Fisher Information

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Introduction

In this article, some properties of Fisher Information will be provided and proved.

1 Definition

Let $X_1, X_2 ... X_n \overset{i.i.d.}{\sim} f(x|\theta)$ and $\boldsymbol{X} = (X_1, X_2 ... X_n)$
Then the Fisher Information for $\boldsymbol{X}$ is defined as

$$\boldsymbol{I}_n(\theta) = E[(\frac{\partial}{\partial\theta} ln f_\theta(\boldsymbol{X}|\theta))^2] \tag{1}$$

2 Properties

2.1 $\boldsymbol{I}_n(\theta) = Var[\frac{\partial}{\partial\theta} ln f_\theta(\boldsymbol{X}|\theta)]$

Proof

Let $Y = \frac{\partial}{\partial\theta} ln f_\theta(\boldsymbol{X}|\theta)$

$$\begin{aligned} E(Y) &= \int_{\mathbb{R}} \frac{\partial}{\partial\theta} ln f_\theta(\boldsymbol{X}|\theta)\ f_\theta(\boldsymbol{X}|\theta)\ dx \\ &= \int_{\mathbb{R}} \frac{\frac{\partial}{\partial\theta} f_\theta(\boldsymbol{X}|\theta)}{f_\theta(\boldsymbol{X}|\theta)}\ f_\theta(\boldsymbol{X}|\theta)\ dx \\ &= \int_{\mathbb{R}} \frac{\partial}{\partial\theta} f_\theta(\boldsymbol{X}|\theta)\ dx \\ &= \frac{\partial}{\partial\theta} \int_{\mathbb{R}} f_\theta(\boldsymbol{X}|\theta)\ dx \\ &= \frac{\partial}{\partial\theta}[1] = 0 \end{aligned}$$

$$\therefore \boldsymbol{I}_n(\theta) = E[Y^2] = Var[Y] + [E(Y)]^2 = Var[Y] = Var[\frac{\partial}{\partial\theta} ln f_\theta(\boldsymbol{X}|\theta)]$$

2.2 $\boldsymbol{I}_n(\theta) = -E[\frac{\partial^2}{\partial\theta^2} ln f_\theta(\boldsymbol{X}|\theta)]$

Proof

$$\begin{aligned}
\frac{\partial^2}{\partial\theta^2} ln f_\theta(\boldsymbol{X}|\theta) &= \frac{\partial}{\partial\theta}[\frac{\partial}{\partial\theta} ln f_\theta(\boldsymbol{X}|\theta)] \\
&= \frac{\partial}{\partial\theta}[\frac{\frac{\partial}{\partial\theta} f_\theta(\boldsymbol{X}|\theta)}{f_\theta(\boldsymbol{X}|\theta)}] \\
&= \frac{[\frac{\partial^2}{\partial\theta^2} f_\theta(\boldsymbol{X}|\theta)]\ f_\theta(\boldsymbol{X}|\theta) - [\frac{\partial}{\partial\theta} f_\theta(\boldsymbol{X}|\theta)]^2}{[f_\theta(\boldsymbol{X}|\theta)]^2} \\
&= [\frac{\frac{\partial^2}{\partial\theta^2} f_\theta(\boldsymbol{X}|\theta)}{f_\theta(\boldsymbol{X}|\theta)}] - [\frac{\frac{\partial}{\partial\theta} f_\theta(\boldsymbol{X}|\theta)}{f_\theta(\boldsymbol{X}|\theta)}]^2
\end{aligned}$$

$$\begin{aligned}
\therefore E[\frac{\partial^2}{\partial\theta^2} ln f_\theta(\boldsymbol{X}|\theta)] &= E[\frac{\frac{\partial^2}{\partial\theta^2} f_\theta(\boldsymbol{X}|\theta)}{f_\theta(\boldsymbol{X}|\theta)}] - E[\frac{\frac{\partial}{\partial\theta} f_\theta(\boldsymbol{X}|\theta)}{f_\theta(\boldsymbol{X}|\theta)}]^2 \\
&= E[\frac{\frac{\partial^2}{\partial\theta^2} f_\theta(\boldsymbol{X}|\theta)}{f_\theta(\boldsymbol{X}|\theta)}] - E[(\frac{\partial}{\partial\theta} ln f_\theta(\boldsymbol{X}|\theta))^2] \qquad (2)
\end{aligned}$$

Follow up, we will have to show that the first part in equation (2) equals zero

$$\begin{aligned}
E[\frac{\frac{\partial^2}{\partial\theta^2} f_\theta(\boldsymbol{X}|\theta)}{f_\theta(\boldsymbol{X}|\theta)}] &= \int_{\mathbb{R}} \frac{\frac{\partial^2}{\partial\theta^2} f_\theta(\boldsymbol{X}|\theta)}{f_\theta(\boldsymbol{X}|\theta)}\ f_\theta(\boldsymbol{X}|\theta)\ dx \\
&= \frac{\partial^2}{\partial\theta^2} \int_{\mathbb{R}} f_\theta(\boldsymbol{X}|\theta)\ dx \\
&= \frac{\partial^2}{\partial\theta^2}[1] = 0
\end{aligned}$$

$\therefore \boldsymbol{I}_n(\theta) = -E[\frac{\partial^2}{\partial\theta^2} ln f_\theta(\boldsymbol{X}|\theta)]$

2.3 $\boldsymbol{I}_n(\theta) = n\boldsymbol{I}(\theta)$

Let $\boldsymbol{I}(\theta) = E[(\frac{\partial}{\partial\theta} ln f_\theta(\boldsymbol{X}_1|\theta))^2] = Var[\frac{\partial}{\partial\theta} ln f_\theta(\boldsymbol{X}_1|\theta)]$

Proof

$$\begin{aligned}
f_\theta(\boldsymbol{X}|\theta) &= E[(\frac{\partial}{\partial\theta} ln f_\theta(\boldsymbol{X}|\theta))^2] \\
&= Var[\frac{\partial}{\partial\theta} ln f_\theta(\boldsymbol{X}|\theta)] \\
&= Var[\frac{\partial}{\partial\theta} \sum_{i=1}^{n} ln f_\theta(X_i|\theta)] \\
&= \sum_{i=1}^{n} Var[\frac{\partial}{\partial\theta} ln f_\theta(X_i|\theta)] \\
&= nVar[\frac{\partial}{\partial\theta} ln f_\theta(X_i|\theta)] \\
&= n\boldsymbol{I}(\theta)
\end{aligned}$$