(-1)^{n+1}\dfrac{x^n}{n};\quad |x|\leq 1\quad x\neq -1$

$\ln(1-x)=-x-\dfrac{x^2}{2}-\dfrac{x^3}{3}-\ldots-\dfrac{x^n}{n};\quad |x|<1$

$\dfrac{1}{1-x}=1+x+x^2+\ldots+x^n+\dfrac{x^n}{1-x};\ |x|<1;\quad \dfrac{1}{(1-x)^2}=1+2x+3x^2+\ldots+nx^n+R;\ |x|<1;\quad \underset{n\to\infty}{R}\to 0$

$e^x=1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\ldots+\dfrac{x^n}{n!}+\ldots;\ \sin x=x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}\ldots;\ \cos x=1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+\ldots$

$\tan x=x+\dfrac{x^3}{3}+\dfrac{2x^5}{15}+\ldots;$ $\cot x=\dfrac{1}{x}-\dfrac{x}{3}-\dfrac{x^3}{45}-\dfrac{2x^5}{245}+\ldots$

$\sec x=1+\dfrac{x^2}{2}+\dfrac{5x^4}{24}+\dfrac{61x^6}{720}+\ldots;$ $\csc x=\dfrac{1}{x}+\dfrac{x}{6}+\dfrac{7}{360}x^3+\dfrac{31}{15120}x^5+\ldots$

$\sin^{-1}x=x+\dfrac{x^3}{2.3}+\dfrac{1\cdot 3x^5}{2\cdot 4\cdot 5}+\dfrac{1\cdot 3\cdot 5x^7}{2\cdot 4\cdot 6\cdot 7}+\ldots;\quad |x|<1;\quad \tan^{-1}x=x-\dfrac{x^3}{3}+\dfrac{x^5}{5}-\dfrac{x^7}{7}+\ldots;\quad |x|\leq 1;$

$\tan^{-1}x=\dfrac{\pi}{2}-\dfrac{1}{x}+\dfrac{1}{3x^3}-\dfrac{1}{5x^5}+\ldots;\quad x>1;$ $\tan^{-1}x=-\dfrac{\pi}{2}-\dfrac{1}{x}+\dfrac{1}{3x^3}-\dfrac{1}{5x^5}+\ldots;\quad x<-1$

$\cot^{-1}x=\tan^{-1}(x^{-1});\ \sec^{-1}x=\cos^{-1}(x^{-1});$ $\csc^{-1}x=\sin^{-1}(x^{-1});\ -1\leq x\leq 1;$

$\sinh x=x+\dfrac{x^3}{3!}+\dfrac{x^5}{5!}+\ldots;$ $\cosh x=1+\dfrac{x^2}{2!}+\dfrac{x^4}{4!}+\ldots$

$\tanh x=x-\dfrac{x^3}{3}+\dfrac{2x^5}{15}-\dfrac{17x^7}{315}+\ldots;\quad |x|<\dfrac{\pi}{2};$ $\coth x=\dfrac{1}{x}+\dfrac{x}{3}-\dfrac{x^3}{45}+\dfrac{2x^5}{945}+\ldots$

$\operatorname{sech} x=1-\dfrac{x^2}{2}+\dfrac{5x^4}{24}-\dfrac{61x^6}{720}+\ldots;$ $\operatorname{csch} x=\dfrac{1}{x}-\dfrac{x}{6}+\dfrac{7x^3}{360}-\dfrac{31x^5}{15120}+\ldots$

$\sinh^{-1}x=x-\dfrac{1}{2\cdot 3}x^3+\dfrac{1\cdot 3}{2\cdot 4\cdot 5}x^5-\dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}x^7+\ldots;\quad |x|<1$

$\quad=\ln 2x+\dfrac{1}{2\cdot 2x^2}-\dfrac{1\cdot 3}{2\cdot 4\cdot 4x^4}+\dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6x^6}+\ldots;\quad |x|>1$

$\cosh^{-1}x=\ln 2x-\dfrac{1}{2\cdot 2x^2}-\dfrac{1\cdot 3}{2\cdot 4\cdot 4x^4}-\dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6x^6}+\ldots;\quad |x|>1$

$\tanh^{-1}x=x+\dfrac{x^3}{3}+\dfrac{x^5}{5}+\dfrac{x^7}{7}+\ldots;\quad |x|<1;\quad \coth^{-1}x=\dfrac{1}{x}+\dfrac{1}{3x^3}+\dfrac{1}{5x^5}+\dfrac{1}{7x^7}+\ldots;\quad |x|>1$

$\cos^3 x=1-\dfrac{3}{2}x^2+\dfrac{7}{8}x^4-\dfrac{x^6}{8}+\ldots;\quad \sin^3 x=x^3-\dfrac{1}{2}x^5+\dfrac{3}{4}x^7-\dfrac{1}{216}x^9+\ldots$

Useful Trigonometric Identities

FORMULAS FOR ADDITION AND SUBTRACTION

$\sin(a \pm b) = \sin a \, \cos b \pm \cos a \, \sin b\,;\qquad \cos(a \pm b) = \cos a \, \cos b \mp \sin a \, \sin b$

$$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b} \qquad \cot(a \pm b) = \frac{\cot a \cot b \mp 1}{\cot b \pm \cot a} \qquad \tan\left(\tfrac{\pi}{4} \pm b\right) = \frac{1 \pm \tan b}{1 \mp \tan b} = \frac{\cos b \pm \sin b}{\cos b \mp \sin b}$$

WERNER'S FORMULAS

$\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]\,;\qquad \sin a \cos b = \frac{1}{2}[\sin(a+b) + \sin(a-b)]$

$\cos a \cos b = \frac{1}{2}[\cos(a+b) + \cos(a-b)]$

INVERSE OF WERNER'S FORMULAS

$\sin a + \sin b = 2\sin(\frac{a+b}{2})\cos(\frac{a-b}{2})\,;\qquad \sin a - \sin b = 2\cos(\frac{a+b}{2})\sin(\frac{a-b}{2})$

$\cos a + \cos b = 2\cos(\frac{a+b}{2})\cos(\frac{a-b}{2})\,;\qquad \cos a - \cos b = -2\sin(\frac{a+b}{2})\sin(\frac{a-b}{2})$

OTHER FORMULAS TRANSFORMING SUMS INTO PRODUCTS OF FUNCTIONS

$\tan a + \tan b = \sin(a+b) \,/\, (\cos a \cos b)\,;\qquad \tan a - \tan b = \sin(a-b) \,/\, (\cos a \cos b)$

$\cot a + \cot b = \sin(a+b) \,/\, (\sin a \sin b)\,;\qquad \cot a - \cot b = -\sin(a-b) \,/\, (\sin a \sin b)$

DUPLICATION FORMULAS

$\sin 2a = 2\sin a \, \cos a\,;\qquad \cos 2a = 2\cos^2 a - 1 = 1 - 2\sin^2 a = \cos^2 a - \sin^2 a$

$\tan 2a = 2\tan a \,/\, (1 - \tan^2 a)\,;\qquad \cot 2a = (\cot^2 a - 1) \,/\, (2\cot a)$

TRIPLICATION FORMULAS

$\sin 3a = 3\sin a - 4\sin^3 a\,;\qquad \cos 3a = 4\cos^3 a - 3\cos a$

$\tan 3a = (3\tan a - \tan^3 a) \,/\, (1 - 3\tan^2 a)\,;\qquad \cot 3a = (\cot^3 a - 3\cot a) \,/\, (3\cot^2 a - 1)$

BISECTION FORMULAS

$\sin \frac{a}{2} = \pm\sqrt{\frac{1}{2}(1 - \cos a)}\,;\qquad \cos \frac{a}{2} = \pm\sqrt{\frac{1}{2}(1 + \cos a)}$

$\tan \frac{a}{2} = \pm\sqrt{\frac{1}{2}\,\frac{(1-\cos a)}{(1+\cos a)}}\,;\qquad \cot \frac{a}{2} = \pm\sqrt{\frac{1}{2}\,\frac{(1+\cos a)}{(1-\cos a)}}$

FORMULAS FOR CONVERTING POWERS INTO MULTIPLE ANGLES

$\sin^2 a = \frac{1}{2}(1 - \cos 2a)\,;\qquad \cos^2 a = \frac{1}{2}(1 + \cos 2a)$

$\sin^3 a = \frac{1}{4}(3\sin a - \sin 3a)\,;\qquad \cos^3 a = \frac{1}{4}(3\cos a + \cos 3a)$

$\sin^4 a = \frac{1}{8}(\cos 4a - 4\cos 2a + 3)\,;\qquad \cos^4 a = \frac{1}{8}(\cos 4a + 4\cos 2a + 3)$

$\sin^5 a = \frac{1}{16}(10\sin a - 5\sin 3a + \sin 5a)\,;\qquad \cos^5 a = \frac{1}{16}(10\cos a + 5\cos 3a + \cos 5a)$

$\sin^6 a = \frac{1}{32}(10 - 15\cos 2a + 6\cos 4a - \cos 6a)\,;\qquad \cos^6 a = \frac{1}{32}(10 + 15\cos 2a + 6\cos 4a + \cos 6a)$