# Basics of Mathematics

Hand Book

General Mathematics

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Part 1

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بسم الله الرحمن الرحيم

# Basics of Mathematics

## Reference

1_(Calculus) J-Stewart 7th Edition
2_Internet
3_(Physics) Cutnell, Johnson 8th Edition
4_Egyptian Secondary School Math Books
5_1st and 2ed stages at Faculty of Science Math Books

## Content

# Basics of Mathematics

## Section $(A)$

### I. Exponent and Arithmetic Operations:

1) $a(b+c) = ab + ac$
2) $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$
3) $\frac{a+c}{b} = \frac{a}{b} + \frac{c}{b}$
4) $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$
5) $x^m \times x^n = x^{m+n}$
6) $\frac{x^m}{x^n} = x^{m-n}$
7) $(xy)^n = x^n y^n$
8) $x^{\frac{1}{n}} = \sqrt[n]{x}$
9) $x^{-n} = \frac{1}{x^n}$
10) $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$
11) $x^{\frac{m}{n}} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$
12) $\sqrt[n]{xy} = \sqrt[n]{x}\sqrt[n]{y}$
13) $(x^m)^n = x^{mn}$
14) $a^x = y \Longrightarrow x = \log_a y$
15) $e^x = y \Longrightarrow x = \ln y$
16) $log_b A = \frac{ln\,A}{ln\,b}$
17) $ln\frac{a}{b} = ln\,a - ln\,b$
18) $ln\,ab = ln\,a + ln\,b$

### II. Factoring, Inequalities and Absolut Value:

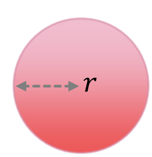
1) $x^2 - y^2 = (x+y)(x-y)$
2) $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$
3) $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$
4) $(x+y)^2 = x^2 + 2xy + y^2$
5) $(x-y)^2 = x^2 - 2xy + y^2$
6) $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
7) $(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$
8) $a < b \text{ and } b < c \Longrightarrow a < c$
9) $a < b \Longrightarrow a + c < b + c$
10) $a < b \text{ and } c > 0 \Longrightarrow ca < cb$
11) $a < b \text{ and } c < 0 \Longrightarrow ca > cb$
12) $|x| = a \Longrightarrow x = a \text{ or } x = -a$
13) $|x| < a \Longrightarrow -a < x < a$
14) $|x| > a \Longrightarrow x > a \text{ or } x < -a$

### III. Lines, Distance and Midpoint Formulas:

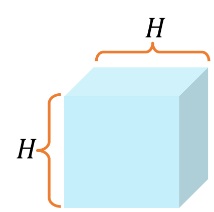
1) Slope of line through $p(x_1, y_1) \Longrightarrow p(x_2, y_2)$ Is $\left\{m = \frac{y_2 - y_1}{x_2 - x_1}\right\}$
2) Point-slope equation of line through $p(x_1, y_1)$ with slope $m$ Is $\{y - y_1 = m(x - x_1)\}$
3) Slope-intercept equation of line with slope $m$ and $y$-intercept $b$ Is $\{y = mx + b\}$
4) Distance between two points $p(x_1, y_1) \Longrightarrow p(x_2, y_2)$ Is $\left\{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\right\}$
5) Midpoint of $\overline{p_1 p_2}$ Is $\left\{\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right\}$

### IV. Circumference, Area and Volume Formulas:

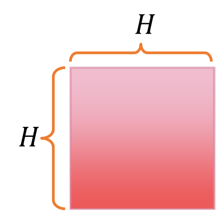
1) Circumference of Rectangle = $2\{L(\text{length}) + H(\text{height})\}$
2) Circumference of Square = $4H \Longrightarrow Fig(A-3)$
3) Circumference of Circle = $2\pi r \Longrightarrow Fig(A-1)$
4) Area of Rectangle = $(H \times L)$
5) Area of Triangle = $\frac{1}{2}\ LH$
6) Area of Square = $H^2 \Longrightarrow Fig(A-3)$
7) Area of Cube = $6H^2 \Longrightarrow Fig(A-2)$
8) Area of block = $6(H \times L)$
9) Area of Circle = $\pi r^2 \Longrightarrow Fig(A-1)$
10) Area of Ball = $4\pi r^2 \Longrightarrow Fig(A-1)$
11) Area of Cylinder = $2\pi r \times H \Longrightarrow Fig(A-5)$
12) Area of Cone = $\pi r\sqrt{r^2 + h^2} \Longrightarrow Fig(A-4)$
13) Length of Arc = $\theta r \Longrightarrow Fig(A-6)$
14) Area of Section Circle = $\frac{1}{2}\theta r^2 \Longrightarrow Fig(A-6)$
15) Volume of Cube = $H^3 \Longrightarrow Fig(A-2)$
16) Volume of Cylinder = $\pi r^2 \times H \Longrightarrow Fig(A-5)$
17) Volume of Ball = $\frac{4}{3}\pi r^3 \Longrightarrow Fig(A-1)$
18) Volume of Pyramid (Cone) = $\frac{1}{3}(A)base\ area \times H$



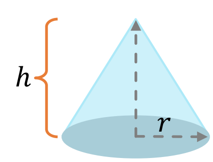
$(A-1)$



$(A-2)$



$(A-3)$



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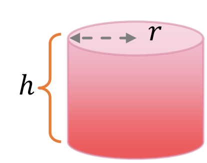
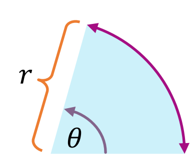


$(A-5)$



$(A-6)$

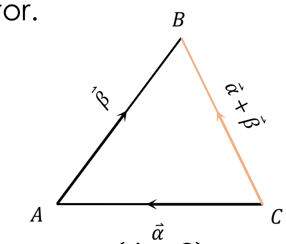
 *Islam Reda Ahmed*

## V. Vectors and Dot, Cross Product:

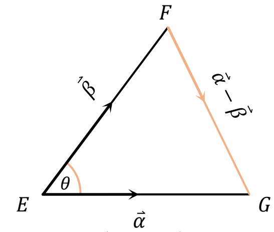
1) If we have a line its start is $A$, and its end is $B$ then its vector is $\overrightarrow{AB}$.
If the start is $B$ and end is $A$, then its vector is $\overrightarrow{BA}$. $\Longrightarrow Fig(A-7)$


$(A-7)$

1. The Equivalent Vectors have the same Length and Direction of Vector.
2. Unit Vector $\underline{\hat{\beta}} = \frac{\vec{\beta}}{\|\vec{\beta}\|}$ that its $\|\underline{\hat{\beta}}\|$ is equal to one.
3. $\overrightarrow{BA} \neq \overrightarrow{AB}$
4. $If\ \vec{\beta} = (x_1, y_1)\ and\ \vec{\alpha} = (x_2, y_2)\ then\ \vec{\beta} + \vec{\alpha} = (x_1 + x_2, y_1 + y_2)$
5. $\vec{\beta} + \vec{\alpha} = \vec{\alpha} + \vec{\beta}$
6. $\vec{\beta} + (\vec{\alpha} + \vec{\gamma}) = (\vec{\beta} + \vec{\alpha}) + \vec{\gamma} = \vec{\beta} + \vec{\alpha} + \vec{\gamma}$
7. $k\vec{\beta} = (kx, ky)\ and\ k \in \mathbb{R}$
8. $if\ \vec{\beta} = k\vec{\alpha}\ and\ k > 0\ then\ (\vec{\beta}\ \backslash\backslash\ \vec{\alpha})\ and\ have\ the\ same\ direction.$
9. $if\ \vec{\beta} = k\vec{\alpha}\ and\ k < 0\ then\ (\vec{\beta}\ \backslash\backslash\ \vec{\alpha})\ and\ have\ opposite\ direction.$
10. $if\ \overrightarrow{AB}\ act\ on\ \vec{\beta}\ and\ \overrightarrow{CA}\ act\ on\ \vec{\alpha}\ then\ \overrightarrow{AB} + \overrightarrow{CA} = \vec{\beta} + \vec{\alpha}\ \Longrightarrow Fig(A-8)$
11. $if\ \overrightarrow{EF}\ act\ on\ \vec{\beta}\ and\ \overrightarrow{EG}\ act\ on\ \vec{\alpha}\ then\ \overrightarrow{EG} + \overrightarrow{EF} = \vec{\alpha} - \vec{\beta}\ \Longrightarrow Fig(A-9)$
12. $if\ \vec{\alpha} + \vec{\beta} = 0\ then\ \vec{\beta} = -\vec{\alpha}$
13. $if\ \vec{\beta} = (x, y)\ then\ \|\vec{\beta}\| = \beta = \sqrt{x^2 + y^2}\ it\ proportional\ to\ vector\ length.$

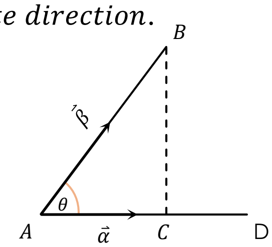


$(A-8)$



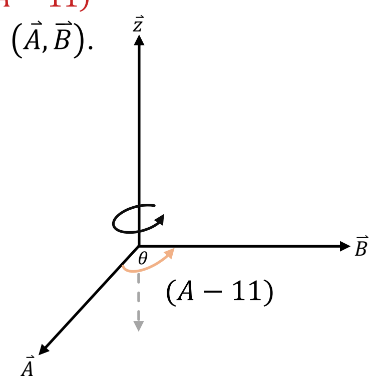
$(A-9)$

2) If the angle between $\overrightarrow{EF}$ $(\vec{\beta})$ and $\overrightarrow{EG}$ $(\vec{\alpha})$ is $(\theta)$ so The dot product would be
$\vec{\beta} \odot \vec{\alpha} = \beta\alpha\cos\theta$, Where $\beta = \|\vec{\beta}\|\ and\ \alpha = \|\vec{\alpha}\|\ \Longrightarrow Fig(A-10)$
1. $If\ 0° < \theta < 90°\ then\ \vec{\beta} \odot \vec{\alpha}\ is\ positive,\ but\ if\ 90° < \theta < 180°\ then\ \vec{\beta} \odot \vec{\alpha}\ is\ Negative.$
2. $If\ \theta = 90°\ then\ \vec{\beta} \odot \vec{\alpha} = 0,\ they\ are\ perpendicular.$
3. $If\ \theta = 0°\ they\ are\ parallel\ with\ the\ same\ direction,\ but\ at\ 180°\ are\ opposite\ direction.$
4. $\vec{\beta} \odot \vec{\alpha} = \vec{\alpha} \odot \vec{\beta}$ and $\vec{\beta} \odot \vec{\beta} = \beta^2$
5. $\vec{\gamma} \odot (\vec{\alpha} + \vec{\beta}) = \vec{\gamma} \odot \vec{\alpha} + \vec{\gamma} \odot \vec{\beta}$
6. $(c\vec{\beta}) \odot \vec{\alpha} = c\,(\vec{\beta} \odot \vec{\alpha}) \Longrightarrow (c\vec{\beta}) \odot (b\vec{\alpha}) = cb\,(\vec{\beta} \odot \vec{\alpha})$
7. $The\ projected\ AC = \beta\cos\theta\ , It's\ a\ scaler\ value.$
8. $If\ \vec{\beta} = a_1\vec{x} + b_1\vec{y}\ and\ \vec{\alpha} = a_2\vec{x} + b_2\vec{y},\ then\ \vec{\beta} \odot \vec{\alpha} = a_1b_1 + a_2b_2$

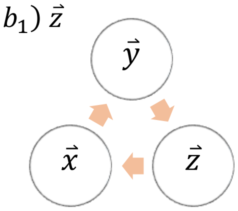


$(A-10)$

3) If the angle between $\vec{A}$ and $\vec{B}$ is $(\theta)$, so $\vec{A} \times \vec{B} = (AB\sin\theta)\,\vec{z} \Longrightarrow Fig(A-11)$
Where $\vec{z}$ is a unit vector perpendicular to the plane which contain $(\vec{A}, \vec{B})$.
1. $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$ using right hand base.
2. $\|\vec{A} \times \vec{B}\| = AB\sin\theta$
3. $\vec{z} = \frac{\vec{A}\times\vec{B}}{AB\sin\theta}$
4. If $\vec{A}$ is parallel to $\vec{B}$, then $\vec{A} \times \vec{B} = \vec{0}$ , so $\vec{A} \times \vec{A} = \vec{0}$
5. $\vec{C} \times (\vec{A} + \vec{B}) = \vec{C} \times \vec{A} + \vec{C} \times \vec{B}$
6. The geometrical meaning of cross product is equal to area of parallelogram where $\vec{B}\ and\ \vec{A}$ are two sides of it.
7. $If\ \vec{A} = a_1\vec{x} + b_1\vec{y}\ and\ \vec{B} = a_2\vec{x} + b_2\vec{y},\ then\ \vec{A} \times \vec{B} = \begin{vmatrix} a_1 & a_2 \\ b_2 & b_2 \end{vmatrix}\ \vec{z} = (a_1b_2 - a_2b_1)\,\vec{z}$
8. Right hand group is a group of vectors where are all perpendicular to each other's and cross product from first vector to second vector is equal to third vector. $\Longrightarrow Fig(A-12)$
9. $(\vec{x} \times \vec{y} = \vec{z}\,), (\vec{y} \times \vec{z} = \vec{x})\ and\ (\vec{z} \times \vec{x} = \vec{y}\,)$
10. $(\vec{y} \times \vec{z} = -\vec{z}\,), (\vec{z} \times \vec{y} = -\vec{x})\ and\ (\vec{x} \times \vec{z} = -\vec{y}\,)$



$(A-11)$



$(A-12)$

## VI. Angels Rules:

1) $\frac{rad}{\pi} = \frac{\theta}{180^\circ} \Longrightarrow 1rad = \frac{180^\circ}{\pi}$

2) $\sin\alpha = \frac{Opposite(a)}{Hypotinuse(c)} = \frac{1}{\csc\alpha} = \frac{e^{i\alpha}-e^{-i\alpha}}{2i}$

3) $\cos\alpha = \frac{Adjacent(b)}{Hypotinuse(c)} = \frac{1}{\sec\alpha} = \frac{e^{i\alpha}+e^{-i\alpha}}{2}$

4) $\tan\alpha = \frac{Opposite(a)}{Adjacent(b)} = \frac{1}{\cot\alpha} = \frac{e^{i\alpha}-e^{-i\alpha}}{i(e^{i\alpha}+e^{-i\alpha})}$

5) $\sin\alpha\sin\beta = 0.5\{\cos(\alpha-\beta) - \cos(\alpha+\beta)\}$

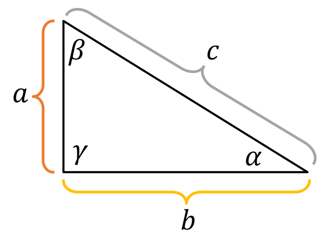
6) $\cos\alpha\cos\beta = 0.5\{\cos(\alpha+\beta) + \cos(\alpha-\beta)\}$

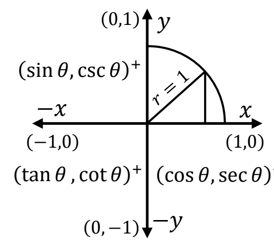
7) $\sin\alpha\cos\beta = 0.5\{\sin(\alpha+\beta) + \sin(\alpha-\beta)\}$

8) $\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$

9) $\cos(\alpha\pm\beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

10) $\tan(\alpha\pm\beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$




| $\theta$ | Radians | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ |
|---|---|---|---|---|
| 0° | 0 | 0 | 1 | 0 |
| 30° | $\pi/6$ | $1/2$ | $\sqrt{3}/2$ | $1/\sqrt{3}$ |
| 45° | $\pi/4$ | $1/\sqrt{2}$ | $1/\sqrt{2}$ | 1 |
| 60° | $\pi/3$ | $\sqrt{3}/2$ | $1/2$ | $\sqrt{3}$ |
| 90° | $\pi/2$ | 1 | 0 | – |
| 180° | $\pi$ | 0 | −1 | 0 |

11) $\sin\alpha + \sin\beta = 2\left\{\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right\}$

12) $\cos\alpha + \cos\beta = 2\left\{\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right\}$

13) $\sin\alpha - \sin\beta = 2\left\{\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)\right\}$

14) $\cos\alpha - \cos\beta = -2\left\{\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)\right\}$

15) $\sin(\alpha+\beta)\sin(\alpha-\beta) = \sin^2\alpha - \sin^2\beta$

16) $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$

17) $\sin 2\alpha = 2\sin\alpha\cos\alpha \Longrightarrow \sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}$

18) $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha \Longrightarrow \cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}$

19) $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} \Longrightarrow \tan\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$

20) $\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$

21) $\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$

22) $\tan\left(\frac{\pi}{2}-\theta\right) = \cot\theta$

23) $\csc\left(\frac{\pi}{2}-\theta\right) = \sec\theta$

24) $\sec\left(\frac{\pi}{2}-\theta\right) = \csc\theta$

25) $\cot\left(\frac{\pi}{2}-\theta\right) = \tan\theta$

26) $\sin(-\theta) = -\sin\theta$

27) $\cos(-\theta) = \cos\theta$

28) $\tan(-\theta) = -\tan\theta$

29) $\csc(-\theta) = -\csc\theta$

30) $\sec(-\theta) = \sec\theta$

31) $\cot(-\theta) = -\cot\theta$

32) $If\ \sin\alpha = \cos\beta \Longrightarrow \beta + \alpha = \frac{\pi}{2} + 2n\pi$

33) $If\ \csc\alpha = \sec\beta \Longrightarrow \beta - \alpha = \frac{\pi}{2} + 2n\pi$

34) $If\ \tan\alpha = \cot\beta \Longrightarrow \beta + \alpha = \frac{\pi}{2}\ or\ \frac{3\pi}{2}$

35) $\sin^2\alpha + \cos^2\alpha = 1$

36) $\sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha)$

37) $\cos^2\alpha = \frac{1}{2}(1+\cos 2\alpha)$

38) $\cot^2\alpha + 1 = \csc^2\alpha$

39) $\tan^2\alpha + 1 = \sec^2\alpha$

40) $\sin^{-1}\alpha + \cos^{-1}\alpha = \frac{\pi}{2}$

41) $\tan^{-1}\alpha + \cot^{-1}\alpha = \frac{\pi}{2}$

42) $a^2 = b^2 + c^2 - 2bc\cos\alpha$

43) $c^2 = b^2 + a^2 - 2ab\cos\gamma$

44) $b^2 = a^2 + c^2 - 2ac\cos\beta$

45) $\frac{\sin\alpha}{a} = \frac{\sin\alpha}{b} = \frac{\sin\alpha}{c} = \frac{1}{2r}$

46) $\frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{\cos(\alpha+\beta)+\cos(\alpha-\beta)} = \tan\alpha$

47) $\frac{\tan\alpha+\tan\beta}{\tan(\alpha+\beta)} + \frac{\tan\alpha-\tan\beta}{\tan(\alpha-\beta)} = 2$

48) $\frac{\sin(\alpha-\beta)}{\sin\alpha\sin\beta} = \cot\beta - \cot\alpha$

49) $\frac{\sin(\alpha+\beta)}{\sin(\alpha-\beta)} = \frac{\tan\alpha+\tan\beta}{\tan\alpha-\tan\beta}$

50) $\frac{a-b}{a+b} = \frac{\tan 0.5(\alpha-\beta)}{\tan 0.5(\alpha+\beta)}$

51) $\frac{a+b}{c} = \frac{\cos 0.5(\alpha-\beta)}{\sin 0.5(\gamma)}$

52) $\frac{a-c}{a+c} = \frac{\tan 0.5(\alpha-\gamma)}{\tan 0.5(\alpha+\gamma)}$

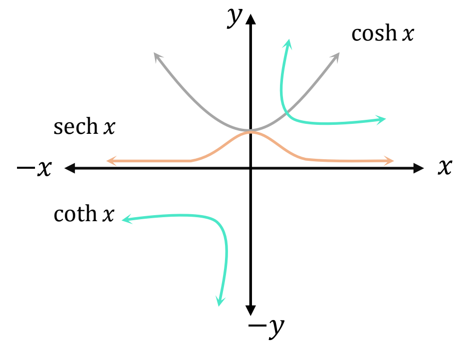
53) $\frac{b-c}{b+c} = \frac{\tan 0.5(\beta-\gamma)}{\tan 0.5(\beta+\gamma)}$

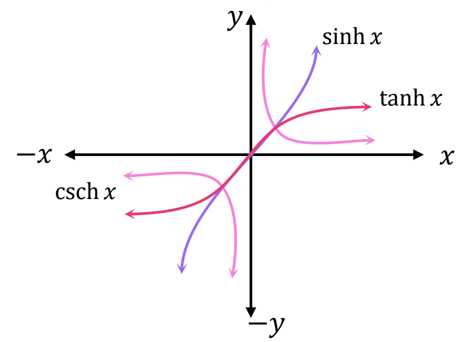
54)

55)

## VII. Hyperbolic Functions:

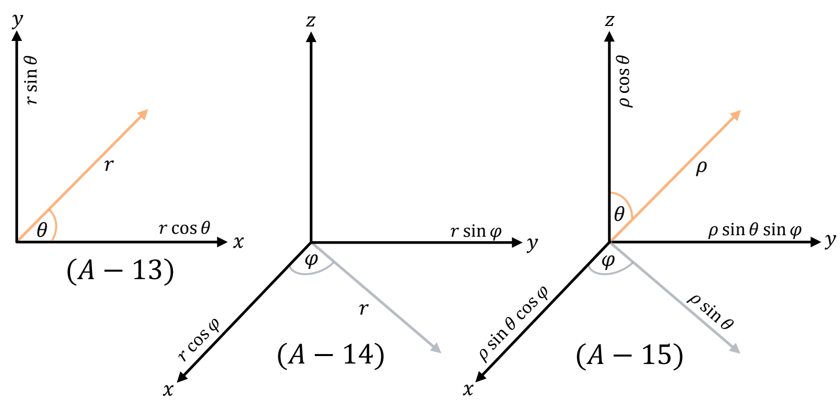
1) $\sinh x = \frac{e^x - e^{-x}}{2} = \frac{1}{\operatorname{csch} x}$
2) $\cosh x = \frac{e^x + e^{-x}}{2} = \frac{1}{\operatorname{sech} x}$
3) $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1}{\coth x}$
4) $\cosh x + \sinh x = e^x$
5) $\cosh x - \sinh x = e^{-x}$
6) $\cosh^2 x - \sinh^2 x = 1$
7) $1 - \tanh^2 x = \operatorname{sech}^2 x$
8) $\coth^2 x - 1 = \operatorname{csch}^2 x$
9) $\sinh(x \pm y) = \cosh(x \mp y) = \sinh x \cosh y \pm \cosh x \sinh y$
10) $\sinh 2x = 2 \sinh x \cosh x$
11) $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$
12) $\sinh^{-1} x = \ln(x + \sqrt{1 + x^2})$
13) $\cosh^{-1} x = \ln(x + \sqrt{1 - x^2})$
14) $\tanh^{-1} x = \ln(\frac{\sqrt{1+x^2}}{\sqrt{1-x^2}})$




## VIII. Polar, Cylinder, Spherical Axis and Jacobin:

1) Polar axis $\Longrightarrow Fig(A-13)$
   $x = r\cos\theta$
   $y = r\sin\theta$
2) Cylinder axis $\Longrightarrow Fig(A-14)$
   $x = r\cos\varphi$
   $y = r\sin\varphi$
   $z = z$
3) Spherical axis $\Longrightarrow Fig(A-15)$
   $x = \rho\cos\varphi\sin\theta$
   $y = \rho\sin\varphi\sin\theta$
   $z = \rho\cos\theta$



$(A-13)$ $(A-14)$ $(A-15)$

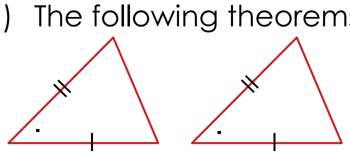
4) Jacobin
   If we have two functions $f(x,y)$ and $g(x,y)$,

$$\text{Then Jacobin } J\frac{\partial(f,g)}{\partial(x,y)} = \begin{vmatrix} f_x & f_y \\ g_x & g_y \end{vmatrix}$$

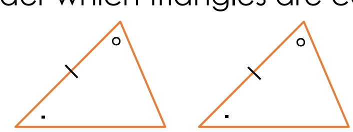
   1. Spherical coordinates Jacobin $\{\rho^2 \sin\theta\}$
   2. Cylinder and polar coordinates Jacobin $\{r\}$

## IX. Geometry:

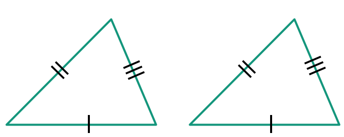
1) The following theorems present conditions under which triangles are congruent.



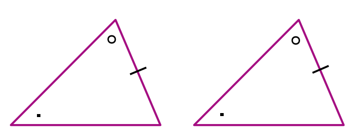
SAS congruence requires the congruence of two sides and the angle between those sides.



ASA congruence requires the congruence of two angles and the side between those angles.

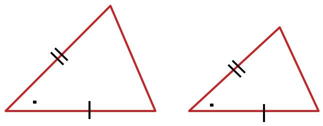


SSS congruence requires the congruence of all three sides.

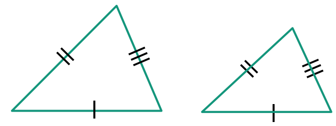


AAS congruence requires the congruence of two angles and a side which is not between.

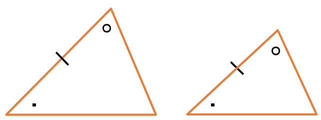
2) The following theorems present conditions under which triangles are similar.



SAS similarity requires the proportionality of two sides and the congruence of the angle.

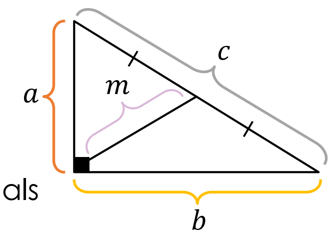


SSS similarity requires the proportionality of all three sides.



ASA similarity requires the congruence of two angles and the side between those angles.

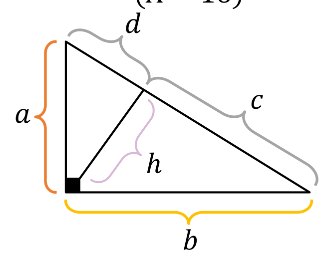
3) Pythagorean theorem: $c^2 = a^2 + b^2 \Longrightarrow\ c = \sqrt{a^2 + b^2}\ \Longrightarrow Fig(A-16)$
4) The ratio between the surface of two similar triangles is equal to the ratio between a square with two symmetrical sides.
$\Longrightarrow$ The ratio between the circumference of two similar triangles equals the ratio between two symmetrical sides.



$(A-16)$

5) Euclid theorem $\Longrightarrow Fig(A-17)$

$$a^2 = d \times (d + c)$$
$$b^2 = c \times (d + c)$$
$$h^2 = d \times c$$
$$h \times (d + c) = a \times b$$



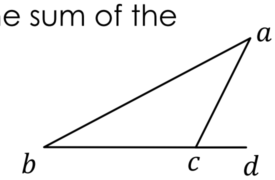
$(A-17)$

6) Height: The formula for the length of a height of a triangle is derived from Heron's formula for the area of a triangle.
$h = \frac{2\sqrt{s(s-a)(s-b)(s-(d+c))}}{d+c}$ , where $s = \frac{1}{2}(a + b + (d + c))$
7) Median: The formula for the length of a median of a triangle is $m = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}$
8) Angle Bisector: The formula for the length of an angle bisector of a triangle supposed to be

$$m = \sqrt{ab\left(1 - \frac{c^2}{(a+b)^2}\right)}$$

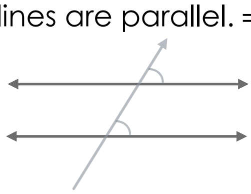
9) Exterior angle equality: The measure of an external angle is equal to the sum of the measures of the two non-adjacent interior angles. $\Longrightarrow Fig(A-18)$

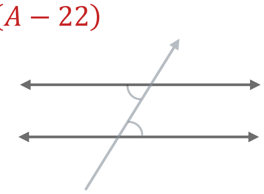
$$\angle\, dca = \angle\, cba + \angle\, cab$$

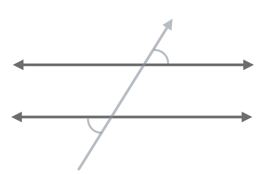


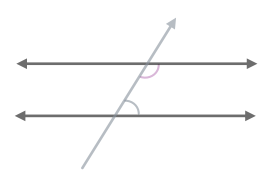
$(A-18)$

10) There is four ways to define a plane,
   1. By three points are not in strait line.
   2. By line and one point.
   3. By two intersecting lines.
   4. By two parallel lines.
11) If two lines cut by a transversal have congruent corresponding angles, then the lines are parallel. $\Longrightarrow Fig(A-19)$
$\Longrightarrow$ If two lines cut by a transversal have congruent alternate interior angles congruent, then the lines are parallel. $\Longrightarrow Fig(A-20)$
$\Longrightarrow$ If two lines cut by a transversal have congruent alternate exterior angles, then the lines are parallel. $\Longrightarrow Fig(A-21)$
$\Longrightarrow$ If two lines cut by a transversal have supplementary consecutive interior angles, then the lines are parallel. $\Longrightarrow Fig(A-22)$


$(A-19)$


$(A-20)$


$(A-21)$


$(A-22)$

12) Two lines are perpendicular if the product of their slopes is $-1$.

## Section $(B)$

### I. Second order and some shapes Equations:

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

1) At $y = 0$ and $b = 2g$ then the equation $(ax^2 + bx + c = 0)$ Its solution would be like

$$Solutions(A, B) = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}$$

$$if\ b^2 + 4ac = ??$$

1. (Negative No) $(A, B)$ must be Imaginary numbers.
2. (Positive No) $(A, B)$ must be real numbers.
3. (Zero) $(A, B)$ must be equal real numbers.

$$A + B = \frac{-b}{a} \quad and \quad AB = \frac{c}{a}$$

2) Line equation $y = mx + c$
   $r = sec\,\theta\ (or)\ r = c\ csc\,\theta\ \ (or)\ \theta = c$
3) Circle equation $(x - \alpha)^2 + (y - \beta)^2 = a^2$, Where $a = \sqrt{g^2 + f^2 - c}$
   $r\ =\ a\ (or)\ r = c\ a\cos\theta\ (or)\ r = c\ a\sin\theta$
4) Parabola equation $(x - \alpha)^2 = 4a(y - \beta)$ Its Head is $(\alpha, \beta)$
   $r = (1 + \cos\theta)$
5) Ellipse equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ Its eccentricity is $e = \sqrt{1 + \frac{b^2}{a^2}}$
   $r = \frac{L}{1+e\cos\theta}$ and $L = \frac{b^2}{a}$
6) Hyperbola equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
   $r^2 = a^2 \sec 2\theta$
7) Cardioid equation $\{r_h = (1 \pm \cos\theta)\}\{r_v = (1 \pm \sin\theta)\}$
8) Four Leafed Rose Equation $\{r = a\cos 2\theta\}\{r\ = a\sin 2\theta\}$
9) Lemniscate equation $\{r_h^2 = a^2 \cos\theta\}\{\ r_v^2 = a^2 \sin\theta\}$

### II. Partial Fraction:

1) $\frac{g(x)}{f(x)} = \frac{g(x)}{(x-a_1)(x-a_2)\dots(x-a_n)} = \frac{b_1}{x-a_1} + \frac{b_2}{x-a_2} + \cdots + \frac{b_n}{x-a_n}$

$$g(x) = b_1(x - a_2)(x - a_3)\dots(x - a_n) + b_2(x - a_1)(x - a_3)\dots$$

For example:

$$\frac{2x^2 - 3}{(x^2 - 4)(x - 1)} = \frac{2x^2 - 3}{(x - 2)(x + 2)(x - 1)} = \frac{b_1}{(x - 2)} + \frac{b_2}{(x + 2)} + \frac{b_3}{(x - 1)}$$

$$2x^2 - 3 = b_1(x + 2)(x - 1) + b_2(x - 2)(x - 1) + b_3(x - 2)(x + 2)$$

$$b_1 = \frac{5}{4}\ \&\ b_2 = \frac{5}{12}\ \&\ b_3 = \frac{1}{3}$$

2) $\frac{g(x)}{f(x)} = \frac{g(x)}{(X-a_1)(x-a_2)^r} = \frac{a}{(X-a_1)} + \frac{b}{(X-a_2)} + \frac{c}{(x-a_2)^2} + \cdots + \frac{d}{(x-a_2)^r}$

For example:

$$\frac{x^2 + 3x + 1}{(x + 3)(x - 1)^2} = \frac{a}{(x + 3)} + \frac{b}{(x - 1)} + \frac{c}{(x - 1)^2} \to \frac{x^2 + 3x + 1}{(x + 3)(x - 1)^2} = \frac{a(x - 1)^2 + b(x - 1)(x + 3) + c(x + 3)}{(x + 3)(x - 1)^2}$$

$$x^3 + 3x + 1 = a(x - 1)^2 + b(x - 1)(x + 3) + c(x + 3)$$

$$a = \frac{1}{16}\ \&\ b = \frac{15}{16}\ \&\ c = \frac{5}{4}$$

3) $\frac{g(x)}{f(x)} = \frac{g(x)}{(ax^2+bx+c)(ry^2+sy+t)} = \frac{b_1x+c_1}{(ax^2+bx+c)} + \frac{b_2x+c_2}{(ry^2+sy+t)}$

For example:

$$\frac{x+8}{x(x-2)(x+2)(x^2+4)} = \frac{a}{x} + \frac{b}{x-2} + \frac{c}{x+2} + \frac{dx+e}{x^2+4}$$
$$a = \frac{0+8}{0-16} = -\frac{1}{2} \ \& \ b = \frac{2+8}{2(4+4)(2+2)} = \frac{5}{23} \ \& \ c = \frac{-2+8}{-2(4+4)(-4)} = \frac{3}{32}$$
$$dx+e = \frac{x+8}{x(x^2-4)} (at\ x^2 = -4) = \frac{x+8}{x(-4-4)} = \frac{x^2+8x}{-8x^2} = \frac{-4+8x}{-8(-4)} = \frac{1}{4}x - \frac{1}{8}$$

4) Combination from (2) & (3)

For example:

$$\frac{x^2+2}{x(x^2+1)^2} = \frac{a}{x} + \frac{bx+c}{(x^2+1)^2} + \frac{dx+e}{(x^2+1)}$$
$$x^2+2 = a(x^2+1)^2 + (bx+c)x + (dx+e)(x^2+1)x$$
$$a = 2 \ \& \ d = -2 \ \& \ b = -1 \ \& \ c, e = 0$$

## III. Matrices and Determinate:

1) A set of $mn$ numbers (real or complex), arranged in a rectangular formation (array or table) having m rows and n columns and enclosed by a square bracket [ ] is called $m \times n$ matrix $(read\ \text{“}m\ by\ n\ matrix\text{”})$

$$A = \begin{bmatrix} a_{11} & a_{12} & ---a_{1n} \\ a_{21} & a_{22} & ---a_{2n} \\ - & - & ---- \\ a_{m1} & a_{m2} & ---a_{mn} \end{bmatrix}$$

The letters $a_{ij}$ stand for real numbers. Note that $a_{ij}$ is the element in the $i^{th}$ row and $j^{th}$ column of the matrix. Thus the matrix A is sometimes denoted by simplified form as $(a_{ij})$ or by $\{a_{ij}\}$ , A $=(a_{ij})$. Matrices are usually denoted by capital letters A, B, C etc. and its elements by small letters a, b, c etc.

2) Order of matrices is the ordered pair having as first component the number of rows and as second component the number of columns in the matrix as $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is a matrix of order (2 × 3).

3) The principal diagonal of a square matrix is the ordered set of elements $a_{ij}$ where $i = j$, extending from the upper left-hand corner to the lower right-hand corner of the matrix like $A = \begin{bmatrix} 5 & & \\ & 4 & \\ & & 8 \end{bmatrix}$ then the principal diagonal is 5, 4 and 8.

4) A square matrix in which all elements are zero except those in the main or principal diagonal is called a diagonal matrix. Some elements of the principal diagonal may be zero but not all.

5) Scaler Matrix is a diagonal matrix in which all the diagonal elements are same, is called a scalar matrix like $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

⟹ Unit matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

⟹ Adding or subtract operation $if\ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\ and\ B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}\ then\ A \pm B = \begin{bmatrix} a \pm e & b \pm f \\ c \pm g & d \pm h \end{bmatrix}$

⟹ Multiplying $if\ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\ then\ aA = \begin{bmatrix} a \times a & a \times b \\ a \times c & a \times d \end{bmatrix}$

⟹ $Row\ \times column\ if\ A = \begin{bmatrix} a \\ b \end{bmatrix} and\ B = [c \quad d]\ then\ A \times B = (a \times c) + (b \times d) = ac + bd$

⟹ Multiplying Matrices $if\ A_{(k \times n)} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\ and\ B_{(m \times k)} = \begin{bmatrix} g & h \\ i & j \\ k & l \end{bmatrix}$

$$then( A \times B)_{(m\times n)} = \begin{bmatrix} (a \quad b \quad c)\begin{pmatrix} g \\ i \\ k \end{pmatrix} & (a \quad b \quad c)\begin{pmatrix} h \\ j \\ l \end{pmatrix} \\ (d \quad e \quad f)\begin{pmatrix} g \\ i \\ k \end{pmatrix} & (d \quad e \quad f)\begin{pmatrix} h \\ j \\ l \end{pmatrix} \end{bmatrix}$$

⟹ Transpose of a matrix $if\ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\ then\ A^t = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

⟹ Matrix determinate $if\ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\ then\ det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

⟹ Inverse of matrix for $(2 \times 2)$ $if\ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\ then\ A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} / \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} / ad - bc$

6) Leibniz was the first who create determinate to easily solving liner equations.

$if\ a_1x + b_1y = c_1\ and\ a_2x + b_2y = c_2\ then\ x = \frac{\Delta_x}{\Delta}\ and\ y = \frac{\Delta_y}{\Delta}$

$$\frac{\Delta_x}{\Delta} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}\ and\ \frac{\Delta_y}{\Delta} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}\ which\ \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

7) Expansion of determinates

5. Using minor and cofactor $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$

6. Sarrus method

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \begin{matrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{matrix} = a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - a_3b_2c_1 - b_3c_2a_1 - c_3a_2b_1$$

7. Triangles method

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix} - \begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix} = (a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3) - (a_3b_2c_1 + b_3c_2a_1 + c_3a_2b_1)$$

8) Properties of determinate

1. $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

2. $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix} = -\begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$

3. $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = \begin{vmatrix} a_1 & c_1 & c_1 \\ a_2 & c_2 & c_2 \\ a_3 & c_3 & c_3 \end{vmatrix} = 0$

4. $\begin{vmatrix} ma_1 & mb_1 & mc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & mc_1 \\ a_2 & b_2 & mc_2 \\ a_3 & b_3 & mc_3 \end{vmatrix} = m \times \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}\ (or)\ \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 \pm nb_1 \pm mc_1 & b_1 & c_1 \\ a_2 \pm nb_2 \pm mc_2 & b_2 & c_2 \\ a_3 \pm nb_3 \pm mc_3 & b_3 & c_3 \end{vmatrix}$

5. $\begin{vmatrix} n_1 \pm a_1 & b_1 & c_1 \\ n_2 \pm a_2 & b_2 & c_2 \\ n_3 \pm a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} n_1 & b_1 & c_1 \\ n_2 & b_2 & c_2 \\ n_3 & b_3 & c_3 \end{vmatrix} \pm \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

6. $if\ a_1x + b_1y = c_1\ , a_2x + b_2y = c_2\ and\ a_3x + b_3y = c_3\ are\ intersecting\ then\ \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

7. Line equation which passes through $(x_1, y_1)\ and\ (x_2, y_2)\ is\ \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

8. If triangle heads are $(x_1, y_1)\ , (x_2, y_2)\ and\ (x_3, y_3)$ then triangle area is $\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

## IV. Permutations and Combinations:

1) The fundamental counting principle : If there are $n_1$ different objects in one set and $n_2$ different objects in a second set, then the number of ways of choosing one object from each set is $n_1 \times n_2$.
   And so on, for $k$ sets, then the number of ways of choosing one object from each set is $n_1 \times n_2 \times n_3 \times \ldots \times n_k$.
2) Factorial notation $n!$ is read as $n$ factorial. The factorial sign (!) means to take the product of all natural numbers less than or equal to the given number.
$$n! = n(n-1)(n-2)\ldots\ldots\ldots \times 3 \times 2 \times 1$$
3) The number of permutations of $n$ distinct objects taken $r$ at a time is
$$P_r^n = n(n-1)(n-2)\ldots\ldots\ldots(n-r+1)$$
4) The number of permutations of $n$ objects with $r$ identical objects is $\frac{n!}{r!}$
5) In a set of $n$ objects with $n_1$ objects of one kind, $n_2$ objects of another kind, $n_3$ objects of another kind, and so on, for $k$ kinds of objects, the number of permutations is
$$\frac{n!}{n_1!\,n_2!\,n_3!\ldots n_k!} \text{ where } n_1 + n_2 + n_3 + \cdots + n_k = n$$
6) Properties
   $\Rightarrow n! = n \times (n-1)! = n \times (n-1) \times (n-2)!$
   $\Rightarrow P_r^n = \dfrac{n!}{(n-r)!}, n \ge r$
   $\Rightarrow 0! = 1 \text{ and } P_0^n = 1$
7) The number of combinations of $n$ different objects taken $r$ at a time is $C_r^n = \frac{P_r^n}{r!} = \frac{n!}{(n-r)!\,r!}$
   Also could be written as $\binom{n}{r} = \frac{n(n-1)(n-2)(n-3)\ldots(n-r+1)}{1.2.3.4.5\ldots r}$
8) Properties
   $\Rightarrow \binom{n}{r} = \binom{n}{n-r} \text{ where } n \ge r$
   $\Rightarrow \binom{n}{n} = \binom{n}{0} = 1$
   $\Rightarrow \binom{n}{r} = \binom{n}{k} \text{ that means } k = r \text{ (or) } r + k = n$
   $\Rightarrow \dfrac{\binom{n}{r}}{\binom{n}{r-1}} = \dfrac{n-r+1}{r}$
   $\Rightarrow \binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$

## V. Binomial Theorem:

1) At expansion of $(x+y)^n$, the first power of $x$ is $x^n$ and the exponent of the power decreases by 1 for each subsequent term, with the last power being $x^0$. The first power of $y$ is $y^0$ and the exponent of the power increases by 1 for each subsequent term, with the last power being $y^n$. The sum of the exponents in each term is $n$.
2) Compare the expansion of the binomial power with Pascal's triangle.

| | | |
|---|---|---|
| $(x+y)^0 = 1$ | 1 | $0C_0$ |
| $(x+y)^1 = x + y$ | 1 1 | $1C_0 \quad 1C_1$ |
| $(x+y)^2 = x^2 + 2xy + y^2$ | 1 2 1 | $2C_0 \quad 2C_1 \quad 2C_2$ |
| $(x+y)^3 = x^3 + 3x^2y + 3y^2x + y^3$ | 1 3 3 1 | $3C_0 \quad 3C_1 \quad 3C_2 \quad 3C_3$ |
| $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4y^3x + y^4$ | 1 4 6 4 1 | $4C_0 \quad 4C_1 \quad 4C_2 \quad 4C_3 \quad 4C_4$ |

   The coefficients of the expansion of $(x+y)^n$ are the terms of row $(n+1)$ of Pascal's triangle.
3) Summation of coefficient $\sum co = 2^n$

$\Longrightarrow (x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \cdots + y^n \text{ where } n > 0$

$\Longrightarrow (x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 \dots + \binom{n}{r}x^{n-r}y^r + \cdots + \binom{n}{n}y^n \text{ where } n > 0$

$\Longrightarrow (x-y)^n = \binom{n}{0}x^n - \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 \dots + \binom{n}{r}x^{n-r}(-y)^r + \cdots + \binom{n}{n}(-y)^n \text{ where } n > 0$

$\Longrightarrow (1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 \dots + \binom{n}{r}x^r + \cdots + \binom{n}{n}x^n \text{ where } n > 0$

$\Longrightarrow (1-x)^n = 1 - \binom{n}{1}x + \binom{n}{2}x^2 - \binom{n}{3}x^3 \dots + \binom{n}{r}(-x)^r + \cdots + \binom{n}{n}(-x)^n \text{ where } n > 0$

4) The general term, or $r^{th}$ term in the expansion of $(x+y)^n$ is $T_{r+1} = (-1)^k \binom{n}{r} x^{n-r}y^r$

5) Mid-term of $(x+y)^n \begin{cases} \text{if } n \text{ is even then there is one mid term } \frac{n}{2}+1 \\ \text{if } n \text{ is odd then there is two mid terms } \frac{n+1}{2}, \frac{n+3}{2} \end{cases}$

6) The ratio between term and previous term at the expansion of $(ax+by)^n$ is $\frac{T_{r+1}}{T_r} = \frac{(n-r+1)by}{r \cdot ax}$

7) The ratio between coefficient of term and previous term coefficient is $\frac{coefficient\ T_{r+1}}{coefficient\ T_r} = \frac{(n-r+1)a}{r \cdot b}$

8) To find the term which contain $x^r$ we suppose that term is the general term after that we simplifying the general term, then compare the power of general term with $r$ to find $r$ value.
For example:

Find the term which contain $x^8$ at the expansion of $\left(2x^2 + \frac{1}{x}\right)^{10}$

$$\text{Suppose that } x^8 = T_{r+1} \text{ then } T_{r+1} = \binom{10}{k}\left(\frac{1}{x}\right)^r (2x^2)^{10-r}$$

$$\binom{10}{k} \cdot x^{-r} \cdot 2^{10-r} \cdot x^{20-2r} = \binom{10}{k} \cdot 2^{10-r} \cdot x^{20-3r}$$

$$\text{then } 20 - 3r = 8 \Longrightarrow 12 = 3r$$

$$\therefore r = 4 \text{ then the term contain } x^8 \text{ is } T_{4+1} = T_5$$

$\Longrightarrow (x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \cdots \text{ where } n \in \mathbb{R}$

$\Longrightarrow (1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 \dots + \frac{n(n+1)\dots(n+r-1)}{r!}x^r + \cdots$
$\text{where } n \in \mathbb{R} \text{ and } -1 < x < 1$

$\Longrightarrow (1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 \dots + \frac{n(n+1)\dots(n+r-1)}{r!}(-x)^r + \cdots$
$\text{where } n \in \mathbb{R} \text{ and } -1 < x < 1$

9) The general term, or $r^{th}$ term in the expansion of $(x+y)^n$ where $n \in \mathbb{R}$ is $T_{r+1} = \frac{n(n-1)\dots(n-r+1)}{r!}x^{n-r}y^r$

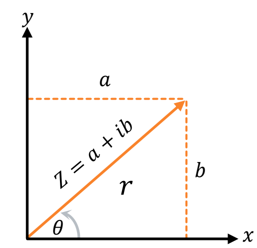
10) Expansion of some important Infinite sequences

1. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \cdots ;\ -\infty < x < \infty$
2. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^{n-1}\frac{x^{2n}}{(2n)!} + \cdots ;\ -\infty < x < \infty$
3. $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{x^{2n-1}}{(2n-1)!} + \cdots ;\ -\infty < x < \infty$
4. $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots ;\ -\infty < x < \infty$
5. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots ;\ -\infty < x < \infty$
6. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n\frac{x^{2n-1}}{2n-1} + \cdots ;\ -1 \le x \le 1$
7. $\frac{1}{1 \mp x} = 1 \pm x + x^2 \pm x^3 + x^4 \pm \cdots + (-1)^n x^n + \cdots ; -1 < x < 1$
8. $\ln|1+x| = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots ;\ -1 < x < 1$
9. $\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots + \frac{x^{2n-1}}{2n-1} + \cdots ; -1 < x < 1$

## VI. Complex Numbers:

The Imaginary number $i$ is a result solving equation $x^2+1=0$ where $i=\sqrt{-1}\ and\ i^2=-1$
The complex number $Z=a+ib$ consist of two parts $(a)$ which is the real part and $(b)$ which is the Imaginary part.

1) The conjugate of complex number $Z=a+ib$ is $\overline{Z}=a-ib.$
2) The conjugate of real number is itself and opposite happens if $z$ conjugate equal $z$ then $z$ is a real Number.
3) Properties
   If $Z=a+ib, Z_1=a_1+ib_1\ and\ Z_2=a_2+ib_2$ are complex numbers, then
   $\Longrightarrow Z+\overline{Z}=2a=2R(Z)$
   $\Longrightarrow Z-\overline{Z}=2ib=2i\ Im(Z)$
   $\Longrightarrow \overline{Z_1+Z_2}=\overline{Z_1}+\overline{Z_2}\ and\ \overline{\left(\frac{Z_1}{Z_2}\right)}=\frac{\overline{Z_1}}{\overline{Z_2}}$
   $\Longrightarrow \overline{\overline{Z}}=Z$
4) Operations
   $\Longrightarrow (a_1+ib_1)+(a_2+ib_2)=(a_1+a_2)+i(b_1+b_2)$
   $\Longrightarrow (a_1+ib_1)-(a_2+ib_2)=(a_1-a_2)+i(b_1-b_2)$
   $\Longrightarrow (a_1+ib_1)(a_2+ib_2)=(a_1a_2-b_1b_2)+i(a_1b_2+a_2b_1)$
   $\Longrightarrow \frac{a_1+ib_1}{a_2+ib_2}=\frac{a_1+ib_1}{a_2+ib_2}\cdot\frac{a_2-ib_2}{a_2-ib_2}=\frac{(a_1a_2+b_1b_2)+i(a_1b_2-a_2b_1)}{a_2^2+b_2^2}$
5) The absolute value $|Z|=|a+ib|=r=\sqrt{a^2+b^2}$
6) Properties
   $\Longrightarrow |Z|^2=Z\overline{Z}$ and $|Z|=|\overline{Z}|$
   $\Longrightarrow |Z_1Z_2|=|Z_1||Z_2|$ and $\left|\frac{Z_1}{Z_2}\right|=\frac{|Z_1|}{|Z_2|}$
   $\Longrightarrow |Z_1\pm Z_2|\le|Z_1|+|Z_2|$
   $\Longrightarrow |Z_1-Z_2|\ge\big||Z_1|-|Z_2|\big|$



$(B-1)$

7) Polar trigonometric form $Z=r(\cos\theta+i\sin\theta)$ where $\theta=\tan^{-1}\frac{b}{a}$ called argument $arg(Z)$
8) $if\ Z_1=r_1(\cos\theta_1+i\sin\theta_1)\ and\ Z_2=r_2(\cos\theta_2+i\sin\theta_2)\ \Longrightarrow Fig(B-1)$
   $then\ \ Z_1Z_2=r_1r_2(\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2))$
9) $\frac{Z_1}{Z_2}=\frac{r_1(\cos\theta_1+i\sin\theta_1)}{r_2(\cos\theta_2+i\sin\theta_2)}=\frac{r_1}{r_2}(\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2))$
10) De-Moiver theorem $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta\ \ if\ n\ is\ positive$
11) $(\cos\theta+i\sin\theta)^{\frac{p}{q}}=\cos\frac{p\theta+2n\pi}{q}+i\sin\frac{p\theta+2n\pi}{q}\ ; n\le q-1$
12) Euler's formula $\Longrightarrow e^{i\theta}=1+i\theta+\frac{(i\theta)^2}{2!}+\frac{(i\theta)^3}{3!}+\cdots=1+i\theta-\frac{\theta^2}{2!}+\frac{i\theta^3}{3!}-\cdots=1-\left(\frac{\theta^2}{2!}-\frac{\theta^4}{4!}+\frac{\theta^6}{6!}\right)+$
   $i\left(\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\frac{\theta^7}{7!}\right)=\cos\theta+i\sin\theta$
13) $Z=re^{i\theta}, \left(e^{i\theta}\right)^n=e^{in\theta}\ and\ \ln Z=\ln r+\ln e^{i(\theta+2\pi n)}=\ln r+i(\theta+2\pi n)$
14) Cubic root for 1 is $1=\cos 0+i\sin 0\ \ then\ \sqrt[3]{1}=\sqrt[3]{\cos\theta+i\sin\theta}=\cos\frac{2n\pi}{3}+i\sin\frac{2n\pi}{3}$
   $1=\cos 0+i\sin 0\ ,\omega=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}=-\frac{1}{2}+\frac{\sqrt{3}}{2}i\ and\ \omega^2=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}=\frac{1}{2}-\frac{\sqrt{3}}{2}i$
   $\Longrightarrow 1+\omega+\omega^2=0\ ,\omega^{3n}=1\ ,\omega^{3n+1}=\omega\ and\ \ \omega^{3n+2}=\omega^2$
   $\Longrightarrow \omega-\omega^2=\omega^2-\omega=\pm\sqrt{3}i$
   $\Longrightarrow \frac{1}{\omega}=\omega^2\ and\ \frac{1}{\omega^2}=\omega$

## VII. Logical formula and connectives:

Proposition or statement: is a sentence could be true or false and couldn't be both.

Negative of statement: $\neg p$ is called negative of $p$ if

| $p$ | $\neg p$ | $\neg(\neg p)$ |
|---|---|---|
| F | T | F |
| T | F | T |

### 1) Connectives

1. Conjunction $(and)$

| $p$ | $q$ | $p \wedge q$ |
|---|---|---|
| T | T | T |
| F | T | F |
| T | F | F |
| F | F | F |

2. Disjunction $(or)$

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| T | T | T |
| F | T | T |
| T | F | T |
| F | F | F |

3. Conditionals $(if\ then)$

| $p$ | $q$ | $p \to q$ |
|---|---|---|
| T | T | T |
| F | T | T |
| T | F | F |
| F | F | T |

4. Bi Conditionals $(iff)$

| $p$ | $q$ | $p \leftrightarrow q$ |
|---|---|---|
| T | T | T |
| F | T | F |
| T | F | F |
| F | F | T |

### 2) Properties

1. $p \wedge p = p$
2. $p \wedge q = q \wedge p$
3. $p \wedge (q \wedge r) = (p \wedge q) \wedge r$
4. $p \vee p = p$
5. $p \vee q = q \vee p$
6. $p \vee (q \vee r) = (p \vee q) \vee r$
7. $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$
8. $\neg(p \wedge q) = \neg q \vee \neg p$
9. $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$
10. $\neg(p \vee q) = \neg q \wedge \neg p$
11. $p \to p \vee q$
12. $p \wedge q \to p$
13. $p \to q == \neg p \vee q$
14. $p \to q = \neg(p \wedge \neg q)$
15. $p \to q = \neg q \to \neg p$
16. $p \leftrightarrow q = (p \to q) \wedge (q \to p)$
17. $p \leftrightarrow q = (\neg p \vee q) \wedge (\neg q \vee p) = (p \wedge q) \vee (\neg p \wedge \neg q)$
18. $p \vee \neg p = \tau\ (totology)$
19. $p \to p = \tau$
20. $p \wedge q \to p = \tau$
21. $p \to p \vee q = \tau$
22. $p \wedge p = \Im\ (contradiction)$
23. $p \leftrightarrow \neg p = \Im$
24. $\tau \wedge p = p$
25. $\tau \vee p = \tau$
26. $\Im \wedge p = \Im$
27. $\Im \vee p = p$

### 3) Proof Techniques

1. Using true or false tables

For example:

Proof that $p \leftrightarrow q = (p \wedge q) \vee (\neg p \wedge \neg q)$

| $p$ | $q$ | $\neg p$ | $\neg q$ | $p \wedge q$ | $\neg p \wedge \neg q$ | $(p \wedge q) \vee (\neg p \wedge \neg q)$ | $p \leftrightarrow q$ |
|---|---|---|---|---|---|---|---|
| T | T | F | F | T | F | T | T |
| F | T | T | F | F | F | F | F |
| T | F | F | T | F | F | F | F |
| F | F | T | T | F | T | T | T |

2. Direct proof

For example:

$$Giving \begin{cases} a \\ b \\ (a \vee b) \to c \end{cases}$$

$$Result \longrightarrow c$$

$$\langle (a \wedge b) \wedge [(a \vee b) \to c] \rangle \to c$$

From the defines of $(\wedge)$ each of parts are true so $(a \wedge b) \to \tau$ and $(a \vee b) \to c \to \tau$

That's mean From the defines of $(\vee)$ $a$ and $b$ are true so that from defines of $(\to)$ $c$ is true

3. Indirect proof

For example:

If $n$ is natural no and $n^2$ $is\ even$ proof that $n\ is\ even\ to.$

$$Let\ n^2 = a\ and\ n = b\ \ we\ need\ to\ proof\ that\ a \to b$$

$$lets\ proof\ \neg a \to \neg b$$

$$Suppose\ that\ \neg b\ is\ true\quad \therefore b\ \ is\ false\ so\ that\ n\ is\ odd$$
$$\therefore n = 2k + 1\ where\ k\ is\ integer\ No.\ so\ that\ \ n^2 = 2(2k^2 + 2k) + 1 = 2k' + 1$$
$$where\ 2k^2 + 2k\ is\ integer\ No\ to.$$
$$\therefore n^2\ is\ odd\ so\ that\ \neg a\ is\ true$$

4. Arithmetical method

If statement $(a)$ is true we will use $No$ $1$ to call it, but if it was false we will use $0$ to call it. We could change true or false symbols by $1$ $or$ $0$ at table method. Now we could see adding or multiplying $1{,}0$ properties

$0 + 0 = 0$ $\quad$ $1 + 1 = 0\ (carry\ 1)$ $\quad$ $1 \cdot 0 = 0$ $\quad$ $1 \cdot 1 = 1$

$0 + 1 = 1$ $\quad$ $1 + 0 = 1$ $\quad$ $0 \cdot 1 = 0$ $\quad$ $0 \cdot 0 = 0$

Now we will define connectives using this method

$\neg a = 1 + a$ $\quad$ $a \vee b = a + b + ab$ $\quad$ $a \wedge b = ab$

$a \to b = 1 + a + ab$ $\quad$ $a \leftrightarrow b = 1 + a + b$

Note that $nb = \begin{cases} 0\ if\ n\ is\ even \\ 1\ if\ n\ is\ odd \end{cases}$ and $b^2 = b^3 = \cdots = b$

For example:

$$a \to b = 1 + a + ab$$
$$a \wedge (a \to b) = a(1 + a + ab) = a + a^2 + a^2b = a + a + ab = 2a + ab = ab$$
$$also\ [a \wedge (a \to b)] \to b = 1 + ab + abb = 1 + ab + ab^2 = 1 + ab + ab = 1 + 2ab = 1$$

So that the statement is logically true

## VIII. Set theory, Relations and Functions:

Set: It consist of finite or infinite things which all well-defined and have a special property. We will use capital letters to refers to it. There is two ways to write a set First: to write all of set Elements without repeat, separating them by "," and all written between curled brackets { }.
Second: to write set properties in Mathematical way between curled brackets { }. Here's some of important sets

$\mathbb{Z} = \{0, \pm1, \pm2, \pm3, \pm4, \pm5, \dots\}\ Integers\ set$ $\quad$ $\mathbb{N} = \{0,1,2,3,4,5, \dots\}\ Natural\ numbers\ Set$

$\mathbb{E} = \{0, \pm2, \pm4, \pm6, \pm8, \dots\}\ Even\ Integers\ Set$ $\quad$ $\mathbb{R} = \{all\ sets\}\ Real\ numbers\ set$

$\mathbb{Q} = \left\{x = \dfrac{p}{q}\ where\ p, q\ \in \mathbb{Z}, q \neq 0\right\}\ Rational\ numbers$

Also empty set $\phi$ this set is part from any other set. If it's not so there is Element is not exist at other sets which is false because $\phi$ is empty.
Power set: A set of all particular sets from the main set. If $A = \{a, b, c\}$, then the power set is $P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a.b\}, \{a,c\}, \{b,c\}, A\}$.

### Operations on Sets

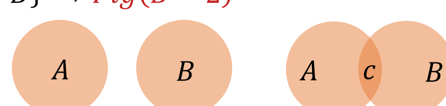
1) Union: The union of two sets $A\ and\ B\ is\ A \cup B = \{x : x \in A\ or\ x \in B\} \Longrightarrow Fig(B-2)$

$A \cup B = B \cup A$ $\quad$ $A \cup U = U\ Universal\ Set$

$A \cup (B \cup C) = (A \cup B) \cup C$ $\quad$ $A \subset A \cup B\ \&\ B \subset A \cup B$

$A \cup A = A$ $\quad$ $A \cup B = B \leftrightarrow A \subseteq B$

$A \cup \varphi = A$



$A \cup B$ $\quad$ $c \subset A \cup B$

$(B-2)$

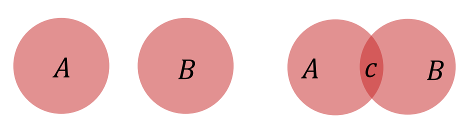
2) Intersecting: The Intersecting of two sets $A\ and\ B\ is\ A \cup B = \{x : x \in A\ and\ x \in B\} \Longrightarrow Fig(B-3)$

$A \cap B = B \cap A$ $\quad$ $A \cap B \subset A\ \&\ A \cap B \subset B$

$A \cap (B \cap C) = (A \cap B) \cap C$ $\quad$ $A \cap B = A \leftrightarrow A \subset B$

$A \cap A = A$ $\quad$ $A \cap U = A$

$A \cap \varphi = \varphi$ $\quad$ $A \cap B = B \leftrightarrow A \subseteq B$



$A \cap B = \varphi$ $\quad$ $A \cap B = c$

$(B-3)$

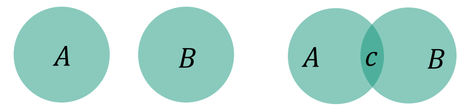
Note that the complement of set $A$ is $\bar{A} = A^c = \{x : x \in U \wedge x \notin A\}$

3) Difference: The difference of two sets $A\ and\ B\ is\ A \cup B = \{x: x \in A\ and\ x \notin B\}. \Longrightarrow Fig(B-4)$

$A - A = \varphi$ $\quad A - B = A - (A \cap B)$

$A - \varphi = A$ $\quad A \cap (B - C) = (A \cap B) - (A \cap C)$

$\varphi - A = \varphi$ $\quad A - B = \varphi \leftrightarrow A \subseteq B$




$A - B = A$ $\quad A - B = A - c$

$(B-4)$

4) Symmetric deference: The symmetric deference of two sets $A\ and\ B\ is\ A \,\Delta\, B = (A \cup B) - (A \cap B)$

$A \,\Delta\, B = B \,\Delta\, A$

Relation: If we have two sets $A\ and\ B$ the Operations (Union, Intersecting and Difference) between sets result is a relation between $A\ and\ B$ even Cartesian product. If $A = \{1,2\}\ and\ B = \{3,4,5\}\ then\ the\ relation\ A \times B = \{(1,3),(1,4),(1,5),(2,3),(2,4),(2,5)\}$.

$A \times B \neq B \times A$ $\quad A \times B = B \times A \leftrightarrow A = B$

$A \times B = \varphi \leftrightarrow A = \varphi\ or\ B = \varphi$ $\quad A \times (B \cap C) = (A \times B) \cap (A \times C)$

$(x,y) \in A \times B \leftrightarrow x \in A\ and\ y \in B$ $\quad A \times (B \cup C) = (A \times B) \cup (A \times C)$

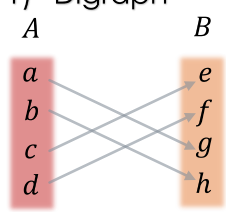
$A \times B = A \times C \leftrightarrow B = C$ $\quad (A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

Binary relation: If $\rho$ is a particular set from $A \times B$ where element $(a,b) \in \rho$ we say that $(a)\ is\ on\ relation\ with\ (b)\ under\ effect\ of\ \rho$.

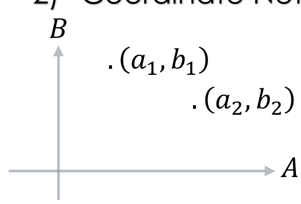
The set $D_\rho = \{a \in A: (a,b) \in \rho\}$ is called the $domain\ of\ relation$. The Set $R_\rho = \{b \in B: (a,b) \in \rho\}$ is called the $range\ of\ relation$.

Graph a Relation

1) Digraph




2) Coordinate Net




3) Using Matrix

$A\ and\ B$ contain $m, n$ Elements and $\rho$ is a relation from $A \to B$ so we could write $\rho$ as a Matrix $M_\rho =$

$$[m_{ij}] : m_{ij} = \begin{cases} 1\ if (a_i, b_j) \in \rho \\ 0 if (a_i, b_j) \notin \rho \end{cases}$$

Composition of relation: If $\rho$ is a relation from $A \to B$ and $\rho'$ is a relation from $B \to C$ then the composition of relations $\rho' \cdot \rho = \{(a,c) \in A \times C :\ \exists b \in B\ ;\ (a,b) \in \rho \wedge (b,c) \in \rho'\}$

Equivalence relation: If $\rho$ is known for set $A$ as an equivalent relation so

1. $(a\ \rho\ a\ \forall a \in A)\ reflexive\ relation.$
2. $(a\ \rho\ b \to b\ \rho\ a)\ symmetric\ relation.$
3. $(a\ \rho\ b\ and\ b\ \rho\ c\ \to a\ \rho\ c\ )\ transitive\ relation.$

Equivalent classes: Let $\rho$ is known for set $A$ as an equivalent relation so the equivalent class is $a \in A\ writen\ as\ [a] = \{x \in A : a\ \rho\ x\}$.

Function: It is like relations but if $S\ and\ T$ are sets which function $f: S \to T$ is a relation from $S\ to\ T$ The condition is for every Element $s \in S$ there's exist one image (one arrow) $t \in T$. All elements that $t = f(s)\ for\ some\ s \in S$ is called $Image\ of\ f\ (Im\ f \subseteq T)$.

Composition of mapping: $If\ g: T \to U\ and\ f: S \to T\ which\ S, U\ and\ T\ are\ not\ empty\ sets, so\ the\ composition\ (g \cdot f) = g(f(x))\ \forall x \in S.$

Inverse Image: $If\ \tau: S \to T\ the\ inverse\ image\ with\ respect\ to\ \tau\ is\ \tau^{-1}(t) = \{s \in S :\ \tau(s) = t\} \subset S\ \forall s \in S.$

Surjective or $onto$ mapping: $If\ \tau: S \to T\ is\ onto\ mapping\ that's\ meaning\ \ \forall t \in T\ \exists s \in S : t = f(s).$

Injective function $1-1$: $If\ \tau: S \to T\ is\ one\ to\ one\ that's\ meaning\ s_1 \neq s_2 \to f(s_1) \neq f(s_2).$

Correspondence bijective mapping: $If\ it\ was\ onto\ and\ one\ to\ one.$

$g(t) = s\ \leftrightarrow f(s) = t$ $\quad A \subset B \to f^{-1}(A) \subset f^{-1}(B)$

$A \subset B \to f(A) \subset f(B)$ $\quad f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$

$f(A \cup B) = f(A) \cup f(B)$ $\quad f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$

$f(A \cap B) = f(A) \cap f(B)$ $\quad f^{-1}(A - B) = f^{-1}(A) - f^{-1}(B)$

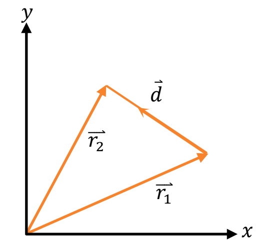
$f(A - B) = f(A) - f(B)$ $\quad A \subset f^{-1}(f(A))\ and\ f^{-1}(f(B)) \subset B$

# Basics of Mathematics

## Section ($C$)

### I. Speed, Acceleration and Momentum:

1) The displacement vector is the shortest line from the start to end $(\vec{d})$.
2) Position vector is line from $(0,0)$ to the object $(\vec{r})$.
3) The relation between displacement and position Vector is $\vec{d} = \overrightarrow{r_2} - \overrightarrow{r_1} \Rightarrow Fig(C-1)$
4) The velocity is the displacement by unit Time $\vec{v} = \frac{\vec{d}}{t}$
5) The speed is the distance by unit time $v = \frac{D}{t}$
6) Mean (average) velocity $\overrightarrow{v_m} = \frac{\vec{d}}{\Delta t} = \frac{\overrightarrow{r_2} - \overrightarrow{r_1}}{t_2 - t_1}$
7) Relative velocity $\overrightarrow{v_{AB}} = \overrightarrow{v_B} - \overrightarrow{v_A}$ with the same direction.
8) Relative velocity $\overrightarrow{v_{AB}} = \overrightarrow{v_B} + \overrightarrow{v_A}$ with the opposite direction.
9) The constant velocity (speed) means the object moves equal displacements by equal Times.



$(C-1)$

10) The acceleration $\vec{a} = \frac{\overrightarrow{v_2} - \overrightarrow{v_1}}{t_2 - t_1}$
    If the object is slowing down, then the acceleration has a negative sign.
    If the object is going faster, then the acceleration has a positive sign.
    If the object moves with constant velocity (speed), then the acceleration is zero.
11) Momentum is the velocity (speed) times mass $P = mv$
12) Kinetic energy it's the energy that the object has as a result of its motion $KE = \frac{1}{2}mv^2$

    $$but\ at\ the\ Momentum\ P = mv \Rightarrow KE = \frac{P^2}{2m}$$
13) Potential energy it's the energy that the object has as a result of its height
    $PE = mgh = Pgt$

### II. Equations of Kinematic:

At the concept of velocity and acceleration $v = \frac{\Delta d}{\Delta t}$ & $a = \frac{\Delta v}{\Delta t}$ the equation of motion describe the speed, acceleration, distance, and time which body takes or have.

$$\begin{cases} v_f = v_i + at \\ d = v_i t + \frac{1}{2}at^2 \\ v_f^2 = v_i^2 + 2ad \end{cases}$$ at vectorial Form the equation would be $$\begin{cases} \overline{v_f} = \overline{v_i} + \bar{a}t \\ \overline{r_f} = \overline{r_i} + \overline{v_i}t + \frac{1}{2}\bar{a}t^2 \\ \overline{v_f^2} = \overline{v_i^2} + 2a(\overline{r_f} - \overline{r_i}) \end{cases}$$

The Acceleration $(a)$ changes to $(g)$ at Freely Falling Object Problems where $(g)$ is the Earth Gravity acceleration $\left(9.8\ \frac{m}{s^2}\right)$.

### III. Newton's Laws:

1) In the absence of external forces and when viewed from an inertial reference frame, an object at rest remains at rest and an object in motion continues in motion with a constant velocity (that is, with a constant speed in a straight line).
2) When viewed from an inertial reference frame, the acceleration of an object is directly proportional to the net force acting on it and inversely proportional to its mass $\sum \bar{f} = m\bar{a}$ or $\bar{f} = \frac{d}{dt}(m\bar{v})$ where $\frac{d}{dt}(m\bar{v})$ is The rate of change for Momentum by Time.
3) If two objects interact, the force $\overline{f_{12}}$ exerted by object 1 on object 2 is equal in magnitude and opposite in direction to the force $\overline{f_{21}}$ exerted by object 2 on object 1 $\overline{f_{12}} = -\overline{f_{21}}$

 Islam Reda Ahmed

## IV. Projectile Motion:

At $(X)$ Axis observer the Acceleration is Zero but at $(Y)$ axis it has $(-g)$ Value $\Longrightarrow Fig(C-2)$

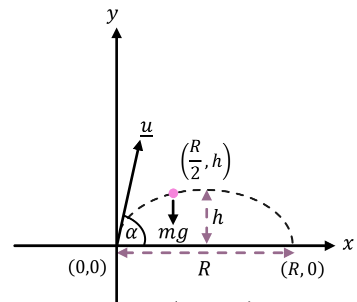
| | |
|---|---|
| $\ddot{x} = \frac{dx}{dt} = 0$ (Integrate ↓) | $\ddot{y} = \frac{dx}{dt} = -g$ (Integrate ↓) |
| $\dot{x} = c_1$ | $\dot{y} = -gt + c_2$ |

At the Initial Conditions

| | |
|---|---|
| $c_1 = u\cos\alpha$ | $c_2 = u\sin\alpha$ |
| $\dot{x} = u\cos\alpha$ (Integrate ↓) | $\dot{y} = -gt + u\sin\alpha$ (Integrate ↓) |
| $x = ut\cos\alpha + c_3$ | $y = -\frac{1}{2}gt^2 + u\,t\sin\alpha + c_4$ |

At the Initial Conditions

| | |
|---|---|
| $c_3 = 0$ | $c_4 = 0$ |
| $\therefore x = ut\cos\alpha$ | $\therefore y = -\frac{1}{2}gt^2 + u\,t\sin\alpha$ |



$(C-2)$

1) At $\dot{y} = 0$ the time to arrive at maximum Height is $\left\{ t_1 = \frac{u\sin\alpha}{g} \right\}$.
2) Maximum Height by Substituting about $t = t_1$ at $y$ Equation then $\left\{ h = \frac{1}{g}\, u^2\, \sin^2\alpha \right\}$.
3) Flying time by Substituting about $t = \tau$ at $y$ Equation then $\left\{ \tau = \frac{2}{g} u\sin\alpha = 2t_1 \right\}$.
4) Horizontal Range by Substituting about $t = \tau$ at $x$ Equation then $\left\{ R = \frac{1}{g} u^2 \sin 2\alpha \right\}$.
5) Maximum Horizontal Range $\left\{ R = \frac{1}{g} u^2 \right\}$.
6) Equation of Projectile Motion by Substituting about $t$ from $x$ to $y$

$$\left\{ y = u\left(\frac{x}{u\cos\alpha}\right)\sin\alpha - \frac{gx^2}{2u^2\cos^2\alpha} = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha} \right\}$$

## V. Projectile Motion with Resistance:

From newton's law $f = ma = -mg\underline{j} - R(\underline{i} + \underline{j})$ for $R = km\underline{v}$ . Hence $\because$ $\underline{a} = \ddot{x}\underline{i} + \ddot{y}\underline{j}$ ; $\underline{v} = \dot{x}\underline{i} + \dot{y}\underline{j}$

$\Longrightarrow Fig(C-3)$

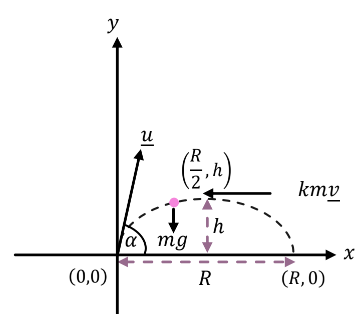
| | |
|---|---|
| $\not{m}\ddot{x} = \frac{d\dot{x}}{dt} = -k\not{m}\dot{x}$ (Integrate ↓) | $\not{m}\ddot{y} = \frac{dx}{dt} = -k\not{m}\left(\frac{g}{k} + \dot{y}\right)$ (Integrate ↓) |
| $\ln\dot{x} = -kt + c_1$ | $\ln\left(\frac{g}{k} + \dot{y}\right) = -kt + c_2$ |

At the Initial Conditions

| | |
|---|---|
| $c_1 = \ln u\cos\alpha$ | $c_2 = \ln\left(\frac{g}{k} + u\sin\alpha\right)$ |
| $\dot{x} = u\cos\alpha\, e^{-kt}$ (Integrate ↓) | $\dot{y} = \left(\frac{g}{k} + u\sin\alpha\right)e^{-kt} - \frac{g}{k}$ (Integrate ↓) |
| $x = \frac{1}{-k} u\cos\alpha\, e^{-kt} + c_3$ | $y = \frac{-1}{k}\left(\frac{g}{k} + u\sin\alpha\right)e^{-kt} - \frac{gt}{k} + c_4$ |

At the Initial Conditions

| | |
|---|---|
| $c_3 = \frac{u\cos\alpha}{k}$ | $c_4 = \frac{1}{k}\left(\frac{g}{k} + u\sin\alpha\right)$ |
| $\therefore x = \frac{1}{k} u\cos\alpha\,(1 + e^{-kt})$ | $\therefore y = \frac{1}{k}\left(\frac{g}{k} + u\sin\alpha\right)(1 - e^{-kt}) - \frac{gt}{k}$ |



$(C-3)$

1) At $\dot{y} = 0$ the time to arrive at maximum Height is $\left\{ t_1 = \frac{1}{k}\ln\left(1 + \frac{ku\sin\alpha}{g}\right) \right\}$.
2) Max Height by Substituting about $t = t_1$ at $y$ Equation then $\left\{ h = \frac{u\sin\alpha}{k} - \frac{g}{k^2}\ln\left(1 + \frac{ku\sin\alpha}{g}\right) \right\}$.
3) Flying time by Substituting about $t = \tau$ at $y$ Equation then $\left\{ \tau = \left(\frac{1}{k} + \frac{u\sin\alpha}{g}\right)(1 - e^{-k\tau}) \right\}$.
4) Horizontal Range by Substituting about $t = \tau$ at $x$ Equation then $\left\{ R = \frac{u\tau\cos\alpha}{\left(1 + \frac{ku\sin\alpha}{g}\right)} \right\}$.
5) By Substituting about $t$ from $x$ to $y \Longrightarrow \left\{ y = x\tan\alpha + \frac{gx}{ku\cos\alpha} + \frac{g}{k^2}\ln\left(1 - \frac{kx}{u\cos\alpha}\right) \right\}$.

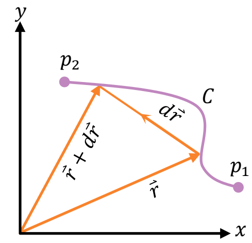
## VI. Work, Power and Energy:

1) Work done by any Force $w = \int \vec{f} \odot d\vec{r}$ it's a scaler magnitude $\Longrightarrow$ $Fig(C-4)$

   Work units $\{Joule = Newton \times Meter\}$ $and$ $\{Erg = Dine \times cm\}$ $thus\ Joule = 10^7 Erg.$

   $w = \int_c \bar{f} \cdot d\bar{r} = \int_c \bar{f} \cdot \frac{d\bar{r}}{dt} dt$ but $\bar{f} = m\bar{a} = m\frac{d\bar{v}}{dt}$ and $\frac{d\bar{r}}{dt} = \bar{v}$

   $w = \int_c \bar{f} \cdot d\bar{r} = \int_c m\frac{d\bar{v}}{dt} \cdot \bar{v} dt \Longrightarrow w = m\int_{p_1}^{p_2} \bar{v} d\bar{v} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 = KE_2 - KE_1$




$(C-4)$

2) Power is the rate of change for Work by Time $P_w = \frac{dw}{dt} = \frac{d}{dt}(f.r)$

   If $f$ is constant Then $P_w = f \cdot \frac{dr}{dt} = f \cdot v$

   Power units $\{Erg/s\}, \{k\ watt\}$ $and$ $\{Horse = 0.735\ k\ watt = 735\ watt\}.$

3) The sum of Energy $(KE, PE)$ at any point during the motion $KE_1 + PE_1 = KE_2 + PE_2$

   $\Longrightarrow \frac{1}{2}mv_1^2 + mgh_1 = \frac{1}{2}mv_2^2 + mgh_2$

## VII. Circular and Central Motion:

1) An acceleration of this nature is called a centripetal acceleration $\left\{a_c = \frac{v^2}{r}\right\}$.

2) The period of a particle in uniform circular motion is $\left\{T = \frac{2\pi r}{v}\right\}$.

3) The angular speed $\left\{\omega = \frac{2\pi}{T}\right\}$.

4) Another form for acceleration $\{a_c = r\omega^2\}$.

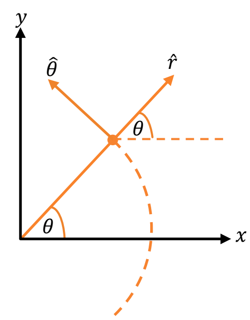
5) Driving the acceleration of this kind in polar coordinates $\Longrightarrow$ $Fig(C-5)$

   $\hat{r} = cos\,\theta\, \underline{i} + sin\,\theta\, \underline{j}$ $and$ $\hat{\theta} = -sin\,\theta\, \underline{i} + cos\,\theta\, \underline{j}$

   $\frac{d\hat{r}}{d\theta} = \hat{\theta}$ $and$ $\frac{d\hat{\theta}}{d\theta} = -\hat{r}$ $\Longrightarrow \frac{d\hat{r}}{dt} = \frac{d\hat{r}}{d\theta} \cdot \frac{d\theta}{dt} = \dot{\theta}\hat{\theta}$ $and$ $\frac{d\hat{\theta}}{dt} = \frac{d\hat{\theta}}{d\theta} \cdot \frac{d\theta}{dt} = -\dot{\theta}\hat{r}$

   $at\ the\ Position\ vector\ \bar{r} = r\hat{r}\ then\ Velocit\ vector\ \bar{v} = \frac{d\bar{r}}{dt} = \frac{d}{dt}(r\hat{r})$

   $\bar{v} = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \Longrightarrow v = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}$




$(C-5)$

   $\bar{a} = \frac{d\bar{v}}{dt} = \frac{d\dot{r}}{dt}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \frac{dr}{dt}\dot{\theta}\hat{\theta} + r\frac{d\dot{\theta}}{dt}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt} = \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - \dot{r}^2\hat{r}$

   $\bar{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} \Longrightarrow a = \sqrt{(\ddot{r} - r\dot{\theta}^2)^2 + (2\dot{r}\dot{\theta} + r\ddot{\theta})^2}$

   $\bar{a} = a_r\hat{r} + a_\theta\hat{\theta}\ for\ a_r = \ddot{r} - r\dot{\theta}^2\ and\ a_\theta = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$

6) The central motion is the motion where the object take to move under force always directed to constant point called half radius force.

   $f_r = m(\ddot{r} - r\dot{\theta}^2)$ and $f_\theta = \frac{m}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0 \Longrightarrow r^2\dot{\theta} = constant = h$

   $\left\{h = 2v_a\ where\ \left(v_a = \frac{1}{2}r^2\dot{\theta}\right)\ Areal\ Velocity\right\}$ and $\{(h = rv)\ where\ (rv)\ is\ Moment\ of\ Velocity\ \}$

7) Velocity of central motion $\dot{r} = \frac{dr}{d\theta} = \frac{dr}{du} \cdot \frac{du}{d\theta} \cdot \frac{d\theta}{dt} = \frac{-1}{u^2}\frac{du}{d\theta}\dot{\theta}$ and $r = \frac{1}{u} \Longrightarrow \dot{r}^2 = h^2\left(\frac{du}{d\theta}\right)^2$

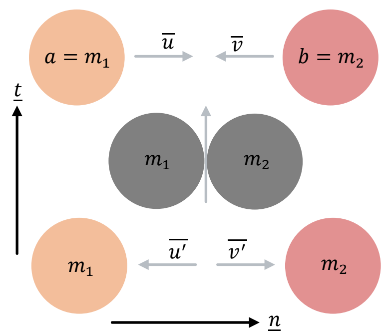
   $\because r^2\dot{\theta}^2 = \frac{(r^2\dot{\theta})^2}{r^2} = h^2u^2$ but $\bar{v}^2 = \dot{r}^2 + r^2\dot{\theta}^2 = h^2\left\{u^2 + \left(\frac{du}{d\theta}\right)^2\right\}$

8) Force of central motion $\dot{r} = \frac{-1}{u^2}\frac{du}{d\theta}\dot{\theta} = -h\frac{du}{d\theta}$ and $\ddot{r} = -h\frac{d^2u}{d\theta^2}\dot{\theta} = -h\frac{d^2u}{d\theta^2}\frac{h}{r^2}$

   $\ddot{r} = h^2u^2\frac{d^2u}{d\theta^2}$ and $r\dot{\theta}^2 = \frac{(r^2\dot{\theta})^2}{r^3} = h^2u^3$ but $f = -(\ddot{r} - r\dot{\theta}^2) = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right)$

## VIII. Impulse and Collisions:

1) The Impulse acting on object is defined as $\overline{I} = \int_0^\tau \overline{f} dt = m \int_0^\tau \frac{d\overline{v}}{dt} dt = (m\overline{v})_\tau - (m\overline{v})_0$
   So that the Impulse is the rate of change for momentum by time.
2) Collisions are often classified according to whether the total kinetic energy changes during the collision.
   8. Elastic collision: One in which the total kinetic energy of the system after the collision is equal to the total kinetic energy before the collision.
   9. Inelastic collision: One in which the total kinetic energy of the system is not the same before and after the collision; if the objects stick together after colliding, the collision is said to be completely inelastic $\Longrightarrow$ $Fig(C-6)$



$(C-6)$

We will describe only the first section.

$\Longrightarrow$ As the conservation of Momentum and Newton third law says

$\overline{f_{ab}} = -\overline{f_{ba}}$ then $\frac{d}{dt}(m_1\overline{u} + m_2\overline{v}) = 0 \Longrightarrow m_1\overline{u} + m_2\overline{v} = c$

$\therefore m_1\overline{u}\cdot\overline{n} + m_2\overline{v}\cdot\overline{n} = m_1\overline{u'}\cdot\overline{n} + m_2\overline{v'}\cdot\overline{n}$

But $\overline{u'}\cdot\overline{t} = \overline{u}\cdot\overline{t}$ and $\overline{v'}\cdot\overline{t} = \overline{v}\cdot\overline{t}$ the Momentum at $\overline{t}$ direction doesn't change.

$\Longrightarrow$ Center of mass Velocity $\overline{q} = \frac{m_1\overline{u} + m_2\overline{v}}{m_1 + m_2}$

$\Longrightarrow$ If $\overline{w}$ is the relative velocity for $(b)$ to $(a)$ then $\overline{w} = \overline{u} - \overline{v}$

So that $\overline{u} = \overline{q} + \frac{\mu}{m_1}\overline{w}$ and $\overline{v} = \overline{q} - \frac{\mu}{m_2}\overline{w}$ where $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$ and $\mu$ called reduced mass

$\Longrightarrow$ The Kinetic energy $KE = \frac{1}{2}m\overline{u}\cdot\overline{u} + \frac{1}{2}m\overline{v}\cdot\overline{v} = \frac{1}{2}(m_1 + m_2)q^2 + \frac{1}{2}\mu\omega^2$

Where $\frac{1}{2}(m_1 + m_2)q^2$ the center of mass Kinetic energy : $\frac{1}{2}\mu\omega^2$ relative motion Kinetic energy.

$\Longrightarrow$ After collision $\overline{q} = \overline{q'}$ and $\overline{w'}\cdot\overline{t} = \overline{w}\cdot\overline{t}$

So that the only variable after colliding is $\overline{w}\cdot\overline{n}$ so that $-e\,\overline{w'}\cdot\overline{n} = \overline{w}\cdot\overline{n}$ where $(e)$ is called recoil coefficient.

$\Longrightarrow \overline{w'} = (\overline{w}\cdot\overline{t})\overline{t} - e(\overline{w}\cdot\overline{n})\overline{n}$

$\Longrightarrow \overline{u'}\cdot\overline{n} = \overline{q'}\cdot\overline{n} + \frac{\mu}{m_1}\overline{w'}\cdot\overline{n} = \overline{q}\cdot\overline{n} - \frac{e\mu}{m_1}\overline{w}\cdot\overline{n}$

$\Longrightarrow \overline{v'}\cdot\overline{n} = \overline{q'}\cdot\overline{n} - \frac{\mu}{m_2}\overline{w'}\cdot\overline{n} = \overline{q}\cdot\overline{n} + \frac{e\mu}{m_2}\overline{w}\cdot\overline{n}$

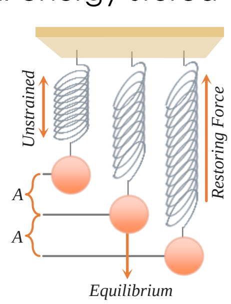
There is another way to describe collision using newton third law.

$m_1\overline{u'} - m_1\overline{u} = \overline{I} \Longrightarrow (1)$ and $m_2\overline{v'} - m_2\overline{v} = \overline{-I} \Longrightarrow (2)$

By adding $(1) + (2) \Longrightarrow m_1\overline{u'} + m_2\overline{v'} = m_1\overline{u} + m_2\overline{v}$

## IX. Simple Harmonic Motion:

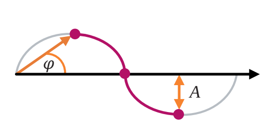
If there is a spring attached with light mass and we pushing the mass up, we will see that the mass will start to vibrating up and down under restoring force and potential energy stored on the spring. $\Longrightarrow$ $Fig(C-7)$



$(C-7)$

1) Restoring force $\left\{f_s = -kx \Longrightarrow a_x = -\frac{k}{m}x \text{ and } \omega = \sqrt{\frac{k}{m}} = 2\pi f = \frac{2\pi}{T}\right\}$.
2) The position of the particle as a function of time $\{x(t) = A\cos(\omega t + \varphi)\}$. $\Longrightarrow$ $Fig(C-8)$
3) The Velocity $\left\{\frac{dx}{dt} = v = -\omega A \sin(\omega t + \varphi)\right\}$.
4) The Acceleration $\left\{\frac{dv}{dt} = a = -\omega^2 A \cos(\omega t + \varphi)\right\}$.

5) The period $\left\{T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}\right\}$ and Frequency $\left\{f = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}\right\}$.
6) Kinetic Energy stored $\left\{K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2A^2sin^2(\omega t + \varphi)\right\}$.
7) Potential Energy stored $\left\{U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2cos^2(\omega t + \varphi)\right\}$.
8) The total energy of a simple harmonic $\left\{E = \frac{1}{2}kA^2\right\}$.



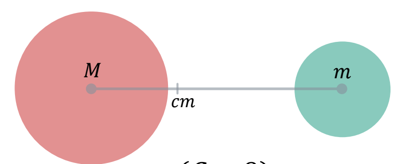
$(C - 8)$

## X. Center of Mass:

Consider a system consisting of a pair of particles that have different masses and are connected by a light, rigid rod. The position of the center of mass of a system can be described as being the average position of the system's mass. The center of mass of the system is located somewhere on the line joining the two particles and is closer to the particle having the larger mass. If a single force is applied at a point on the rod above the center of mass, the system rotates clockwise. If the force is applied at a point on the rod below the center of mass, the system rotates counterclockwise. If the force is applied at the center of mass, the system moves in the direction of the force without rotating. The center of mass of an object can be located with this procedure. $\Longrightarrow Fig(C - 9)$

$$\bar{r}_{cm} = \frac{1}{M}\int \bar{r}\, dm$$

1) $X\ Axis\ Component\ \bar{x} = \frac{\int x\, dm}{\int dm}$
2) $Y\ Axis\ Component\ \bar{y} = \frac{\int y\, dm}{\int dm}$
3) $Z\ Axis\ Component\ \bar{z} = \frac{\int z\, dm}{\int dm}$



$(C - 9)$

$$(dm)\ could\ be\ replaced\ by\ one\ of\ three \begin{cases} At\ Length\ Section\ dm = \lambda\, dl \\ At\ Area\ Section\ dm = \sigma\, dA \\ At\ Volume\ Section\ dm = \rho\, dV \end{cases} \quad Where\ \lambda, \sigma\ and\ \rho\ are\ the\ density$$

## XI. Moment of Inertia Tensor:

When the rotation of rigid object around the fixed axis the amount of angular momentum $\bar{l}$ proportional to the angular speed $\bar{\omega}$ is written as $\bar{l} = I\bar{\omega}$

$$\begin{bmatrix} l_x \\ l_y \\ l_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

Where $I$ depends on the body spindle and called Moment of Inertia Tensor

1) Moment of Inertia Tensor $\bar{\bar{I}} = \begin{vmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{vmatrix}$ but in tow dimension $\bar{\bar{I}} = \begin{vmatrix} I_{xx} & I_{xy} \\ I_{yx} & I_{yy} \end{vmatrix}$
2) The components $\{I_{xx} = \int(y^2 + z^2)\, dm\}$ , $\{I_{yy} = \int(x^2 + z^2)\, dm\}$ and $\{I_{zz} = \int(x^2 + y^2)dm\}$
3) The components $\{I_{xy} = I_{yx} = \int -xy\, dm\}$ , $\{I_{zx} = I_{xz} = \int -xz\, dm\}$ and $\{I_{yz} = I_{zy} = \int -yz\ dm\}$
4) In tow dimensions $\{I_{xx} = \int y^2\, dm\}$ , $\{I_{yy} = \int x^2\ dm\}$ and $\{I_{yx} = I_{xy} = \int -xy\, dm\}$

## Section $(D)$

### I. Infinite Series $cgt\ or\ dgt$ test:

If $\sum a_n$ is Seri and $\lim_{n\to\infty} a_n \neq 0$ then Seri is $dgt$ but if $\lim_{n\to\infty} a_n = 0$ then Seri is $cgt$.

When adding, subtract or multiplying constant to Seri it doesn't effect on its $dgt\ or\ cgt$.

1) Some important series
   1. Telescopic series $\sum \frac{1}{n(n+1)}\ cgt\ series\ and\ equal\ to\ 1$
   2. Harmonic series $\sum \frac{1}{n}\ dgt\ series$
   3. Geometric series $\sum r^{n-1}\ cgt\ when\ 1 > |x|\ and\ equal\ to\ \frac{1}{1-r}$
   4. P series $\sum \frac{1}{n^p}\ cgt\ when\ p > 1$
   5. If $\sum a_n\ is\ cgt\ and\ cgt\ is\ cgt\ then\ \sum a_n \pm b_n\ is\ cgt$
   6. If $\sum a_n\ is\ dgt\ and\ \sum b_n\ is\ cgt\ then\ \sum a_n \pm b_n\ is\ dgt$
2) Test for series with positive terms
   1. Comparison test: If $\sum a_n$ is $cgt$ and $\sum b_n$ need for test so that if $\sum a_n \geq \sum b_n \Rightarrow \sum b_n\ is\ cgt$ But if $\sum a_n$ is $dgt$ and $\sum b_n$ need for test so that if $\sum a_n \leq \sum b_n \Rightarrow \sum b_n\ is\ dgt$.
   2. Quotient test: If $\sum a_n$ and $\sum b_n$ are series which $\lim_{n\to\infty} \frac{a_n}{b_n} = A \neq 0\ or\ \infty$ then both are $dgt$ or both $cgt$ but if $A = \infty\ and\ \sum b_n\ is\ dgt\ so\ \sum a_n\ is\ dgt$ if else test fails.
   3. Integral test: Suppose that function $a_n = f(x)$ is defined for $x \geq 1$ so if the function is positive, continues and decreasing function then if $\int_1^\infty f(x)$ is exist so $\sum a_n\ is\ cgt$ but if $\int_1^\infty f(x)$ is not exist so $\sum a_n\ is\ dgt$.
   4. $n^{th}$ Root test: If the series $\sum a_n$ need for test and has positive terms Where $\lim_{n\to\infty} \sqrt[n]{a_n} = N$ so that if $N < 1$ the series is $cgt$ but if $N > 1$ the series is $dgt$ and test fails if $N = 1$.
   5. Ratio test: If the series $\sum a_n$ need for test and has positive terms Where $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = K$ so that if $K < 1$ the series is $cgt$ but if $K > 1$ the series is $dgt$ and test fails if $K = 1$.
   6. Raabe's test: If the series $\sum a_n$ need for test and has positive terms Where $\lim_{n\to\infty} n\left(1 - \frac{a_{n+1}}{a_n}\right) = R$ so that if $R > 1$ the series is $cgt$ but if $R < 1$ the series is $dgt$ and test fails if $K = 1$.
   7. Morgan's test: : If the series $\sum a_n$ need for test and has positive terms Where $\lim_{n\to\infty} \ln\left\{1 - n\left(1 - \frac{a_{n+1}}{a_n}\right)\right\} = M$ so that if $M > 1$ the series is $cgt$ but if $M < 1$ the series is $dgt$ and test fails if $M = 1$.
3) Test for Alternating series

   Leibniz test: The series $\sum_1^\infty (-1)^{n-1} a_n\ is\ cgt$ if $a_{k+1} \leq a_k : \forall k \geq 1$ and $\lim_{n\to\infty} a_n = 0$.

### II. Limits Rules:

1) $\lim_{x\to a} c = c \Rightarrow \lim_{x\to a} cf(x) = c \lim_{x\to a} f(x)$
2) $\lim_{x\to\infty} e^x = \infty \Rightarrow \lim_{x\to-\infty} e^x = 0$
3) $\lim_{x\to\infty} \ln x = \infty \Rightarrow \lim_{x\to 0} \ln x = -\infty$
4) $\lim_{x\to a}\{g(x) \pm f(x)\} = \lim_{x\to a} g(x) \pm \lim_{x\to a} f(x)$
5) $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \lim_{x\to a} g(x)$
6) $\lim_{x\to a}\{f(x)\}^n = \left\{\lim_{x\to a} f(x)\right\}^n \Rightarrow \lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$
7) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$
8) $\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1} \Rightarrow \lim_{x\to a} \frac{x^n - a^n}{x^m - a^m} = \frac{n}{m} a^{n-m}$
9) $\lim_{a\to 0} \frac{(x+a)^n - x^n}{a} = nx^{n-1}$
10) $\lim_{x\to 0} \frac{\sin x}{x} = 1 \Rightarrow \lim_{x\to 0} \frac{\sin ax}{\sin bx} = \frac{a}{b} \Rightarrow \lim_{x\to 0} \frac{\tan x}{x} = 1$
11) If $\lim_{x\to a} f(x) = 0$, $\lim_{x\to a} g(x) = \infty\ then\ \lim_{x\to a}\{1 + f(x)\}^{g(x)} = e^k\ where\ k = \lim_{x\to a} f(x)g(x)$

## III. Some Important Integrations and Derivatives:

1) $f'(x) = \lim\limits_{h \to \infty} \frac{f(x+h)-f(x)}{h}$

2) $\frac{d}{dx}\{f(x)\}^n = nf'(x)\{f(x)\}^{n-1}$

3) $\left(\frac{f}{g}\right)' = \frac{gf'-fg'}{g^2}$

4) $\frac{d}{dx}\ln f(x) = \frac{f'(x)}{f(x)}$

5) $(fg)' = f'g + fg'$

6) $\frac{d}{dx}\{f(g(x))\} = f'\{g(x)\}g'(x)$

| Function | Derivatives | Function | Derivatives | Function | Integrations |
|---|---|---|---|---|---|
| $\sin x$ | $\cos x$ | $\sin^{-1} x$ | $\frac{1}{\sqrt{1-x^2}}$ | $\frac{dx}{\sqrt{a^2-x^2}}$ | $\sin^{-1}(\frac{x}{a}) + c$ |
| $\cos x$ | $-\sin x$ | $\cos^{-1} x$ | $\frac{-1}{\sqrt{1-x^2}}$ | $\frac{dx}{\sqrt{a^2+x^2}}$ | $\sinh^{-1}(\frac{x}{a}) + c$ |
| $\tan x$ | $\sec^2 x$ | $\tan^{-1} x$ | $\frac{1}{1+x^2}$ | $\frac{dx}{\sqrt{x^2-a^2}}$ | $\cosh^{-1}(\frac{x}{a}) + c$ |
| $\cot x$ | $-\csc^2 x$ | $\sec^{-1} x$ | $\frac{1}{x\sqrt{x^2-1}}$ | $\frac{dx}{a^2+x^2}$ | $\frac{1}{a}\tan^{-1}(\frac{x}{a}) + c$ |
| $\sec x$ | $\sec x \tan x$ | $\csc^{-1} x$ | $\frac{-1}{x\sqrt{x^2-1}}$ | $\frac{dx}{a^2-x^2}$ | $\frac{1}{a}\tanh^{-1}(\frac{x}{a}) + c$ |
| $\csc x$ | $-\csc x \cot x$ | $\cot^{-1} x$ | $\frac{-1}{1+x^2}$ | $\frac{dx}{x\sqrt{a^2+x^2}}$ | $\frac{1}{a}\operatorname{csch}^{-1}(\frac{x}{a}) + c$ |
| $\sinh x$ | $\cosh x$ | $\sinh^{-1} x$ | $\frac{1}{\sqrt{1+x^2}}$ | $\frac{dx}{x\sqrt{x^2-a^2}}$ | $\frac{1}{a}\sec^{-1}(\frac{x}{a}) + c$ |
| $\cosh x$ | $\sinh x$ | $\cosh^{-1} x$ | $\frac{1}{\sqrt{x^2-1}}$ | $\frac{dx}{x\sqrt{a^2-x^2}}$ | $\frac{1}{a}\operatorname{sech}^{-1}(\frac{x}{a}) + c$ |
| $\tanh x$ | $\operatorname{sech}^2 x$ | $\tanh^{-1} x$ | $\frac{1}{1-x^2}$ | $\sin x$ | $-\cos x + c$ |
| $\coth x$ | $-\operatorname{csch}^2 x$ | $\operatorname{sech}^{-1} x$ | $\frac{-1}{x\sqrt{1-x^2}}$ | $\cos x$ | $\sin x + c$ |
| $\operatorname{sech} x$ | $-\operatorname{csch} x \coth x$ | $\operatorname{csch}^{-1} x$ | $\frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$ | $\sec^2 x$ | $\tan x + c$ |
| $\operatorname{csch} x$ | $-\operatorname{sech} x \tanh x$ | $\coth^{-1} x$ | $\frac{1}{1-x^2}$ | $\csc^2 x$ | $-\cot x + c$ |
| $a^x$ | $a^x \ln a$ | $x^n$ | $nx^{n-1}$ | $\tan x$ | $\ln\{\cos x\} + c$ |
| $e^{ax}$ | $a^n e^{ax}$ | $\ln x$ | $\frac{1}{x}$ | $\cot x$ | $-\ln\{\sin x\} + c$ |
| $\log_a x$ | $\frac{1}{x \ln a}$ | $\log x$ | $\frac{1}{x \ln 10}$ | $\sec x$ | $\ln\{\sec x + \tan x\} + c$ |
| | | | | $\csc x$ | $\ln\{\csc x - \cot x\} + c$ |
| | | | | $\sec x \tan x$ | $\sec x + c$ |
| | | | | $\csc x \cot x$ | $-\csc x + c$ |
| | | | | $\sinh x$ | $\cosh x + c$ |
| | | | | $\cosh x$ | $\sinh x + c$ |
| | | | | $\operatorname{sech}^2 x$ | $\tanh x + c$ |
| | | | | $\operatorname{csch}^2 x$ | $-\coth x + c$ |
| | | | | $\operatorname{sech} x \tanh x$ | $-\operatorname{sech} x + c$ |
| | | | | $\operatorname{csch} x \coth x$ | $-\operatorname{csch} x + c$ |
| | | | | $\frac{1}{x}$ | $\ln x + c$ |
| | | | | $x^n$ | $\frac{x^{n+1}}{n+1} + c$ |
| | | | | $a^x$ | $\frac{a^x}{\ln a} + c$ |
| | | | | $\ln x$ | $-x + x\ln x + c$ |

$$\int \sec^n\theta\, d\theta = \frac{\sec^{n-1}\theta \sin\theta}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2}\theta\, d\theta$$

$$\int \csc^n\theta\, d\theta = \frac{-1}{n-1}\cot\theta \csc^{n-2}\theta + \frac{n-2}{n-1}\int \csc^{n-2}\theta\, d\theta$$

$$\int \cot^n\theta\, d\theta = \frac{-1}{n-1}\cot^{n-1}\theta - \int \cot^{n-2}\theta\, d\theta$$

$$\int \sin^n\theta\, d\theta = \frac{1}{n}\left\{-\sin^{n-1}\theta\cos\theta + (n-1)\int \sin^{n-1}\theta\, d\theta\right\}$$

$$\int \cos^n\theta\, d\theta = \frac{1}{n}\left\{\cos^{n-1}\theta \sin\theta + (n-1)\int \cos^{n-2}\theta\, d\theta\right\}$$

$$\int \tan^n\theta\, d\theta = \frac{1}{n-1}\tan^{n-1}\theta - \int \tan^{n-2}\theta\, d\theta$$

$$\int \sin^n\theta \cos^m\theta\, d\theta = -\frac{\cos^{m+1}\theta \sin^{n-1}\theta}{m+n} + \frac{n-1}{m+n}\int \sin^{n-2}\theta \cos^m\theta\, d\theta$$

$$= \frac{\cos^{m-1}\theta \sin^{n+1}\theta}{m+n} + \frac{m-1}{m+n}\int \sin^n\theta \cos^{m-2}\theta\, d\theta$$

## IV. Definite Integral:

### 1) Advantages

1. $\int_a^b g(x) \pm f(x)dx = \int_a^b g(x)dx \pm \int_a^b f(x)dx$
2. $c\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
3. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$
4. $\int_{-a}^a f(x)dx = 0 \quad if\ f(x)\ is\ odd$
5. $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx \quad if\ f(x)\ is\ even$

### 2) Wallace First Rule

$$\int_0^{\frac{\pi}{2}} cos^n\theta d\theta = \int_0^{\frac{\pi}{2}} sin^n\theta d\theta = \frac{(n-1)(n-3)\ldots..2\ or\ 1}{n(n-2)(n-4)\ldots\ldots 2\ or\ 1}\begin{cases}0.5\pi\ if\ n\ is\ even\\ 1\ if\ n\ is\ odd\end{cases}$$

### 3) Wallace Second Rule

$$\int_0^{\frac{\pi}{2}} sin^m\theta cos^n\theta d\theta = \frac{\{(m-1)(m-3)\ldots..2\ or\ 1\}\{(n-1)(n-3)\ldots..2\ or\ 1\}}{\{(m+n)(m+n-2)(m+n-4)\ldots..2\ or\ 1\}}\begin{cases}0.5\pi\ if\ m,n\ is\ even\\ 1\ if\ m,n\ is\ odd\end{cases}$$

## V. Integration Techniques:

### 1) Integration by parts

$$\int udv = uv - \int vdu$$

For example:

$$\int x\sin x\,dx = -x\cos x - \int -\cos x\,dx = -x\cos x + \sin x + c$$

### 2) Integral by Substituting

At this technique we choose a simpler variable to substitute on function.

For example:

$$\int 2x\,e^{x^2}dx = ??\quad let\ x^2 = y \implies dy = 2xdx$$

$$\int e^y dy = e^y + c\ \ and\ \ y = x^2$$

### 3) Trigonometric Substitution

| Function | Substitution | Identity |
|---|---|---|
| $\sqrt{a^2 - x^2}$ | $x = a\,sin\,\theta \quad \frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$ | $1 - sin^2\theta = cos^2\theta$ |
| $\sqrt{a^2 + x^2}$ | $x = a\,tan\,\theta \quad \frac{-\pi}{2} < \theta < \frac{\pi}{2}$ | $1 + tan^2\theta = sec^2\theta$ |
| $\sqrt{x^2 - a^2}$ | $x = a\,sec\,\theta \quad 0 \le \theta < \frac{\pi}{2}$ | $sec^2\theta - 1 = tan^2\theta$ |

1. $\int \sqrt{a^2 - x^2}dx = \frac{1}{2}\left\{x\sqrt{a^2 - x^2} + a^2 sin^{-1}\frac{x}{a}\right\} + c$
2. $\int \sqrt{a^2 + x^2}dx = \frac{1}{2}\left\{x\sqrt{a^2 + x^2} + a^2 sinh^{-1}\frac{x}{a}\right\} + c$
3. $\int \sqrt{x^2 - a^2}dx = \frac{1}{2}\left\{x\sqrt{x^2 - a^2} - a^2 cosh^{-1}\frac{x}{a}\right\} + c$

### 4) Integration by Partial Fractions

To integrate any rational function by expressing it as a sum of simpler fractions.

For example:

$$\int \frac{x+5}{x^2+x-2}dx = \int \left(\frac{2}{x-1} - \frac{1}{x+2}\right)dx = 2\ln|x-1| - \ln|x+2| + c$$

### 5) Numerator is Derivative of Denominator

$$\int \frac{dx}{x} = \ln x + c \quad or \int \frac{dx}{\sqrt{x}} = x + c$$

## VI. Laplace Transform:

$$L\{f(x)\} = \bar{f}(p) = \int_0^\infty f(x)\, e^{-sx} dx \quad and \quad s > 0$$

| Function | Transform | Function | Transform |
|---|---|---|---|
| $c$ | $\frac{1}{s}$ | $\sinh \alpha t$ | $\frac{\alpha}{s^2 - \alpha^2}$ |
| $t^n$ | $\frac{n!}{s^{n+1}}$ | $\cosh \alpha t$ | $\frac{s}{s^2 - \alpha^2}$ |
| $e^{\alpha t}$ | $\frac{1}{s - \alpha}$ | $e^{\alpha t} f(t)$ | $f(s - \alpha)$ |
| $\sin \alpha t$ | $\frac{\alpha}{s^2 + \alpha^2}$ | $t^n f(t)$ | $(-1)^n f^n(s)$ |
| $\cos \alpha t$ | $\frac{s}{s^2 + \alpha^2}$ | $t^{-\frac{1}{2}}$ | $\sqrt{\frac{\pi}{s}}$ |

## VII. Partial Differentiation and Taylor Expansion:

1) Total Differentiation $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$
2) $f_x{}' = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z) - f(x,y,z)}{\Delta x}$
3) $f_y{}' = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y, z) - f(x,y,z)}{\Delta y}$
4) $f_z{}' = \lim_{\Delta z \to 0} \frac{f(x, y, z+\Delta z) - f(x,y,z)}{\Delta z}$

   For example:

$$if\ u = x^3 + 3x^2 y - 3xy^2 - z^3\ find\ u'_x / u'_y / u'_{xy} / u'_{yx}$$

$u'_x = 3x^2 + 6xy - 3y^2$ $\quad u'_y = 3x^2 - 6xy$ $\quad u'_{xy} = 6x - 6y$ $\quad u'_{yx} = 6x - 6y$

5) Laplace (Harmonic) equation $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$
6) Taylor expansion for function $f(x, y)$ at point $f(x_0, y_0)$ as Infinity power series Is

$$f(x,y) = f(x_0,y_0) + \left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y}\right) f(x_0,y_0) + \frac{1}{2!}\left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y}\right)^2 f(x_0,y_0) + \frac{1}{3!}\left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y}\right)^3 f(x_0,y_0) + \frac{1}{4!}\left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y}\right)^4 f(x_0,y_0) + \cdots + R_n$$

## VIII. Differential Vector:

1) Nabla ($\overline{\nabla}$)
   1. (gradient)$\overline{\nabla} f(x, y, z) = \frac{\partial f}{\partial x} \underline{i} + \frac{\partial f}{\partial y} \underline{j} + \frac{\partial f}{\partial z} \underline{k}$
   2. (divergent)$\overline{\nabla} \cdot f(x, y, z) = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}$
   3. (curl)$\overline{\nabla} \wedge f(x, y, z) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$
   4. (laplace)$\nabla^2 f(x, y, z) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = (gradient)(divergent) f(x, y, z)$
2) Properties
   1. $\overline{\nabla} r = \hat{r} \implies \overline{\nabla} r^n = n r^{n-1} \hat{r} \implies \overline{\nabla} \ln r = \frac{\hat{r}}{r}\ and\ \overline{\nabla} \frac{1}{r} = \frac{-\hat{r}}{r^2}$
   2. $\overline{\nabla} \varphi A = \varphi \overline{\nabla} A + A \overline{\nabla} \varphi$
   3. $\overline{\nabla} \cdot r = 3 \implies \overline{\nabla} \cdot \varphi A = \varphi \overline{\nabla} \cdot A + A \overline{\nabla} \cdot \varphi\ and\ (\bar{A} \cdot \overline{\nabla}) \bar{r} = \bar{A}$
   4. $\overline{\nabla} \wedge (\varphi \bar{A}) = \overline{\nabla} \varphi \wedge \bar{A} + (\overline{\nabla} \wedge \bar{A}) \varphi$
   5. $\overline{\nabla} \cdot (\bar{A} \wedge \bar{B}) = \bar{B} \cdot (\overline{\nabla} \wedge \bar{A}) - \bar{A} \cdot (\overline{\nabla} \wedge \bar{B})$
   6. $\varphi \nabla^2 \beta - \beta \nabla^2 \varphi = \overline{\nabla} \cdot (\beta \overline{\nabla} \varphi - \varphi \overline{\nabla} \beta)$

## IX. Improper Integral:

There are three kinds of Improper Integral

- If the boundaries for integral $\int_a^b f(x)dx \ \ (a = \infty\ or - \infty\ and\ b = \infty\ or - \infty)\ or\ once.$
- If function is undefined for point $s$ which called singular point $\lim_{x\to s} f(x)\ \ doesn't\ exist.$
- $3^{rd}$ kind is combination between $(1)\ and\ (2).$

### 1) First case

Special Improper Integrations from first kind like

- Geometric or Exponential: $\int_a^\infty e^{-tx}dx\ \ which\ t\ is\ constant\ is\ cgt\ if\ t > 0\ and\ dgt\ if\ t < 0$
- The $p$ integration: $\int_a^\infty \frac{1}{x^p}dx\ \ is\ cgt\ if\ p > 1\ and\ dgt\ if\ p \le 1$

$dgt\ or\ cgt$ test for this kind

1. Comparison test: $If\ g(x) \ge 0\ ; \forall\ x \ge a\ and\ \int_a^\infty g(x)dx\ \ is\ cgt\ which\ f(x) \le g(x)$ $so\ \int_a^\infty f(x)dx\ \ is\ cgt\ to. but\ if\ g(x) \ge 0\ ; \forall\ x \ge a\ and\ \int_a^\infty g(x)dx\ \ is\ dgt\ which\ f(x) \ge g(x)\ so$ $\int_a^\infty f(x)dx\ \ is\ dgt\ to$
2. Quotient test: Suppose $\int_a^\infty f(x)dx$ is an Improper Integral where $f(x) \ge 0\ and\ g(x) \ge 0$. If $\lim_{x\to\infty}\frac{f(x)}{g(x)} = A\ \ so\ that\ if\ A \ne 0\ or\ \infty\ then\ \int_a^\infty f(x)dx\ \ and\ \int_a^\infty g(x)dx\ \ both\ cgt\ or\ dgt\ but\ if\ A = 0$ $and\ \int_a^\infty g(x)dx\ \ is\ cgt\ so\ \int_a^\infty f(x)dx\ \ is\ cgt. Also\ if\ A = \infty\ and\ \int_a^\infty g(x)dx\ \ is\ dgt$ $so\ \int_a^\infty f(x)dx\ \ is\ dgt.$
3. Series test: $\int_a^\infty f(x)dx\ \ is\ dgt\ or\ cgt\ if\ \sum a_n\ \ is\ dgt\ or\ cgt\ where\ f(n) = a_n$

### 2) Second case

Special Improper Integrations from second kind like

$\int_a^b \frac{1}{(b-x)^p}dx\ \ and\ \int_a^b \frac{1}{(x-a)^p}dx\ \ both\ cgt\ if\ 0 < p < 1\ and\ dgt\ if\ p \ge 1\ also\ non\ improper\ if\ p \le 0$

$dgt\ or\ cgt$ test for this kind

1. Comparison test: $If\ g(x) \ge 0\ ; \forall\ a < x \le b\ and\ \int_a^b g(x)dx\ \ is\ cgt\ which\ 0 < f(x) \le g(x)$ $so\ \int_a^b f(x)dx\ \ is\ cgt\ to. but\ if\ g(x) \ge 0\ ; \forall\ a < x \le b\ and\ \int_a^b g(x)dx\ \ is\ dgt\ which\ \ \ f(x) \ge$ $g(x)\ so\ \int_a^b f(x)dx\ \ is\ dgt\ to$
2. Quotient test: Suppose $f(x) \ge 0\ and\ g(x) \ge 0\ \forall\ a < x \le b$. If $\lim_{x\to a}\frac{f(x)}{g(x)} = A\ \ so\ that\ if\ A \ne$ $0\ or\ \infty\ then\ \int_a^b f(x)dx\ \ and\ \int_a^b g(x)dx\ \ both\ cgt\ or\ dgt\ but\ if\ A = 0$ $and\ \int_a^b g(x)dx\ \ is\ cgt\ so\ \int_a^b f(x)dx\ \ is\ cgt. Also\ if\ A = \infty\ and\ \int_a^b g(x)dx\ \ is\ dgt$ $so\ \int_a^b f(x)dx\ \ is\ dgt.$

### 3) Third kind

This kind is a Combination between first and second kind and could tested by them tests.

## X. Double, Triple and liner Integral:

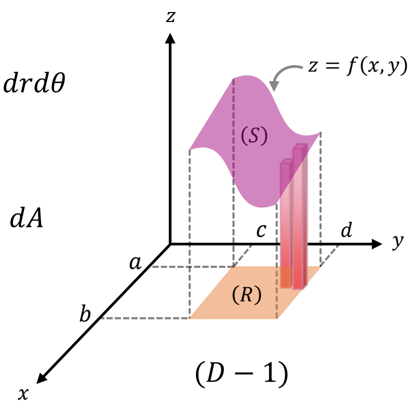
We consider a Function of two variables defined on a closed Rectangle $R = [a,b] \times [c,d] = \{(x,y) \in R\ |\ a \le x \le b\ , c \le y \le d\}$ and we first suppose that $f(x,y) \ge 0$ The graph of $(f)$ is a surface Let $(S)$ be the solid that lies above and under the graph of $(f)$ that is $S = \{f(x,y,z) \in R^3\ |0 \le z \le f(x,y), (x,y) \in R\}\ \Rightarrow Fig(D-1)$

1) Double integral equal the volume between surface $(S)$ and Panel $z = 0\ (or)\ y = 0\ (or)\ x = 0$

$$\iint_R f(x,y)\ dA = \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k,y_k)\ \Delta A_k = \int_c^d\int_a^b f(x,y)\ dxdy$$

2) To change from Cartesian to Polar coordinates we use

$$\iint_R f(x,y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr d\theta$$



(D − 1)

3) The surface $(S)$ area by double Integral

$$A(S) = \iint_D \sqrt{\{f_x(x,y)\}^2 + \{f_y(x,y)\}^2 + 1}\ dA$$

For example:

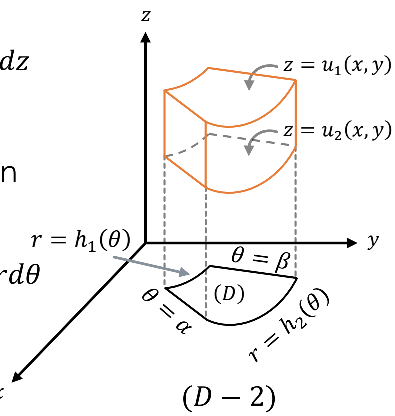
$$find \int_0^2 \int_{x^2}^{2x} (x^3 + 4y)\, dydx$$

$$\int_0^2 \left( \int_{x^2}^{2x} (x^3 + 4y)\, dy \right) dx = \int_0^2 \left( x^3 y + 2y^2 \Big|_{x^2}^{2x} \right) dx$$

$$\int_0^2 \{(2x^4 + 8x^2) - (x^5 + 2x^4)\}\, dx = \int_0^2 8x^2 + x^5\, dx = \frac{8}{3}x^3 - \frac{1}{6}x^5 \Big|_0^2 = \left(\frac{64}{3} - \frac{32}{6}\right) - 0 = 16$$

4) Triple Integral also like double Integral but for three variables and if region $(B)$ as a rectangular box $B = \{(x,y,z) | a \le x \le b, c \le y \le d, r \le z \le s\}$ then

$$\iiint_B f(x,y,z)\, dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\, dxdydz$$



(D − 2)

5) Triple Integral at cylinder coordinates

If $E$ is defined as $E = \{(x,y,z) | (x,y) \in D, u_1(x,y) \le z \le u_2(x,y)\}$ then

Triple Integral for this region $\Longrightarrow Fig(D-2)$

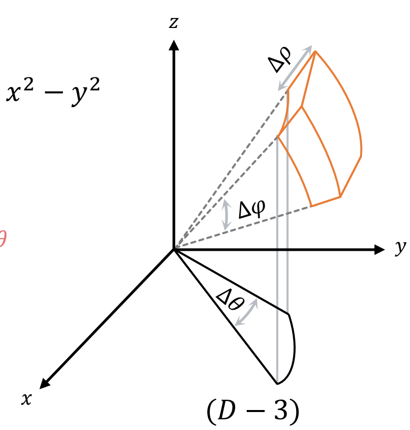
$$\iiint_E f(x,y,z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dzdrd\theta$$

6) Triple Integral at spherical coordinates

If $E$ is defined as $E = \{(\rho,\theta,\varphi) |\ a \le \rho \le b, \alpha \le \theta \le \beta, c \le \varphi \le d\}$

Then Triple Integral for this region $\Longrightarrow Fig(D-3)$

$$\iiint_E f(x,y,z)\, dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\varphi\cos\theta, \rho\sin\varphi\sin\theta, \rho\cos\varphi)\rho^2 \sin\varphi\, d\rho d\theta d\varphi$$



(D − 3)

For example:

Find the Volume of Paraboloid $z = 4 - x^2 - y^2$

$$V = \iiint_{V(x,y,z)} dxdydz = 4 \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{4-x^2-y^2} dxdydz$$

$$4 \int_0^2 \int_0^{\sqrt{4-x^2}} 4 - x^2 - y^2\, dydx = 4 \int_0^{\frac{\pi}{2}} \int_0^2 (4 - \rho^2)\, \rho\, d\rho d\theta$$

$$4 \int_0^{\frac{\pi}{2}} \left( 2\rho^2 - \frac{\rho^4}{4} \Big|_0^2 \right) d\theta = 4 \int_0^{\frac{\pi}{2}} 4 d\theta = 8\pi$$

7) Liner Integral gives the curve or line length

$\int_c p(x,y)dx + q(x,y)dy$ $(or)$ $\int_{(a_1,b_1)}^{(a_2,b_2)} p\, dx + q\, dy$

For example:

$$find \int_c (2x - y + 4)dx + (3x + 5y - 6)dy\ \ from\ (3,2)\ to\ (3,0)\ at\ counterclockwise\ direction$$

$$\int_{(3,0)}^{(3,2)} (2x - y + 4)dx + (3x + 5y - 6)dy = \int_0^2 (5y + 3)\, dy = 16$$

8) Green's theorem in the plane $\oint pdx + qdy = \iint_R \left( \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \right) dA$

## XI. Reference for some Integrations:

### A) Integrations involving $\sqrt{a^2+x^2}$ , $a>0$

1. $\int x^2\sqrt{a^2+x^2}\,dx=\frac{x}{8}(a^2+2x^2)\sqrt{a^2+x^2}-\frac{a^4}{8}ln\left(x+\sqrt{a^2+x^2}\right)+c$

2. $\int \sqrt{a^2+x^2}\,dx=\frac{x}{2}\sqrt{a^2+x^2}+\frac{a^2}{2}ln\left(x+\sqrt{a^2+x^2}\right)+c$

3. $\int \sqrt{a^2+x^2}/x\,dx=\sqrt{a^2+x^2}+a\ln\left|\left(a+\sqrt{a^2+x^2}\right)/x\right|+c$

4. $\int 1/\sqrt{a^2+x^2}\,dx=\ln\left(x+\sqrt{a^2+x^2}\right)+c$

5. $\int \sqrt{a^2+x^2}/x^2\,dx=-\sqrt{a^2+x^2}/x+a\ln\left|a+\sqrt{a^2+x^2}\right|+c$

6. $\int 1/\left(x\sqrt{a^2+x^2}\right)dx=-\frac{1}{a}\ln\left|\left(\sqrt{a^2+x^2}+a\right)/x\right|+c$

7. $\int x^2/\sqrt{a^2+x^2}\,dx=\frac{x}{2}\sqrt{a^2+x^2}-\frac{a^2}{2}ln\left(x+\sqrt{a^2+x^2}\right)+c$

8. $\int 1/\left(x^2\sqrt{a^2+x^2}\right)dx=-\left(\sqrt{a^2+x^2}\right)/(a^2x)+c$

9. $\int 1/(a^2+x^2)^{3/2}\,dx=x/a^2\sqrt{a^2+x^2}+c$

### B) Integrations involving $\sqrt{a^2-x^2}$ , $a>0$

10. $\int x^2\sqrt{a^2-x^2}\,dx=\frac{x}{8}(2x^2-a^2)\sqrt{a^2-x^2}-\frac{a^4}{8}\sin^{-1}\left(\frac{x}{a}\right)+c$

11. $\int \sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c$

12. $\int \sqrt{a^2-x^2}/x\,dx=\sqrt{a^2-x^2}-\ a\ln\left|\left(a+\sqrt{a^2+x^2}\right)/x\right|+c$

13. $\int \sqrt{a^2-x^2}/x^2\,dx=(-1/x)\sqrt{a^2+x^2}-\sin^{-1}\left(\frac{x}{a}\right)+c$

14. $\int x^2/\sqrt{a^2-x^2}\,dx=\frac{-x}{2}\sqrt{a^2-x^2}-\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c$

15. $\int 1/\left(x^2\sqrt{a^2-x^2}\right)dx=-\left(\sqrt{a^2-x^2}\right)/(a^2x)+c$

16. $\int 1/\left(x\sqrt{a^2-x^2}\right)dx=(-1/a)\ln\left|\left(\sqrt{a^2-x^2}+a\right)/x\right|+c$

17. $\int 1/(a^2-x^2)^{3/2}\,dx=x/a^2\sqrt{a^2-x^2}+c$

18. $\int (a^2-x^2)^{3/2}\,dx=\frac{-x}{8}(2x^2-5a^2)\sqrt{a^2-x^2}+\frac{3a^4}{8}\sin^{-1}\left(\frac{x}{a}\right)+c$

### C) Integrations involving $\sqrt{x^2-a^2}$ , $a>0$

19. $\int x^2\sqrt{x^2-a^2}\,dx=\frac{x}{8}(2x^2-a^2)\sqrt{x^2-a^2}-\frac{a^4}{8}ln\left(x+\sqrt{x^2-a^2}\right)+c$

20. $\int \sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}ln\left(x+\sqrt{x^2-a^2}\right)+c$

21. $\int \sqrt{x^2-a^2}/x\,dx=\sqrt{x^2-a^2}-a\cos^{-1}\frac{a}{|x|}+c$

22. $\int 1/\sqrt{x^2-a^2}\,dx=\ln\left(x+\sqrt{x^2-a^2}\right)+c$

23. $\int \sqrt{x^2-a^2}/x^2\,dx=-\sqrt{x^2-a^2}/x+a\ln\left|a+\sqrt{x^2-a^2}\right|+c$

24. $\int 1/\left(x^2\sqrt{x^2-a^2}\right)dx=\left(\sqrt{x^2-a^2}\right)/(a^2x)+c$

25. $\int x^2/\sqrt{a^2+x^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}+\frac{a^2}{2}ln\left(x+\sqrt{x^2-a^2}\right)+c$

26. $\int 1/(x^2-a^2)^{3/2}\,dx=-x/a^2\sqrt{x^2-a^2}+c$

### D) Integrations involving $\sqrt{2ax-x^2}$ , $a>1$

27. $\int x\sqrt{2ax-x^2}\,dx=\frac{2x^2-ax-3a^2}{6}\sqrt{2ax-x^2}+\frac{a^3}{2}\cos^{-1}\left(\frac{a-x}{a}\right)+c$

28. $\int \sqrt{2ax-x^2}\,dx=\frac{x-a}{2}\sqrt{2ax-x^2}+\frac{a^2}{2}\cos^{-1}\left(\frac{a-x}{a}\right)+c$

29. $\int \sqrt{2ax-x^2}/x\,dx=\sqrt{2ax-x^2}+a\cos^{-1}\left(\frac{a-x}{a}\right)+c$

30. $\int 1/\sqrt{2ax-x^2}\,dx=\cos^{-1}\left(\frac{a-x}{a}\right)+c$

31. $\int \sqrt{2ax-x^2}/x^2\,dx=-2\sqrt{a^2+x^2}/x-a\cos^{-1}\left(\frac{a-x}{a}\right)+c$

32. $\int x/\sqrt{2ax-x^2}\,dx=-\sqrt{a^2+x^2}+a\cos^{-1}\left(\frac{a-x}{a}\right)+c$

33. $\int 1/\left(x\sqrt{2ax-x^2}\right)dx=-\left(\sqrt{2ax-x^2}\right)/(ax)+c$

34. $\int x^2/\sqrt{2ax-x^2}\,dx=\frac{-(x+3a)}{2}\sqrt{2ax-x^2}+\frac{3a^2}{2}\cos^{-1}\left(\frac{a-x}{a}\right)+c$

## E) Integrations involving $a + bx$

35. $\int x^n\sqrt{a+bx}\,dx = \frac{2}{b(2n+3)}\left\{x^n(a+bx)^{3/2} - na\int x^{n-1}\sqrt{a+bx}\,dx\right\} + c$

36. $\int x^2/(a+bx)\,dx = \frac{1}{2b^3}\{(a+bx)^2 - 4a(a+bx) - 2a^2\ln(a+bx)\} + c$

37. $\int x/(a+bx)\,dx = \frac{1}{b^2}\{a+bx - a\ln(a+bx)\} + c$

38. $\int 1/x(a+bx)\,dx = \frac{1}{a}\ln\left(\frac{x}{a+bx}\right) + c$

39. $\int 1/x^2(a+bx)\,dx = \frac{-1}{ax} + \frac{b}{a^2}\ln\left(\frac{a+bx}{x}\right) + c$

40. $\int x/(a+bx)^2\,dx = \frac{a}{b^2(a+bx)} + \frac{1}{b^2}\ln|a+bx| + c$

41. $\int 1/x(a+bx)^2\,dx = \frac{1}{a(a+bx)} - \frac{1}{a^2}\ln\{(a+bx)/x\} + c$

42. $\int x^2/(a+bx)^2\,dx = \frac{1}{b^3}\left\{a+bx - \frac{a^2}{a+bx} - 2a\ln(a+bx)\right\} + c$

43. $\int x\sqrt{a+bx}\;dx = \frac{2}{15b^3}(3bx-2a)(a+bx)^{3/2} + c$

44. $\int x^2/\sqrt{a+bx}\;dx = \frac{2}{15b^3}(8a^2+3b^2x^2-4abx)\sqrt{a+bx} + c$

45. $\int x/\sqrt{a+bx}\;dx = \frac{2}{3b^2}(bx-2a)\sqrt{a+bx} + c$

46. $\int 1/x\sqrt{a+bx}\,dx = \frac{2}{\sqrt{-a}}\tan^{-1}\sqrt{\frac{a+bx}{-a}} + c \text{ at } a<0$

47. $\int 1/x\sqrt{a+bx}\,dx = \frac{2}{\sqrt{a}}\ln\left|\frac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}\right| + c \text{ at } a>0$

48. $\int \sqrt{a+bx}/x\,dx = 2\sqrt{a+bx} + a\int 1/x\sqrt{a+bx}\,dx + c$

49. $\int \sqrt{a+bx}/x^2\,dx = \frac{-\sqrt{a+bx}}{x} + \frac{b}{2}\int 1/x\sqrt{a+bx}\,dx$

50. $\int 1/x^n\sqrt{a+bx}\;dx = \frac{\sqrt{a+bx}}{(an-1)x^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int\frac{1}{x^{n-1}\sqrt{a+bx}}dx + c$

51. $\int x^n/\sqrt{a+bx}\;dx = \frac{2x^n\sqrt{a+bx}}{b(2n+1)} - \frac{2na}{b(2n+1)}\int\frac{x^{n-1}}{\sqrt{a+bx}}dx + c$

## F) Trigonometric Forms

52. $\int sin^2 x\,dx = \frac{1}{2}x - \frac{1}{4}sin\,2x + c$

53. $\int cos^2 x\,dx = \frac{1}{2}x + \frac{1}{4}sin\,2x + c$

54. $\int tan^2 x\,dx = tan\,x - x + c$

55. $\int cot^2 x\,dx = -cot\,x - x + c$

56. $\int sin^3 x\,dx = \frac{-1}{3}(2+sin^2 x)\,cos\,x + c$

57. $\int cos^3 x\,dx = \frac{1}{3}(2+cos^2 x)\,sin\,x + c$

58. $\int tan^3 x\,dx = \frac{1}{2}tan^2 x + ln|cos\,x| + c$

59. $\int cot^3 x\,dx = -\frac{1}{2}cot^2 x - ln|sin\,x| + c$

60. $\int sec^3\theta\,d\theta = \frac{1}{2}sec\,x\,tan\,x + \frac{1}{2}ln|sec\,x + tan\,x| + c$

61. $\int csc^3\theta\,d\theta = -\frac{1}{2}csc\,x\,cot\,x + \frac{1}{2}ln|csc\,x - cot\,x| + c$

62. $\int x^n\,sin\,x\,dx = -x^n\,cos\,x + n\int x^{n-1}\,cos\,x\,dx + c$

63. $\int x^n\,cos\,x\,dx = x^n\,sin\,x - n\int x^{n-1}\,sin\,x\,dx + c$

64. $\int sin\,ax\,sin\,bx\,dx = \frac{sin(a-b)x}{2(a-b)} - \frac{sin(a+b)x}{2(a+b)} + c$

65. $\int cos\,ax\,cos\,bx\,dx = \frac{sin(a-b)x}{2(a-b)} + \frac{sin(a+b)x}{2(a+b)} + c$

66. $\int sin\,ax\,cos\,bx\,dx = -\frac{cos(a-b)x}{2(a-b)} - \frac{cos(a+b)x}{2(a+b)} + c$

## G) Invers Trigonometric Forms

67. $\int \sin^{-1}x\,dx = x\sin^{-1}x + \sqrt{1-x^2} + c$

68. $\int \cos^{-1}x\,dx = x\cos^{-1}x - \sqrt{1-x^2} + c$

69. $\int \tan^{-1}x\,dx = x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + c$

70. $\int x^n\sin^{-1}x\,dx = \frac{1}{n+1}\left\{x^{n+1}\sin^{-1}x - \int\frac{x^{n+1}}{\sqrt{1-x^2}}dx\right\} + c$

71. $\int x^n\cos^{-1}x\,dx = \frac{1}{n+1}\left\{x^{n+1}\cos^{-1}x + \int\frac{x^{n+1}}{\sqrt{1-x^2}}dx\right\} + c$

72. $\int x^n\tan^{-1}x\,dx = \frac{1}{n+1}\left\{x^{n+1}\tan^{-1}x - \int\frac{x^{n+1}}{1+x^2}dx\right\} + c$

### H) Exponential and Logarithmic Forms

73. $\int xe^{ax}dx = \frac{1}{a^2}(ax-1)e^{ax} + c$

74. $\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}\{a\cos bx + b\sin bx\} + c$

75. $\int x^n e^{ax}dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}dx + c$

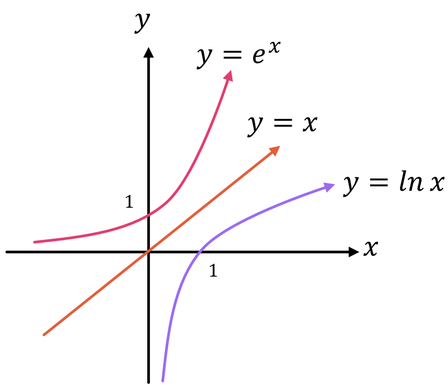
76. $\int \ln x\, dx = x\ln x - x + c$

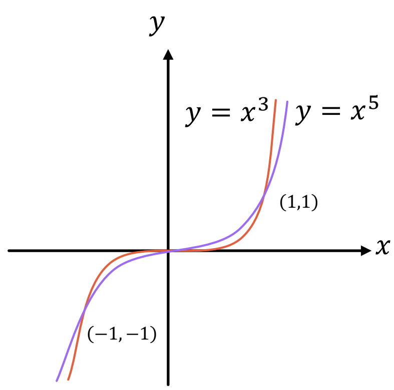
77. $\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}\{a\sin bx - b\cos bx\} + c$

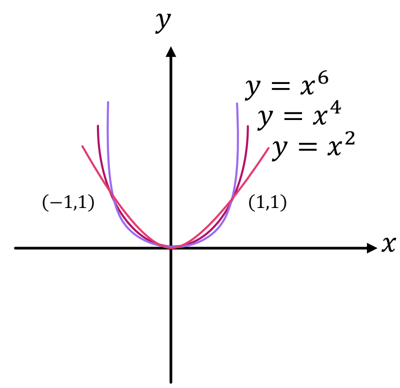
78. $\int x^n \ln x\, dx = \frac{x^{n+1}}{(n+1)^2}\{(n+1)\ln x - 1\} + c$

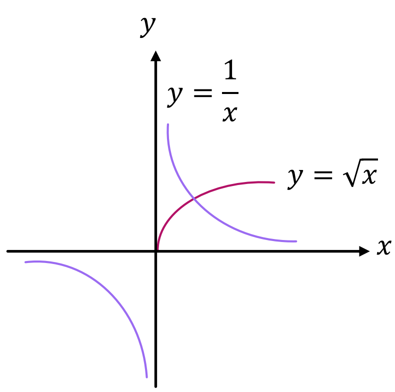
79. $\int \frac{1}{x\ln x}dx = \ln\ln x + c$

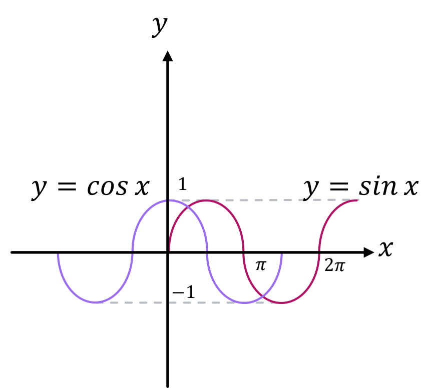
## XII. Some Function's Curves:

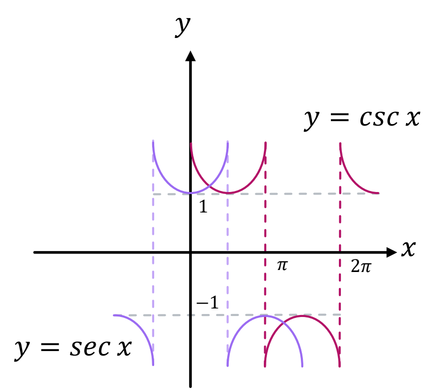


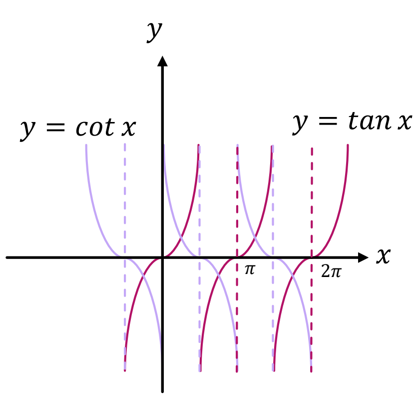



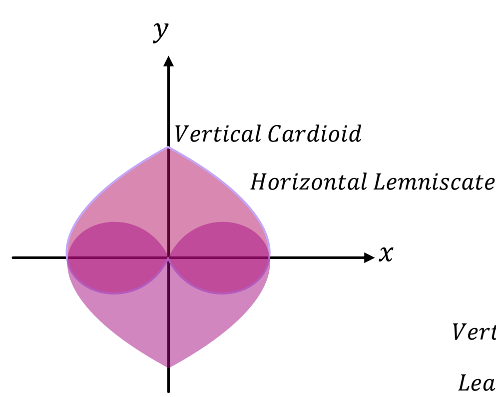



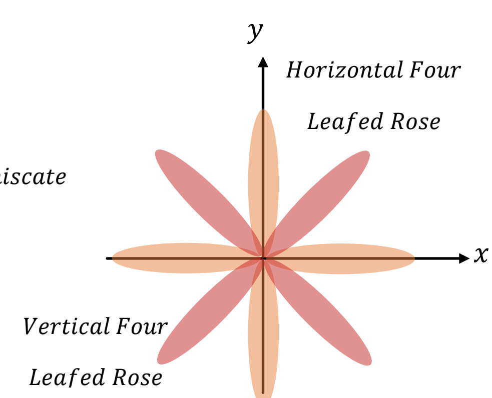



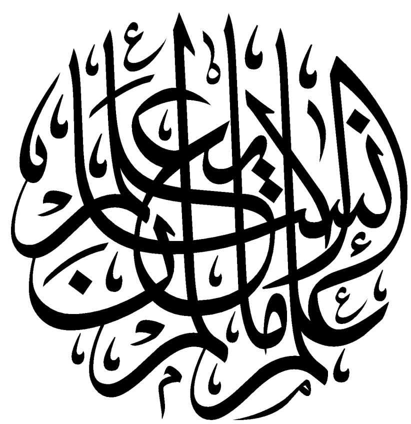

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