MOHAMED BACHIR UNIVERSITY - BBA **Analysis 1**
Faculty : Mathematics and informatics **First year LMD**
Department : Operational research **2023/2024 (semester 1)**

Series of exercises $n°4$

Limits and continuity

Exercise 1:

- Calculate when they exist the following limits:

1. $\lim_{x\to 0} \frac{x^m}{x^n} = 0, (m, n \in \mathbb{N}^\star)$, 2. $\lim_{x\to 1} \sin \frac{x-1}{(x^n-1)}, (n \in \mathbb{N}^\star)$ 3. $\lim_{x\to \pi} \frac{\sin^2 x}{1+\cos x}$,

4. (⋇) $\lim_{x\to 0} \frac{\sqrt[3]{1+x^2}-1}{x^2}$, 5. $\lim_{x\to 0^+} 2x\ln(x+\sqrt{x})$, 6. $\lim_{x\to +\infty} \frac{e^x}{x^n} (n \in \mathbb{N})$,

7. $\lim_{x\to +\infty} \sin(x+1)\ln|x+1|$, 8.(⋇) $\lim_{x\to 0^+} \frac{\ln(3x+1)}{2x}$,

9. $\lim_{x\to +\infty} \left(\frac{x+1}{x-3}\right)^x$, 10. $\lim_{x\to 0^+} (1+x)^{\ln(x)}$, 11. $\lim_{x\to 0} xE(\frac{1}{x})$

12. (⋇) $\lim_{x\to +\infty} [\ln(1+e^{-x})]^{\frac{1}{x}}$, 13. (⋇) $\lim_{x\to 0^+} \left(\frac{a^x+b^x}{2}\right)^{\frac{1}{x}}, (a, b) \in (\mathbb{R}_+^\star)^2$.

- Using the definition of the limit of a real function, demonstrate that:

1) $\lim_{x\to 0} x\sin\frac{1}{x} = 0$, 2) $\lim_{x\to 1} \frac{1}{(1-x)^2} = +\infty$, 3) (⋇) $\lim_{x\to 1} \frac{x+2}{x-1} = +\infty$, 4) $\lim_{x\to +\infty} \frac{1}{x^2} = 0$,

Exercise 2: Let $f : \mathbb{R} \to \mathbb{R}$ a periodic function with period T. Demonstrate that:

1. If f continues on $\mathbb{R}$, so f is bounded on $\mathbb{R}$.
2. Si f admits a limit $l (l \in \mathbb{R})$ at $+\infty$, so f is a constant.

Exercise 3: Let $\alpha \in \mathbb{R}^\star$ and $\mathbb{R} \to \mathbb{R}$ a function defined by:

$$f(x) = \begin{cases} x^2 + \frac{\sqrt{(x+\alpha)^2}}{x+\alpha}, & x \neq -\alpha \\ 0, & x = -\alpha \end{cases}$$

- Determine the set of points where it is continuous.

Exercise 4:

Let be the functions:

$$f(x) = \begin{cases} \frac{x^2 \sin\frac{1}{x}}{e^x - 1}, & x < 0 \\ ax+b, & 0 \le x \le 1 \\ \sqrt{x+3}, & x > 1 \end{cases}, g(x) = \begin{cases} 1 + x^2\cos(x^2), & x < 0 \\ 1, & x = 0 \\ \frac{\ln(1+\sin x)}{x}, & x > 0 \end{cases}$$

1) Determine the values of a and b so that f is continuous on $\mathbb{R}$.

2) Demonstrate that g is continuous on $x_0 = 0$.

3) Demonstrate that the equation $g(x) = -\frac{\pi}{2}$ has a solution in $\left[0, -\sqrt{\pi}\right]$.

Exercise 5:

- Calculate the following limits using the equivalent functions:

$$1)\ \lim_{x\to 0}\frac{\tan 2x}{x},\quad 2)\ \lim_{x\to 0}\frac{2-\cos x-\cos 2x}{\tan^2 x}.\quad 3)(\circledast)\ \lim_{x\to 0^+}\ln x.\ln\left[1+\ln(1+x)\right].$$

$$4)\ \lim_{x\to 0}\frac{x\ln x}{\sin(2x)},\quad 5)\ \lim_{x\to 0}\frac{\ln(1+(\sin x)^2)}{\tan\frac{x}{2}}.\quad 6)(\circledast)\ \lim_{x\to_< e}\ln(e-x).\ln(\ln(x)).$$

$$7)\lim_{x\to+\infty}\left(\frac{\ln(1+x)}{\ln(x)}\right)^{x\ln x},\ 8)\ \lim_{x\to\frac{\pi}{2}} e^{\frac{1}{2x-\pi}(\ln(\sin x))},\ 9)(\circledast)\lim_{x\to+\infty}x(e^{\frac{1}{x}}-1), 10)\lim_{x\to 0}\frac{\tan x-\sin x}{\sin x(\cos 2x-\cos x)}.$$

Exercise 6: Say whether the following functions can be extended by continuity on $\mathbb{R}$:

$$1)\ f(x)=\frac{(x+1)|x-1|}{x-1},\quad 2)(\circledast)\ g(x)=\frac{1-\cos\sqrt{|x|}}{|x|}.\quad 3)(\circledast)\ h(x)=e^{\frac{-1}{x^2}}.$$

$$4)\ K(x)=x\sin\frac{1}{x},\quad 5)(\circledast)\ l(x)=\sin\frac{1}{x}.\quad 6)(\circledast)\ m(x)=\cos x\cos\frac{1}{x}.$$

$$7)(\circledast)\ f_1(x)=\frac{x^2}{|x|},\quad 8)\ f_2(x)=\frac{1}{1+e^{\frac{1}{x}}},\quad 9)\ f_3(x)=\frac{\ln(1+x^\alpha)}{e^x-1},\alpha>0$$

Exercise 7:

Let be the function f defined on $[0,+\infty[$ by $f(x)=\sqrt{x}$.

1. Demonstrate that $\forall x, x' \in [0,+\infty[: |\sqrt{x}-\sqrt{x'}| \le \sqrt{|x-x'|}$,then deduce that the function f is uniformly continuous on $[0,+\infty[$.
2. Demonstrate that the function g defined on $\mathbb{R}$ by $g(x) = x + \sin x$ is uniformly continuous on $\mathbb{R}$.

Exercise 8:

Let the function :$f(x) = \dfrac{x+1}{x+2}$.

1) Demonstrate that the function f is uniformly continuous on $[0,+\infty[$.

2) Show that any continuous application f of a segment $[a,b]$ in itself admits a fixed point.

Reminder

- $\sin x \sim x$, • $\tan x \sim x$, • $1-\cos x \sim \dfrac{x^2}{2}$
- $\ln(1+x) \sim x$ • $(e^x-1)\sim x$, • $(1+x)^\alpha - 1 \sim \alpha x$

NB : questions that end with ($\circledast$) left to students to answer.