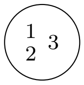
Discrete Mathematics Cheat Sheet

Set Theory

Definitions

Set Definition: A *set* is a collection of objects called *elements*

Visual Representation: 

List Notation: $\{1, 2, 3\}$

Characteristics

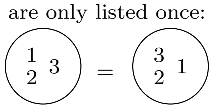
Sets can be finite or infinite.

Finite: $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Infinite: $\mathbb{Z}^+ = \{1, 2, 3, 4, \ldots\}$

Dots represent an implied pattern that continues infinitely

Repeated Elements are only listed once: $\{a, b, a, c, b, a\} = \{a, b, c\}$

Sets are **Unordered:** 

$\{3, 2, 1\} = \{1, 2, 3\} = \{2, 1, 3\}$

Common Sets

Natural Numbers: $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$

Positive Integers: $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$

Integers: $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

Rational Numbers: $\mathbb{Q} = \{\ldots, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \ldots\}$

Elements and Cardinality

Elements are the things contained in the set.
Let $C = \{Yellow, Blue, Red\}$

$Yellow \in C$: **Yellow** is an element of C.

$Green \notin C$: **Green** is not an element of C.

Cardinality refers to the number of elements in the set.

$|C| = 3$: The **Cardinality** (size) of C is 3.

The Empty Set

$\emptyset = \{\}$: The empty set is a set with no elements.

$|\emptyset| = |\{\}| = 0$: The cardinality of the empty set is 0.

$\{\emptyset\} \neq \emptyset$:

$\{\emptyset\} = \{\{\}\}$

$|\{\emptyset\}| = 1$: The set contains the empty set.

Set Builder Notation

Elements in the list are defined as variables.

$X = \{expression \mid rule\}$

If $Desk = \{drink, laptop, microphone\}$

Set Builder Notation defines the set as:
$Desk = \{x \mid x\, is\, on\, the\, desk\}$

Let $E = \{2n \mid n \in \mathbb{Z}\}$

Reads as: The set of all things with form $2n$ such that n is an element of $\mathbb{Z}$

$2n$ is an expression that defines the **form** of the elements.

$n \in \mathbb{Z}$ defines a rule for elements appearing in the set.

$\mid$ is read "such that" and separates the expression from the rule.

Examples:

$E = \{2n \mid n \in \mathbb{Z}\}$: A set containing even integers

$\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0\}$: A set containing rational numbers (m and n are integers and n is not zero)

Ordered Pairs

An **Ordered Pair** is any *list* of things enclosed in parentheses: (x, y).

$(1, 2) \neq (2, 1)$

Cartesian Products

AKA **Cross Product**
Given 2 sets, A and B, a Cartesian Product is denoted by $A \times B$.
The Cartesian Product is a *set* of Ordered Pairs where the first element comes from A and the second element comes from B.

$A \times B = \{(a, b) \mid a \in A, b \in B\}$

Let $X = \{0, 1, 2\}$ and $Y = \{0, 1\}$

$X \times Y = \{(0, 0), (0, 1), (1, 0), (1, 1), (2, 0), (2, 1)\}$
$Y \times X = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2)\}$

3-tuple: $A \times B \times C = \{(a, b, c) \mid a \in A, b \in B, c \in C\}$
n-tuple:
$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}$
$\emptyset \times A = \emptyset$

Cartesian Product Cardinality

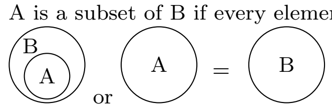
The cardinality of a cross product is the product of the cardinalities of each set.
If $|A| = m$ and $|B| = n$, then $|A \times B| = m \times n$
Let $|X| = 3$ and $|Y| = 2$.

$|X \times Y| = 3 \times 2 = 6$

$|\emptyset| \times |A| = 0$

Subsets

A is a subset of B if every element in A is also in B.

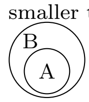


$A \subseteq B$: A is a subset of B.

$\{a, b\} \subseteq \{a, b, c\}$
$\{c, d\} \subseteq \{c, d\}$
$\{a\} \not\subseteq \{\{a\}\}$: The element a is not an element of the second set.
$\emptyset \subseteq \{x, y, z\}$: The empty set is a subset of every set.

Proper Subsets

A is a **proper** subset of B if every element in A is also in B *and* A is smaller than B



$A \subset B$: A is a *proper* subset of B.

$\{a, b\} \subset \{a, b, c\}$
$\{c, d\} \not\subset \{c, d\}$
$\emptyset \subset \{x, y, z\}$

Power Sets

A Power Set of a set A is the set containing *all possible* subsets of A.
Let $A = \{a, b\}$

$\mathbb{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

$\mathbb{P}(\emptyset) = \emptyset$

Power Sets Cardinality

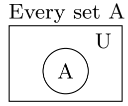
If $|A| = n$, $then |\mathbb{P}(A)| = 2^n$
Let $|A| = 2$

$|\mathbb{P}(A)| = 2^2 = 4$

Set Operations

Universes

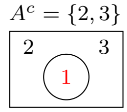
Every set A exists within some universe U.



Complement

The complement of a set A is everything outside of A that is in the Universe.
A^c (or $\bar{A}$) $= \{a \in U\, and\, a \notin A\}$
Let $A = \{1\}$ and $U = \{1, 2, 3\}$.

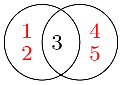
$A^c = \{2, 3\}$



Intersection

The intersection of sets A and B is every element that occurs in both A **and** B.
$A \cap B = \{x \mid x \in A \, and \, x \in B\}$
Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$

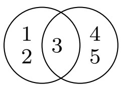
$A \cap B = \{3\}$



Union

The union of sets A and B is every element that occurs in either A **or** B.
$A \cup B = \{x \mid x \in A \, or \, x \in B\}$
Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$

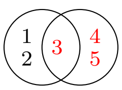
$A \cup B = \{1, 2, 3, 4, 5\}$



Difference

The difference between two sets, A-B, is every element from A **minus** any element that appears in B.
$A \setminus B = \{x \mid x \in A \, and \, x \notin B\}$
Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$

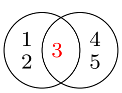
$A \setminus B = \{1, 2\}$



Symmetric Difference

The symmetric difference of sets A and B is every element that is **exclusively** in A **or** B (i.e. every element from A or B that is not in both).
$A \oplus B = \{x \mid x \in A \, xor \, x \in B\}$
Let $A = \{1, 2, 3\}$ and $B = \{3, 4, 5\}$

$A \oplus B = \{1, 2, 4, 5\}$



Indexed Sets

Indexed Set Notation is used to shorten long strings of intersections and unions.
$\bigcap_{i=1}^{n} A_i = A_1 \cap A_1 \cap A_2 \cap \cdots \cap A_n$
$\bigcup_{i=1}^{n} A_i = A_1 \cup A_1 \cup A_2 \cup \cdots \cup A_n$

Well Ordering Principle

An Axiom: Any non-empty subset of the natural numbers ($\mathbb{N}$) has a *least* element.
$\mathbb{N} = \{1, 2, 3, ..., \infty\}$
Let $A = \{1, 4, 9\} (Note that A \subset \mathbb{N})$

1 is the least element.

$B = \{i, j, k \mid i, j, k \in \mathbb{N}\}$

i, j, or k will be the least element.

When extended to $\mathbb{Z}$, the axiom does not hold since $\mathbb{Z}$ contains $-\infty$.

Logic

Definitions

A **statement** is a declarative sentence that is either true (1) or false (0).
Examples:

Milk is white.

$|\emptyset| = 0$

Humans are just fish with legs.

A **proposition** represents the idea behind a statement.
A single proposition can be expressed by multiple statements.
Example:

The statements "It is cloudy." and "It is not sunny." both capture the same proposition.

Notation:

Capital letters (P, Q, R, etc.) are used to represent a specific proposition.

Lowercase letters (p, q, r, etc.) are used for general proofs and do not represent a specific proposition.

A **well-formed formula** (WFF) is an expression involving propositions and compound propositions that conform to the syntax of propositional logic.

Example:

The statements "It is cloudy." and "It is not sunny." both capture the same proposition.

Connectives and Truth Tables

All **connectives** take a truth value and output a new truth value.

A **truth table** shows all possible combinations of truth conditions.

A proposition, P, can either be true (1) or false (0).

P
1
0

Negation ($\neg$)

$\neg$P is read as "Not P" and negates the truth value of P.

If P is "It is raining", then $\neg$P is "It is not raining."

P	$\neg P$
1	0
0	1

Mathematically: $\neg P = 1 - P$

Conjunction ($\wedge$)

$P \wedge Q$ is read as "P and Q".

If P is "It is raining" and Q is "It is cloudy", then $P \wedge Q$ is only true if if is raining and it is cloudy.

P	Q	$P \wedge Q$
1	1	1
1	0	0
0	1	0
0	0	0

Mathematically: $P \wedge Q = min(P, Q)$

Disjunction ($\vee$)

$P \vee Q$ (or $P + Q$) is read as "P or Q".

If P is "It is raining" and Q is "It is cloudy", then $P \vee Q$ is true if it is raining or it is cloudy.

P	Q	$P \vee Q$
1	1	1
1	0	1
0	1	1
0	0	0

Mathematically: $P \vee Q = max(P, Q)$

Conditional ($\Rightarrow$)

$P \Rightarrow Q$ is read as "If P, then Q".

If P is "It is sunny" and Q is "I'm wearing sunscreen", then $P \Rightarrow Q$ means "If it is sunny then I'm wearing sunscreen."

Ask the question: When am I lying about wearing sunscreen?

If it is sunny (P = 1) and I'm not wearing sunscreen (Q = 0), then I have lied.

P	Q	$P \Rightarrow Q$
1	1	1
1	0	0
0	1	1
0	0	1

Grammatically:

If P, then Q.
Whenever P, then also Q.
For P, it is necessary that Q.
P is a sufficient condition for Q.
Q if/whenever P.
Q, provided that P.
For Q, it is sufficient that P.
Q is a necessary condition for P.
P only if Q.

Note that $P \Rightarrow Q \neq Q \Rightarrow P$

P	Q	$Q \Rightarrow P$
1	1	1
1	0	1
0	1	0
0	0	1

Mathematically: $P \Rightarrow Q \ iff \ P \leq Q$

Biconditional ($\iff$)

$P \iff Q$ is read as "P if and only if Q".

If P is "a is even" and Q is "a is divisible by 2", then $P \iff Q$ is true if both P and Q are true or if Q and P are false.

Equivalent to $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$

P	Q	$P \iff Q$
1	1	1
1	0	0
0	1	0
0	0	1

Mathematically: $P \iff Q \ iff \ P = Q$

Sheffer Stroke ($\uparrow$)

$P \uparrow Q$ is read as "P nand Q".

Equivalent to $\neg(P \wedge Q)$

P	Q	$P \uparrow Q$
1	1	0
1	0	1
0	1	1
0	0	1

Logic Laws

Logical equivalences can be used to reduce complex formulas into simpler ones.

$\top$: A **Tautology** is always true:
$p \vee \neg p$
$\bot$: A **Contradiction** is always false:
$p \wedge \neg p$

$\top$	$\bot$
1	0
1	0
.	.
.	.
1	0

Identity Law

The identity of the proposition remains.

$p \wedge \top = p$
$p \vee \bot = p$

p	$\top$	$\bot$	$p \wedge \top$	$p \vee \bot$
1	1	0	1	1
1	1	0	1	1
0	1	0	0	0
0	1	0	0	0

Domination Law

The $\top$ or $\bot$ dominates the proposition.

$p \vee \top = \top$
$p \wedge \bot = \bot$

P	$\top$	$\bot$	$p \vee \top$	$p \wedge \bot$
1	1	0	1	0
1	1	0	1	0
0	1	0	1	0
0	1	0	1	0

Double Negation Law

$\neg\neg p = p$

p	$\neg p$	$\neg\neg p$
1	0	1
1	0	1
0	1	0
0	1	0

DeMorgan's Law

$\neg(p \wedge q) = \neg p \vee \neg q$

$\neg(p \vee q) = \neg p \wedge \neg q$

Distribute the negation ($\neg$) and flip the connective ($\wedge$ or $\vee$)

p	q	$\neg p$	$\neg q$	$(p \wedge q)$	$\neg(p \wedge q)$	$(\neg p \vee \neg q)$
1	1	0	0	1	0	0
1	0	0	1	0	1	1
0	1	1	0	0	1	1
0	0	1	1	0	1	1

Distributive Law

$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

When the $\wedge$ is outside the parentheses and the $\vee$ is inside, or vice versa.

Absorption Law

$p \wedge (p \vee r) = p$

$p \vee (p \wedge r) = p$

When the connectives are flipped and the p is in both.

Commutative Law

$p \wedge q = q \wedge p$

$p \vee q = q \vee p$

Associative Law

$p \wedge (q \wedge r) = (p \wedge q) \wedge r$

$p \vee (q \vee r) = (p \vee q) \vee r$

Order can be changed when the connectives are the same (when the connectives are different, the Distributive law applies)

Inverse Law

$p \wedge \neg p = \bot$

$p \vee \neg p - \top$

The inverses result in a contradiction or tautology.

Conditional Law

$p \Rightarrow q = \neg p \vee q$

The inverses result in a contradiction or tautology.

p	$\neg p$	q	$p \Rightarrow q$	$\neg p \vee q$
1	0	1	1	1
1	0	0	0	0
0	1	1	1	1
0	1	0	1	1

Converse, Inverse, and Contrapositive

There are three terms related to conditionals ().
Converse:

If $p \Rightarrow q$, then the converse is $q \Rightarrow p$

Reverse the order of the propositions.

Inverse:

If $p \Rightarrow q$, then the inverse is $\neg p \Rightarrow \neg q$

Negate each proposition.

Contrapositive:

If $p \Rightarrow q$, then the inverse is $\neg a \Rightarrow \neg p$

Reverse the order and negate each proposition.

Take the contrapositive and the inverse.

Converse, Inverse, and Contrapositive Logical Equivalence

The conditional is logically equivalent to the contrapositive

$p \Rightarrow q$	$\neg q \Rightarrow \neg p$	*Steps*
$\neg p \vee q$	$\neg\neg q \vee \neg p$	*Conditional Law*
	$q \vee \neg p$	*Double Negative*
	$\neg p \vee q$	*Associative Law*
$\neg p \vee q$	$\neg p \vee q$	*qed*

The converse is logically equivalent to the inverse.

$q \Rightarrow p$	$\neg p \Rightarrow \neg q$	*Steps*
$\neg q \vee p$	$\neg\neg p \vee \neg q$	*Conditional Law*
	$p \vee \neg q$	*Double Negative*
	$\neg q \vee$	*Associative Law*
$\neg q \vee p$	$\neg q \vee p$	*qed*

Rules of Inference

The primary method of proofs in philosophical logic.

Definitions

A set of **premises** $(p_1, p_2, \ldots, p_n)$ prove some **conclusion** (q) in an **argument**:

$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \Rightarrow q$

The argument is **valid** if whenever each premise is true, the conclusion is also true.
Example:

Let R = "It is raining." and W = "I will get wet."

Premise 1: If it is raining, I will get wet $(R \Rightarrow W)$.

Premise 2: It is raining (R).

Conclusion: I will get wet (W).

Step	*Grammatically*	*Logically*
Premise 1	*If it is raining, I will get wet.*	$R \Rightarrow W$
Premise 2	*It is raining.*	R
Conclusion	*I will get wet.*	W

The truth table of a valid argument is a tautology.

R	W	$R \Rightarrow W$	$((R \Rightarrow W) \wedge R)$	$((R \Rightarrow W) \wedge R) \Rightarrow W)$
P_2	Q	P_1	$P_1 \wedge P_2$	$(P_1 \wedge P_2) \Rightarrow Q$
1	1	1	1	1
1	0	0	0	1
0	1	1	1	1
0	0	1	0	1

Modus Ponens (MPP)

Affirming the antecedent.

$p \Rightarrow q$
p
$\therefore q$

Modus Tollens (MTT)

Denying the consequent.

$p \Rightarrow q$
$\neg q$
$\therefore \neg p$

Since $p \Rightarrow q$ is equivalent to $\neg q \Rightarrow \neg p$, MTT is equivalent to MPP on the contrapositive.

$\neg q \Rightarrow \neg p$
$\neg q$
$\therefore \neg p$

Hypothetical Syllogism (HS)

Transitivity.

$p \Rightarrow q$
$q \Rightarrow r$
$\therefore p \Rightarrow r$

Disjuntive Syllogism (DS)

$p \vee q$
$\neg q$
$\therefore p$

Addition

Or Induction.

p
$\therefore p \vee q$

Simplification

And Elimination.

$p \wedge q$
$\therefore p$
$\therefore q$

Conjunction

And Introduction.

p
q
$\therefore p \wedge q$

Predicate Logic

Predicate logic uses variables and allows forms that are not statements.
The truth value of predicates depends on the value of variable **terms**.
$G(x, y)$ x is greater than y.

$G(x, y)$ is an **open statement** since it does not have a truth value.

$G(2, 1)$ is a closed statement and is true since 2 is greater than 1.

$G(3, 6$ is a closed statement and is false.

Quantifiers

$\forall_x$: Universal Quantifier

For all x, x is P.

$\forall_x P(x) = (P(1) \wedge P(2) \wedge \cdots \wedge P(n))$

$\exists_x$: Existential Quantifier

There exists some x such that x is P.

$\exists_x P(x) = (P(1) \vee P(2) \vee \cdots \vee P(n))$

Sentences

For every real number n, there exists a real number m such that $m^2 = n$.

$\forall_x \in \mathbb{R} \quad \exists_m \in \mathbb{R} \mid m^2 = n$

Given two rationals x and y, $\sqrt{xy}$ will be rational.

$\forall_x \in \mathbb{Q} \quad \forall_y \in Q \quad \sqrt{xy} \in \mathbb{Q}$

Negating Quantifiers

$\forall_x P(x) = \neg\exists_x \neg[P(x)]$

$\exists_x P(x) = \neg\forall_x \neg[P(x)]$

$\neg\forall_x P(x) = \exists_x \neg[p(x)]$

$\neg\exists_x P(x) = \forall_x \neg[P(x)]$

Equivalence trick:

$\neg\forall_x P(X)$

$-\forall_x + P(x)$

$+\exists_x - P(x)$

$\exists_x \neg P(X)$

Counting

To do

Proofs

To do

Relations and Functions

To do

Number Theory

To do