Extended Essay Mathematics

SOLVING A ONE DIMENSIONAL WAVE EQUATION USING SEPARATION OF VARIABLES

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Introduction

Calculus, or the study of continuous change, is an area within mathematics that I find especially intriguing since it gave me a new perspective on the subject as a whole. Derivatives in particular was one of my first introductions to the very abstract thinking that is used by both mathematicians and physicists when tackling complex problems. The paradoxical property of measuring the change in an essentially non-existent distance between two points and thus deriving a new function, made me realize what a powerful tool imagination is within mathematics and how useful it can be even if it intuitively doesn't make sense. A differential equation is a system that relates a function to its own derivative and is a very important concept within physics as they're used to describe several phenomena. Differential equations are divided into two forms: Ordinary (ODE) and partial (PDE), where the former contains the derivatives of a function that has only one independent variable, in contrast to the latter which can be in respect to several independent variables. One famous example of a PDE is the one dimensional wave equation, which contains both a spatial and time variable. I've worked with simpler wave equations before in physics, so I decided to investigate what's required to solve a more complex problem. There are many ways to solve the equation, but I chose to investigate the following: Can you solve the one dimensional wave equation using the method of separation of variables? Separation of variables allows you to rewrite the equation in such a way that all expressions containing the same variable are on the same side of the equation and can thus be separated into two different ODEs.

Setting up the problem

The homogenous one-dimensional undamped wave equation describes the propagation of wave in one dimension between two fixed points where the amplitude remains constant. In this case, consider a piece of thin flexible string with a length of L and of negligible weight. When a pulse travels through the string, it will oscillate back and forth between the fixed points in the manner of a transverse standing wave. Thus the solution of the equation $u(x,t)$ is the displacement function that describes the vertical displacement of the string at any horizontal position of *x*, *§1*where $0 < x < L$, and at any time *t*, where $t > 0$. This is because there can't be any motion at the position where the string is clamped and since we're measuring from the start of the wave motion time can only be positive. The equation itself is:

$$u_{tt} = c^2 u_{xx}$$

Where u_{tt} is the second derivative of $u(x,t)$ with respect to time, i.e. the vertical acceleration, and correspondingly u_{xx} is the second derivative of $u(x,t)$ with respect to position. The constant coefficient $c^2$is the horizontal propagation speed. It's given by the equation $c^2 = T/\rho$ where T is the force of tension exerted on the string and ρ is mass density (mass per unit length) of the string.

To solve this differential equation, some boundary conditions are essential. Boundary conditions are used to restrain an equation at its extreme points, and used together with the domain of a function it can be used to find the appropriate general solution to the problem. In this case the boundary conditions are $u(0,t) = 0$ and $u(L,t) = 0$, which describes how there can be no vertical displacement at the ends of the string since they are clamped and therefore are held motionless at all time. The boundary conditions will be necessary to disregard the trivial solutions (i.e. solutions that will always lead to 0) for this equation and are thus essential in finding the eigenvalues and the corresponding eigenfunctions for the solution. Eigenfunctions are a form of eigenvectors that when acted upon by a linear operator, is only multiplied by some scalar, the eigenvalue. Eigenvectors never change direction and are perpendicular, two properties that will prove useful later on.

The general solution to the equation found with the boundary conditions will contain some unknown coefficients that one can find using some initial conditions. The initial condition describes the vertical displacement when $t = 0$, i.e. before the wave motion has started. The equation has two initial conditions since it involves a second derivative. The initial conditions in this case are $u(x,0) = f(x) = 2x$, which describes the initial vertical displacement as a function of x, and $u_t(\mathrm{x},0) = \mathrm{g(x)} = 2$, which describes the initial vertical velocity as a function of x (though it is a constant in this case). The exact values of the initial conditions are arbitrary and just meant as an example when solving the differential equation.

The wave problem is given by (Tseng. 2012a):

$$u_{tt} = c^2 u_{xx}$$

$$0 < x < L \qquad\qquad t > 0$$

$$u(0,t) = 0 \qquad\qquad u(L,t) = 0$$

$$u(x,0) = f(x) = 2x \qquad\qquad u_t(\mathrm{x},0) = \mathrm{g(x)} = 2$$

Separating the variables

The method of separation of variables involves rewriting the equation $u(x,t)$ in such a way that there is only one independent variable on each side of the equation. The expression can then be separated into two different ODEs. The function $u(x,t)$ is linear and homogeneous so I will assume that the solution can be written in the special form as the product of the spatial and time variable (Tseng. 2012b):

$$u(x,t) = X(x)T(t)$$

Where X is a function of the spatial variable *x* and T is a function of the time variable *t*. This might seem as an odd assumption to make but this allows the function $u(x,t)$ to be split up in the time-axis and the position-axis. Consequently the partial derivatives of $u(x,t)$ can be expressed as:

$$u_{xx}(x,t) = X''T \qquad\qquad u_{tt}(x,t) = XT''$$

Where $''$ denotes the function being a second derivative with respect to the corresponding variable. Substituting the expressions into the original wave equation $u_{tt} = c^2 u_{xx}$ yields:

$$XT'' = c^2 X''T$$

Now the variables can be separated by bringing all the *x*-terms to one side and all *t*-terms to the other.

$$\frac{X''}{X} = \frac{T''}{c^2 T}$$

I assume that $X \neq 0$ and $T \neq 0$ in order to avoid arriving at the trivial solution.

The expression on the left hand side depends only on the spatial variable *x*, and similarly the right hand side depends only on the time variable *t*. Thus no matter you tweak either of each variable, the two ratios will always be equal to the same constant. Let this constant be $-\lambda$. I chose a negative value to simplify further calculations as the two ODEs will now only obtain positive values.

$$\frac{X''}{X} = \frac{T''}{c^2 T} = -\lambda$$

The ratios between *x* and *t* will always be equal to the same constant which means that they can be split up into two ODEs:

$\frac{X''}{X} = -\lambda$ => $X'' + \lambda X = 0$ and $\frac{T''}{c^2 T} = -\lambda$ => $T'' + c^2\lambda T = 0$

The original equation has been broken up into two ODEs.

- A second order linear homogeneous ODE with respect to *x:* $X'' + \lambda X = 0$
- A second order linear homogeneous ODE with respect to *t:* $T'' + c^2\lambda T = 0$

The equations are linear since they appear in a linear fashion independent variable and all its derivatives are not dependent on some coefficients in terms of the independent variable (i.e. the terms are not multiplied or squared with one another).

Comparing the ODEs to the most general form of second order linear differential equations (Dawkins. 2019a):

$$y'' + p(t)y' + q(t)y = g(t)$$

In both cases, $g(t) = 0$, which means that both ODEs are homogeneous. Furthermore, in both cases $q(t)$ is equal to some constant which means that the equations are second order linear homogeneous ODEs with constant coefficients.

Now that the PDE has been separated into two ODEs, I can rewrite the boundary conditions in the same fashion:

$$U(0,t) = 0 \Rightarrow X(0)T(t) = 0 \qquad \text{So: } X(0) = 0 \text{ or } T(t) = 0$$

$$U(L,t) = 0 \Rightarrow X(L)T(t) = 0 \qquad \text{So: } X(L) = 0 \text{ or } T(t) = 0$$

I will ignore the boundary condition $T(t) = 0$ since it will lead to the trivial solution.

In summary: By assuming the solution to $u(x,t)$ can be written in a special form, I've been able to separate the original PDE into two separate ODEs using the separation constant $-\lambda$. I've also established that the two second order ODEs are linear and homogenous. Furthermore I've also rewritten the boundary conditions in the same fashion as the ODEs:

$$X'' + \lambda X = 0 \qquad X(0) = 0 \text{ and } X(L) = 0$$

$$T'' + c^2\lambda T = 0$$

Eigenvalues and Eigenfunctions within boundary conditions

This section will be devoted to solving the second order linear homogeneous ODE with constant coefficients that is in respect to the spatial variable from the previous section:

$$X'' + \lambda X = 0 \qquad\qquad X(0) = 0 \text{ and } X(L) = 0$$

The equation is restricted by two boundary conditions that equation has to fulfill which makes it a boundary value problem (BVP). An important detail to note is that the boundary conditions are homogeneous since they both equal to zero, just like the ODE was homogeneous since $g(t) = 0$. This to ensure that solutions to the problem actually exists, as otherwise the boundary conditions could lead to a trivial solution or no solution at all.

By inspecting the problem further I see that $\frac{d^2}{dx^2}$ is a differential operator that acts upon the function X to produce a new scalar multiple of itself, λX. Recognizing that this is the same relation of eigenvectors and eigenvalues in linear algebra (Dawkins. 2019b):

$$A\vec{x} = \lambda\vec{x}$$

Where A is a $n\ x\ n$ square matrix and $\vec{x}$ is an eigenvector in the form of $n\ x\ 1$ column vector. It's clear that I am dealing with an eigenvalue problem. Eigenvalue problems include either an eigenfunction or vector with some operator that turns it into some new entity which is the scalar multiple of the same vector or function. The goal is thus to find the eigenvalue of λ which corresponds to the eigenfunction of X (which also acts an eigenvector) so that the expression holds true. Eigenvectors has the property of only changing in scale and never in direction, which will become useful when I reach the general solution as our eigenfunctions will have the same vector direction regardless of how it is multiplied or summed up.

To find the eigenfunction and corresponding eigenvalue, the roots of the characteristic polynomial is needed. To find the roots I will compare the BVP with the general form of linear homogeneous second order differential equations with constant coefficients (Dawkins. 2019a):

$$ay'' + by' + cy = 0$$

Where *a*, *b* and *c* corresponds respectively to some constant. The solutions for these equations are made up of exponential functions and therefore we can substitute our solution $y = e^{rx}$ with r as an unknown constant since e^{x} has the property of being equal to its own derivative:

$$ay'' + by' + cy = 0 \quad => \quad ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$$

Factorizing out $e^{rx}$of the equation:

$$e^{rx}(ar^2 + br + c) = 0$$

Since e^{rx} is an exponent, it can never be equal to zero:

$$ar^2 + br + c = 0$$

So the characteristic equation for the general form of a second order linear homogenous equation is a simple quadratic:

$$ay'' + by' + cy = 0 \quad => \quad ar^2 + br + c = 0$$

So a comparison between the ODE and its characteristic equation yields:

$$X'' + \lambda X = 0 \quad => \quad r^2 + \lambda = 0$$

Subsequently the characteristic root is:

$$r = \sqrt{-\lambda}$$

The characteristic equation for our polynomial is $r^2 + \lambda = 0$ and each of its roots will result in a different solution. The roots depend on the discriminant which in my case is:

$$b^2 - 4ac \quad => \quad -4\lambda$$

There are three possible types of roots depending on the value of the discriminant (Tseng 2012b):

1. Distinct real roots when $-4\lambda > 0$, so $\lambda < 0$
2. Two equal roots when $-4\lambda = 0$, so $\lambda = 0$
3. Complex roots when $-4\lambda < 0$, so $\lambda > 0$

I need to examine each of these scenarios which I'll substitute into the characteristic equation to obtain the different roots. The different roots will be used to find the general solution that I can use together with the boundary conditions to disregard any trivial solutions.

Case 1. $\lambda < 0$

Let's denote $\sqrt{-\lambda}$ as σ

$$\sqrt{-\lambda} = \sigma \qquad => \qquad \lambda = -\sigma^2$$

The characteristic equation and root subsequently becomes:

$$r^2 - \sigma^2 = 0 \qquad => \qquad r = \pm \sigma$$

The general solution for two distinct real roots is (Tseng. 2012b):

$$X(x) = C_1 e^{\sigma x} + C_2 e^{-\sigma x}$$

Applying the first boundary condition:

$$X(0) = C_1 e^0 + C_2 e^0 = 0$$

$$C_1 = -C_2$$

C_1 and C_2 can thus be rewritten as C and $-C$ respectively. Applying the second boundary condition:

$$X(L) = Ce^{\sigma L} - Ce^{-\sigma L} = 0$$

$$C(e^{\sigma L} - e^{-\sigma L}) = 0$$

This equation can only hold true for real roots if $C = 0$ which will only give a trivial solution and can thus be discarded.

Case 2. $\lambda = 0$

$$r^2 = 0 \qquad => \qquad r = 0$$

There are two equal roots so the general solution is (Tseng. 2012b):

$$X(x) = C_1 + C_2 x$$

Applying the first boundary condition:

$$X(0) = C_1 = 0$$

Substituting $C_1 = 0$ and applying the second boundary condition gives the expression:

$$X(L) = C_2 L = 0$$

Since $L > 0$ the equation only holds true if $C_2 = 0$, hence the general solution is:

$$X(x) = 0 + 0x = 0$$

Which is trivial and can be discarded.

Case 3. $\lambda > 0$

Let's denote $\sqrt{\lambda}$ as σ

$$\sqrt{\lambda} = \sigma \quad => \quad \lambda = \sigma^2$$

$$r^2 + \sigma^2 = 0 \quad => \quad r = \pm \sigma i$$

The general solution for complex roots is (Tseng. 2012b):

$$X(x) = C_1 \cos(\sigma x) + C_2 \sin(\sigma x)$$

Applying the first boundary condition:

$$X(0) = C_1 \cos(0) + C_2 \sin(0) = 0$$

$$C_1 = 0$$

Applying second boundary condition:

$$X(L) = 0 = C_1 \, cos(\sigma L) + C_2 \, sin(\sigma L)$$

$$C_2 \, sin(\sigma L) = 0$$

$\sigma \neq 0$, and C_1 and C_2 cannot both be equal to zero as it would give the trivial solution, hence:

$$\sin(\sigma L) = 0$$

Since $\sin(\pi) = 0$, the equation implies that σL is equal to π or any multiple, n, of π. Therefore:

$$\sigma L = n\pi$$

$$\sigma = \frac{n\pi}{L}$$

Thus the eigenvalue for the first BVP is:

$$\lambda = \sigma^2 \qquad => \qquad \lambda = \frac{n^2\pi^2}{L^2}$$

To each eigenvalue there is an eigenfunction that corresponds to the solution. In this case the eigenfunction that corresponds to the spatial second order ODE with coefficient $\lambda = \frac{n^2\pi^2}{L^2}$ is a periodic function:

$$X_n(x) = \sin\frac{n\pi x}{L}$$

Where *n* is any positive integer. The eigenfunction was found by substituting the value of σ into the general solution to the complex roots, and since $C_1 = 0$, I was left with just sinus function. Though I dropped the constant C_2 since the eigenfunction in itself is the only matter of interest, and because the solution will hold true regardless of the constant in this case.

In summary I recognized that the spatial ODE was both a BVP and an eigenvalue problem where I had to find the eigenvalue and corresponding eigenfunction. By finding the characteristic roots of the polynomial and discarding any trivial solution using the boundary conditions, I deduced that the solution was an infinite set of eigenvalues and corresponding eigenfunctions which are respectively:

$$\lambda = \frac{n^2\pi^2}{L^2} \qquad \text{And} \qquad X_n = \sin\frac{n\pi x}{L} \qquad \text{Where } n = 1, 2, 3 \ldots$$

General solution:

In the previous section I found the eigenvalue and corresponding eigenfunction for the spatial ODE. I will solve the second order linear homogeneous ODE with respect to time:

$$T'' + c^2\lambda T = 0$$

The two ODEs are connected since they make up the same system of simultaneous equations. Consequently I can substitute the eigenvalue $\lambda = \frac{n^2\pi^2}{L^2}$ of the spatial variable into the time variable ODE:

$$T'' + c^2 \frac{n^2\pi^2}{L^2} T = 0$$

It's a second order homogenous linear equation with constant coefficients. So its characteristic equation has a pair of purely complex roots (Tseng. 2012a):

$$r = \pm \frac{cn\pi}{L} i \qquad \text{Where } n = 1, 2, 3 \ldots$$

In the previous section I determined that the general solution is then:

$$T(t) = C_1 \cos(\sigma t) + C_2 \sin(\sigma t) \qquad \text{Where } \sigma = \frac{cn\pi}{L}$$

So the corresponding eigenfunction is a simple harmonic function with constants A_n and B_n:

$$T_n(t) = A_n \cos \frac{cn\pi t}{L} + B_n \sin \frac{cn\pi t}{L} \qquad \text{Where } n = 1, 2, 3 \ldots$$

I have now obtained the two eigenfunctions which correspond to the eigenvalue the two simultaneous equations. These eigenfunctions propagate independently in their respective axis, position and time, and are thus not affected by each other. Recall that I assumed the solution to $u(x,t)$ could be in the form of the product of the spatial and time variable:

$$u(x,t) = X(x)T(t)$$

The equation is a linear system, and thus the principles of superposition applies, which are the properties of additivity, $F(x_1 + x_2) = F(x_1) + F(x_2)$, and homogeneity, $F(ax) = aF(x)$ (Wikipedia. 2019). In other words any multiple, sum, difference, sum of the multiples, or difference of the multiples of any two solutions will also be a solution. Thus taking the sum of the product of each of our eigenfunctions of ODEs yields the general solution to the one-dimensional linear homogeneous wave equation (Tseng. 2012a):

$$u(x,t) = \sum_{n=1}^{\infty} (A_n \cos \frac{cn\pi t}{L} + B_n \sin \frac{cn\pi t}{L}) \sin \frac{n\pi x}{L}$$

I was able to do this due to the property of eigenvectors being orthogonal, or in other terms, being perpendicular to each other. Thus what I've essentially done is combine two eigenvectors that propagates in their own respective axes, unchanged by one another, into one general periodic wave equation. It is helpful to visualize this in a three dimensional space, as despite it being called a one-dimensional wave, it's actually monitored in three axes: Time, horizontal position and the resultant vertical displacement of the former two:

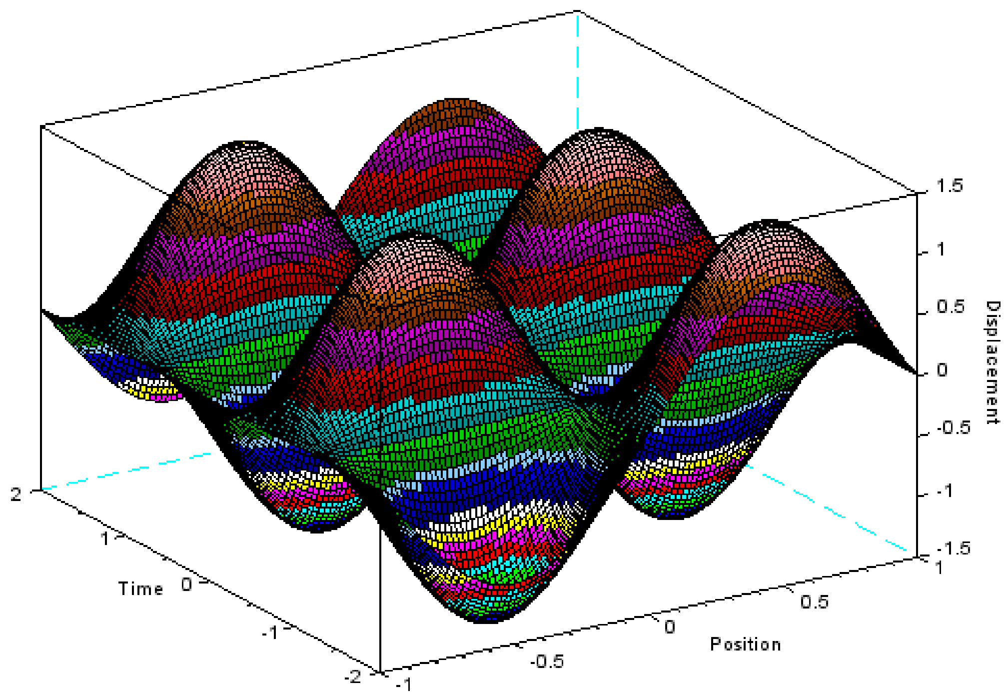

Figure 1 showing the function $u(x,t) = (A_n \cos\frac{cn\pi t}{L} + B_n \sin\frac{cn\pi t}{L}) \sin\frac{n\pi x}{L}$ for $= L = A_n = B_n = n = 1$. Where *u* is the resultant vertical displacement, *x* is the position and *t* is the time. Note the graph is only valid in this case for $0 \leq t$ and $0 \leq x$. The colours represent different levels of displacement of the function and in this case are more there to make the graph easier to look at and understand. Made using Scilab.

Solving for coefficients

The equation from the previous section is an incomplete solution, since I have two unknown coefficients: A_n and B_n. To determine these I need to introduce the initial conditions that describe the initial vertical displacement and velocity as two different functions of the horizontal position:

$$u(x,0) = f(x) = 2x \qquad\qquad u_t(\mathrm{x},0) = \mathrm{g(x)} = 2$$

These initial conditions will be used to solve for the coefficients A_n and B_n respectively.

<u>Solving for</u>: A_n

Applying the first boundary condition for initial position: $u(x,0) = f(x)$:

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = f(x)$$

The solution to $f(x)$ will be an infinite series of sines and coefficients of A_n. At first glance this equation might be impossible to solve, however that is not the case. By recognizing that this is an odd Fourier sine series, I can make use of the very special property of our eigenfunctions being orthogonal to give the answer to A_n. A function is mutually orthogonal when (Dawkins. 2019c):

$$\int_a^b f_i(x) f_j(x) dx = f(x) = \begin{cases} 0 & i \neq j \\ c > 0 & i = j \end{cases}$$

So at some defined interval, the scalar product of the integral of two functions will always equal zero when the multiples of *j* and *i* are not equal to one another. I'll use this feature by multiplying each side with the odd periodic function of $sin\frac{m\pi x}{L}$ where *m* is some positive integer, and then integrate both sides within interval $0 \leq x \leq L$.

$$\int_0^L f(x) \sin\frac{m\pi x}{L} dx = \sum_{n=1}^{\infty} A_n \int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L} dx$$

Note that both the series and the coefficient A_n were factored out of the integral. It is not always possible to do this but in this particular case it is. I will now use the feature of the eigenfunctions' orthogonality with the interval $0 \leq x \leq L$ to find the non-zero solution: (Dawkins. 2019d)

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \begin{cases}\frac{L}{2} & n = m\\ 0 & n \neq m\end{cases}$$

Thus the case when $n \neq m$ can be discarded as it will only lead to the trivial zero-constant solution. This useful feature of the Fourier series has allowed me to narrow down an infinite number of possible solutions to only one possible integral. It's helpful to think of this property in terms of eigenvectors. When two vectors are perpendicular their scalar product is always equal to zero:

$$a \cdot b = |a||b| \cos 90 = 0$$

Where a and b are vectors and $|a|$ and $|b|$ are their respective scalar magnitudes. By the same way the eigenfunctions $sin\frac{n\pi x}{L}$ and $sin\frac{m\pi x}{L}$ will be perpendicular when they have different periods and cancel out due to the principal of superposition, and because of their orthogonality this will hold true for every $n \neq m$.

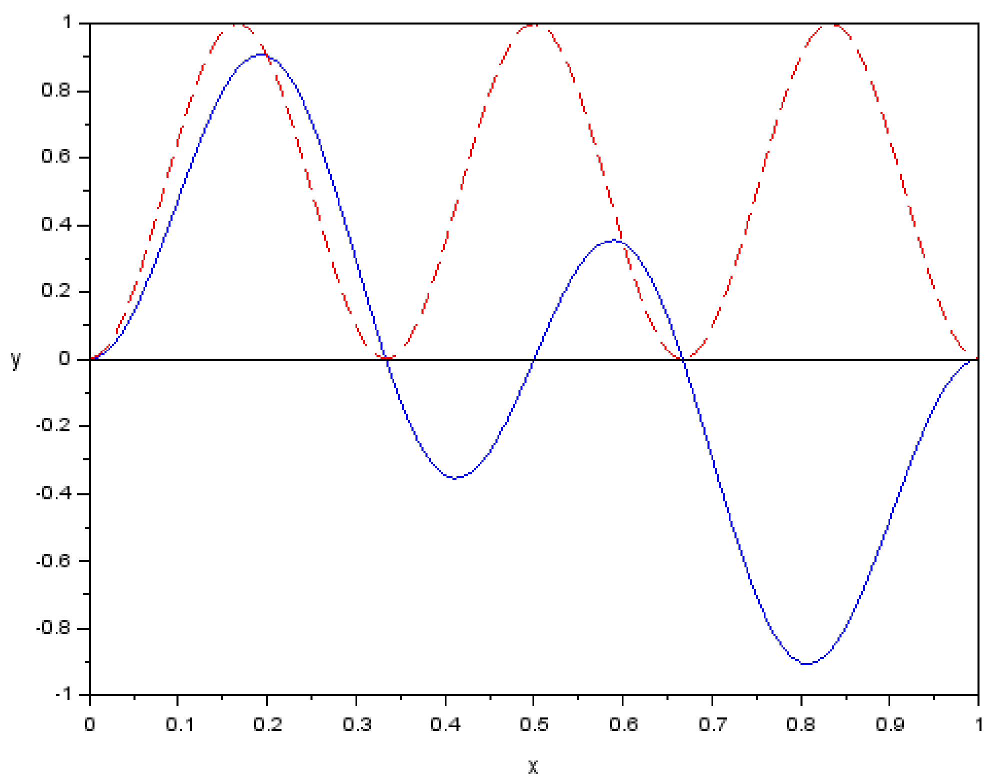

Figure 2 showing $y = \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}$ for $L = 1$, $n = 3$ and $m = 2$ (blue solid line) and $n = m = 3$ (red dashed line). The area beneath the blue curve will be zero and the area beneath the red curve will be a non-zero value. Made using Scilab.

The answer can thus be substituted into the orthogonal integral as the answer to $f(x)$:

$$\int_0^L f(x)\sin\frac{m\pi x}{L}\,dx = A_m\frac{L}{2}$$

$$m = n \qquad => \qquad A_m = A_n$$

$$f(x) = 2x$$

By rearranging the equation and substituting in the values, I have obtained an expression for the definite integral of A_n that can be solved:

$$A_n = \frac{4}{L}\int_0^L x\sin\frac{n\pi x}{L}\,dx$$

The appropriate method for solving this integral is by integration of parts (Dawkins. 2019e):

$$\int_a^b u\,dv = uv|_a^b - \int_a^b v\,du$$

Where u and v are two different functions of the same variable x. To use this method I must identify the two functions u and dv which in this case are x and $\sin\frac{n\pi x}{L}$.

$$u = x, \qquad \frac{dv}{dx} = \sin\frac{n\pi x}{L} \quad => \quad dv = \sin\frac{n\pi x}{L}\,dx$$

Taking the derivative of u to find du:

$$\frac{du}{dx} = 1 \quad => \quad du = dx$$

Taking the integral of dv to find v.

$$v = \int dv = \int \sin\frac{n\pi x}{L}\,dx$$

Solving this integral using substitution:

$$\mu = \frac{n\pi x}{L} \quad => \quad \frac{d\mu}{dx} = \frac{n\pi}{L} \quad => \quad L\frac{d\mu}{n\pi} = dx$$

Then the expression becomes:

$$v = \int \sin\frac{n\pi x}{L}dx = \int \sin\mu\, L\frac{d\mu}{n\pi} = \frac{L}{n\pi}\int \sin\mu\, d\mu$$

This is a standard integral. Solve this then undo the substitution.

$$v = \frac{L}{n\pi}\int \sin\mu\, d\mu = -\frac{L}{n\pi}\cos\mu = -\frac{L}{n\pi}\cos\frac{n\pi x}{L}$$

Usually the answer to a non-definite integral should have some arbitrary constant $+C$ at the end of the expression. However this is not necessary when integrating by parts since the constants will cancel out by the end of the expression anyway. Now that I have obtained *du* and *v* I can now substitute them into the original expression.

$$A_n = uv|_a^b - \int_a^b v\, du = \frac{4}{L}\left[-x\frac{L}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^L - (-\frac{L}{n\pi}\int_0^L \cos\frac{n\pi x}{L}\ dx)\right]$$

Recall that the integral on the right hand side can be solved using substitution:

$$\mu = \frac{n\pi x}{L} \quad => \quad L\frac{d\mu}{n\pi} = dx$$

$$-\frac{L}{n\pi}\int_0^L \cos\frac{n\pi x}{L}dx = -\frac{L^2}{n^2\pi^2}\int_0^L \cos\mu\, d\mu = -\frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}\Big|_0^L$$

Since cosine is also a standard integral. With the right hand solved and with some rearrangements I can finalize the expression:

$$A_n = \frac{4}{L}\int_0^L x\sin\frac{n\pi x}{L}dx = \frac{4}{L}\left[-x\frac{L}{n\pi}\cos\frac{n\pi x}{L} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}\right]_0^L$$

I've changed the notation of evaluation from two separate brackets to one large encircling bracket for clarity. The change is arbitrary and doesn't affect the results. Evaluating the expression will yield the result:

$$A_n = \frac{4}{L}\left[\left(-L\frac{L}{n\pi}\cos\frac{n\pi L}{L} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi L}{L}\right) - \left(-0\frac{L}{n\pi}\cos\frac{n\pi 0}{L} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi 0}{L}\right)\right]$$

Solving the expression and rearranging gives a final expression for A_n:

$$A_n = \frac{4L}{n^2\pi^2}\sin n\pi - \frac{4L}{n\pi}\cos n\pi$$

Since $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$ for $n = 1, 2, 3 \ldots$ The expression becomes:

$$A_n = -\frac{4L}{n\pi}(-1)^n \qquad n = 1, 2, 3 \ldots$$

<u>Solving for</u>: B_n

For the other coefficient B_n the second initial condition will be applied: $U_t(\mathrm{x}, 0) = \mathrm{g(x)} = 2.$

To do this the first partial derivative of the general solution with respect to time is needed:

$$u_t(x,t) = \sum_{n=1}^{\infty}(A_n - \frac{cn\pi}{L}\sin\frac{cn\pi t}{L} + B_n\frac{cn\pi}{L}\cos\frac{cn\pi t}{L})\ \sin\frac{n\pi x}{L}$$

Applying the initial condition yields:

$$u_t(x,0) = \sum_{n=1}^{\infty}\frac{cn\pi}{L}B_n\ \sin\frac{n\pi x}{L} = g(x)$$

Again using the same method as when I solved for A_n, I'll multiply each side by $sin\frac{m\pi x}{L}$ and integrate within the interval $0 \leq x \leq L$:

$$\int_0^L g(x)\sin\frac{m\pi x}{L}dx = \sum_{n=1}^{\infty}\frac{cn\pi}{L}B_n\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx$$

Using the same principles, I know that $sin\frac{n\pi x}{L}$ and $sin\frac{m\pi x}{L}$ are mutually orthogonal and the answer to the integral is thus (Dawkins. 2019d):

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx = \begin{cases}\frac{L}{2} & n = m\\ 0 & n \neq m\end{cases}$$

I know from before that $n \neq m$ will only give a trivial zero-constant solution, so the solution must be when $n = m$. Substituting the integral into the solution yields:

$$\int_0^L g(x) \sin \frac{m\pi x}{L} dx = \frac{L}{2} \frac{cn\pi}{L} B_m$$

$$m = n \qquad => \qquad B_m = B_n$$

$$g(x) = 2$$

Rearranging and substituting yields a definite integral for B_n:

$$B_n = \frac{4}{cn\pi} \int_0^L \sin \frac{n\pi x}{L} dx$$

Recall from solving A_n that I have already solved this integral using substitution:

$$B_n = \left[-\frac{4L}{cn^2\pi^2} \cos \frac{n\pi x}{L} \right]_0^L = -\frac{4L}{cn^2\pi^2} \cos \frac{n\pi L}{L} - \left(-\frac{4L}{cn^2\pi^2} \cos \frac{n\pi 0}{L} \right)$$

Solving and rearranging gives B_n:

$$B_n = -\frac{4L}{cn^2\pi^2} (\cos(n\pi) - 1)$$

I already established before that $\cos n\pi = (-1)^n$ so the expression becomes:

$$B_n = -\frac{4L}{cn^2\pi^2} ((-1)^n - 1) \qquad n = 1, 2, 3 \ldots$$

In conclusion the solution to the one-dimensional undamped linear wave equation is:

$$u(x,t) = \sum_{n=1}^{\infty} (A_n \cos \frac{cn\pi t}{L} + B_n \sin \frac{cn\pi t}{L}) \sin \frac{n\pi x}{L}$$

$$A_n = -\frac{4L}{n\pi} (-1)^n \qquad B_n = -\frac{4L}{cn^2\pi^2} ((-1)^n - 1)$$

Where $n = 1, 2, 3 \ldots$

Using these formulas, one can find the particular solution which can be used to predict the vertical displacement of the string at any horizontal position x and at any time t.

Conclusion

In summary there was a lot of steps involved into solving this equation. Firstly under the assumption that the solution of the linear homogenous one dimensional wave equation could be written as the product of the spatial and time variable functions, the PDE was separated into two simultaneous systems of ODEs which were established to contain eigenvalues with corresponding eigenfunctions. Comparing the spatial ODE to the general form of a second order linear homogenous equation, the characteristic equation was found and by using the boundary conditions, the roots to the characteristic polynomial was determined and consequently the eigenvalue with corresponding eigenfunction was determined. Substituting the same eigenvalue into the time dependent ODE yielded the second eigenfunction. Recognizing the linearity of the system and the eigenfunctions to be orthogonal allowed for the principle of superposition to be applied. Taking the infinite sum of the product of the eigenfunctions gave us the general solution to the wave equation with two unknown coefficients. Recognizing the general solution to be a Fourier series, the initial conditions were applied and the orthogonality of the eigenfunctions were used to discard the trivial solution. The unknown coefficients were found to be two separate defined integrals. Through the methods integration by parts and substitution the exact answers to the coefficients were found. Using the general solution together with coefficients, one can find the particular solution which can be used to predict the vertical displacement of the wave at any horizontal position and at any time.

Solving this differential equation required a lot of steps involving many different mathematical areas. Understanding these steps and methods behind it can not only yield a way into solving these kinds of differential equations, but also give a rich knowledge within the different methods, properties and ideas one would use to obtain the solution which can be applied to many other problems. Solving this problem also gives a good look into how challenging problems can be solved by different and abstracts ways of thinking. In my case I solved this equation using a combination of the orthogonality of eigenfunctions, the principle of superposition and the ingenious use of sinus waves that make up the Fourier series, which together could also give a visual understanding for the nature of the solution to this problem. Exploring different methods for solving these kinds of differential equations gives the mathematical foundation that can be built upon to solve more complex equations describing the world around us such as Maxwell's equations or the Schrödinger wave equation.

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Appendix

Scilab Code *Figure 1*:

```
function u=f(x, t)
    u=(cos(%pi*t)+sin(%pi*t))*sin(%pi*x)
endfunction
x=linspace(-1,1,100)
t=linspace(-2,2,200)
u=feval(x,t,f)';
clf
surf(x,t,u)
```

Scilab Code *Figure 2*:

```
xL=0:.01:1;

n=3;g=sin(n*%pi*xL);

m=2;f=sin(m*%pi*xL);

plot(xL,f.*g,'b',xL,xL*0,'k',xL,g.*g,'--r')
```