Mathematical Statistics

SinkingFun

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Hi!

I´m currently preparing for a final test on Mathematical Statistics and as you might have guessed, my doubts are more on the theoretical part of the course.

So, I have worked on three different questions but have not been able to finish any of them, here is what I´ve done so far:

Problem 1

1. The sum of the values obtained in a random sample of size 5 taken from a population with Poisson distribution will be used to test the null hypothesis

H_0: population mean > 2

against the alternative hypothesis:

H_1: population mean ≤ 2

The null hypothesis will be rejected if and only if the sum of the observations is 5 or less.

- Determine:

The power function for the means: 1.5, 2, 2.4, 2.6 and

Obtain a table with probabilities of errors type 1 and 2

My solution:

Let X ~ Poisson(θ), so that E[X] = θ

Then let X_i with i in $\{1, 2, \ldots, 5\}$ be the random sample, so

$Y = \sum_{i=1}^{5} X_i$, and we reject H_0 iff $Y \leq 5$

Then, we know that Y ~ Poisson(5θ) and so our power function will be given by:

$\pi(\theta) = \mathrm{P}(Y \leq 5) = \sum_{y=0}^{5} e^{-5\theta} \frac{(5\theta)^y}{y!}$

i.e. $\pi(\theta) = e^{-5\theta}(1 + 5\theta + \frac{(5\theta)^2}{2!} + \frac{(5\theta)^3}{3!} + \frac{(5\theta)^4}{4!} + \frac{(5\theta)^5}{5!})$

And from here, the first part of the problem would be to get the values of

$\pi(1.5)$, $\pi(2)$, $\pi(2.4)$ and $\pi(2.6)$ right?

But what I don´t get in this problem is how to construct the *classical* table with probabilities of errors Type I and II.

Then, for Problem 2

2. Consider a population with Bernoulli distribution, we have the hypothesis test:

H_0: p = p_0

H_1: p $< p_0$

α

Given a sample of size "n", use the correct theory to find the region of rejection, the test statistic (for the exact test and for a large sample size), and the property that the region has in order to generate the corresponding unilateral alternative hypothesis and specify it

My solution

I followed the process of a similar problem that we saw on class, so here we start by considering $X \sim Bernoulli(p)$ and we note that $\hat{p} = \frac{X}{n}$

Then, our likelyhood function is:

- $L(x_1, ..., x_n) = \binom{n}{X} p^X (1-p)^{n-X}$ with $X \in 0, ..., n$

we let $\omega_0 = [p_0, 1)$ and $\omega_1 = (0, p_0)$ so that:

- $\Omega = \omega_0 \cup \omega_1 = (0, p_0) \cup [p_0, 1) = (0, 1)$

And using the likelyhood ratio we get:

- $\Lambda = \frac{ML_0}{ML} = \frac{p_0^X (1-p_0)^{n-X}}{p^X (1-p)^{n-X}} \leq k_1$ with $k_1 \in (0, 1)$

Taking $ln(\Lambda) = h(\hat{p})$ we get:

- $h(\hat{p}) = \hat{p} ln(p_0) + (1-\hat{p}) ln(1-p_0) - \hat{p} ln(\hat{p}) - (1-\hat{p}) ln(1-\hat{p}) \leq k$ with $k < 0$

We then analyze the behavior of $h(\hat{p})$ with the following 2 limits:

- $\lim_{\hat{p} \to 0} h(\hat{p}) = ln(1-p_0)$, and
- $\lim_{\hat{p} \to 1} h(\hat{p}) = ln(p_0)$

And taking the derivative we get:

- $h(\hat{p})' = ln(\frac{p_0(1-\hat{p})}{\hat{p}(1-p_0)}) \geq 0 \iff \frac{p_0(1-\hat{p})}{\hat{p}(1-p_0)} \geq 1 \iff p_0(1-\hat{p}) \geq \hat{p}(1-p_0)$

hence

- $p_0 \geq \hat{p}$

But from here I don´t know how to continue. I think the next step is to give a size of an arbitrary α , i.e. some sort of $P(X \leq somebound | p = p_0) = \alpha$ in order to find the unilateral interval, but I don´t know how to get there.

Also, I think the wanted test statistic is $Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0 q_0}{n}}}$ but I´m also not sure.

And finally, Problem 3

3. Given a random sample of size "n" taken from a normally distributed population with unknown mean and variance, find the region of rejection and the test statistic (completely specified) to prove the null hypothesis

H_0: $\sigma = \sigma_0$ against the alternative hypothesis:

H_1: $\sigma \neq \sigma_0$

My Solution

I´m actually sort of lost in this one, the only thing I think I´ve figured out is that if we consider $X_i \sim N(\mu, \sigma)$ independent, we then can get

- $\frac{nS_n^2}{\sigma^2} = \sum_{i=1}^{n} (\frac{(x_i - \bar{x})^2}{\sigma^2})$

and then, by the m.g.f, we know that:

- $\sum_{i=1}^{n} (\frac{(x_i - \bar{x})^2}{\sigma^2}) \sim \chi^2_{n-1}$ so that $\frac{nS_n^2}{\sigma^2} = \chi^2_{n-1}$

And from there I don´t know how to continue.

So, I know it is sort of a long post, and I thank you for the time you take on reading this. Any help you can provide will be great for a starting-to-panic student.