1)

[redacted]

Augmented Matrix

$x_1 + 2x_3 = 3$

$-x_2 + x_3 = 1$

$6x_2 + x_3 = -2$

The augmented matrix for this system of equation is $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & -1 & 1 & 1 \\ 0 & 6 & 1 & -2 \end{array}\right]$

Can perform a series of **elementary row operations** to reduce system and solve it:

(1) Interchange rows: $R_i \leftrightarrow R_j$

(2) Add/subtract rows: $R_i \rightarrow R_i \pm R_j$

(3) Multiply by constants: $R_i \rightarrow sR_i$

Taking above matrix as an example.

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & -1 & 1 & 1 \\ 0 & 6 & 1 & -2 \end{array}\right] \underset{\sim}{R_3 \to R_3 + 6R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 7 & 4 \end{array}\right]$$

Now it's in R-E form

Row-Echelon form

(1) Rows with all zero entries appear below rows with any nonzero entries

(2) When comparing two nonzero rows, the first nonzero entry (called the leading entry or pivot) in the upper row is to the left of the leading entry in the lower row.

Note: The goal is to create zeros below the leading entry in each row.

Part 1

Example 1: Reduce to R-E form

$$\begin{bmatrix} 2 & 1 & 4 \\ 1 & 3 & 2 \\ 3 & -1 & 6 \end{bmatrix} \underset{\sim}{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 3 & -1 & 6 \end{bmatrix} \underset{\sim}{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & 0 \\ 3 & -1 & 6 \end{bmatrix} \underset{\sim}{R_3 \to R_3 - 3R_1} \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & 0 \\ 0 & -10 & 0 \end{bmatrix}$$

$$\underset{\sim}{R_3 \to R_3 - 2R_2} \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Gauss Method with Back Substitution

- Reduce augmented matrix to R-E form and solve

Example 2:

$2x - y = 8$

$6x - 5y = 32$

$$\left[\begin{array}{cc|c} 2 & -1 & 8 \\ 6 & -5 & 32 \end{array}\right] \underset{\sim}{R_2 \to R_2 - 3R_1} \left[\begin{array}{cc|c} 2 & -1 & 8 \\ 0 & -2 & 8 \end{array}\right]$$

This aug. matrix now says, $2x - y = 8$, $-2y = 8$ which means $y = \frac{-8}{2} = -4$ $\boxed{y = -4}$

Sub into first eqn: $2x - (-4) = 8$

$\boxed{x = 2}$

Reduced-Row-Echelon form (RREF)

(1) Just like R-E form, except, all pivots are reduced to 1

(2) All entries above and below any pivots are zeros

2)

Example 3: Reduce to RREF and solve

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 2 & 1 & -2 \end{array}\right] \underset{\sim}{R_3 \to R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & -3 \end{array}\right] \underset{R_3 \to R_3 - R_2}{\overset{R_1 \to R_1 - R_2}{\sim}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & -5 \end{array}\right]$$

This says $x_1 - x_3 = -1$
$x_2 + x_3 = 2$
$0 = -5$ ← Clearly this is impossible.

∴ There are no values of x_1, x_2, x_3 that can satisfy all three equations.
∴ No solution

PART II

Consistent/Inconsistent Solutions

- A system having no solutions is inconsistent
- A system that has at least one solution is consistent
- # of pivots = rank of matrix, A [denoted rank(A)]

Free variables

↳ Need free variables when column(s) do(es) not have pivot.

Example 4: Describe all solutions

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 1 \\ 0 & 1 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Is this is R-E form? Yes! We can look at the pivots and analyze and obtain a solution set.

↳ So, we have two pivots, one in column 1, and one in column 2. This means x_1 and x_2 are "basic variables".

↳ We have no pivots in columns 3 and 4. So, x_3 and x_4 are "free variables", also known as parameters.

Our sys. of eqns is $x_1 + 2x_3 = 1$ (1)
$x_2 + x_3 + 3x_4 = -2$ (2)

Since x_3, x_4 are free variables, let $x_3 = s$, $x_4 = r$, $s, r \in \mathbb{R}$.

Sub into (1) and (2): $x_1 + 2s = 1$
$x_2 + s + 3r = -2$

Solve for x_1 and x_2: $x_1 = 1 - 2s$
$x_2 = -2 - s - 3r$

, our solution vector $\vec{x} = \begin{bmatrix} 1-2s \\ -2-s-3r \\ s \\ r \end{bmatrix}$

$$\vec{x} = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} r \quad , s, r \in \mathbb{R}.$$

Determining whether vector is in span of other multiple vectors

Example 5: Is $\vec{b}$ in span$(\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3})$?

$$\vec{b} = \begin{pmatrix} 3 \\ 5 \\ 3 \end{pmatrix}, \overrightarrow{v_1} = \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}, \overrightarrow{v_2} = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}, \overrightarrow{v_3} = \begin{pmatrix} -3 \\ -1 \\ 5 \end{pmatrix}$$

Recall, $\vec{b}$ is in span of these vectors if $\vec{b} = r_1\vec{v_1} + r_2\vec{v_2} + r_3\vec{v_3}$

i.e. $r_1\begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} + r_2\begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix} + r_3\begin{bmatrix} -3 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 3 \end{bmatrix}$

So, really, we have a system of eqns:

$$\begin{aligned} 0r_1 + r_2 - 3r_3 &= 3 \\ 2r_1 + 4r_2 - r_3 &= 5 \\ 4r_1 - 2r_2 + 5r_3 &= 3 \end{aligned}$$

We want to check if this system is consistent or not. (Consistent = $\vec{b}$ is in sp$(\vec{v_i})$.)

$$\left[\begin{array}{ccc|c} 0 & 1 & -3 & 3 \\ 2 & 4 & -1 & 5 \\ 4 & -2 & 5 & 3 \end{array}\right] \sim \cdots \sim \left[\begin{array}{ccc|c} 2 & 4 & -1 & 5 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & -23 & 23 \end{array}\right]$$

This is now in R-E form.

Since we have a consistent system, we can conclude that $\vec{b}$ is in sp$(\vec{v_1}, \vec{v_2}, \vec{v_3})$.

Homogenous System

A system of linear equations is homogenous if the right-hand-side of all the equations is zero. For example,

$$\begin{aligned} 2x - y &= 0 \\ 6x - 5y &= 0 \end{aligned}$$

Linear Independence

Vectors are linearly independent if, when you put into a matrix, all columns have a pivot.

- The columns with pivots correspond to vectors that are linearly independent.

ex. $\begin{bmatrix} | & | \\ \vec{v_1} & \vec{v_2} \\ | & | \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$ ← both columns have pivot, ∴ both lin. indep.

Basis for solution set

The solution vectors for the solution set form a basis.

4)

Example 6: Is this set linearly independent? $\left\{\begin{pmatrix}1\\-3\\2\end{pmatrix},\begin{pmatrix}2\\-5\\3\end{pmatrix},\begin{pmatrix}4\\0\\1\end{pmatrix}\right\}$ in R^3

$$\begin{bmatrix}1&2&4\\-3&-5&0\\2&3&1\end{bmatrix}\sim\begin{bmatrix}1&2&4\\0&1&12\\0&-1&-7\end{bmatrix}\sim\begin{bmatrix}1&2&4\\0&1&12\\0&0&5\end{bmatrix}$$

Can see there's a pivot in each column.
$\therefore$ All vectors in the set are lin. indep.

Example 7: Consider

$$x_1+2x_2+x_3+2x_4=0$$
$$x_1-x_2+2x_3-2x_4=0$$

a) Find the RREF of the matrix
b) How many pivots are there? What is rank(A)?
c) How many parameters are there?
d) Find a vector equation for the solution space.
e) Find a basis for the solution space.

a) $\begin{bmatrix}1&2&1&2\\1&-1&2&-2\end{bmatrix}\sim\begin{bmatrix}1&2&1&2\\0&-3&1&-4\end{bmatrix}\sim\begin{bmatrix}1&0&5/3&-2/3\\0&1&-1/3&4/3\end{bmatrix}$

b) There are 2 pivots $\therefore \text{rank}(A)=2$

c) There are 2 free variables (i.e. 2 parameters) since there are two columns without pivots.

d) Since x_3, x_4 are free variables,
let $x_3=s$, $x_4=r$, $s,r\in\mathbb{R}$

$x_1=-\frac{5}{3}s+\frac{2}{3}t$
$x_2=\frac{1}{3}s-\frac{4}{3}t$

$$\therefore \vec{x}=\begin{bmatrix}-\frac{5}{3}s+\frac{2}{3}t\\ \frac{1}{3}s-\frac{4}{3}t\\ s\\ t\end{bmatrix}=\begin{bmatrix}-5/3\\1/3\\1\\0\end{bmatrix}s+\begin{bmatrix}2/3\\-4/3\\0\\1\end{bmatrix}t \quad s,t\in\mathbb{R}$$

e) Basis $=\left\{\begin{bmatrix}-5/3\\1/3\\1\\0\end{bmatrix},\begin{bmatrix}2/3\\-4/3\\0\\1\end{bmatrix}\right\}$