Proof*. 1. Choose $m = \left[\frac{n}{2}\right] + 2$ and consider the Banach space $X = H^m(U) \cap H_0^1(U)$. According to Theorem 3 in §5.3.2 $X \subset C^2(\bar{U})$. We define the linear, compact operator $A : X \to X$ by setting $Af = u$, where u is the unique solution of

$$\text{(23)} \qquad \begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$$

Next define the cone

$$C = \{u \in X \mid u \geq 0 \text{ in } U\}.$$

According to the maximum principle, $A : C \to C$.

2. Hereafter fix any function $w \in C$, $w \not\equiv 0$. Employing the strong maximum principle and Hopf's Lemma, we deduce

$$\text{(24)} \qquad v > 0 \text{ in } U, \quad \frac{\partial v}{\partial \nu} < 0 \text{ on } \partial U$$

for $v = A(w)$.

Remember that $w = 0$ on ∂U. So in view of (24) there exists a constant $\mu > 0$ so that

$$\text{(25)} \qquad \mu v \geq w \quad \text{in } U.$$

3. Fix $\epsilon > 0$, $\eta > 0$, and consider then the equation

$$\text{(26)} \qquad u = \eta A[u + \epsilon w]$$

for the unknown $u \in C$. We claim that

(27) if (26) has a solution u, then $\eta \leq \mu$.

To verify this assertion, suppose in fact $u \in C$ solves (26). We compute

$$u \geq \eta A[\epsilon w] = \eta \epsilon v \geq \frac{\eta}{\mu} \epsilon w,$$

according to (25). Hence

$$u \geq \eta A u \geq \frac{\eta^2 \epsilon}{\mu} A w = \frac{\eta^2 \epsilon}{\mu} v \geq \left(\frac{\eta}{\mu}\right)^2 \epsilon w.$$

*Omit on first reading.

Continuing, we deduce

$$u \geq \left(\frac{\eta}{\mu}\right)^k \epsilon w \quad (k = 1, \dots),$$

a contradiction unless $\eta \leq \mu$. This observation confirms the assertion (27).

4. Define

$$S_\epsilon := \{u \in C \mid \text{ there exists } 0 \leq \eta \leq 2\mu \text{ such that } u = \eta A[u + \epsilon w]\}.$$

We next assert

$$\text{(28)} \qquad S_\epsilon \text{ is unbounded in } X.$$

For otherwise we could apply Schaefer's fixed point theorem (to be proved later, as Theorem 4 in §9.2.2), to deduce that the equation

$$u = 2\mu A[u + \epsilon w]$$

has a solution, in contradiction to (27).

5. Owing to (28), there exist

$$\text{(29)} \qquad 0 \leq \eta_\epsilon \leq 2\mu$$

and $v_\epsilon \in C$, with $\|v_\epsilon\|_X \geq 1$, satisfying

$$\text{(30)} \qquad v_\epsilon = \eta_\epsilon A[v_\epsilon + \epsilon w].$$

Renormalize by setting

$$\text{(31)} \qquad u_\epsilon := \frac{v_\epsilon}{\|v_\epsilon\|_X}.$$

Using (29)–(31) and the compactness of the operator A, we obtain a subsequence $\epsilon_k \to 0$ so that

$$\eta_{\epsilon_k} \to \eta \text{ and } u_{\epsilon_k} \to u \text{ in } X.$$

Then (31) implies

$$\text{(32)} \qquad \|u\|_X = 1, \ u \in C.$$

Since $u_\epsilon = \eta_\epsilon A\left[u_\epsilon + \frac{\epsilon w}{\|v_\epsilon\|_X}\right]$, we deduce in the limit that $u = \eta A u$. In view of (32), $\eta > 0$. We may consequently rewrite the above to read

$$\begin{cases} Lw_1 = \lambda_1 w_1 & \text{in } U \\ w_1 = 0 & \text{on } \partial U, \end{cases}$$

for $\lambda_1 = \eta$, $u = w_1$. Thus λ_1 is a real eigenvalue for the operator L, taken with zero boundary conditions, and $w_1 \geq 0$ is a corresponding eigenfunction. In view of the strong maximum principle and Hopf's Lemma, we have

$$w_1 > 0 \text{ in } U, \; \frac{\partial w_1}{\partial \nu} < 0 \text{ on } \partial U. \tag{33}$$

Additionally, we know w_1 is smooth, owing to the regularity theory in §6.3.

6. All expressions occurring in steps 1–5 above are real. Suppose now $\lambda \in \mathbb{C}$ and u is a complex-valued solution of

$$\begin{cases} Lu = \lambda u & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases} \tag{34}$$

Now choose any smooth function $w : U \to \mathbb{R}$, with $w > 0$ in U, and set $v := \frac{u}{w}$. We compute

$$\begin{aligned} \lambda v &= \frac{1}{w} L(vw) \quad \text{by (34)} \\ &= Lv - cv - \frac{2}{w} \sum_{i,j=1}^{n} a^{ij} w_{x_j} v_{x_i} + \frac{v}{w} Lw. \end{aligned} \tag{35}$$

Writing

$$Kv := -\sum_{i,j=1}^{n} a^{ij} v_{x_i x_j} + \sum_{i=1}^{n} b_i' v_{x_i}$$

for $b_i' := b^i - \frac{2}{w} \sum_{j=1}^{n} a^{ij} w_{x_j}$ $(1 \leq i \leq n)$, we deduce from (35) that

$$Kv + \left(\frac{Lw}{w} - \lambda \right) v = 0 \quad \text{in } U. \tag{36}$$

Take complex conjugates:

$$K\bar{v} + \left(\frac{Lw}{w} - \bar{\lambda} \right) \bar{v} = 0 \quad \text{in } U. \tag{37}$$

Next we compute

$$K(|v|^2) = K(v\bar{v}) = \bar{v}Kv + vK\bar{v} - 2 \sum_{i,j=1}^{n} a^{ij} v_{x_i} \bar{v}_{x_j} \leq \bar{v}Kv + vK\bar{v}, \tag{38}$$

since

$$\sum_{i,j=1}^{n} a^{ij} \xi_i \bar{\xi}_j = \sum_{i,j=1}^{n} a^{ij} (\operatorname{Re}(\xi_i) \operatorname{Re}(\xi_j) + \operatorname{Im}(\xi_i) \operatorname{Im}(\xi_j)) \geq 0$$

for $\xi \in \mathbb{C}^n$. Combining (36)–(38), we discover

$$K(|v|^2) \leq 2\left(\operatorname{Re}\lambda - \frac{Lw}{w}\right)|v|^2.$$

Now choose

$$w := w_1^{1-\epsilon} \tag{39}$$

for $0 < \epsilon < 1$. Then

$$Lw = \frac{(1-\epsilon)}{w_1^{\epsilon}}Lw_1 + \frac{\epsilon(1-\epsilon)}{w_1^{1+\epsilon}}\sum_{i,j=1}^{n} a^{ij}w_{1,x_i}w_{1,x_j} + \epsilon c w_1^{1-\epsilon} \geq (1-\epsilon)\lambda_1 w.$$

Consequently

$$K(|v|^2) \leq 2(\operatorname{Re}\lambda - (1-\epsilon)\lambda_1)|v|^2 \quad \text{in } U.$$

Thus if $\operatorname{Re}(\lambda) \leq (1-\epsilon)\lambda_1$, then $K(|v|^2) \leq 0$ in U. As $v = 0$ on ∂U, according to (33) and (39), we deduce from the maximum principle that $v = \frac{u}{w} = 0$ in U. Thus $u \equiv 0$ in U and so λ cannot be an eigenvalue. This conclusion obtains for each $\epsilon > 0$, and so $\operatorname{Re}\lambda \geq \lambda_1$ if λ is any complex eigenvalue.

7. Finally, let u be any (possibly complex-valued) solution of

$$\begin{cases} Lu = \lambda_1 u & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases} \tag{40}$$

Since $\operatorname{Re}(u)$ and $\operatorname{Im}(u)$ also solve (40), we may as well suppose from the outset u is real-valued. Replacing u by $-u$ if needs be, we may also suppose $u > 0$ somewhere in U. Now set

$$\chi := \sup\{\mu > 0 \mid w_1 - \mu u \geq 0 \text{ in } U\}. \tag{41}$$

Then $0 < \chi < \infty$. Write $v = w_1 - \chi u$; so that $v \geq 0$ in U and

$$\begin{cases} Lv = \lambda_1 v \geq 0 & \text{in } U \\ v = 0 & \text{on } \partial U. \end{cases}$$

Now if v is not identically zero, the strong maximum principle and Hopf's Lemma imply

$$v > 0 \text{ in } U, \quad \frac{\partial v}{\partial \nu} < 0 \text{ on } \partial U.$$

Thus

$$v - \epsilon u \geq 0 \quad \text{in } U \text{ for some } \epsilon > 0,$$

and so

$$w_1 - (\chi + \epsilon)u \geq 0 \quad \text{in } U,$$

a contradiction to (41). Hence $v \equiv 0$ in U, and so u is a multiple of w_1. □