Fluid Dynamics Equation Sheet

1 Conservation Equations

1.1 Integral Form

1.1.1 Mass

$$\frac{\partial}{\partial t}\oiiint \rho \mathrm{d}V + \oiint \rho \mathbf{u}\cdot \mathrm{d}\mathbf{S} = 0$$

1.1.2 Momentum

$$\frac{\partial}{\partial t}\oiiint \rho \mathbf{u} \mathrm{d}V + \oiint \rho\left(\mathbf{u}\cdot \mathrm{d}\mathbf{S}\right)\mathbf{u} = \oiiint \rho \mathbf{F}\mathrm{d}V - \oiint p\mathrm{d}\mathbf{S} + \mathbf{F}_{viscous}$$

1.1.3 Energy

$$\frac{\partial}{\partial t}\oiiint \rho\left(e + \tfrac{1}{2}\mathbf{u}\cdot\mathbf{u}\right)\mathrm{d}V + \oiint \rho\left(e + \tfrac{1}{2}\mathbf{u}\cdot\mathbf{u}\right)\mathbf{u}\cdot \mathrm{d}\mathbf{S} = \oiiint \dot{q}\rho \mathrm{d}V + \dot{Q}_{viscous} - \oiint p\mathbf{u}\cdot\mathrm{d}\mathbf{S} + \oiiint \rho\left(\mathbf{f}\cdot\mathbf{u}\right)\mathrm{d}V + \dot{W}_{viscous}$$

1.2 Differential Form

1.2.1 Mass

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\left(\rho\mathbf{u}\right) = 0$$

1.2.2 Momentum

$$\begin{aligned}\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) &= \rho\mathbf{F} - \nabla p + \nabla\cdot\boldsymbol{\tau}\\ &= \rho\mathbf{F} - \nabla p + \mu\nabla^2\mathbf{u} + \tfrac{1}{3}\mu\nabla\left(\nabla\cdot\mathbf{u}\right)\\ &+ \kappa\nabla\left(\nabla\cdot\mathbf{u}\right) + 2\left(\nabla\mu\right)\cdot\nabla\mathbf{u} + \left(\nabla\mu\right)\times\left(\nabla\times\mathbf{u}\right)\\ &- \tfrac{2}{3}\left(\nabla\mu\right)\left(\nabla\cdot\mathbf{u}\right) + \left(\nabla\kappa\right)\left(\nabla\cdot\mathbf{u}\right)\end{aligned}$$

$$\kappa = \lambda + \tfrac{2}{3}\mu \qquad \text{(bulk viscosity)}$$

$$\boldsymbol{\tau} = \mu\left[\nabla\mathbf{u} + \left(\nabla\mathbf{u}\right)^{\intercal}\right] + \left(\kappa - \tfrac{2}{3}\mu\right)\left(\nabla\cdot\mathbf{u}\right)\mathbf{I}$$

$$\mathbf{I} = \sum_{i=1}^{3}\sum_{j=1}^{3}\delta_{ij}$$

1.2.3 Energy

$$\rho\frac{De}{Dt} = -p\left(\nabla\cdot\mathbf{u}\right) + \nabla\cdot\left(k\nabla T\right) + \boldsymbol{\tau}:\nabla\mathbf{u}$$

$$\boldsymbol{\tau}:\nabla\mathbf{u} = \sum_{i=1}^{3}\sum_{j=1}^{3}\tau_{ij}\left(\nabla\mathbf{u}\right)_{ij}$$

1.2.4 Vorticity Transport

$$\frac{\partial\boldsymbol{\Omega}}{\partial t} + \nabla\times\left(\boldsymbol{\Omega}\times\mathbf{u}\right) = \nabla\times\mathbf{F} + \frac{\nabla\rho\times\nabla p}{\rho^2} + \nu\nabla^2\boldsymbol{\Omega} + \left(\nabla\nu\right)\times\left[\tfrac{3}{2}\nabla\left(\nabla\cdot\mathbf{u}\right) - \nabla^2\boldsymbol{\Omega}\right]$$

$$\nabla\times\left(\boldsymbol{\Omega}\times\mathbf{u}\right) = \boldsymbol{\Omega}\left(\nabla\cdot\mathbf{u}\right) + \left(\mathbf{u}\cdot\nabla\right)\boldsymbol{\Omega} - \left(\boldsymbol{\Omega}\cdot\nabla\right)\mathbf{u}$$

$$\boldsymbol{\Omega} = \nabla\times\mathbf{u} \qquad \text{and} \qquad \nu = \mu/\rho$$

2 Vector Calculus Identities

2.1 Properties of vector product

$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$$
$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = -\mathbf{A}\times(\mathbf{C}\times\mathbf{B}) = -(\mathbf{B}\times\mathbf{C})\times\mathbf{A}$$

2.2 Differentiation of vectors

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \frac{\mathrm{d}A_1}{\mathrm{d}t}\mathbf{i} + \frac{\mathrm{d}A_2}{\mathrm{d}t}\mathbf{j} + \frac{\mathrm{d}A_3}{\mathrm{d}t}\mathbf{k}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{A}+\mathbf{B}) = \frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\phi\mathbf{A}\right) = \frac{\mathrm{d}\phi}{\mathrm{d}t}\mathbf{A} + \phi\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{A}\cdot\mathbf{B}) = \frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\cdot\mathbf{B} + \mathbf{A}\cdot\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{A}\times\mathbf{B}) = \frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\times\mathbf{B} + \mathbf{A}\times\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{A}\times\mathbf{B}\cdot\mathbf{C}) = \frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\times\mathbf{B}\cdot\mathbf{C} + \mathbf{A}\times\frac{\mathrm{d}\mathbf{B}}{\mathrm{d}t}\cdot\mathbf{C} + \mathbf{A}\times\mathbf{B}\cdot\frac{\mathrm{d}\mathbf{C}}{\mathrm{d}t}$$

2.3 First Derivative Identities

2.3.1 Distributive Properties

$$\nabla\left(\psi+\phi\right) = \nabla\psi + \nabla\phi$$
$$\nabla(\mathbf{A}+\mathbf{B}) = \nabla\mathbf{A} + \nabla\mathbf{B}$$
$$\nabla\cdot(\mathbf{A}+\mathbf{B}) = \nabla\cdot\mathbf{A} + \nabla\cdot\mathbf{B}$$
$$\nabla\times(\mathbf{A}+\mathbf{B}) = \nabla\times\mathbf{A} + \nabla\times\mathbf{B}$$

2.3.2 Product Rule for Multiplication by a Scalar

$$\nabla\left(\psi\phi\right) = \phi\nabla\psi + \psi\nabla\phi$$
$$\nabla(\psi\mathbf{A}) = (\nabla\psi)\,\mathbf{A}^{\intercal} + \psi\nabla\mathbf{A} = \nabla\psi\otimes\mathbf{A} + \psi\nabla\mathbf{A}$$
$$\nabla\cdot(\psi\mathbf{A}) = \psi\nabla\cdot\mathbf{A} + (\nabla\psi)\cdot\mathbf{A}$$
$$\nabla\times(\psi\mathbf{A}) = \psi\nabla\times\mathbf{A} + (\nabla\psi)\times\mathbf{A}$$
$$\nabla^2(fg) = f\nabla^2 g + 2\nabla f\cdot\nabla g + g\nabla^2 f$$

2.3.3 Quotient rule for division by a scalar

$$\nabla\left(\frac{\psi}{\phi}\right) = \frac{\phi\nabla\psi - \psi\nabla\phi}{\phi^2}$$
$$\nabla\left(\frac{\mathbf{A}}{\phi}\right) = \frac{\phi\nabla\mathbf{A} - \nabla\phi\otimes\mathbf{A}}{\phi^2}$$
$$\nabla\cdot\left(\frac{\mathbf{A}}{\phi}\right) = \frac{\phi\nabla\cdot\mathbf{A} - \nabla\phi\cdot\mathbf{A}}{\phi^2}$$
$$\nabla\times\left(\frac{\mathbf{A}}{\phi}\right) = \frac{\phi\nabla\times\mathbf{A} - \nabla\phi\times\mathbf{A}}{\phi^2}$$

2.3.4 Dot Product Rule

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\,\mathbf{B} + (\mathbf{B}\cdot\nabla)\,\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A})$$
$$\tfrac{1}{2}\nabla(\mathbf{A}\cdot\mathbf{A}) = (\mathbf{A}\cdot\nabla)\,\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{A})$$

2.3.5 Cross Product Rule

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = (\nabla\times\mathbf{A})\cdot\mathbf{B} - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$
$$\begin{aligned}\nabla\times(\mathbf{A}\times\mathbf{B}) &= \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\,\mathbf{A} - (\mathbf{A}\cdot\nabla)\,\mathbf{B} \\ &= \nabla\cdot(\mathbf{B}\mathbf{A}^{\intercal} - \mathbf{A}\mathbf{B}^{\intercal})\end{aligned}$$
$$\mathbf{A}\times(\nabla\times\mathbf{B}) = (\nabla\mathbf{B})\cdot\mathbf{A} - (\mathbf{A}\cdot\nabla)\,\mathbf{B}$$
$$\begin{aligned}(\mathbf{A}\times\nabla)\times\mathbf{B} &= (\nabla\mathbf{B})\cdot\mathbf{A} - \mathbf{A}(\nabla\cdot\mathbf{B}) \\ &= \mathbf{A}\times(\nabla\times\mathbf{B}) + (\mathbf{A}\cdot\nabla)\,\mathbf{B} - \mathbf{A}(\nabla\cdot\mathbf{B})\end{aligned}$$

2.4 Second Derivative Identities

2.4.1 Divergence of curl is zero

$$\nabla\cdot(\nabla\times\mathbf{A}) = 0$$

2.4.2 Divergence of gradient is Laplacian

$$\Delta\psi = \nabla^2\psi = \nabla\cdot(\nabla\psi)$$

2.4.3 Divergence of divergence is not defined

$$\nabla\cdot(\nabla\cdot\mathbf{A}) \text{ is undefined}$$

2.4.4 Curl of gradient is zero

$$\nabla\times\left(\nabla\phi\right) = 0$$

2.4.5 Curl of curl

$$\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$$

Here ∇^2 is the vector Laplacian operating on the vector field **A**.

2.4.6 Curl of divergence is not defined

$$\nabla\times(\nabla\cdot\mathbf{A}) \text{ is undefined}$$

2.5 Third Derivative Identities

$$\nabla^2(\nabla\psi) = \nabla[\nabla\cdot(\nabla\psi)] = \nabla\left(\nabla^2\psi\right)$$
$$\nabla^2(\nabla\cdot\mathbf{A}) = \nabla\cdot[\nabla(\nabla\cdot\mathbf{A})] = \nabla\cdot\left(\nabla^2\mathbf{A}\right)$$
$$\nabla^2(\nabla\times\mathbf{A}) = -\nabla\times[\nabla\times(\nabla\times\mathbf{A})] = \nabla\times\left(\nabla^2\mathbf{A}\right)$$

3 Vector Differential Invariants

$$\nabla\psi \equiv \text{grad}\ \psi = \frac{\mathbf{i}_i}{h_i}\frac{\partial\psi}{\partial q_i}$$

$$\nabla\cdot\mathbf{u} \equiv \text{div}\ \mathbf{u} = \frac{1}{h_1h_2h_3}\left[\frac{\partial(u_1h_2h_3)}{\partial q_1}+\frac{\partial(u_2h_3h_1)}{\partial q_2}+\frac{\partial(u_3h_1h_2)}{\partial q_3}\right]$$

$$\nabla\times\mathbf{u} \equiv \text{curl}\ \mathbf{u} = \frac{\mathbf{i}_1}{h_2h_3}\left[\frac{\partial(u_3h_3)}{\partial q_2}-\frac{\partial(u_2h_2)}{\partial q_3}\right]+\cdots$$

$$\nabla\cdot\nabla\psi = \nabla^2\psi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial q_1}\right)+\cdots\right]$$

$$\nabla(\nabla\cdot\mathbf{u}) = \frac{\mathbf{i}_1}{h_1}\frac{\partial}{\partial q_1}\left\{\frac{1}{h_1h_2h_3}\left[\frac{\partial(u_1h_2h_3)}{\partial q_1}+\frac{\partial(u_2h_3h_1)}{\partial q_2}+\frac{\partial(u_3h_1h_2)}{\partial q_3}\right]\right\}+\cdots$$

$$\nabla\times(\nabla\times\mathbf{u}) = \frac{\mathbf{i}_1}{h_2h_3}\left\langle\frac{\partial}{\partial q_2}\left\{\frac{h_3}{h_1h_2}\left[\frac{\partial(u_2h_2)}{\partial q_1}-\frac{\partial(u_1h_1)}{\partial q_2}\right]\right\}-\frac{\partial}{\partial q_3}\left\{\frac{h_2}{h_3h_1}\left[\frac{\partial(u_1h_1)}{\partial q_3}-\frac{\partial(u_3h_3)}{\partial q_1}\right]\right\}\right\rangle+\cdots$$

$$\begin{aligned}\nabla^2\mathbf{u} = \mathbf{i}_1\Bigg[&\nabla^2u_1-u_1h_1\nabla^2\frac{1}{h_1}\\
&+\frac{u_1h_1}{h_1}\frac{\partial}{\partial q_1}\left(\nabla^2q_1\right)+\frac{u_2h_2}{h_1}\frac{\partial}{\partial q_1}\left(\nabla^2q_2\right)+\frac{u_3h_3}{h_1}\frac{\partial}{\partial q_1}\left(\nabla^2q_3\right)\\
&-\frac{2}{h_1^2}\frac{\partial(1/h_1)}{\partial q_1}\frac{\partial}{\partial q_1}(u_1h_1)-\frac{2}{h_2^2}\frac{\partial(1/h_1)}{\partial q_2}\frac{\partial}{\partial q_1}(u_2h_2)-\frac{2}{h_3^2}\frac{\partial(1/h_1)}{\partial q_3}\frac{\partial}{\partial q_1}(u_3h_3)\\
&+\frac{1}{h_1}\frac{\partial\left(1/h_1^2\right)}{\partial q_1}\frac{\partial}{\partial q_1}(u_1h_1)+\frac{1}{h_1}\frac{\partial\left(1/h_2^2\right)}{\partial q_1}\frac{\partial}{\partial q_2}(u_2h_2)+\frac{1}{h_1}\frac{\partial\left(1/h_3^2\right)}{\partial q_1}\frac{\partial}{\partial q_3}(u_3h_3)\Bigg]+\cdots\end{aligned}$$

$$\nabla\mathbf{u} = \sum_{j=1}^{3}\sum_{k=1}^{3}\frac{\mathbf{i}_j}{h_j}\mathbf{i}_j\left(\mathbf{i}_k\frac{\partial u_k}{\partial q_j}+u_k\frac{\partial\mathbf{i}_k}{\partial q_j}\right); \qquad \frac{\partial\mathbf{i}_k}{\partial q_j}=\frac{\mathbf{i}_j}{h_k}\frac{\partial h_j}{\partial q_k}\quad(j\neq k), \qquad \frac{\partial\mathbf{i}_j}{\partial q_j}=-\frac{\mathbf{i}_k}{h_k}\frac{\partial h_j}{\partial q_k}-\frac{\mathbf{i}_l}{h_l}\frac{\partial h_j}{\partial q_l}$$

$$\begin{aligned}&=\frac{\mathbf{i}_1\mathbf{i}_1}{h_1}\left[\frac{\partial u_1}{\partial q_1}+\frac{u_2}{h_2}\frac{\partial h_1}{\partial q_2}+\frac{u_3}{h_3}\frac{\partial h_1}{\partial q_3}\right]+\frac{\mathbf{i}_1\mathbf{i}_2}{h_1}\left[\frac{\partial u_2}{\partial q_1}-\frac{u_1}{h_2}\frac{\partial h_1}{\partial q_2}\right]+\frac{\mathbf{i}_1\mathbf{i}_3}{h_1}\left[\frac{\partial u_3}{\partial q_1}-\frac{u_1}{h_3}\frac{\partial h_1}{\partial q_3}\right]\\
&+\frac{\mathbf{i}_2\mathbf{i}_1}{h_2}\left[\frac{\partial u_1}{\partial q_2}-\frac{u_2}{h_1}\frac{\partial h_2}{\partial q_1}\right]+\frac{\mathbf{i}_2\mathbf{i}_2}{h_2}\left[\frac{\partial u_2}{\partial q_2}+\frac{u_3}{h_3}\frac{\partial h_2}{\partial q_3}+\frac{u_1}{h_1}\frac{\partial h_2}{\partial q_1}\right]+\frac{\mathbf{i}_2\mathbf{i}_3}{h_2}\left[\frac{\partial u_3}{\partial q_2}-\frac{u_2}{h_3}\frac{\partial h_2}{\partial q_3}\right]\\
&+\frac{\mathbf{i}_3\mathbf{i}_1}{h_3}\left[\frac{\partial u_1}{\partial q_3}-\frac{u_3}{h_1}\frac{\partial h_3}{\partial q_1}\right]+\frac{\mathbf{i}_3\mathbf{i}_2}{h_3}\left[\frac{\partial u_2}{\partial q_3}-\frac{u_3}{h_2}\frac{\partial h_3}{\partial q_2}\right]+\frac{\mathbf{i}_3\mathbf{i}_3}{h_3}\left[\frac{\partial u_3}{\partial q_3}+\frac{u_1}{h_1}\frac{\partial h_3}{\partial q_1}+\frac{u_2}{h_2}\frac{\partial h_3}{\partial q_2}\right]\end{aligned}$$

$$\boldsymbol{\phi} = \sum_{j=1}^{3}\sum_{k=1}^{3}\mathbf{i}_j\mathbf{i}_k\phi_{jk} \qquad \text{(Dyadic)}$$

$$\boldsymbol{\phi}^{\intercal} = \sum_{j=1}^{3}\sum_{k=1}^{3}\mathbf{i}_j\mathbf{i}_k\phi_{kj} \qquad \text{(Dyadic transpose)}$$

4 Cylindrical/Spherical Vector Identities

4.1 Cylindrical

$$
\begin{aligned}
\nabla\psi &= \mathbf{i}_r \frac{\partial \psi}{\partial r} + \mathbf{i}_\theta \frac{1}{r}\frac{\partial \psi}{\partial \theta} + \mathbf{i}_z \frac{\partial \psi}{\partial z} \\
\nabla^2\psi &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2} \\
\nabla\cdot\mathbf{u} &= \frac{1}{r}\frac{\partial}{\partial r}\left(r u_r\right) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \\
\nabla\times\mathbf{u} &= \mathbf{i}_r\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right) + \mathbf{i}_\theta\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) + \mathbf{i}_z\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r u_\theta\right) - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right] \\
\nabla^2\mathbf{u} &= \mathbf{i}_r\left(\nabla^2 u_r - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta} - \frac{u_r}{r^2}\right) + \mathbf{i}_\theta\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2}\right) + \mathbf{i}_z\nabla^2 u_z \\
\nabla\mathbf{u} &= \mathbf{i}_r\mathbf{i}_r\frac{\partial u_r}{\partial r} + \mathbf{i}_r\mathbf{i}_\theta\frac{\partial u_\theta}{\partial r} + \mathbf{i}_r\mathbf{i}_z\frac{\partial u_z}{\partial r} \\
&\quad + \mathbf{i}_\theta\mathbf{i}_r\frac{1}{r}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right) + \mathbf{i}_\theta\mathbf{i}_\theta\frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right) + \mathbf{i}_\theta\mathbf{i}_z\frac{1}{r}\frac{\partial u_z}{\partial \theta} \\
&\quad + \mathbf{i}_z\mathbf{i}_r\frac{\partial u_r}{\partial z} + \mathbf{i}_z\mathbf{i}_\theta\frac{\partial u_\theta}{\partial z} + \mathbf{i}_z\mathbf{i}_z\frac{\partial u_z}{\partial z}
\end{aligned}
$$

4.2 Spherical

$$
\begin{aligned}
\nabla\psi &= \mathbf{i}_R\frac{\partial \psi}{\partial R} + \mathbf{i}_\phi\frac{1}{R}\frac{\partial \psi}{\partial \phi} + \mathbf{i}_\theta\frac{1}{R\sin\phi}\frac{\partial \psi}{\partial \theta} \\
\nabla^2\psi &= \frac{1}{R^2}\left[\frac{\partial}{\partial R}\left(R^2\frac{\partial \psi}{\partial R}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial \psi}{\partial \phi}\right) + \frac{1}{\sin^2\phi}\frac{\partial^2 \psi}{\partial^2 \theta}\right] \\
\nabla\cdot\mathbf{u} &= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 u_R\right) + \frac{1}{R\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi u_\phi\right) + \frac{1}{R\sin\phi}\frac{\partial u_\theta}{\partial \theta} \\
\nabla\times\mathbf{u} &= \mathbf{i}_R\frac{1}{R\sin\phi}\left[\frac{\partial}{\partial \phi}\left(\sin\phi u_\theta\right) - \frac{\partial u_\phi}{\partial \theta}\right] + \mathbf{i}_\phi\frac{1}{R}\left[\frac{1}{\sin\phi}\frac{\partial u_R}{\partial \theta} - \frac{\partial}{\partial R}\left(R u_\theta\right)\right] + \mathbf{i}_\theta\frac{1}{R}\left[\frac{\partial}{\partial R}\left(R u_\phi\right) - \frac{\partial u_R}{\partial \phi}\right] \\
\nabla^2\mathbf{u} &= \mathbf{i}_R\left[\nabla^2 u_R - \frac{2}{R^2}\left(u_R + \frac{\partial u_\phi}{\partial \phi} + u_\phi\cot\phi - \frac{1}{\sin\phi}\frac{\partial u_\theta}{\partial \theta}\right)\right] \\
&\quad + \mathbf{i}_\phi\left[\nabla^2 u_\phi + \frac{1}{R^2}\left(2\frac{\partial u_R}{\partial \phi} - \frac{u_\phi}{\sin^2\phi} - 2\frac{\cos\phi}{\sin^2\phi}\frac{\partial u_\theta}{\partial \theta}\right)\right] \\
&\quad + \mathbf{i}_\theta\left[\nabla^2 u_\theta + \frac{1}{R^2}\left(\frac{2}{\sin\phi}\frac{\partial u_R}{\partial \theta} + 2\frac{\cos\phi}{\sin^2\phi}\frac{\partial u_\phi}{\partial \theta} - \frac{u_\theta}{\sin^2\phi}\right)\right] \\
\nabla\mathbf{u} &= \mathbf{i}_R\mathbf{i}_R\frac{\partial u_R}{\partial R} + \mathbf{i}_R\mathbf{i}_\phi\frac{\partial u_\phi}{\partial R} + \mathbf{i}_R\mathbf{i}_\theta\frac{\partial u_\theta}{\partial R} \\
&\quad + \mathbf{i}_\phi\mathbf{i}_R\frac{1}{R}\left(\frac{\partial u_R}{\partial \phi} - u_\phi\right) + \mathbf{i}_\phi\mathbf{i}_\phi\frac{1}{R}\left(\frac{\partial u_\phi}{\partial \phi} + u_R\right) + \mathbf{i}_\phi\mathbf{i}_\theta\frac{1}{R}\frac{\partial u_\theta}{\partial \phi} \\
&\quad + \mathbf{i}_\theta\mathbf{i}_R\frac{1}{R}\left(\frac{1}{\sin\phi}\frac{\partial u_R}{\partial \theta} - u_\theta\right) + \mathbf{i}_\theta\mathbf{i}_\phi\frac{1}{R}\left(\frac{1}{\sin\phi}\frac{\partial u_\phi}{\partial \theta} - \cot\phi u_\theta\right) + \mathbf{i}_\theta\mathbf{i}_\theta\frac{1}{R}\left(\frac{1}{\sin\phi}\frac{\partial u_\theta}{\partial \theta} + u_R + \cot\phi u_\phi\right)
\end{aligned}
$$

5 Vector Integral Theorems

5.1 Gauss's Divergence Theorem

$$\oiint \mathbf{f} \cdot \mathrm{d}\mathbf{S} = \oiiint \boldsymbol{\nabla} \cdot \mathbf{f} \mathrm{d}V$$

5.2 Green's Theorems

5.2.1 Green's First Theorem

$$\oiint \phi \frac{\partial \psi}{\partial n} \mathrm{d}S = \oiiint \left(\phi \boldsymbol{\nabla}^2 \psi + \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\psi\right) \mathrm{d}V$$

5.2.2 Green's Second Theorem

$$\oiint \left(\phi \frac{\partial \phi}{\partial n} - \psi \frac{\partial \phi}{\partial n}\right) \mathrm{d}S = \oiiint \left(\phi \boldsymbol{\nabla}^2 \psi - \psi \boldsymbol{\nabla}^2 \phi\right) \mathrm{d}V$$

5.2.3 Special Cases

$$\oiint \left(\phi \boldsymbol{\nabla}\phi\right) \cdot \mathrm{d}\mathbf{S} = \oiiint \left[\phi \boldsymbol{\nabla}^2 \phi + \left(\boldsymbol{\nabla}\phi\right)^2\right] \mathrm{d}V$$

$$\oiint \frac{\partial \phi}{\partial n} \mathrm{d}S = \oiiint \boldsymbol{\nabla}^2 \phi \mathrm{d}V$$

5.3 Stokes' Theorem

Let a simple closed curve C be spanned by a surface S. Define the positive normal $\mathbf{n}$ to S, and the positive sense of description of the curve C with line element $\mathrm{d}\mathbf{r}$, such that the positive sense of the contour C is clockwise when we look through the surface S in the direction of the normal. Then, if $\mathbf{f}$ is continuously differentiable vector field defined on S and C with vector element $\mathbf{S} = \mathbf{n}\mathrm{d}S$

$$\oint \mathbf{f} \cdot \mathrm{d}\mathbf{r} = \oiint \boldsymbol{\nabla} \times \mathbf{f} \cdot \mathrm{d}\mathbf{S}$$

where the line integral around C is taken in the positive sense.

5.4 Integral rate of change theorems

5.4.1 Rate of change of volume integral bounded by a moving closed surface.

Let f be a continuous scalar function of position and time t defined throughout the volume $V(t)$, which is itself bounded by a simple closed surface $S(t)$ moving with velocity $\mathbf{v}$. Then the rate of change of the volume integral of f is given by

$$\frac{D}{Dt} \int_{V(t)} f \mathrm{d}V = \int_{V(t)} \frac{\partial f}{\partial t} \mathrm{d}V + \int_{S(t)} f \mathbf{v} \cdot \mathrm{d}\mathbf{S}$$

where $\mathrm{d}\mathbf{S}$ is the outward drawn vector element of area, and

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})$$

5.4.2 Rate of change of flux through a surface.

Let $\mathbf{q}$ be a vector function that may also depend on the time t, and $\mathbf{n}$ be the unit outward drawn normal to the surface S that moves with velocity $\mathbf{v}$. Defining the flux of $\mathbf{q}$ through S as

$$m = \oint_S \mathbf{q} \cdot \mathbf{n} \mathrm{d}S$$

then

$$\frac{Dm}{Dt} = \oint \left[\frac{\partial \mathbf{q}}{\partial t} + \mathbf{v}\left(\boldsymbol{\nabla} \cdot \mathbf{q}\right) + \boldsymbol{\nabla} \times \left(\mathbf{q} \times \mathbf{v}\right)\right] \cdot \mathbf{n} \mathrm{d}S$$

5.4.3 Rate of change of the circulation around a given moving curve.

Let C be a closed curve, moving with velocity $\mathbf{v}$, on which is defined a vector field $\mathbf{q}$. Defining the circulation ζ of $\mathbf{q}$ around C by

$$\zeta = \oint \mathbf{q} \cdot \mathrm{d}\mathbf{r}$$

then

$$\frac{D\zeta}{Dt} = \oint \left[\frac{\partial \mathbf{q}}{\partial t} + \left(\boldsymbol{\nabla} \times \mathbf{q}\right) \times \mathbf{v}\right] \cdot \mathrm{d}\mathbf{r}$$

6 Ideal Flow (irrotational/inviscid/incompressible)

6.1 Governing equations

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{(continuity)}$$

$$\rho \left[\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right] = -\nabla p \qquad \text{(momentum)}$$

6.2 Fundamentals

6.2.1 Streamfunction

May be defined when continuity eq. reduces to two terms. For cartesian coordinates:

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$$

The streamfunction may be defined as:

$$u \equiv \frac{\partial \psi}{\partial y} \qquad \text{and} \qquad v \equiv -\frac{\partial \psi}{\partial x}$$

6.2.2 Velocity potential

$$u \equiv \frac{\partial \phi}{\partial x} \qquad \text{and} \qquad v \equiv \frac{\partial \phi}{\partial y}$$

6.2.3 Bernoulli's equation

If flow is irrotational throughout:

$$p + \tfrac{1}{2}\rho V^2 = \text{const.}$$

else, only valid along streamlines.

6.2.4 Incompressible pressure coefficient

$$C_\mathrm{p} = 1 - \left(\frac{V}{V_\infty} \right)^2$$

7 Viscous Flow

7.1 Governing Equatinos

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{(continuity)}$$
$$\rho (\mathbf{u} \cdot \nabla \mathbf{u}) = \rho \mathbf{F} - \nabla p + \mu \nabla^2 \mathbf{u} \qquad \text{(momentum)}$$

7.2 Boundary layer terms

7.2.1 Reynolds Numbers

$$\text{Re} = \frac{\rho U l}{\mu}$$

7.2.2 Skin criction coefficient

$$C_f(x) = \frac{\tau_w(x)}{\frac{1}{2}\rho U^2} = \frac{2}{\sqrt{\text{Re}}}\left(\frac{\partial u^*}{\partial \bar{y}}\right)_w$$

7.2.3 Drag coefficient

$$C_D = \frac{F_D}{\frac{1}{2}\rho U^2 A} = b\int_{\text{LE}}^{\text{TE}} (-p_u \sin\theta + \tau_u \cos\theta)\, \mathrm{d}S_u + b\int_{\text{LE}}^{\text{TE}} (p_l \sin\theta + \tau_l \cos\theta)\, \mathrm{d}S_l$$

where "u", "l", and "b" denote "upper", "lower", and "width"

7.2.4 Separation point

$$\left(\frac{\partial u}{\partial y}\right)_{-} y = 0$$

7.2.5 Displacement thickness

$$\delta^* = \int_{y=0}^{\infty} \left(1 - \frac{u}{U_e}\right) dy$$

7.2.6 Momentum thickness

$$\theta = \int_{y=0}^{\infty} \frac{u}{U_e}\left(1 - \frac{u}{U_e}\right)$$

7.2.7 Shape factor

$$H = \frac{\delta^*}{\theta}$$

7.3 Blasius equation

Assumptions:

- Incompressible
- Laminar
- Steady
- $\frac{\partial p}{\partial x} = 0$

7.3.1 Equation

$$f''' + \tfrac{1}{2} f f'' = 0 \qquad \text{where} \qquad \eta = \sqrt{\nu x / U}$$

7.3.2 Shear stress

$$\tau_w = \mu \left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)_{y=0} = 0.332 \rho U^2 / \sqrt{\text{Re}_x}$$

7.3.3 Skin friction coefficient

$$C_f \equiv \frac{\tau_w}{\frac{1}{2}\rho U^2} = \frac{0.664}{\sqrt{\text{Re}_x}}$$

7.3.4 Drag on plate

$$F_D = \int_0^L \tau_w \mathrm{d}x = \frac{0.664 \rho U^2 L}{\sqrt{\text{re}_L}}$$

(drag force per unit width)

7.3.5 Drag coefficient

$$C_D = \frac{F_D}{\frac{1}{2}\rho U^2 L} = \frac{1.33}{\sqrt{\text{Re}_L}}$$

7.4 Pohlhausen's Polynomial

$$\frac{u(\eta)}{U_\infty} = \left(2 + \tfrac{1}{6}\Lambda\right)\eta - \tfrac{1}{2}\Lambda\eta^2 + \left(\tfrac{1}{2}\Lambda - 2\right)\eta^3 + \left(1 - \tfrac{1}{6}\Lambda\right)\eta^4$$

$$\text{where:} \qquad \eta = y/\delta \qquad \Lambda = \frac{\rho \delta^2}{\mu}\frac{\mathrm{d}U}{\mathrm{d}x}$$

7.5 Von Karman momentum integral equation

$$\tau_w = \rho \left[\frac{d}{dx}\left(U_e^2\theta\right) + U_e\delta^* \frac{dU_e}{dx} \right]$$

$$\tau_w = \mu \left. \frac{\partial u}{\partial y} \right|_{y\to 0}$$

solve for $\delta(x)$ (boundary layer thickness)

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} \qquad \text{(skin friction coefficient)}$$

$$\mathrm{Re} = \frac{\rho U \ell}{\mu} \quad \text{and} \quad \mathrm{Re}_x = \frac{\rho U x}{\mu}$$

(Reynolds number and x-Reynolds number)

7.6 Thwaites' method

For solving Von Karmar mom. int. eq.:

$$\theta^2 = \frac{0.45\nu}{U_e^2(x)} \int_0^x U_e^5(x')\,\mathrm{d}x' + \frac{\theta_0^2 U_0^6}{U_e^6(x)}$$

$$\lambda \equiv \frac{\theta^2}{\nu}\frac{\mathrm{d}U_e}{\mathrm{d}x} \qquad \text{(solve for } \theta\text{)}$$

$$\tau \equiv \mu \frac{U_e}{\theta} l(\lambda)$$

$$H(\lambda) = \frac{\delta^*}{\theta}$$

(use table to find surface shear stress and displacement thickness)

8 Compressible Flow

8.1 Governing equations

$$\frac{\partial}{\partial t}\oiiint_V \rho \mathrm{d}V + \oiint_S \rho \mathbf{u}\cdot \mathrm{d}\mathbf{S} = 0 \qquad \text{(continuity)}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0$$

$$\frac{\partial}{\partial t}\oiiint \rho \mathbf{u}\mathrm{d}V + \oiint \rho\left(\mathbf{u}\cdot\mathrm{d}\mathbf{S}\right)\mathbf{u} = \oiiint \rho\mathbf{F}\mathrm{d}V - \oiint p\mathrm{d}\mathbf{S} \qquad \text{(momentum)}$$

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = \rho\mathbf{F} - \nabla p$$

8.2 General Definitions

8.2.1 Mach number

$$M \equiv U/c$$

$$\text{where} \qquad c^2 \equiv \left(\frac{\partial p}{\partial \rho}\right)_s$$

8.2.2 Compressibility

$$\tau = -\frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}p} = \frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}p}$$

8.2.3 Isothermal compressibility

$$\tau_T = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_T$$

8.2.4 Isentropic compressibility

$$\tau_s = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_s$$

8.3 Perfect Gas Thermodynamic Relations

8.3.1 Internal Energy/Enthalpy/Thermal eq. of state

$$e = c_\mathrm{v}T \qquad \text{(internal energy)}$$

$$h = c_\mathrm{p}T \qquad \text{(enthalpy)}$$

$$p = \rho RT \qquad \text{(thermal equation of state)}$$

8.3.2 Specific heats

$$c_\mathrm{v} = \frac{R}{\gamma - 1} \qquad c_\mathrm{p} = \frac{\gamma R}{\gamma - 1} \qquad \gamma = c_\mathrm{p}/c_\mathrm{v}$$

8.3.3 Speed of sound

$$c = \sqrt{\gamma RT} = \sqrt{\gamma p/\rho}$$

8.3.4 Entropy change

$$s_2 - s_1 = c_\mathrm{p}\ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right) = c_\mathrm{v}\ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{\rho_2}{\rho_1}\right)$$

8.3.5 Isentropic relations

$$\frac{p_2}{p_1} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma}$$

$$\frac{T_2}{T_2} = \left(\frac{\rho_2}{\rho_1}\right)^{\gamma-1} = \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}$$

8.3.6 Important air properties

$$R = 287\ \mathrm{m}^2/\left(\mathrm{s}^2\mathrm{K}\right)$$

$$c_\mathrm{v} = 717\mathrm{m}^2/\left(\mathrm{s}^2\mathrm{K}\right)$$

$$c_\mathrm{p} = 1004\mathrm{m}^2/\left(\mathrm{s}^2\mathrm{K}\right)$$

$$\gamma = 1.40$$

8.4 One-dimensional isentropic flow

8.4.1 Continuity

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2$$

8.4.2 Stagnation enthalpy

$$h_0 = h + \tfrac{1}{2}u^2 = \text{const.}$$

8.4.3 Stagnation property relations

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2$$
$$\frac{p_0}{p} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma-1}}$$
$$\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{1}{\gamma-1}}$$

8.4.4 Nozzle area relation

$$\frac{A}{A*} = \frac{1}{M}\left[\frac{2}{\gamma + 1}\left(1 + \frac{\gamma - 1}{2} M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}$$

8.4.5 Velocity-area differential relationship

$$\frac{\mathrm{d}u}{u} = -\frac{1}{1 - M^2}\frac{\mathrm{d}A}{A}$$

8.5 Normal shock waves

8.5.1 Mach number, pressure, temperature, and density relations across a shock

$$M_2^2 = \frac{(\gamma - 1) M_1^2 + 2}{2\gamma M_1^2 + 1 - \gamma}$$
$$\frac{p_2}{p_1} = 1 + \frac{2\gamma}{\gamma + 1}\left(M_1^2 - 1\right)$$
$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1) M_1^2}{(\gamma - 1) M_1^2 + 2}$$
$$\frac{T_2}{T_1} = 1 + \frac{2(\gamma - 1)}{(\gamma + 1)^2}\frac{\gamma M_1^2 + 1}{M_1^2}\left(M_1^2 - 1\right)$$

8.5.2 Entropy change

$$\frac{s_2 - s_1}{c_\mathrm{v}} = \ln\left\{\frac{p_2}{p_1}\left(\frac{\rho_1}{\rho_2}\right)^\gamma\right\}$$
$$= \ln\left\{\left[1 + \frac{2\gamma}{\gamma - 1}\left(M_1^2 - 1\right)\right]\left[\frac{(\gamma - 1) M_2^1 + 2}{(\gamma + 1) M_1^2}\right]^\gamma\right\}$$

8.6 Wind Tunnel equations

8.6.1 Mass flow rate

$$\dot{m} = \frac{\rho_0 A^*}{T_0}\sqrt{\frac{\gamma}{R}\left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{\gamma-1}}}$$

8.6.2 Nozzle/Diffuser area ratio

$$\frac{A_2^*}{A_1^*} = \frac{p_0}{p_0'}$$
$$= \left[1 + \frac{2\gamma}{\gamma + 1}\left(M_1^2 - 1\right)\right]^{\frac{1}{\gamma-1}}\left[\frac{(\gamma - 1) M_1^2 + 2}{(\gamma + 1) M_1^2}\right]^{\frac{\gamma}{\gamma-1}}$$

where A_2^*, A_1^*, p_0, p_0', and M_1 are the diffuser throat area, inflow nozzle throat area, stagnation pressure behind the shock, stagnation pressure after the shock, and Mach number upstream of the shock

8.7 Two-dimensional compressible flow

8.7.1 Definitions

$$\delta = \text{deflection angle}$$
$$\sigma = \text{wave angle}$$
$$M_1 = \text{upstream Mach number}$$
$$M_2 = \text{downstream Mach number}$$

8.7.2 Mach wave angle

$$\sin\mu = 1/M$$

8.7.3 Mach number relation

$$M_2^2 \sin^2(\sigma - \delta) = \frac{(\gamma - 1) M_1^2 \sin^2\sigma + 2}{2\gamma M_1^2 \sin^2\sigma + 1 - \gamma}$$

8.7.4 Oblique shockwave angle

$$\tan\delta = 2\cot\sigma\frac{M_1^2\sin^2\sigma - 1}{M_1^2(\gamma - \cos 2\sigma) + 2}$$

8.7.5 Prandtl-Meyer function (expansion fans)

$$\delta_1 + \nu(M_1) = \delta_2 + \nu(M_2) = \text{const.}$$
$$\nu(M) = \sqrt{\frac{\gamma + 1}{\gamma - 1}}\arctan\sqrt{\frac{\gamma - 1}{\gamma + 1}\left(M^2 - 1\right)} - \arctan\sqrt{M^2 - 1}$$

8.8 Thin-airfoil theory

8.8.1 Lift and Drag coefficient

$$
\begin{aligned}
C_L &= \frac{L}{\frac{1}{2}\rho_\infty U_\infty^2 b} \\
&\approx \frac{4\alpha}{\sqrt{M_\infty^2 - 1}} \\
C_D &= \frac{D}{\frac{1}{2}\rho_\infty U_\infty^2 b} \\
&\approx \frac{4\alpha^2}{\sqrt{M_\infty^2 - 1}}
\end{aligned}
$$

8.9 Velocity Potential

$$
\nabla^2\phi = \tfrac{1}{2}M_0^2\left[(\gamma-1)\,\nabla^2\phi\left(\nabla\phi\cdot\nabla\phi\right) + \nabla\phi\cdot\nabla\left(\nabla\phi\cdot\nabla\phi\right)\right]
$$

where

$$
a^2 = a_0^2 - \frac{\gamma-1}{2}V^2 = a_0^2 - \frac{\gamma-1}{2}\left(u^2+v^2+w^2\right)
$$

9 Rocket Propulsion Equations

9.1 Definitions and Fundamentals

9.1.1 Total impulse

$$
\begin{aligned}
I_t &= \int_0^t F\mathrm{d}t \\
&= Ft \quad \text{(for constant thrust)}
\end{aligned}
$$

9.1.2 Specific impulse

$$
\begin{aligned}
I_s &= I_t/\left(m_p g_0\right) \\
&= F/\left(\dot{m} g_0\right) \\
&= F/\dot{w} \\
&= I_t/w
\end{aligned}
$$

9.1.3 Effective exhaust velocity

$$
c = I_s g_0 = F/\dot{m}
$$

9.1.4 Mass ratio MR

$$
m_f/m_0
$$

9.1.5 Propellant mass fraction ζ

$$
\begin{aligned}
\zeta &= m_p/m_0 \\
&= \left(m_0 - m_f\right) = m_p/\left(m_p + m_f\right)
\end{aligned}
$$

where $m_0 = m_p + m_f$

9.1.6 Impulse-to-weight Ratio

$$
\frac{I_t}{w_0} = \frac{I_t}{\left(m_f + m_p\right) g_0} = \frac{I_s}{m_f/m_p + 1}
$$

9.1.7 Thrust F

$$
\begin{aligned}
F &= \frac{\mathrm{d}\left(m v_2\right)}{\mathrm{d}t} = \dot{m} v_2 \quad \text{at sea level} \quad = \frac{\dot{w}}{g_0} v_2 \\
&= \dot{m} v_2 + \left(p_2 - p_3\right) A_2 \\
&= \dot{m} c \\
&= \dot{m} v_2 + p_2 A_2 \quad \text{(in vacuum of space)} \\
&= F_{\text{opt}} + p_1 A_t \left(\frac{p_2}{p_1} - \frac{p_3}{p_1}\right)\epsilon \\
&= \frac{A_t v_t v_2}{V_t} + \left(p_2 - p_3\right) A_2 \\
&= A_t p_1 \sqrt{\frac{2\gamma^2}{\gamma - 1}\left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{\gamma-1}}\left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right]} + \left(p_2 - p_3\right) A_2 \\
&= C_F A_t p_1 \\
&= \dot{m} c^* C_F
\end{aligned}
$$

9.1.8 Thrust Coefficient

$$
\begin{aligned}
C_F &= \frac{F}{p_1 A_t} \\
&= \sqrt{\frac{2\gamma^2}{\gamma - 1}\left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{\gamma-1}}\left[1 - \left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right]} + \frac{p_2 - p_3}{p_1}\frac{A_2}{A_t}
\end{aligned}
$$

9.1.9 Exhaust velocity c

$$
c = v_2 + \left(p_2 - p_3\right) A_2/\dot{m} = I_s g_0
$$

9.1.10 Characteristic velocity c^*

$$
\begin{aligned}
c^* &= p_1 A_t/\dot{m} \\
&= \frac{I_s g_0}{C_f} \\
&= \frac{c}{C_F} \\
&= \frac{1}{\gamma}\sqrt{\frac{\gamma R T_1}{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}}
\end{aligned}
$$

9.1.11 Power of the jet P_{jet}

$$
P_{\text{jet}} = \tfrac{1}{2}\dot{m} v_2^2 = \tfrac{1}{2}\dot{w} g_0 I_s^2 = \tfrac{1}{2} F g_0 I_s = \tfrac{1}{2} F v_2
$$

9.1.12 Power input to a chemical engine

$$
P_{\text{chem}} = \dot{m} Q_R
$$

where Q_R is the heat of combustion per unit of propellant mass.

9.1.13 Power transmitted to vehicle

$$
P_{\text{vehicle}} = Fu \qquad \text{(where } u \text{ is vehicle velocity)}
$$

9.1.14 Internal efficiency η_{p}

$$
\eta_{\text{int}} = \frac{\text{kinetic power in jet}}{\text{available chemical power}} = \frac{\tfrac{1}{2}\dot{m} v^2}{\eta_{\text{comp}} P_{\text{chem}}}
$$

9.1.15 Propulsive efficiency

$$
\begin{aligned}
\eta_p &= \frac{\text{vehicle power}}{\text{vehicle power + residual kinetic jet power}} \\
&= \frac{Fu}{Fu + \tfrac{1}{2}\dot{m}\left(c + u\right)^2} = \frac{2u/c}{1 + \left(u/c\right)^2}
\end{aligned}
$$

9.1.16 Multiple propulsion systems

$$
\begin{aligned}
F_{\text{oa}} &= \sum F = F_1 + F_2 + F_3 + \cdots \\
\dot{m}_{\text{oa}} &= \sum \dot{m} = \dot{m}_1 + \dot{m}_2 + \dot{m}_3 + \cdots \\
(I_s)_{\text{oa}} &= F_{\text{oa}} / (g_0 \dot{m}_{\text{oa}})
\end{aligned}
$$

9.2 Nozzle theory and thermodynamic relations

9.2.1 Stagnation enthalpy

$$h_0 = h + \frac{1}{2}v^2 = \text{const.}$$

9.2.2 Perfect gas law

$$p_x V_x = RT_x$$

9.2.3 Specific heats c and their γ

$$
\begin{aligned}
k &= c_p / c_v \\
c_p - c_v &= R \\
c_p &= \gamma R / (\gamma - 1)
\end{aligned}
$$

9.2.4 Isentropic flow between x and y in nozzle

$$T_x / T_y = \left(p_x / p_y\right)^{\frac{\gamma-1}{\gamma}} = \left(V_y / V_x\right)^{\gamma-1}$$

9.2.5 Stagnation temperature

$$T_0 = T + \tfrac{1}{2} v^2 / c_p$$

9.2.6 Speed of sound & Mach number

$$
\begin{aligned}
a &= \sqrt{\gamma RT} \\
M &= v/a = v / \sqrt{\gamma RT}
\end{aligned}
$$

9.2.7 Isentropic relations (temperature, Mach number, pressure)

$$
\begin{aligned}
T_0 &= T\left[1 + \tfrac{1}{2}(\gamma - 1) M^2\right] \\
M &= \sqrt{\frac{2}{\gamma - 1}\left(\frac{T_0}{T} - 1\right)} \\
p_0 &= p\left[1 + \tfrac{1}{2}(\gamma - 1) M^2\right]^{\frac{\gamma}{\gamma-1}}
\end{aligned}
$$

9.2.8 General area ratio

$$\frac{A_y}{A_x} = \frac{M_x}{M_y}\sqrt{\left[\frac{1 + M_y^2(\gamma - 1)/2}{1 + M_x^2(\gamma - 1)/2}\right]}$$

9.2.9 Exit velocity

$$
\begin{aligned}
v_2 &= \sqrt{2(h_1 - h_2) + v_1^2} \\
&= \sqrt{\frac{2\gamma}{\gamma - 1} RT_1 \left[1 - \left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma}\right] + v_1^2} \\
&= \sqrt{\frac{2\gamma}{\gamma - 1} RT_1 \left[1 - \left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma}\right]} \quad (\text{when } v_1 \approx 0 (\text{chamber})) \\
&= \sqrt{\frac{2\gamma}{\gamma - 1} \frac{R' T_1}{\mathbb{M}} \left[1 - \left(\frac{p_2}{p_1}\right)^{(\gamma-1)/\gamma}\right]}
\end{aligned}
$$

where $\mathbb{M}$ is molecular mass.

9.2.10 Critical/Throat velocity

$$v_t = \sqrt{\frac{2\gamma}{\gamma + 1} RT_1} = \sqrt{\gamma RT_t} = a_t$$

9.2.11 Mass flow rate

$$\dot{m} = \frac{A_t v_t}{V_t} = A_t p_1 \gamma \sqrt{\frac{\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma+1}{\gamma-1}}}{\gamma RT_1}}$$

9.2.12 Ratio between throat and downstream area with pressure p_y

$$\frac{A_t}{A_y} = \frac{V_t v_y}{V_y v_t} = \left(\frac{\gamma + 1}{2}\right)^{\frac{1}{\gamma-1}} \left(\frac{p_y}{p_1}\right)^{1/\gamma} \sqrt{\frac{\gamma + 1}{\gamma - 1}\left[1 - \left(\frac{p_y}{p_1}\right)^{\frac{\gamma-1}{\gamma}}\right]}$$

9.2.13 Velocity of downstream point from throat

$$\frac{v_y}{v_t} = \sqrt{\frac{\gamma + 1}{\gamma - 1}}$$

9.2.14 Cone-shaped nozzle correction factor for exhaust velocity

$$\gamma = \tfrac{1}{2}(1 + \cos\alpha)$$

where α is the cone angle.

9.2.15 Rocket Equation

$$e^{\Delta u / c} = 1/\mathbf{MR} = m_0 / m_f$$

9.3 Chemical Rocket Performance Analysis

9.3.1 Mol fraction

$$X_j = \frac{n_j}{n} \quad \text{and} \quad n = \sum_{j=1}^{m} n_j$$

9.3.2 Effective average molecular mass

$$\mathbb{M} = \frac{\sum_{j=1}^{m} n_j \mathbb{M}_j}{\sum_{j=1}^{m} n_j}$$

9.3.3 Molar specific heat and specific heat ratio

$$\left(C_p\right)_{\text{mix}} = \frac{\sum_{j=1}^{m} n_j \left(C_p\right)_j}{\sum_{j=1}^{m} n_j}$$

$$\gamma_{\text{mix}} = \frac{\left(C_p\right)_{\text{mix}}}{\left(C_p\right)_{\text{mix}} - R'}$$

9.3.4 Heat of reaction

$$\Delta_r H^0 = \sum \left[n_j \left(\Delta_f H^0\right)\right]_{\text{products}} - \sum \left[n_j \left(\Delta_f H^0\right)_j\right]_{\text{reactants}}$$

9.3.5 Gibbs free energy

$$G = \sum_{j=1}^{m} n_j G_j$$

$$G_j = U_j + p_j V_j - T_j S_j = h_j - T_j S_j$$

$$\Delta_r G^0 = \sum_{j=1}^{m} \left[n_j \left(\Delta_f G^0\right)\right]_{\text{products}} - \sum_{j=1}^{r} \left[n_j \left(\Delta_f G^0\right)_j\right]_{\text{reactants}}$$

9.3.6 Heat of reaction

$$\Delta_r H = \sum_{1}^{m} n_j \int_{T_{\text{ref}}}^{T_1} C_{pj} \mathrm{d}T = \sum_{1}^{m} n_J \Delta h_j \Big|_{T_{\text{ref}}}^{T_1}$$

9.4 Solid Propellant Rocket Motor Fundamentals

9.4.1 Mass flow rate

$$\begin{aligned} \dot{m} &= A_b r \rho_b \\ &= \frac{\mathrm{d}\left(\rho_1 V_1\right)}{\mathrm{d}t} + \frac{A_t p_1}{c^*} \\ &= A_b \rho_b a p_1^n \end{aligned}$$

where A_b, r, and ρ_b are the burn area, burn rate, and density of solid propellant before burning

9.4.2 Pressure in steady burning

$$p_1 = K \rho_b r c^* \quad \text{where} \quad K \equiv A_b / A_t$$

9.4.3 Burning rate approximation

$$r = a p_1^n$$

10 Aeroacoustics

10.1 Governing Equations

$$\frac{\partial p'}{\partial t} + \rho_0 \nabla \cdot \mathbf{u}' = 0 \qquad \text{(continuity)}$$
$$\rho_0 \frac{\partial \mathbf{u}'}{\partial t} = -\nabla p' \qquad \text{(momentum)}$$

10.2 General definitions

10.2.1 Frequency

$$f = \frac{\omega}{2\pi} = \frac{1}{T} \rightarrow \omega = \frac{1}{T}$$
$$f = \text{frequency}$$
$$\omega = \text{angular rate}$$
$$T = \text{period}$$

10.2.2 Fourier Series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{2\pi n t}{T}\right) + b_n \sin\left(\frac{2\pi n t}{T}\right)\right]$$
$$a_n = \frac{2}{T}\int_0^T f(t)\cos\left(\frac{2\pi n t}{T}\right)\mathrm{d}t$$
$$b_n = \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2\pi n t}{T}\right)\mathrm{d}t$$
$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi n t}{T}}$$
$$c_n = \frac{1}{T}\int_0^T f(t) e^{i\frac{2\pi n t}{T}}\mathrm{d}t$$
$$c_n = \begin{cases} \frac{a_{-n} + i b_{-n}}{2}; & n \le -1 \\ \frac{a_0}{2} & n = 0 \\ \frac{a_n - i b_n}{2} & n \ge 1 \end{cases}$$

10.2.3 Fourier Transform

$$\bar{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty} f(t) e^{-i\omega t} \qquad \text{(fourier transform)}$$
$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty} \bar{f}(\omega) e^{i\omega t}\mathrm{d}\omega \qquad \text{(inverse fourier transform)}$$

10.2.4 Acoustic Intensity

$$\mathbf{I} = p'\mathbf{u}'$$
$$\langle I \rangle = \frac{A^2}{2\rho_0 a_0} \qquad \text{Units: } \left[\frac{\mathrm{W}}{\mathrm{m}^3}\right]$$

10.2.5 Acoustic Energy

$$e = \frac{p'^2}{2\rho_0 a_0^2} + \frac{1}{2}\rho_0 \mathbf{u}'^2$$
$$\langle e \rangle = \frac{A^2}{2\rho_0 a_0^2} \qquad \text{Units: } \left[\frac{\mathrm{J}}{\mathrm{m}^2}\right] \qquad \text{(energy density)}$$

10.2.6 Root of Square Average (RMS)

$$p'_{\text{rms}} = \sqrt{\frac{1}{T}\int_0^T p(t)'^2 \mathrm{d}t}$$
$$= \frac{A}{\sqrt{2}} \qquad (\text{for } p(t)' = A\cos t)$$

10.2.7 Decibel

$$10\log_{10}\left(\frac{\text{Power}}{\text{Ref power}}\right)$$

10.2.8 Acoustic Power Level

$$\text{APL} = 10\log_{10}\left(\frac{\mathbb{P}_{\text{acoustic}}}{\mathbb{P}_{\text{ref}}}\right) \quad [\text{dB}]$$
$$\mathbb{P} = \mathbb{P}_{\text{ref}} 10^{\text{APL}/10}$$
$$\text{where } \mathbb{P}_{\text{ref}} = 10^{-12}\,[\text{W}] \quad \text{and} \quad \mathbb{P} = \langle I \rangle A$$

where A is the area the sound is going through

10.2.9 Sound Pressure Level

$$\text{SPL} = 10\log_{10}\left(\frac{p'^2_{\text{rms}}}{p^2_{\text{ref}}}\right)$$
$$= 20\log_{10}\left(\frac{p_{\text{rms}}}{p_{\text{ref}}}\right)$$
$$p_{\text{ref}} = 2\times 10^{-5}\,[\text{Pa}]$$

10.2.10 RMS pressure summation

$$(p_{\text{rms}})_{1+2} = \sqrt{\left(p^2_{\text{rms}}\right)_1 + \left(p^2_{\text{rms}}\right)_2} \tag{1}$$

10.2.11 SPL summation

$$\text{SPL}_{1+2+\cdots+n} = 10\log_{10}\left(10^{\text{SPL}_1/10} + 10^{\text{SPL}_2/10} + \cdots + 10^{\text{SPL}_n/10}\right) \tag{2}$$

10.2.12 Impedence

$$Z = i\sigma\omega$$
$$= \frac{p'}{u'}$$
$$m/A = \sigma \qquad \text{(mass/area)}$$

10.2.13 Planar wave transmission & reflection

$$p_t = \frac{2\rho_0 a_0}{2\rho_0 a_0 + i\sigma\omega} p_i \qquad \text{(transmission)}$$
$$p_r = \frac{i\sigma\omega}{2\rho_0 a_0 + i\sigma\omega} p_i \qquad \text{(reflection)}$$
$$\text{where} \quad \sigma = m/A$$

10.2.14 Reflection/Transmission coefficient

$$\alpha_\mathrm{r} \equiv \frac{\langle I\rangle_\mathrm{r}}{\langle I\rangle_\mathrm{i}}$$
$$\alpha_\mathrm{t} \equiv \frac{\langle I\rangle_\mathrm{t}}{\langle I\rangle_\mathrm{i}}$$

10.2.15 Composite surface transmission coefficient

$$\alpha_{\mathrm{t}_\mathrm{eff}} = \frac{\sum_{j=1}^{n} \alpha_{\mathrm{t}_j} A_j}{A_\mathrm{T}} \quad \text{where} \quad A \text{ is surface area}$$

10.2.16 Transmission loss

$$\mathrm{T.L.} = 10\log\left(\frac{1}{\alpha_\mathrm{t}}\right)$$
$$= 10\log\left[\frac{(\sigma\omega)^2}{(2\rho_0 a_0)^2}\right] + 10\log\left[1 + \frac{(2\rho a_0)^2}{(\sigma\omega)^2}\right]$$

10.2.17 Spherical wave equation

$$p = \frac{A}{r} e^{i(\omega t - kr)}$$

10.2.18 Spherical Sound Pressure Level

$$SPL = 20\log\left(\frac{p_\mathrm{rms}}{p_\mathrm{ref}}\right)$$
$$= 20\log\left(\frac{A}{\sqrt{2} p_\mathrm{ref}}\right) - 20\log(r)$$
$$\Delta_\mathrm{SPL} = \mathrm{SPL}_1 - \mathrm{SPL}_2 = 20\log\left(\frac{r_2}{r_1}\right)$$

10.2.19 Spherical acoustic velocity

$$u_r' = \frac{p'(t)}{\rho_0 a_0}\left(1 + \frac{1}{ikr}\right)$$
$$= \frac{A}{\rho_0 a_0 r}\left(1 + \frac{1}{ikr}\right) e^{i(\omega t - kr)}$$

10.2.20 Reverberant Field energy density

$$\langle e\rangle_\mathrm{RF} = \frac{4\pi p_\mathrm{rms}^2}{\rho_0 a_0^2}$$
$$= \frac{2\pi A^2}{\rho_0 a_0^2}$$

10.2.21 Reverberant field intensity

$$\langle I\rangle_\mathrm{RF} = \frac{\pi A^2}{2\rho_0 a_0}$$
$$= \frac{\langle e\rangle_\mathrm{RF}\, a_0}{4}$$

10.2.22 Absorption

$$\langle I\rangle_\mathrm{RF} = \frac{\mathbb{P}(1 - \alpha_\mathrm{A})}{\alpha_\mathrm{A} A_\mathrm{wall}} = \frac{\mathbb{P}}{R}$$
$$\alpha_\mathrm{A} = \frac{\langle I\rangle_\mathrm{a}}{\langle I\rangle_\mathrm{i}} = \frac{\langle I\rangle_\mathrm{i} - \langle I\rangle_\mathrm{r}}{\langle I\rangle_i}$$

10.2.23 Reverberant Field SPL

$$\mathrm{SPL}_\mathrm{RF} = 10\log_{10}\left[\frac{\left(p_\mathrm{rms}^2\right)_\mathrm{RF}}{p_\mathrm{ref}^2}\right]$$
$$= 10\log\left(\frac{4\rho_0 a_0 \mathbb{P}}{R p_\mathrm{ref}^2}\right)$$
$$R = \frac{A_\mathrm{wall}\alpha_\mathrm{A}}{(1 - \alpha_\mathrm{A})}$$

10.2.24 Direct field and Total SPL

$$\mathrm{SPL}_\mathrm{total} = 10\log_{10}\left(\frac{\rho_0 a_0 \mathbb{P}}{p_\mathrm{ref}^2}\right) + 10\log_{10}\left(\frac{1}{4\pi r^2} + \frac{4}{R}\right)$$

10.2.25 Average absorption coefficient

$$\alpha_\mathrm{A} = \frac{\sum \alpha_{\mathrm{A}_i} A_i}{\sum A_i}$$

10.2.26 Average SPL or "equivalent level"

$$\mathrm{SPL}_\mathrm{AVG} = L_\mathrm{EQ} = 10\log_{10}\left(\frac{1}{T}\sum_{i=1}^{N} 10^{\mathrm{SPL}_i/10}\Delta t_i\right)$$

10.2.27 Day/Night level

$$L_\mathrm{DN} = 10\log_{10}\left[\frac{1}{T}\left(\sum_{0700}^{2200} 10^{\mathrm{SPL(t)}/10}\Delta t_i + \sum_{2200}^{0700} 10^{\frac{\mathrm{SPL(10)}+10}{10}}\Delta t_i\right)\right]$$

10.2.28 Permissible time & dose

$$T = \frac{8}{2^{\frac{L-90}{5}}}$$

$$D = \sum_{i=1}^{N} \frac{A_i}{T} = \frac{1}{8} \sum_{i=1}^{N} \Delta t_i 2^{\frac{L_i-90}{5}} \leq 1$$

A_i is the actual time

T is the permissible time (8 hours at 90 dB)

11 Numerical Analysis

11.1 Second Order PDEs classification

11.1.1 General Form

$$A\frac{\partial^2\phi}{\partial x^2}+B\frac{\partial^2\phi}{\partial x\partial y}+C\frac{\partial^2\phi}{\partial y^2}+D\frac{\partial\phi}{\partial x}+E\frac{\partial\phi}{\partial y}+f\phi+G=0$$

$$A\frac{\partial^2\phi}{\partial x^2}+B\frac{\partial^2\phi}{\partial x\partial y}+C\frac{\partial^2\phi}{\partial y^2}=H$$

$$H=-\left(D\frac{\partial\phi}{\partial x}+E\frac{\partial\phi}{\partial y}+f\phi+G\right)$$

11.1.2 Characteristics in physical space

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{\alpha,\beta}=\frac{B\pm\sqrt{B^2-4AC}}{2A}$$

11.1.3 Equation characteristics

elliptic if	$B^2-4AC<0$
parabolic if	$B^2-4AC=0$
hyperbolic if	$B^2-4AC>0$

11.2 Finite Difference Schemes

11.2.1 First derivative

$$\begin{aligned}u_i'&=\frac{1}{\Delta x}\left(-u_{i-1}+u_i\right)+\tfrac{1}{2}\Delta x u''\\&=\frac{1}{\Delta x}\left(-u_i+u_{i+1}\right)-\tfrac{1}{2}\Delta x u_i''\\&=\frac{1}{\Delta x}\left(-u_{i-1}+u_{i+1}\right)-\tfrac{1}{6}\Delta x^2 u_i^{(3)}\\&=\frac{1}{\Delta x}\left(\tfrac{1}{2}u_{i-2}-2u_{i-1}+\tfrac{3}{2}u_i\right)+\tfrac{1}{3}\Delta x^2u_i^{(3)}\\&=\frac{1}{\Delta x}\left(-\tfrac{3}{2}u_i+2u_{i+1}-\tfrac{1}{2}u_{i+2}\right)+\tfrac{1}{3}\Delta x^2u_i^{(3)}\end{aligned}$$

11.3 Stability Analysis

11.3.1 Discrete Perturbation Stability analysis

Consider the parabolic model equation

$$\frac{\partial u}{\partial t}=\alpha\frac{\partial^2u}{\partial x^2}.$$

Using a first order time and second order spatial derivative, this equation may be written as:

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}=\alpha\frac{u_{i+1}^n-2u_i^n+u_{i-1}^n}{(\Delta x)^2}.$$

If a disturbance ϵ at node i and time level n is introduced and we search for solution at time level $n+1$ for all i nodes, the finite difference eq. becomes:

$$\frac{u_i^{n+1}-\left(u_i^n+\epsilon\right)}{\Delta t}=\alpha\frac{u_{i+1}^n-2\left(u_i^n+\epsilon\right)+u_{i-1}^n}{(\Delta x)^2}.$$

If $u^n=0$ for all i the equation reduces to:

$$\frac{u_i^{n+1}-\epsilon}{\Delta t}=\alpha\frac{-2\epsilon}{(\Delta x)^2}$$

or equivalently

$$\begin{aligned}u_i^{n+1}&=\epsilon\left\{1+2\alpha\left[\frac{\Delta t}{(\Delta x)^2}\right]\right\}\\&=\epsilon\left(1-2d\right)\\\rightarrow\frac{u_i^{n+1}}{\epsilon}&=(1-2d)\end{aligned}$$

where $d=\alpha\Delta t/\left(\Delta x\right)^2$ is known as the *diffusion number*. It's required that,

$$\left|\frac{u_i^{n+1}}{\epsilon}\right|\le1$$

or

$$1-2d\le1\qquad\text{and}\qquad1-2d\ge-1$$

11.3.2 Von Neumann Stability Analysis

Solution of finite difference equation is expanded in a Fourier series. The decay or growth of the *amplification factor* determines the stability.

Assume a Fourier component for u_i^n as

$$u_i^n=U^ne^{IP(\Delta x)i}$$

where $I=\sqrt{-1}$, U^n is the amplitude at time n, and P is the wave number in the x-direction, i.e., $\lambda_x=2\pi/P$, where λ_x is the wavelength. Similarly,

$$u_i^{n+1}=U^{n+1}e^{IP(\Delta x)i}\quad\text{and}\quad u_{i\pm1}^n=U^ne^{IP(\Delta x)(i\pm1)}$$

If phase angle $\theta=P\Delta x$ is defined, then

$$\begin{aligned}u_i^n&=U^ne^{I\theta i}\\u_i^{n+1}&=U^{n+1}e^{I\theta i}\\u_{i\pm1}^n&=U^ne^{I\theta(i\pm1)}\end{aligned}$$

Consider again

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}=\alpha\frac{u_{i+1}^n-2u_i^n+u_{i-1}^n}{(\Delta x)^2}$$

or in terms of the diffusion number

$$u_i^(n+1)=u_i^n+d\left(u_{i+1}^n-2u_i^n+u_{i-1}^n\right)$$

Substituting the Fourier component and cancelling out terms of $e^{I\theta i}$, we get

$$U^{n+1}=U^n\left[1+d\left(e^{I\theta}-e^{-I\theta}-2\right)\right]$$

Using the relation $\cos\theta=\left(e^{I\theta}+e^{-I\theta}\right)/2$, we get

$$U^{n+1}=U^n\left[1+2d\left(\cos\theta-1\right)\right]$$

Introducing the amplification factor $U^{n+1}=GU^n$, we get

$$G=1-2d\left(1-\cos\theta\right).$$

Stability requires,

$$|G| \le 1 \qquad \text{and} \qquad |1 - 2d\,(1 - \cos\theta)| \le 1$$

so that

$$1 - 2d\,(1 - \cos\theta) \le 1$$
$$\text{and}$$
$$1 - 2d\,(1 - \cos\theta) \ge -1$$

must be valid for all values of θ. With a maximum value of $(1 - \cos\theta) = 2$, the LHS becomes $1 - 4d$. Solving the new equation gives us a final condition

$$d \le \frac{1}{2} \tag{3}$$

11.3.3 Scheme Requirements

- *Consistency*: A finite difference approximation of a PDE is consistent if the finite difference equation approaches the PDE as the grid size approaches zero.
- *Stability*: A numerical scheme is said to be stable if any error introduced in the finite difference equation does not grow with the solution of the finite difference equation.
- *Convergence*: A finite difference scheme is convergent if the solution of the finite difference scheme approaches that of the PDE as the grid size approaches zero.
- *Lax's equivalence theorem*: For a FDE which approximates a well-posed, linear initial value problem, the necessary and sufficient condition for convergence is that the FDE must be stable and consistent.
- *The conservative (divergent) form of a PDE*: In this formulation of a physical law, the coefficients of the derivatives are either constant or, if variable, their derivatives do not appear anywhere in the equation.
- *Conservative property of a FDE*: If the finite difference approximation of a PDE maintains the integral property of the conservation law over an arbitrary region containing any number of grid points, it is said to possess a conservative property.

11.4 Classification of equations

Elliptic: $$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

Parabolic: $$\frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2}$$

Hyperbolic: $$\frac{\partial^2\phi}{\partial t^2} = a^2\frac{\partial^2\phi}{\partial x^2}$$

12 Useful Identities

12.1 Series

12.1.1 Expansion of asymptotic series to a power

$$
\begin{aligned}
\left(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots\right)^n = {} & x_0^n \\
& + \epsilon n x_0^{n-1} x_1 \\
& + \epsilon^2 \left[\frac{n\,(n-1)}{2!} x_0^{n-2} x_1^2 + n x_0^{n-1} x_2\right] \\
& + \epsilon^3 \left[\frac{n\,(n-1)\,(n-2)}{3!} x_0^{n-3} x_1^3 + n\,(n-1)\, x_0^{n-2} x_1 x_2 + n x_0^{n-1} x_3\right] \\
& + \epsilon^4 \left[\frac{n\,(n-1)\,(n-2)\,(n-3)}{4!} x_0^{n-4} x_1^4 + \frac{n\,(n-1)\,(n-2)}{2!} x_0^{n-3} x_1^2 x_2 + \frac{n\,(n-1)}{2!} x_0^{n-2} \left(x_2^2 + 2 x_1 x_3\right) + n x_0^{n-1} x_4\right]
\end{aligned}
$$