and A satisfies the homogeneous wave equation. The fields are given by.

$$E = -\frac{1}{c}\frac{\partial A}{\partial t}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

It is well known that electromagnetic disturbances propagate with finite speed. Eq^n (3) represent that scalar potential propagates instantaneously everywhere in space. The vector potential satisfies eq^n(11) with its implied finite speed of propagation c. Transverse current extends over all spaces even if J is localized.

Poynting's Theorem and Conservation of Energy and momentum for a system of charged Particles and Electromagnetic Fields :-

The laws of conservation of energy & momentum are important result to establish for electromagnetic field. Conservation of energy, often called Poynting's theorem.

For a single charge q the rate of doing work by external electromagnetic fields E & B is

$$\frac{dW}{dt} = \frac{d}{dt}[\text{force} \times \text{displacement}] = \vec{F} \cdot \frac{d\vec{x}}{dt} = \vec{F} \cdot \vec{v}$$

$$\vec{F} = q\,[\vec{E} + (\vec{v} \times \vec{B})]$$

$$\frac{dW}{dt} = q\,[\vec{E} + (\vec{v} \times \vec{B})] \cdot \vec{v}$$

$$\frac{dW}{dt} = q\vec{E}\vec{v} + q(\vec{v}\times\vec{B}).\vec{v}$$

$\because q(\vec{v}\times\vec{B}).\vec{v} = 0$ div of a curl is zero
Curl(div) = 0

$$\frac{dW}{dt} = q.\vec{E}.\vec{v}$$

Mag. field does no work since the mag. force is perpendicular to the velocity. If there exists a continuous distribution of charge and current the total rate of doing work by the fields in a finite volume V is

$$\int_V \frac{dW}{dt} = \int_V \rho\, d^3x\, (\vec{E}.\vec{v})$$

$$= \int_V \rho \cdot \vec{v}.\vec{E}\, d^3x$$

$$= \int_V \vec{J}.\vec{E}\, d^3x \quad (1)$$

This power represents a conversion of electro magnetic energy into mechanical or thermal thermal energy. It is balanced by rate of decrease of energy in electromagnetic field with in volume V. For this we use Modified Ampere's law to eliminate J.

$$\int_V \vec{J}.\vec{E}\, d^3x = \int_V \left[\vec{E}.(\vec{\nabla}\times\vec{H}) - \vec{E}\frac{\partial\vec{D}}{\partial t}\right] d^3x \quad --(2)$$

Now using vector identity.

$$\vec{\nabla}.(\vec{E}\times\vec{H}) = \vec{H}.(\vec{\nabla}\times\vec{E}) - \vec{E}(\vec{\nabla}\times\vec{H})$$

Since $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$\vec{\nabla}.(\vec{E}\times\vec{H}) = -H\frac{\partial \vec{B}}{\partial t} - \vec{E}.(\vec{\nabla}\times\vec{H})$$

$$\int_V \vec{J}.\vec{E}\, d^3x = -\int_V \left[\vec{\nabla}.(\vec{E}\times\vec{H}) + \vec{H}.\frac{\partial \vec{B}}{\partial t} + \vec{E}.\frac{\partial \vec{D}}{\partial t}\right] d^3x \quad --(3)$$

To proceed further we make two assumptions we assume that the macroscopic medium involved is linear in its electric & mag. properties. Then two time derivative in eq^n(3) can be interpreted acc^n to the equation given below as the time derivative of electrostatics & magnetic energy densities.

$$u = \frac{1}{2}\vec{E}.\vec{D} \quad --(4)$$

$$u = \frac{1}{2}\vec{H}.\vec{B} \quad ---(5)$$

we know that

$\vec{D} = \epsilon\vec{E}$ $\qquad \vec{B} = \mu\vec{H}$

Second assumption is sum of eq^n (4) & (5) represents the total electromagnetic energy for time-varying field. Then total energy density is

$$u = \frac{1}{2}[\vec{E}.\vec{D} + \vec{B}.\vec{H}]$$

$$\frac{\partial u}{\partial t} = \frac{1}{2}\left[\frac{\partial}{\partial t}(\vec{E}.\vec{D}) + \frac{\partial}{\partial t}(\vec{B}.\vec{H})\right]$$

$$= \frac{1}{2}\left[\frac{\partial}{\partial t}(\vec{E}.\epsilon\vec{E}) + \frac{\partial}{\partial t}(\mu\vec{H}.\vec{H})\right]$$

$$= \frac{1}{2}\left[\epsilon\frac{\partial(E^2)}{\partial t} + \mu\frac{\partial(H^2)}{\partial t}\right]$$

$$\frac{\partial u}{\partial t} = \frac{1}{2}\left[\epsilon\ 2\vec{E}\frac{\partial \vec{E}}{\partial t} + \mu\ 2H\frac{\partial H}{\partial t}\right]$$

$$\frac{\partial u}{\partial t} = \left[\vec{E}\cdot\frac{\partial \vec{D}}{\partial t} + \vec{H}\cdot\frac{\partial \vec{B}}{\partial t}\right] \quad \text{---(6)}$$

Now eqn (3) can be written as. also used (6) in (3)

$$\int_V \vec{J}\cdot\vec{E}\, d^3x = -\int_V\left[\vec{\nabla}\cdot(\vec{E}\times\vec{H}) + \frac{\partial u}{\partial t}\right]d^3x \quad \text{---(7)}$$

Since volume 'V' is arbitrary this can be cast into the form of a differential continuity equation or conservation law.

$$\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} = -\vec{J}\cdot\vec{E} \quad \text{---(8)}$$

where vector S, representing energy flow, is called the poynting vector. It is given by.

$$\vec{S} = \vec{E}\times\vec{H} \quad \text{---(9)}$$

and has the dimension of energy/area×time. The poynting vector is arbitrary to the extent that the curl of any vector field can be added to it. The physical meaning of the integral or differential form (7) or (8) is that the time rate of change of electromagnetic energy with in a certain volume plus the energy flowing out through the boundary surfaces of the volume per unit time is equal to −ve of work done by fields on the sources with in volume.

Since matter is ultimately composed

of charged particles (e^- & nuclei), we can think of this rate of conversion as a rate of increase of energy of the charged particle per unit volume. Then we can interpret Poynting's theorem for the microscopic fields as a statement of conservation of energy of combined system of particle & fields.

We denote total energy of particle within volume V as E_{mech} & assume that no particle move out side of the volume, then we have

$$\frac{dE_{mech}}{dt} = \int_V \vec{J}\cdot\vec{E}\, d^3x \quad \text{particle energy}$$

then poynting theorem expresses the conservation of energy for the combined system as

$$\frac{dE}{dt} = \frac{d}{dt}(E_{mech} + E_{field}) \quad \text{--- (10)}$$

where total field energy with in V is

$$E_{field} = \int_V u\, d^3x = \frac{\epsilon_0}{2}\int(\vec{E}^2 + c^2\vec{B}^2)\, d^3x$$

$$E_{field} = \int_V \vec{\nabla}\cdot s\, d^3x$$

$$E_{field} = -\oint \vec{n}\cdot\vec{s}\, da \quad \text{--- (11)}$$

Conservation of linear momentum can be similarly considered. The total electromagnetic force on a charged particle is

$$F = q(\vec{E} + \vec{v}\times\vec{B})$$

If sum of all momenta ~~can be similarly considered~~ of all particles in the volume V is denoted by P_{mech}, from Newton's II^{nd} law.

$\rho . v = J$

$$\frac{dP_{mech}}{dt} = \int_V (\rho E + J \times B)\, d^3x \quad \text{---} (12)$$

We use Maxwell equations to eliminate ρ and J

$$\rho = \epsilon_0 \vec{\nabla}.\vec{E} \qquad \vec{J} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B} - \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

$\frac{1}{\mu_0 \epsilon_0} = c^2$

RHS of above eq^n is

$$= \int \left[\epsilon_0 \vec{\nabla}.\vec{E}\cdot\vec{E} + \epsilon_0 \left[c^2 \vec{\nabla}\times\vec{B} - \frac{\partial \vec{E}}{\partial t} \right] \times \vec{B} \right] d^3x$$

$$\rho\vec{E} + \vec{J}\times\vec{B} \Rightarrow \epsilon_0 \left[\vec{E}.(\vec{\nabla}.\vec{E}) + c^2(\vec{\nabla}\times\vec{B})\times\vec{B} + \vec{B}\times\frac{\partial \vec{E}}{\partial t} \right]$$

$$\because \vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$$

$$\vec{B}\times\frac{\partial \vec{E}}{\partial t} = -\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) + \vec{E}\times\frac{\partial \vec{B}}{\partial t}$$

$\frac{\partial B}{\partial t} = -\vec{\nabla}\times\vec{E}$

$$= \epsilon_0 \left[\vec{E}.(\vec{\nabla}.\vec{E}) + \left\{ -\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) + \vec{E}\times\frac{\partial \vec{B}}{\partial t} \right\} + c^2(\vec{\nabla}\times\vec{B})\times\vec{B} \right]$$

Now adding $c^2 B (\nabla . B) = 0$ to the square braket

$$= \epsilon_0 \left[\vec{E}(\vec{\nabla}.\vec{E}) + c^2\vec{B}(\vec{\nabla}.\vec{B}) - \vec{E}\times(\vec{\nabla}\times\vec{E}) + c^2(\vec{\nabla}\times\vec{B})\times\vec{B} - \frac{\partial}{\partial t}(\vec{E}\times\vec{B}) \right]$$

Now eq^n (12) becomes

$$\frac{dP_{mech}}{dt} + \frac{\partial}{\partial t}\int \epsilon_0 (\vec{E}\times\vec{B})\, d^3x = \epsilon_0 \int \left\{ \vec{E}(\vec{\nabla}.\vec{E}) + c^2 \vec{B}(\vec{\nabla}\vec{B}) - \vec{E}\times(\vec{\nabla}\times\vec{E}) + c^2(\vec{\nabla}\times\vec{B})\times B \right\} d^3x$$

$$\frac{dP_{mech}}{dt} + \frac{d}{dt}\int_V \epsilon_0 (\vec{E}\times\vec{B})\, d^3x = \epsilon_0 \int_V \Big[\vec{E}(\nabla.\vec{E}) - \vec{E}\times(\vec{\nabla}\times\vec{E})$$
$$+ c^2\vec{B}(\vec{\nabla}.\vec{B}) - c^2\vec{B}\times(\vec{\nabla}\times\vec{B})\Big] d^3x \quad \text{---(13)}$$

The volume integral on the left is as the total electro-magnetic momentum P_{field} in the volume V.

$$P_{field} = \epsilon_0 \int_V (\vec{E}\times\vec{B})\, d^3x$$

$B = \mu_0 H$

$$\Rightarrow \quad P_{field} = \mu_0\epsilon_0 \int (\vec{E}\times\vec{H})\, d^3x$$

The integrand can be interpreted as a density of eletromagnetic momentum. We remember that this momentum density is proportional to the energy flux density S, with proportionality constt. c^2.

$$\vec{g} = \frac{1}{c^2}(\vec{E}\times\vec{H})$$

Convert the volume integral on the right side into a surface integral of the normal component of something which can be identified as momentum flow.

Let the cartesian coordinate be denoted by x_α, $\alpha = 1, 2, 3$.

The $\alpha = 1$ component of the electric part of integrand in eqn (13) is as

Now the RHS of eqn (13)

$E(\nabla.E) - E\times(\nabla\times E)$

We know the product rule

$$\nabla(E.E) = 2(E.\nabla)E + 2E\times(\nabla\times E)$$

$$E \times (\nabla \times E) = \frac{1}{2}\nabla(E^2) - (E\cdot\nabla)E$$

Now $E(\nabla\cdot E) - E\times(\nabla\times E) = E(\nabla\cdot E) - \frac{1}{2}\nabla(E^2) + (E\cdot\nabla)E$

$$\Rightarrow E\left(\hat{i}\frac{\partial}{\partial x_1} + \hat{j}\frac{\partial}{\partial x_2} + \hat{k}\frac{\partial}{\partial x_3}\right)(E_1\hat{i} + E_2\hat{j} + E_3\hat{k}) + (E_1\hat{i} + E_2\hat{j} + E_3\hat{k})\cdot$$

$$\left(\hat{i}\frac{\partial}{\partial x_1} + \hat{j}\frac{\partial}{\partial x_2} + \hat{k}\frac{\partial}{\partial x_3}\right)E - \frac{1}{2}\nabla(E^2)$$

$$\Rightarrow E\left[\frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} + \frac{\partial E_3}{\partial x_3}\right] + \left(E_1\frac{\partial}{\partial x_1} + E_2\frac{\partial}{\partial x_2} + E_3\frac{\partial}{\partial x_3}\right)E - \frac{1}{2}\nabla(E^2)$$

$$\Rightarrow \cancel{(E_1\hat{i} + E_2\hat{j} + E_3\hat{k})} + \cancel{\left(E_1\frac{\partial}{\partial x_1} + E_2\frac{\partial}{\partial x_2} + E_3\frac{\partial}{\partial x_3}\right)(E_1\hat{i} + E_2\hat{j} + E_3 k)} - \cancel{\frac{1}{2}\nabla(E^2)}$$

$$\Rightarrow (E_1\hat{i} + E_2\hat{j} + E_3 k)\left(\frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} + \frac{\partial E_3}{\partial x_3}\right) + \left(E_1\frac{\partial}{\partial x_1} + E_2\frac{\partial}{\partial x_2} + E_3\frac{\partial}{\partial x_3}\right)(E_1\hat{i} + E_2\hat{j} + E_3\hat{k}) - \frac{1}{2}\left(\hat{i}\frac{\partial}{\partial x_1} + \hat{j}\frac{\partial}{\partial x_2} + \hat{k}\frac{\partial}{\partial x_3}\right)(E_1^2 + E_2^2 + E_3^2)$$

taking component of i^{th}.

$$\Rightarrow \sum_i E_1\left[\frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} + \frac{\partial E_3}{\partial x_3}\right] + \left[E_1\frac{\partial E_1}{\partial x_1} + E_2\frac{\partial E_1}{\partial x_2} + E_3\frac{\partial E_1}{\partial x_3}\right] - \frac{1}{2}\frac{\partial}{\partial x_1}(E_1^2 + E_2^2 + E_3^2)$$

$$\Rightarrow \sum_i 2E_1\frac{\partial E_1}{\partial x_1} + E_1\frac{\partial E_2}{\partial x_2} + E_2\frac{\partial E_1}{\partial x_2} + E_1\frac{\partial E_3}{\partial x_3} + E_3\frac{\partial E_1}{\partial x_3} - \frac{1}{2}\frac{\partial}{\partial x_1}(E_1^2 + E_2^2 + E_3^2)$$

$$\Rightarrow \sum_i \frac{\partial (E_1^2)}{\partial x_1} + \frac{\partial}{\partial x_2}(E_1E_2) + \frac{\partial}{\partial x_3}(E_1E_3) - \frac{1}{2}\frac{\partial}{\partial x_1}(E_1^2 + E_2^2 + E_3^2)$$

$$[E(\nabla\cdot E) - E\times(\nabla\times E)]_\alpha = \sum_\beta \frac{\partial}{\partial x_\beta}\left(E_\alpha E_\beta - \frac{1}{2}E\cdot E\,\delta_{\alpha\beta}\right)$$

$$\frac{dP_{mech}}{dt} + \frac{d}{dt}\int_V \epsilon_0$$

$$\frac{d}{dt}\left[P_{mech} + P_{field}\right]_\alpha = \epsilon_0 \int_V \sum_{\beta=1}^{3} \frac{\partial}{\partial x_\beta}\Big\{\left(E_\alpha E_\beta - \frac{1}{2}\vec{E}.\vec{E}\,\delta_{\alpha\beta}\right)$$

$$+ c^2\left(B_\alpha B_\beta - \frac{1}{2}\vec{B}.\vec{B}\,\delta_{\alpha\beta}\right)\Big\}\, d^3x$$

We define a quantity called Maxwell Stress Tensor $T_{\alpha\beta}$ of Second Rank.

$$T_{\alpha\beta} = \epsilon_0\left[E_\alpha E_\beta + B_\alpha B_\beta - \frac{1}{2}\left(E.E + c^2 B.B\right)\delta_{\alpha\beta}\right]$$

$$\frac{d}{dt}\left[P_{mech} + P_{field}\right]_\alpha = \sum_\beta \int_V \frac{\partial}{\partial x_\beta} T_{\alpha\beta}\, d^3x$$

Using Gauss Div. theorem

$$\frac{d}{dt}\left[P_{mech} + P_{field}\right]_\alpha = \oint_S \sum_\beta T_{\alpha\beta}\, n_\beta\, da$$

where n is normal to surface.

Statement :- The work done on the charges by electromagnetic force is equal to the decrease in energy stored in the field, less the Energy that flowed out through the surface.

The energy per unit time, per unit area transported by fields is called the Poynting Vector.